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ABSTRACT

A Riemannian manifold (M^n, g) of quasi-constant curvature is defined. It is shown that an (M^n, g) in association with other class of manifolds gives rise, under certain conditions, to a manifold of quasi-constant curvature. Some observations on how a manifold of quasi-constant curvature accounts for a pseudo Ricci-symmetric manifold and quasi-umbilical hypersurface are made.

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1. INTRODUCTION

A Riemannian manifold $(M^n, g; n > 3)$ is said to be of *quasi-constant curvature* [3] if the (M^n, g) is conformally flat with its curvature tensor R satisfying the condition

$$\begin{aligned} R(X,Y)Z &= a[Xg(Y,Z) - Yg(X,Z)] + b[XB(Y)B(Z) - YB(X)B(Z) \\ &\quad + B(X)g(Y,Z)\mu - B(Y)g(X,Z)\mu] \end{aligned} \tag{1}$$

where X, Y, Z are vector fields, a, b are differentiable functions on (M^n, g), B is a non-zero 1-form, and μ a parallel unit vector field connected by

$$B(X) = g(\mu, X) \tag{2}$$

for every vector field X.

It is to be noted that a, b are *associated scalars*, B is the *associated* 1-form, and μ designates the *generator* of the manifold.

Manifolds of quasi-constant curvature were studied by Adati [2-5] and Wang [5]. They dealt with a manifold of quasi-constant curvature and an S-manifold, quasi-umbilical hypersurfaces, and also manifolds admitting a concircular vector field, an almost paracompact metric structure and a pseudo-subprojective space.

In this context we are tempted to define a pseudo Ricci-symmetric manifold.

Pseudo Ricci-symmetric manifold

A non-flat Riemannian manifold $(M^n, g; n > 2)$ whose Ricci tensor S of type $(0,2)$ satisfies the relations

$$\begin{aligned} (\nabla_X S)(Y,Z) &= 2A(X)S(Y,Z) + A(Y)S(X,Z) + A(Z)S(Y,X) \\ S &\neq 0 \end{aligned} \tag{3}$$

where ∇ denotes the operator of covariant derivative along the vector field X with respect to the metric g and A is a non-zero associated 1-form, is called a *pseudo Ricci-symmetric manifold* [6].

We shall denote an n-manifold of this kind by PRS_n.

It was shown [7] that a conformally flat pseudo Ricci-symmetric manifold $(M^n, g; n > 3)$ has quasi-constant curvature in the case for which (i) $a + b = 0$, (ii) B is the associated form A of PRS_n, and (iii) the generator is the unit vector field ρ with $A(X) = g(\rho, X)$ for every vector field X.

This article contains an approach, in spirit to quasi-normal spaces [9], to pseudo-subprojective manifold and manifold admitting a concircular vector field to fit in with manifold of quasi-constant curvature and focuses on pseudo Ricci-symmetric manifolds and quasi-umbilical hypersurface.

2. CONDITIONS THAT A MANIFOLD HAS QUASI-CONSTANT CURVATURE

We touch upon, in this section, certain conditions for a manifold to be an (M^n, g) of quasi-constant curvature. They are furnished through the emergence of a number of theorems.

THEOREM 1 *Let an $(M^n, g; n > 3)$ be a conformally flat pseudo Ricci-symmetric manifold with associated 1-form A and scalar curvature r and let its associated scalars a, b satisfy the condition $a + b = 0$ and the unit vector field ρ be given as its generator by $A(X) = g(\rho, X)$ for every vector field X. Then the (M^n, g) is a manifold of quasi-constant curvature.*

PROOF It is known [6] that the scalar curvature r of a conformally flat PRS_n-manifold is non-zero.

From the relation $dr(X) = 2rA(X)$ it readily follows that if r is a non-zero constant then $r = 0$, since $A(X) \neq 0$. This guarantees that for a conformally flat PRS_n, r is neither zero nor a non-zero constant. Since the vector field ρ given by $A(X) = g(\rho, X)$ is assumed to be unit, we have

$$g(\rho, \rho) = 1 \tag{4}$$

In a conformally flat PRS_n the curvature tensor R of type $(0, 4)$ has the form

$$\begin{aligned} R(X,Y,Z,W) &= \frac{r}{(n-1)(n-2)}[g(X,W)g(Y,Z) - g(Y,W)g(X,Z)] \\ &\quad -\frac{r}{(n-1)(n-2)}[T(Y)T(Z)g(X,W) - T(X)T(Z)g(Y,W) \\ &\quad + T(X)T(W)g(Y,Z) - T(Y)T(W)g(X,Z)] \end{aligned} \tag{5}$$

with

$$R(X,Y,Z,W) = g[R(X,Y)Z, W]$$

and

$$T(X) = \frac{A(X)}{\sqrt{A(\rho)}} .$$

The latter gives

$$T(X) = A(X) , \tag{6}$$

on account of $A(X) = g(\rho, X)$ and (4).

Accordingly, (5) reduces to, on using (6),

$$\begin{aligned} R(X,Y,Z,W) &= \frac{r}{(n-1)(n-2)}[g(X,W)g(Y,Z) - g(Y,W)g(X,Z)] \\ &\quad -\frac{r}{(n-1)(n-2)}[A(Y)A(Z)g(X,W) - A(X)A(Z)g(Y,W) \\ &\quad + A(X)A(W)g(Y,Z) - A(Y)A(W)g(X,Z)] \end{aligned} \tag{7}$$

Hence we find, from (7),

$$\begin{aligned} R(X,Y)Z &= \frac{r}{(n-1)(n-2)}[Xg(Y,Z) - Yg(X,Z)] \\ &\quad -\frac{r}{(n-1)(n-2)}[XA(Y)A(Z) - YA(X)A(Z) \\ &\quad + A(X)g(Y,Z)\rho - A(Y)g(X,Z)\rho] \end{aligned}$$

which has the form of (1) with

$$\frac{r}{(n-1)(n-2)} = a, \quad -\frac{r}{(n-1)(n-2)} = b; \quad A = B; \quad \rho = \mu .$$

The defining property of an (M^n, g) to have quasi-constant curvature is thus exhibited. This completes the proof.

QED

We now restrict our attention to *pseudo-subprojective manifold.*

Pseudo-subprojective manifold

Consider a Riemannian manifold (M^n, g) with positive definite metric.

If (M^n, g) is subprojective then the Christoffel symbols for the canonical coordinate system x^h with respect to g may be written

$$\left\{ {h \atop i\ j} \right\} = u_{ij}\, x^h ,$$

u_{ij} being a tensor field [1].

This means that x^h is a non-parallel torse-forming vector field.

A necessary and sufficient condition that the (M^n, g)-manifold be *subprojective* is that

$$(M^n, g) \text{ be conformal to flat space} \tag{8a}$$

$$H_{ij} = \rho g_{ij} + \nabla_i \rho \, \nabla_j \sigma \tag{8b}$$

where H_{ij} is given by

$$H_{ij} = R_{ij} - \frac{R}{2(n-1)} g_{ij}$$

and

$$\sigma = \sigma(\rho) \; .$$

Put $\nabla_i \rho = \lambda u_i$ where u_i is a unit vector field and λ is some function.
Then (8b) becomes

$$H_{ij} = \rho g_{ij} + \kappa u_i u_j \; . \tag{9}$$

We assert, in virtue of (8a), that u_i is a gradient and u^i concircular.

Also, with the aid of (9) the curvature tensor R^h_{ijk} is expressible as

$$R^h_{ijk} = -2\rho(\delta^h_i g_{kj} - \delta^h_j g_{ik}) - \kappa[(\delta^h_i u_j - \delta^h_j u_i)u_k + (g_{kj}u_i - g_{ik}u_j)u^h] \tag{10}$$

The vector field u^h corresponds to the one in the canonical coordinate system x^h. Consequently, *the subprojective manifold has quasi-constant curvature* with the generator u^h as concircular.

We now seek the case for *pseudo subprojective manifold.*

Suppose that $\rho = c$, a constant. Then (9) has the form

$$H_{ij} = c g_{ij} + \kappa u_i u_j \; . \tag{9a}$$

In view of (8a), u^i is parallel and so $\kappa = -2c$ [5].
Eq.(10) is then written

$$R^h_{ijk} = -2c(\delta^h_i g_{kj} - \delta^h_j g_{ik}) + 2c[(\delta^h_i u_j - \delta^h_j u_i)u_k + (g_{kj}u_i - g_{ik}u_j)u^h] \; .$$

An (M^n, g) for which (8a) and (9a) are satisfied is called *pseudo-subprojective manifold.*

Hence (M^n, g) is a manifold of quasi-constant curvature for which the generator is parallel.

This inherits

THEOREM 2 *A pseudo-subprojective (M^n, g)-manifold is of quasi-constant curvature.*

Next, we are able to state

THEOREM 3 *If an (M^n, g)-manifold admits a concircular vector field with $Z^*(X,Y)Z = 0$ then the manifold has quasi-constant curvature.*

PROOF We have

$$\begin{aligned} Z^*(X,Y)Z &= R(X,Y)Z + \rho^2[(Xg(Y,Z) - Yg(X,Z)] + \beta[X\eta(Y)\eta(Z) \\ &\quad -Y\eta(X)\eta(Z) + \eta(X)g(Y,Z)\xi - \eta(Y)g(X,Z)\xi] = 0 \; , \end{aligned}$$

where Z^* is a tensor field of type (1,4) on M^n.
Therefore,

$$\begin{aligned} R(X,Y)Z &= -\rho^2[Xg(Y,Z) - Yg(X,Z)] - \beta[X\eta(Y)\eta(Z) \\ &\quad -Y\eta(X)\eta(Z) + \eta(X)g(Y,Z)\xi - \eta(Y)g(X,Z)\xi] \; . \end{aligned}$$

This has the same form as (1).

We note the relations [3]

$$\begin{aligned} S'(X',Y') &= S(X,Y) + [(n-1)\rho^2 + \beta]g(X,Y) + (n-2)\beta\eta(X)\eta(Y), \\ r' &= r + (n-1)(n\rho^2 + 2\beta) \; , \end{aligned}$$

where S' and r' are respectively the Ricci tensor and scalar curvature of M^{n-1}.

Hence it follows, because $S'(X',Y') = 0, r' = 0$,

$$\begin{aligned} S(X,Y) &= -[(n-1)\rho^2 + \beta]g(X,Y) - (n-2)\beta\eta(X)\eta(Y), \\ r &= -(n-1)(n\rho^2 + 2\beta) \; . \end{aligned}$$

Let $H(X,Y) = -\frac{1}{n-2}\,[S(X,Y) - \frac{r}{2(n-1)}\,g(X,Y)]$.

This simplifies to

$$H(X,Y) = \frac{1}{2}\,\rho^2 g(X,Y) + \beta\eta(X)\eta(Y)$$

Since $\nabla_X \rho = \beta\eta(X)$ and $\nabla_X \beta = \lambda\eta(X)$, λ being some function, we have

$$\begin{aligned} \nabla_Z H(X,Y) &= \nabla_Z \left[\frac{1}{2}\,\rho^2 g(X,Y) + \beta\eta(X)\eta(Y)\right] \\ &= \rho\nabla_Z \rho g(X,Y) + \nabla_Z \beta\eta(X)\eta(Y) \\ &= \rho\beta\eta(Z)g(X,Y) + \lambda\eta(Z)\eta(X)\eta(Y) \end{aligned}$$

This leads us to

$$\nabla_Z H(X,Y) - \nabla_X H(Z,Y) = 0$$

The proof is now complete.

3. CASE FOR PSEUDO RICCI-SYMMETRIC MANIFOLDS

We shall now devote to the establishment of a theorem on an (M^n, g)-manifold of quasi-constant curvature which under certain conditions accounts for a pseudo Ricci-symmetric manifold.

THEOREM 4 *A manifold $(M^n, g; n > 3)$ of quasi-constant curvature for which the conditions* (i) $a + b = 0$, (ii) $B(X) = X \cdot \log \sqrt{r}$, (iii) $\nabla_X \mu = -X + B(X)\mu$ *(a, b, B, μ have their usual meaning) are satisfied for every vector field X is pseudo Ricci-symmetric.*

PROOF It follows from (1) that

$$\begin{aligned} R(X,Y,Z,W) &= a[g(X,Z)g(Y,W) - g(Y,Z)g(X,W)] \\ &\quad -b[B(X)B(Z)g(Y,W) - B(Y)B(Z)g(X,W) \\ &\quad +B(Y)B(W)g(X,Z) - B(X)B(W)g(Y,Z)] \end{aligned}$$

Transvecting this we have, by (i),

$$S(Y,Z) = (n-2)b[g(Y,Z) - B(Y)B(Z)] \ , \tag{11}$$

S being the non-zero Ricci tensor of type $(0,2)$.

A further transvection yields

$$r = (n-1)(n-2)b \tag{12}$$

Evidently,

$$X \cdot r = (n-1)(n-2)X \cdot b \tag{13}$$

From (ii)

$$B(X) = \frac{1}{2r}\ X \cdot r$$

Combining this with (12) and (13) we see that

$$B(X) = \frac{1}{2}\ \frac{X \cdot b}{b} \tag{14}$$

Now

$$(\nabla_X S)(Y,Z) = \nabla_X S(Y,Z) - S(\nabla_X Y, Z) - S(Y, \nabla_X Z) \tag{15}$$

Substituting from (11) and (14) in (15) gives

$$\begin{aligned} (\nabla_X S)(Y,Z) &= \nabla_X (n-2)b([g(Y,Z) - B(Y)B(Z)] - [g(\nabla_X Y, Z) \\ &\quad -B(\nabla_X Y)B(Z) - [g(Y, \nabla_X Z) - B(Y)B(\nabla_X Z)]) \\ &= \frac{X \cdot b}{b}\ S(Y,Z) - (n-2)b[B(Z)(\nabla_X B)(Y) - B(Y)(\nabla_X B)(Z)] \\ &= 2B(X)S(Y,Z) - (n-2)b[B(Z)(\nabla_X B)(Y) - B(Y)(\nabla_X B)(Z)] \end{aligned} \tag{16}$$

We note that

$$(\nabla_X g)(Y,\mu) = \nabla_X g(Y,\mu) - g(\nabla_X Y, \mu) - g(Y, \nabla_X \mu)$$

which, by (2) and (iii), takes the form

$$\begin{aligned} (\nabla_X g)(Y,\mu) &= \nabla_X g(Y,\mu) - g(\nabla_X Y,\mu) - g(Y, -X + B(X)\mu) \\ &= \nabla_X B(Y) - B(\nabla_X Y) - g(Y, -X + B(X)\mu) \\ &= (\nabla_X B)(Y) + g(X,Y) - g(Y, B(X)\mu) \end{aligned}$$

This admits

$$\begin{aligned} (\nabla_X B)(Y) &= (\nabla_X g)(Y,\mu) - g(X,Y) + g(Y, B(X)\mu) \\ &= -g(X,Y) + B(X)B(Y) \end{aligned}$$

Thus (16) can be expressed

$$\begin{aligned} (\nabla_X S)(Y,Z) &= 2B(X)S(Y,Z) + (n-2)b[g(X,Y)B(Z) + g(X,Z)B(Y) \\ &\quad -2B(X)B(Y)B(Z)] \\ &= 2B(X)S(Y,Z) + B(Z)[S(X,Y) + (n-2)bB(X)B(Y)] \\ &\quad +B(Y)[S(X,Z) + (n-2)bB(X)B(Z)] \\ &\quad -2(n-2)bB(X)B(Y)B(Z) \\ &= 2B(X)S(Y,Z) + B(Y)S(X,Z) + B(Z)S(Y,X) \end{aligned} \tag{17}$$

Therefore, with the fact of the identicality of (17) and (3) we may look upon an $(M^n, g; n > 3)$ of quasi-constant curvature having certain conditions as pseudo Ricci-symmetric.

4. QUASI-UMBILICAL HYPERSURFACE

We consider a hypersurface V_n in a Riemannian V_{n+1}.

A V_n is said to be a *quasi-umbilical hypersurface* if there exist two functions u, v on the V_n and a unit vector field λ_i such that

$$h_{ij} = ug_{ij} + v\lambda_i\lambda_j \tag{18}$$

where h_{ij} is the second fundamental form of V_n.

Let us discuss two cases:

Case I Let $u = 0$ identically. In this case V_n is a *cylindrical hypersurface*.

Case II If $v = 0$ identically then V_n is called a *totally umbilical hypersurface*.

We now define a tensor field T_{ij} on the hypersurface V_n by

$$T_{ij} = L_{hk}B_i^h B_j^k$$

where

$$L_{hk} = -\frac{1}{2}\,(u^2 + c)g_{hk} - v(\nabla_h u)(\nabla_k u)$$

for some constant c, and B_i^h are n linearly independent vectors tangent to V_n connected by

$$g_{ij} = g_{\alpha\beta}B_i^\alpha B_j^\beta\ ,$$

g_{ij} and $g_{\alpha\beta}$ being the first fundamental tensor of V_n and the fundamental (metric) tensor of V_{n+1} respectively.

If a quasi-umbilical hypersurface V_n having (18) satisfies [8]

$$T_{ij} = -\frac{1}{2}\,(u^2 + c)g_{ij} + uh_{ij}$$

for some constant c then V_n is termed as a *quasi-umbilical hypersurface of type c*.

It was deduced [10]: *If V_n is a conformally flat hypersurface of a conformally flat V_{n+1}, then V_n is a quasi-umbilical hypersurface.*

In order that a V_{n+1} of quasi-constant curvature has to be a conformally flat manifold we may, concerned with this result, compute

THEOREM 5 *If V_n is a hypersurface of V_{n+1} of quasi-constant curvature then V_n is a quasi-umbilical hypersurface.*

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