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FOR THE VECTOR  $k$ -CONSTRAINED KP  
HIERARCHY**

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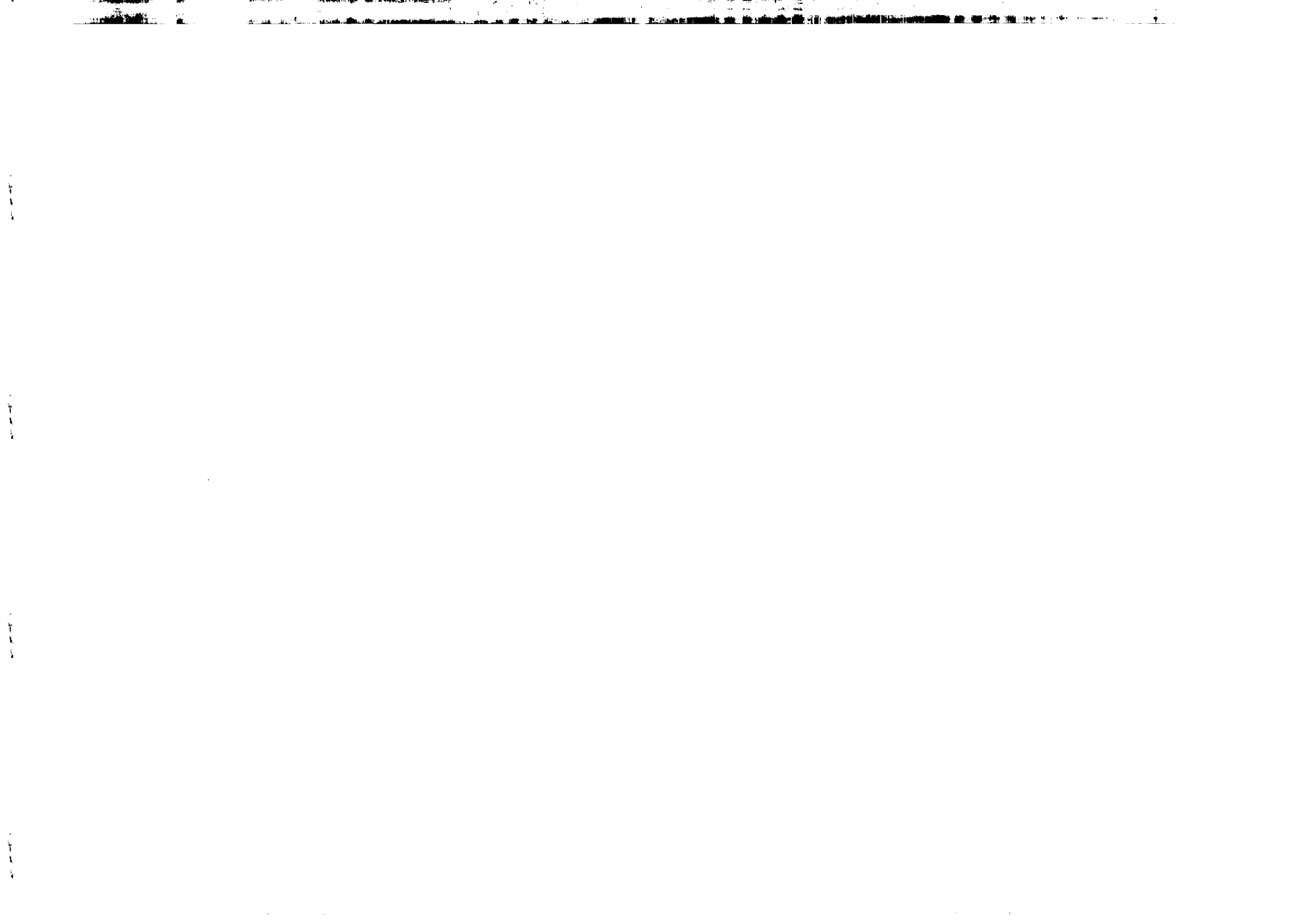


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**WRONSKIAN TYPE SOLUTIONS  
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**ABSTRACT**

Motivated by a relation of the 1-constrained Kadomtsev-Petviashvili (KP) hierarchy with the 2 component KP hierarchy, the tau-functions of the vector  $k$ -constrained KP hierarchy are constructed by using an analogue of the Baker-Akhiezer ( $m+1$ ) point function. These tau-functions are expressed in terms of Wronskian type determinants.

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**1. Introduction.**

In recent years, a series of papers have been devoted to the study of a class of integrable systems which are constrained from the Kadomtsev-Petviashvili (KP) hierarchy [1-6], as in Ref[5] we call this class of integrable systems the *vector  $k$ -constrained KP hierarchy*. For arbitrary given positive integers  $k, m$ , the vector  $k$ -constrained KP hierarchy can be expressed as [3-5]:

$$L_{t_n}^k = \{B_n, L^k\}, \quad (1.1a)$$

$$q_{i,t_n} = B_n q_i, \quad (1.1b)$$

$$r_{i,t_n} = -B_n^* r_i, \quad (1.1c)$$

$$i = 1, 2, \dots, m; n \geq 2,$$

where the micro-differential operator  $L$  is defined as

$$L = \partial + u_2 \partial^{-1} + u_3 \partial^{-2} + \dots, \quad (1.2)$$

which satisfies the following condition

$$L^k = B_k + \sum_{i=1}^m q_i \partial^{-1} r_i, \quad (1.3)$$

$u_i$ 's,  $q_i$ 's and  $r_i$ 's are functions of the variable  $t = (t_1, t_2, \dots)$ ,  $\partial = \frac{\partial}{\partial x}$  with  $x = t_1$ ,  $B_n$  is the differential part of the micro-differential operator  $L^n$ ,  $B_n^*$  is the operator adjoint to  $B_n$ , and  $\partial^{-1} r_i$  is defined as

$$\partial^{-1} r_i = r_i \partial^{-1} - r_{i,x} \partial^{-2} + r_{i,xx} \partial^{-3} - \dots. \quad (1.4)$$

The hierarchy of equations in (1.1) can be represented in terms of the dynamical variables  $u_2, u_3, \dots, u_k$  and  $q_i, r_i (i = 1, 2, \dots, m)$ . When  $m = 1$ , we call the hierarchy of equations in (1.1) the  *$k$ -constrained KP hierarchy*. These integrable systems are proved to possess the Lax pair representations, the bi-Hamiltonian structures and the bilinear representations [3-5, 7], and they are also equivalent to some integrable systems which are

closely related to the study of  $W$ -algebras, multi-matrix models[8-11] and topological field theory[12].

In Ref[5] we studied the solutions of the vector  $k$ -constrained KP hierarchy (1.1) by employing the bilinear method, we showed how to obtain their rational and soliton-like solutions starting from the solutions of the KP hierarchy, these rational and soliton-like solutions can be expressed by vertex operators. In Ref[13], we studied the Wronskian structure of the solutions of the  $k$ -constrained KP hierarchy, we proved that the Yajima-Oikawa equation which is the first flow in the 2-constrained KP hierarchy has solutions in generalized double Wronskian form, then based on this and some other facts we conjectured that the general  $k$ -constrained KP hierarchy also possess solutions in generalized double Wronskian form, and we also conjectured the form of these solutions.

The purpose of the present paper is to construct the Wronskian type solutions for the vector  $k$ -constrained KP hierarchy (1.1), and by the way to prove our conjecture given in Ref[13]. We shall employ the method presented in Ref.[14], where soliton-like solutions for some integrable systems were constructed, these integrable systems are equivalent to the first flows of the vector  $k$ -constrained KP hierarchy (1.1) with  $k = 1, 2, 3, 4$ , this method arises from the algebraic-geometric method in soliton theory[15,16], it starts from the construction of an analogue of the Baker-Akhiezer function for the KP hierarchy(the Baker-Akhiezer 1-point function) by solving certain algebraic linear system, and then imposes on the constructed function some self-consistency conditions to obtain the solutions for the relevant integrable systems. To construct the Wronskian type solutions for the vector  $k$ -constrained KP hierarchy (1.1), we shall start from the construction of an analogue of the Baker Akhiezer  $m+1$ -point function instead of starting from the construction of the analogue of Baker-Akhiezer 1-point function, the advantage of our construction lies in the fact that we can obtain solutions of the whole hierarchy (1.1) in a straightforward way and can express these solutions in a simple form by using some Wronskian type determinants. The motivation of our construction comes from the relation of the 1-constrained KP hierarchy(i.e. the Ablowitz-Kaup-Newell-Segur(AKNS) hierarchy, see Ref [3]) with the 2-component KP hierarchy, this relation enables us to

construct the solutions of the 1-constrained KP hierarchy from the solutions of the 2-component KP hierarchy in a much more straightforward way than to construct the solutions of the 1-constrained KP hierarchy by imposing some constraints on solutions of the KP hierarchy, as was done in Ref[5, 14].

In Section 2 we explain our motivation by the construction of the double Wronskian solutions for the first flow of the 1-constrained KP hierarchy, in Section 3 we construct the solutions for the vector  $k$ -constrained KP hierarchy (1.1), in Section 4 we construct the tau-functions of the hierarchy (1.1) and show that they can be expressed in Wronskian type determinants. in Section 5 we give some concluding remarks.

## 2. Double Wronskian solutions for the first flow of the 1-constrained KP hierarchy as a motivation.

In this section, we construct the double Wronskian solutions of the first flow of the 1-constrained KP hierarchy by using its relation with the 2-component KP hierarchy, and thus to give a hint to the construction of the Wronskian type solutions for the general vector  $k$ -constrained KP hierarchy. For the convenience of our further construction of Wronskian type solutions, we will state this well known relation[17,18] in the language of the method given in Ref[14].

Let us assume that the functions  $\psi_1^{(i)}(x, y, \lambda)$ ,  $\psi_2^{(i)}(x, y, \zeta)$  have the following forms:

$$\psi_1^{(i)}(x, y, \lambda) = (\delta_1^i + \sum_{j=1}^M u_{1j}^{(i)}(x, y)\lambda^{-j})e^{x_1\lambda + y_1\lambda^2}, \quad (2.1a)$$

$$\psi_2^{(i)}(x, y, \zeta) = (\delta_2^i + \sum_{j=1}^N u_{2j}^{(i)}(x, y)\zeta^{-j})e^{x_2\zeta + y_2\zeta^2}, \quad (2.1b)$$

where  $x = (x_1, x_2)$ ,  $y = (y_1, y_2)$ ,  $\delta_j^i$  is the Kronecker-delta function, and  $\lambda, \zeta$  are two parameters. These functions give rise an analogue of the Baker-Akhiezer 2-point function[15,16]. To specify the coefficients  $u_{ij}^{(i)}$ 's, let us impose the following linear conditions on the functions  $\psi_j^{(i)}$

$$a_l\psi_1^{(i)}(x, y, \lambda_l) + b_l\psi_2^{(i)}(x, y, \zeta_l) = 0, \quad (2.2)$$

$$l = 1, 2, \dots, M + N; \quad i = 1, 2$$

where  $a_l, b_l$  are some given constants. Let us define

$$U^{(i)} = (u_{11}^{(i)}, \dots, u_{1M}^{(i)}, u_{21}^{(i)}, \dots, u_{2N}^{(i)})^T, \quad (2.3)$$

and rewrite the linear conditions in (2.2) as

$$AU^{(i)} = F^{(i)}, \quad i = 1, 2, \quad (2.4)$$

where  $A$  and  $F^{(i)}$ 's are  $(M+N) \times (M+N)$  and  $(M+N) \times 1$  matrices respectively. We assume that the matrix  $A$  is not identically degenerate, since the determinant of  $A$  is a smooth function of  $x, y$ , we can assume in what follows that  $A$  is non-degenerate in some open domain, and  $(x, y)$  belongs to this domain.

LEMMA 2.1. Denote  $u_{11}^{(i)} = r_{i1}, u_{21}^{(i)} = r_{i2}$ , then the functions  $\psi_1^{(i)}, \psi_2^{(i)}$  satisfy the following system of linear equations:

$$\psi_{j,x_2}^{(1)} = r_{12}\psi_j^{(2)}, \quad (2.5a)$$

$$\psi_{j,x_1}^{(2)} = r_{21}\psi_j^{(1)}, \quad (2.5b)$$

$$\psi_{j,y_1}^{(1)} = \psi_{j,x_1x_1}^{(1)} - 2r_{11,x_1}\psi_j^{(1)}, \quad (2.6a)$$

$$\psi_{j,y_2}^{(1)} = \psi_{j,x_2x_2}^{(1)} - 2r_{12,x_2}\psi_j^{(2)}, \quad (2.6b)$$

$$\psi_{j,y_1}^{(2)} = \psi_{j,x_1x_1}^{(2)} - 2r_{21,x_1}\psi_j^{(1)}, \quad (2.6c)$$

$$\psi_{j,y_2}^{(2)} = \psi_{j,x_2x_2}^{(2)} - 2r_{22,x_2}\psi_j^{(2)}, \quad (2.6d)$$

PROOF: From the form of  $\psi_j^{(i)}$  we see that  $\phi_j := \psi_{j,x_2}^{(1)} - r_{12}\psi_j^{(2)}$  has the form  $(\sum_{j=1}^M v_{1j}(x, y)\lambda^{-j})e^{r_1\lambda + r_2\lambda^2}$  for  $j = 1$  and has the form  $(\sum_{j=1}^N v_{2j}(x, y)\lambda^{-j})e^{y_1\zeta + y_2\zeta^2}$  for  $j = 2$ , and from (2.2) we see that  $\phi_j$ 's satisfy

$$a_l\phi_1(x, y, \lambda_l) + b_l\phi_2(x, y, \zeta_l) = 0, \quad 1 \leq l \leq M+N. \quad (2.7)$$

If we define

$$V = (v_{11}, \dots, v_{1M}, v_{21}, \dots, v_{2N})^T, \quad (2.8)$$

then conditions in (2.7) can be written as

$$AV = 0, \quad (2.9)$$

thus from the non-degeneracy of  $A$  we obtain that  $\phi_j = 0$ , which proved the identity (2.5a), the identities (2.5b) and (2.6a)-(2.6d) can be proved similarly. The lemma was proved.

From the equalities  $\psi_{2,x_1}^{(2)} = r_{21}\psi_2^{(1)}, \psi_{1,x_2}^{(1)} = r_{12}\psi_1^{(2)}$  and the form of  $\psi_j^{(i)}$ 's we can easily see that  $r_{22,x_1} = r_{11,x_2} = r_{12}r_{21}$ . This fact and the compatibility conditions of the linear equations in (2.5) and (2.6) lead to

THEOREM 2.1. The functions  $r_{ij}$  satisfy the following system of equations

$$r_{12,y_1} = r_{12,x_1x_1} - 2r_{12}r_{11,x_1}, \quad (2.10a)$$

$$r_{12,y_2} = -r_{12,x_2x_2} + 2r_{12}r_{22,x_2}, \quad (2.10b)$$

$$r_{21,y_1} = -r_{21,x_1x_1} + 2r_{21}r_{11,x_1}, \quad (2.10c)$$

$$r_{21,y_2} = r_{21,x_2x_2} - 2r_{21}r_{22,x_2}, \quad (2.10d)$$

$$r_{11,x_2} = r_{12}r_{21}, \quad (2.10e)$$

$$r_{22,x_1} = r_{12}r_{21}. \quad (2.10f)$$

The equations in (2.10) are the simplest non-trivial flows of the 2-component KP hierarchy[17,18]. If we introduce the following new variables:

$$\tilde{t}_1 = x_1 - x_2, \quad \tilde{t}_1 = x_1 + x_2, \quad \tilde{t}_2 = y_1 - y_2, \quad \tilde{t}_2 = y_1 + y_2, \quad (2.11)$$

then equations in (2.10) lead to the Davey-Stewartson system[17], and if we require that  $r_{ij}$ 's are independent of  $\tilde{t}_i (i = 1, 2)$ , then the equations in (2.10) lead to

$$\partial_{\tilde{t}_2} r_{12} = \partial_{\tilde{t}_1}^2 r_{12} + 2(r_{12}r_{21})r_{12}, \quad (2.12a)$$

$$\partial_{\tilde{t}_2} r_{21} = -\partial_{\tilde{t}_1}^2 r_{21} - 2(r_{12}r_{21})r_{21}, \quad (2.12b)$$

which is the first non-trivial flow in the 1-constrained KP hierarchy[3].

LEMMA 2.2. Let us assume  $\lambda_l = \zeta_l$  for  $1 \leq l \leq M + N$  in (2.2), then  $r_{ij}$ 's constructed from (2.1)-(2.4) are independent of  $\hat{t}_1$  and  $\hat{t}_2$ .

PROOF: Since  $\lambda_l = \zeta_l$ , we can eliminate the factor  $e^{\frac{1}{2}\hat{t}_1\lambda_l + \frac{1}{2}\hat{t}_2\lambda_l^2}$  from the linear systems in (2.2), and obtain linear systems which do not depend on  $\hat{t}_1$  and  $\hat{t}_2$ . Thus the  $r_{ij}$ 's which are constructed from (2.1)-(2.4) are independent of  $\hat{t}_1$  and  $\hat{t}_2$ . We proved the lemma.

Let us define now the following two vectors:

$$\varphi = (a_1 \lambda_1^{-M} e^{\frac{1}{2}\hat{t}_1\lambda_1 + \frac{1}{2}\hat{t}_2\lambda_1^2}, \dots, a_{M+N} \lambda_{M+N}^{-M} e^{\frac{1}{2}\hat{t}_1\lambda_{M+N} + \frac{1}{2}\hat{t}_2\lambda_{M+N}^2})^T, \quad (2.13a)$$

$$\phi = (b_1 \lambda_1^{-N} e^{-\frac{1}{2}\hat{t}_1\lambda_1 - \frac{1}{2}\hat{t}_2\lambda_1^2}, \dots, b_{M+N} \lambda_{M+N}^{-N} e^{-\frac{1}{2}\hat{t}_1\lambda_{M+N} - \frac{1}{2}\hat{t}_2\lambda_{M+N}^2})^T, \quad (2.13b)$$

and we also denote  $s = \frac{1}{2}\hat{t}_1$ , then from Lemma 2.2 and the linear conditions in (2.2) we have

THEOREM 2.2. The system of equations in (2.12) has the following double Wronskian solution[19,20]:

$$r_{12} = \frac{\rho}{\tau}, \quad r_{21} = \frac{\sigma}{\tau}, \quad (2.14)$$

where

$$\rho = -\det(\varphi, \partial_s \varphi, \dots, \partial_s^M \varphi, \phi, \partial_s \phi, \dots, \partial_s^{N-2} \phi), \quad (2.15a)$$

$$\sigma = -\det(\varphi, \partial_s \varphi, \dots, \partial_s^{M-2} \varphi, \phi, \partial_s \phi, \dots, \partial_s^N \phi), \quad (2.15b)$$

$$\tau = \det(\varphi, \partial_s \varphi, \dots, \partial_s^{M-1} \varphi, \phi, \partial_s \phi, \dots, \partial_s^{N-1} \phi), \quad (2.15c)$$

Remark 2.1 Under the condition  $\lambda_l = \zeta_l (1 \leq l \leq M + N)$ , we know that the functions  $\psi_j^{(i)}(x, y, \lambda) e^{-x_2 \lambda - y_2 \lambda^2}$ ,  $i, j = 1, 2$  do not depend on  $\hat{t}_1$  and  $\hat{t}_2$ , so we can denote them as  $\varphi_j^{(i)}(t, \lambda)$ , where  $t = (t_1, t_2)$ , then instead of condition (2.2) we have

$$a_l \varphi_1^{(i)}(t, \lambda_l) + b_l \varphi_2^{(i)}(t, \lambda_l) = 0, \quad i = 1, 2, \quad 1 \leq l \leq M + N. \quad (2.16)$$

We also define

$$\hat{\tau}(t) = c(t)\tau(t), \quad \hat{\rho}(t) = c(t)\rho(t), \quad \hat{\sigma}(t) = c(t)\sigma(t), \quad (2.17)$$

where  $c(t) = e^{\frac{1}{2}\sum_{l=1}^{M+N}\lambda_l + \frac{1}{2}\sum_{l=1}^{M+N}\lambda_l^2}$ . Then the functions  $\varphi_j^{(i)}(t, \lambda)$ ,  $\hat{\tau}(t)$ ,  $\hat{\rho}(t)$ ,  $\hat{\sigma}(t)$  can be written as functions of  $e^{t_1\lambda_l + t_2\lambda_l^2}$ ,  $1 \leq l \leq M + N$ . We now replace the term  $e^{t_1\lambda_l + t_2\lambda_l^2}$  in these functions by the term  $e^{\sum_{n=1}^{\infty} t_n \lambda^n}$  and denote  $t = (t_1, t_2, \dots, t_n, \dots)$ , then we can express  $\varphi_1^{(i)}$  in the following form(see theorem 4.1):

$$\varphi_1^{(1)}(t, \lambda) = \frac{\hat{\tau}(t - \epsilon(\lambda))}{\hat{\tau}(t)} e^{\sum_{n=1}^{\infty} t_n \lambda^n}, \quad \varphi_1^{(2)}(t, \lambda) = \frac{\hat{\sigma}(t - \epsilon(\lambda))}{\lambda \hat{\tau}(t)} e^{\sum_{n=1}^{\infty} t_n \lambda^n}, \quad (2.18)$$

where  $\epsilon(\lambda) = (\frac{1}{\lambda}, \frac{1}{2\lambda^2}, \frac{1}{3\lambda^3}, \dots)$ . The above formulas show us that  $\varphi_1^{(i)}(t, \lambda)$ ,  $i = 1, 2$  are the wave functions of the 1-constrained KP hierarchy defined in Ref[5], this fact gives us a motivation for our further construction of the Wronskian type solutions of the vector  $k$ -constrained KP hierarchy.

### 3. Solutions for the vector $k$ -constrained KP hierarchy (1.1).

Let's define the following functions:

$$\varphi_1^{(i)}(t, \lambda) = (\delta_1^i + \sum_{\nu=1}^{M_j} u_{1\nu}^{(i)}(t) \lambda^{-\nu}) e^{\xi(t, \lambda)}, \quad (3.1a)$$

$$\varphi_j^{(i)}(t, \lambda) = \delta_j^i + \sum_{\nu=1}^{M_j} u_{j\nu}^{(i)}(t) \lambda^{-k\nu}, \quad (3.1b)$$

$$2 \leq j \leq m+1, \quad 1 \leq i \leq m+1$$

where  $M_j$ ,  $1 \leq j \leq m+1$  are some positive integers,  $\lambda$  is a parameter,  $t = (t_1, t_2, \dots, t_n, \dots)$ , and

$$\xi(t, \lambda) = \sum_{l=1}^{\infty} t_l \lambda^l. \quad (3.2)$$

We specify the coefficients  $u_{j\nu}^{(i)}$ 's by imposing the following linear conditions on  $\varphi_j^{(i)}$ :

$$\sum_{\nu=1}^{m+1} a_{l\nu} \varphi_\nu^{(i)}(t, \lambda_l) = 0, \quad 1 \leq l \leq M, \quad 1 \leq i \leq m+1, \quad (3.3)$$

where  $M = \sum_{\mu=1}^{m+1} M_\mu$ , and  $a_{l\nu}$ 's are some given constants. If we denote

$$U^{(i)} = (u_{11}^{(i)}, \dots, u_{1M_1}^{(i)}, u_{21}^{(i)}, \dots, u_{2M_2}^{(i)}, \dots, u_{(m+1)1}^{(i)}, \dots, u_{(m+1)M_{(m+1)}}^{(i)})^T, \quad (3.4)$$

then the linear conditions in (3.3) can be written as

$$AU^{(i)} = F^{(i)}, \quad (3.5)$$

where  $A$  and  $F^{(i)}$ 's are  $M \times M$  and  $M \times 1$  matrices respectively. As in Section 2 we assume in what follows that the matrix  $A$  is non-degenerate in certain open domain, and we assume that  $t$  belongs to this domain, then the linear systems of equations in (3.5) uniquely determine the functions  $\varphi_j^{(i)}$ 's, which give an analogue of the Baker-Akhiezer  $(m+1)$ -point function(see remark 2.1).

We define now the following operators:

$$P = 1 + \sum_{\nu=1}^{M_1} u_{1\nu}^{(1)}(t) \partial^{-\nu}, \quad (3.6a)$$

$$L = P \partial P^{-1}, \quad (3.6b)$$

where  $\partial = \frac{\partial}{\partial t}$ . Let's also denote

$$q_i(t) = u_{(i+1)1}^{(1)}, \quad r_i(t) = u_{1i}^{(i+1)}, \quad 1 \leq i \leq m. \quad (3.7)$$

Then we have

**LEMMA 3.1.** *The functions  $\varphi_j^{(i)}$ 's defined by (3.1)-(3.5) satisfy the following system of linear equations:*

$$B_k \varphi_j^{(1)} + \sum_{\mu=1}^m q_\mu \varphi_j^{(\mu+1)} = \lambda^k \varphi_j^{(1)}, \quad (3.8a)$$

$$\varphi_{j,t_1}^{(\nu+1)} = r_\nu \varphi_j^{(1)}, \quad (3.8b)$$

$$\varphi_{j,t_n}^{(1)} = B_n \varphi_j^{(1)}, \quad (3.8c)$$

$$\varphi_{j,t_n}^{(\nu+1)} = A_n^{(\nu)} \varphi_j^{(1)}, \quad (3.8d)$$

$$1 \leq \nu \leq m, \quad n \geq 2,$$

where  $B_n = (L^n)_+$  is the differential part of the micro-differential operator  $L^n$ .  $A_n^{(\nu)}$  is a  $(n-1)$ -th order differential operator which has the form

$$A_n^{(\nu)} = \sum_{l=0}^{n-1} f_{nl}^{(\nu)} \partial^l, \quad (3.9)$$

- 9 -

and is uniquely defined by the following relations:

$$\partial A_n^{(\nu)} = r_\nu B_n - (B_n^* r_\nu), \quad n \geq 2, \quad (3.10)$$

here  $B_n^*$  is the operator adjoints to  $B_n$ .

PROOF: Let's denote

$$\Phi_j = B_k \varphi_j^{(1)}(t, \lambda) - \lambda^k \varphi_j^{(1)}(t, \lambda) + \sum_{\mu=1}^m q_\mu \varphi_j^{(\mu+1)}(t, \lambda). \quad (3.11)$$

From the definition of  $\varphi_1^{(1)}(t, \lambda)$  and  $P$  we have  $\varphi_1^{(1)} = P e^{\xi(t, \lambda)}$ , so from (3.6b) we obtain

$$L^k \varphi_1^{(1)} = \lambda^k \varphi_1^{(1)}. \quad (3.12)$$

By using (3.12) and (3.1) we see that  $\Phi_1$  has the form

$$\left( \sum_{\nu=1}^{M_1} w_{1\nu}(t) \lambda^{-\nu} \right) e^{\xi(t, \lambda)}. \quad (3.13)$$

On the other hand, from (3.1) we see that  $\Phi_j$  for  $2 \leq j \leq m+1$  has the form

$$\sum_{\nu=1}^{M_j} w_{j\nu} \lambda^{-k\nu}, \quad (3.14)$$

The linear conditions in (3.3) lead to

$$\sum_{\nu=1}^{m+1} a_{l\nu} \Phi_\nu(t, \lambda_l) = 0, \quad 1 \leq l \leq M \quad (3.15)$$

this turns out to be a homogenous linear system for the unknowns  $w_{l\nu}$ 's due to (3.13) and (3.14), from the non-degeneracy of the coefficient matrix  $A$ , we obtain that  $\Phi_j = 0$ .

In a similar way, we can show that the equalities (3.8b), (3.8c) hold, and we can show that there exist operators  $A_n^{(\nu)}$ 's of the form (3.9) such that (3.8d) holds. To specify the operators  $A_n^{(\nu)}$ 's, we use the compatibility conditions of (3.8b) and (3.8d), which lead to

$$\partial A_n^{(\nu)} = r_\nu B_n + r_{\nu, t_n}. \quad (3.16)$$

- 10 -

We denote  $B_n$  by  $\sum_{\mu=0}^n b_\mu \partial^\mu$ , then substitute (3.9) into the equality (3.16) we obtain

$$f_{n(n-1)}^{(\nu)} = r_\nu b_n, \quad f_{nl}^{(\nu)} = r_\nu b_{l+1} - \partial_l f_{n(l+1)}^{(\nu)}, \quad 0 \leq l \leq n-2, \quad (3.17a)$$

$$r_{\nu, t_n} = -B_n^* r_\nu. \quad (3.17b)$$

thus we proved the lemma.

Relations in (3.8) coincide with the Lax representation of the vector  $k$ -constrained KP hierarchy[5], from which we have the following theorem:

**THEOREM 3.1.** *The operator  $L$  and the functions  $q_i(t)$ ,  $r_i(t)$ ,  $1 \leq i \leq m$  which are constructed in this section from (3.1)-(3.7) gave a solution of the vector  $k$ -constrained KP hierarchy (1.1).*

**PROOF:** From (3.8a)-(3.8c) and (3.12) we see that  $L$  satisfies (1.1a) and (1.3), and (3.17b) shows that  $r_i(t)$  satisfies (1.1c). To prove that  $q_i$ 's satisfy equations in (1.1b), let us take the derivative of both sides of (3.8a) with respect to  $t_n$ , by using (3.8c) we have

$$\partial_{t_n}(B_k \varphi_j^{(1)}) + \sum_{\mu=1}^m q_{\mu, t_n} \varphi_j^{(\mu+1)} + \sum_{\mu=1}^m q_\mu \varphi_{j, t_n}^{(\mu+1)} = \lambda^k B_n \varphi_j^{(1)}, \quad (3.18)$$

we assume that  $2 \leq j \leq m+1$ , then by using (3.1b) and equating the coefficients of  $\lambda^0$  in both sides of (3.18) we obtain  $q_{j-1, t_n} = B_n q_{j-1}$ ,  $2 \leq j \leq m+1$ . The Theorem is proved.

In the next section, we show that the solutions of the vector  $k$ -constrained KP hierarchy constructed in this section can be expressed in terms of tau-functions which are some Wronskian type determinants.

#### Section 4. Wronskian type tau-functions of the vector $k$ -constrained KP hierarchy (1.1).

Let's keep in mind the notations of Section 3, and define

$$\phi_1 = (a_{11} \lambda_1^{-M_1} e^{\frac{1}{2} \xi(t, \lambda_1)}, a_{21} \lambda_2^{-M_1} e^{\frac{1}{2} \xi(t, \lambda_2)}, \dots, a_{M_1} \lambda_M^{-M_1} e^{\frac{1}{2} \xi(t, \lambda_M)})^T, \quad (4.1a)$$

$$\phi_j = (a_{1j} \lambda_1^{-k M_j} e^{-\frac{1}{2} \xi(t, \lambda_1)}, a_{2j} \lambda_2^{-k M_j} e^{-\frac{1}{2} \xi(t, \lambda_2)}, \dots, a_{M_j} \lambda_M^{-k M_j} e^{-\frac{1}{2} \xi(t, \lambda_M)})^T, \quad (4.1b)$$

$$2 \leq j \leq m+1$$

We denote  $s = \frac{t}{2}$  as in section 2, and define the following two matrices

$$R = (\phi_1, \partial_s \phi_1, \dots, \partial_s^{M_1} \phi_1, \phi_2, \partial_s^k \phi_2, \dots, \partial_s^{k(M_2-1)} \phi_2, \dots, \phi_{m+1}, \partial_s^k \phi_{m+1}, \dots, \partial_s^{k(M_{m+1}-1)} \phi_{m+1}), \quad (4.2a)$$

$$Q = (\phi_1, \partial_s \phi_1, \dots, \partial_s^{M_1-2} \phi_1, \phi_2, \partial_s^k \phi_2, \dots, \partial_s^{k(M_2-1)} \phi_2, \dots, \phi_{m+1}, \partial_s^k \phi_{m+1}, \dots, \partial_s^{k(M_{m+1}-1)} \phi_{m+1}). \quad (4.2b)$$

We also define

$$\tau(t) = e^{\frac{1}{2} \sum_{i=1}^M \xi(t, \lambda_i)} \det(R(n_0+1)), \quad (4.3a)$$

$$\rho_j(t) = (-1)^{n_j - M_j + k(M_j+1)} e^{\frac{1}{2} \sum_{i=1}^M \xi(t, \lambda_i)} \det(R(n_j+1)), \quad (4.3b)$$

$$\sigma_j(t) = (-1)^{n_j - M_j + k M_j + 1} e^{\frac{1}{2} \sum_{i=1}^M \xi(t, \lambda_i)} \det(Q(n_j-1, \partial_s^{k M_j+1} \phi_{j+1})), \quad (4.3c)$$

$$1 \leq j \leq m, \quad n_i = \sum_{l=1}^{i+1} M_l, \quad 0 \leq i \leq m,$$

where  $R(n_j+1)$  is a matrix obtained from the matrix  $R$  by removing its  $(n_j+1)$ -th column, and  $Q(n_j-1, \partial_s^{k M_j+1} \phi_{j+1})$  is a matrix obtained from  $Q$  by adding to the matrix  $Q$  the column  $\partial_s^{k M_j+1} \phi_{j+1}$  behind its  $(n_j-1)$ -th column  $\partial_s^{k(M_j+1)} \phi_{j+1}$ .

**THEOREM 4.1.** *The functions  $q_i(t)$ ,  $r_i(t)$  for  $i = 1, 2, \dots, m$  and  $\varphi_1^{(j)}$  for  $j = 1, 2, \dots, m+1$  have the following expression:*

$$q_i(t) = \frac{\rho_i(t)}{\tau(t)}, \quad r_i(t) = \frac{\sigma_i(t)}{\tau(t)}, \quad (4.4)$$

$$\varphi_1^{(1)}(t, \lambda) = \frac{\tau(t - \epsilon(\lambda))}{\tau(t)} e^{\xi(t, \lambda)}, \quad (4.5a)$$

$$\varphi_1^{(j)}(t, \lambda) = \frac{\sigma_{j-1}(t - \epsilon(\lambda))}{\lambda \tau(t)} e^{\xi(t, \lambda)}, \quad (4.5b)$$

$$1 \leq i \leq m, \quad 2 \leq j \leq m+1$$

where  $\epsilon(\lambda) = (\frac{1}{\lambda}, \frac{1}{2\lambda^2}, \frac{1}{3\lambda^3}, \dots)$ .

**PROOF:** The relations in (4.4) are evident from (3.1)-(3.7) and the definitions given in (4.1)-(4.3). The relations in (4.5) can be proved by using the fact that[18]

$$\begin{aligned} & e^{\frac{1}{2} \xi(t, \lambda_i)} \partial_s^j (a_{11} \lambda_1^{-M_1} e^{\frac{1}{2} \xi(t, \lambda_1)} - \lambda^{-1} e^{\frac{1}{2} \xi(t, \lambda_i)} \partial_s^{j+1} (a_{11} \lambda_1^{-M_1} e^{\frac{1}{2} \xi(t, \lambda_1)})) \\ &= e^{\frac{1}{2} \xi(t - \epsilon(\lambda), \lambda_i)} \partial_s^j (a_{11} \lambda_1^{-M_1} e^{\frac{1}{2} \xi(t - \epsilon(\lambda), \lambda_1)}), \end{aligned} \quad (4.6)$$



the theorem is proved.

From (4.4),(4.5) and Ref[5] we see that  $\tau(t)$ ,  $\rho_i(t)$ ,  $\sigma_i(t)$ ,  $i = 1, 2, \dots, m$  satisfy the bilinear equations of the vector  $k$ -constrained KP hierarchy given in Ref[5], thus they are the tau-functions of the vector  $k$ -constrained KP hierarchy. From (4.5) we also see that the operator  $P$  defined in (3.6a) can be expressed by the function  $\tau(t)$  as follows:

$$P = \sum_{\nu=0}^{M_1} \frac{p_\nu(-\tilde{\partial})\tau(t)}{\tau(t)} \partial^{-\nu}, \quad (4.7)$$

where  $p_\nu(t)$ 's are the Schur polynomials defined by

$$e^{\xi(t,\lambda)} = \sum_{\nu=0}^{\infty} p_\nu(t)\lambda^\nu, \quad (4.8)$$

and  $\tilde{\partial} = (\partial_{t_1}, \frac{1}{2}\partial_{t_2}, \frac{1}{3}\partial_{t_3}, \dots)$ . So the operator  $L$  is also determined by the function  $\tau(t)$ , this fact together with (4.4) implies that the tau-functions  $\tau(t)$ ,  $\rho_i(t)$ ,  $\sigma_i(t)$ ,  $i = 1, 2, \dots, m$  characterize a solution of the vector  $k$ -constrained KP hierarchy (1.1).

When  $m = 1$ , the tau-functions have the following form:

$$\tau(t) = c_1(t)\det(\phi_1, \partial_s \phi_1, \dots, \partial_s^{M_1-1} \phi_1, \phi_2, \partial_s^k \phi_2, \dots, \partial_s^{k(M_2-1)} \phi_2), \quad (4.9a)$$

$$\rho_1(t) = c_2(t)\det(\phi_1, \partial_s \phi_1, \dots, \partial_s^{M_1} \phi_1, \phi_2, \partial_s^k \phi_2, \dots, \partial_s^{k(M_2-2)} \phi_2), \quad (4.9b)$$

$$\sigma_1(t) = c_3(t)\det(\phi_1, \partial_s \phi_1, \dots, \partial_s^{M_1-2} \phi_1, \phi_2, \partial_s^k \phi_2, \dots, \partial_s^{kM_2} \phi_2), \quad (4.9c)$$

where  $c_1(t) = e^{\frac{1}{2}\sum_{i=1}^M \xi(t,\lambda_i)}$ ,  $c_2(t) = (-1)^{(k+1)M_2-k} c_1(t)$ ,  $c_3(t) = (-1)^{(k+1)M_2+1} c_1(t)$ .

From the bilinear equations of the  $k$ -constrained KP hierarchy[5] we see that if  $\tau(t)$ ,  $\rho_1(t)$ ,  $\sigma_1(t)$  are its tau-functions, then we can define another set of tau-functions for the  $k$ -constrained KP hierarchy as follows:

$$\hat{\tau}(t) = \varepsilon_1 \tau(-t), \quad \hat{\rho}_1(t) = \varepsilon_2 \rho_1(-t), \quad \hat{\sigma}_1 = \varepsilon_2 \sigma_1(-t), \quad (4.10)$$

where  $\varepsilon_i = \pm 1$ . It is then not hard to see that the solution of the  $k$ -constrained KP hierarchy obtained from the tau-functions in (4.9) is equivalent to the solution constructed from the tau-functions which were given in the conjecture of Ref [13], thus we also proved this conjecture.

## Section 5. Concluding remarks.

The fact that the 1-constrained KP hierarchy can be reduced from the 2-component KP hierarchy enables us to construct the double Wronskian solutions of the 1-constrained KP hierarchy starting from an analogue of the Baker-Akhiezer 2-point functions, this construction in turn motivated us to construct the Wronskian type solutions of the vector  $k$ -constrained KP hierarchy starting from an analogue of the Baker-Akhiezer  $(m+1)$ -point function. We remark here that the reduction from the 2-component KP hierarchy to the 1-constrained KP hierarchy is much more straightforward than the reduction from the KP hierarchy to the 1-constrained KP hierarchy, it only needs to require that the dynamical variables of the 2-component KP hierarchy do not depend on some independent variables, which is similar to the reduction from the KP hierarchy to the Korteweg-de Vries(KdV) hierarchy. We finally remark that our construction of the Wronskian type solutions of the vector  $k$ -constrained KP hierarchy suggests that there maybe exists certain analogue of the 2+1-dimensional  $(m+1)$ -component KP hierarchy from which the vector  $k$ -constrained KP hierarchy can be straightforwardly reduced, and it also suggests us a relatively convenient way to construct the algebraic-geometric solutions of the vector  $k$ -constrained KP hierarchy by using certain analogue of the Baker-Akhiezer  $(m+1)$ -point function. These subjects will be discussed in other publications.

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