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A COMPANION MATRIX FOR 2-D POLYNOMIALS

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ABSTRACT

In this paper, a matrix form analoguous to the companion matrix which is often encountered in the theory of one dimensional (1-D) linear systems is suggested for a class of polynomials in two indeterminates and real coefficients, here referred to as two dimensional (2-D) polynomials. These polynomials arise in the context of 2-D linear systems theory. Necessary and sufficient conditions are also presented under which a matrix is equivalent to this companion form.

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1. Introduction

Canonical forms play an important role in the modern theory of linear systems. One particular form that has proved to be very useful for 1-D linear systems is the so-called companion matrix which is associated with its characteristic polynomial. Barnett [3] showed that many of the concepts encountered in 1-D linear systems theory can be nicely linked via the companion matrix.

It is therefore worthwhile to seek a form of matrix which can be associated with 2-D polynomials and which can play a role similar to that of its 1-D counterpart.

In this paper, a matrix form which can be regarded as a 2-D companion form is presented. The characteristic polynomial of the matrix is in the form which arises from 2-D linear first order discrete equations e.g. those describing 2-D image processing systems as suggested by Roesser [5]. The condition of equivalence to the Smith form as given by Frost and Boudellioua [2] is used to obtain necessary and sufficient conditions for the equivalence of a matrix to the 2-D companion form.

2. Statement of the Problem

Let $$d(z_1,z_2) = \sum_{j=0}^{n_2} P_j(z_1) z_2^{n_2-j} = \sum_{i=0}^{n_1} Q_i(z_2) z_1^{n_1-i} \tag{2.1}$$

where $P_0(z_1)$ and $Q_0(z_2)$ are monic polynomials in z_1 and z_2 and have degrees n_1 and n_2 respectively. Also $P_j(z_1), j=1,2,\ldots,n_2$ have degrees less or equal to n_1 and $Q_i(z_2), i=1,2,\ldots,n_1$ have degrees less or equal to n_2.

The problem is to find a $(n_1+n_2)\times(n_1+n_2)$ matrix in the block form :

$$A \equiv \begin{bmatrix} A_1 & A_2 \\ A_3 & A_4 \end{bmatrix} \tag{2.2}$$

where A_1 is $n_1 \times n_1$, A_2 is $n_1 \times n_2$, A_3 is $n_2 \times n_1$ and A_4 is $n_2 \times n_2$ which has a form which is similar to the 1-D companion matrix and such that the determinant of the characteristic matrix :

$$zI_{n_1+n_2} - A \equiv \begin{bmatrix} z_1 I_{n_1} - A_1 & -A_2 \\ -A_3 & z_2 I_{n_2} - A_4 \end{bmatrix} \quad (2.3)$$

is given by the polynomial $d(z_1, z_2)$.
Furthermore under what necessary and sufficient conditions a matrix A, in a general form, is equivalent to this companion form.

The matrix A often presented in the literature as a 2-D companion form see e.g. [6] is one in which A_1 and A_4 are in companion forms but A_2 and A_3 have no special forms. In the following, a 2-D companion form is presented in which A_1 and A_4 are in companion forms and moreover A_2 is such that the overall matrix A, like its 1-D counterpart, has all the elements above the diagonal zero except for the elements on the superdiagonal which are all equal to 1.

3. A Companion Form for 2-D Polynomials

Proposition 1:

Given a 2-D polynomial $d(z_1, z_2)$ given by (2.1), then a 2-D companion matrix associated with $d(z_1, z_2)$ is given by :

$$F \equiv \left[\begin{array}{c|c} F_1 & F_2 \\ \hline F_3 & F_4 \end{array}\right] =$$

$$\left[\begin{array}{cccc|cccc}
0 & 1 & \cdots & 0 & 0 & 0 & \cdots & 0 \\
0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\
\vdots & \vdots & \cdots & \vdots & \vdots & \vdots & \cdots & \vdots \\
0 & 0 & \cdots & 1 & 0 & 0 & \cdots & 0 \\
f_1(n_1,1) & f_1(n_1,2) & \cdots & f_1(n_1,n_1) & 1 & 0 & \cdots & 0 \\
\hline
f_3(1,1) & f_3(1,2) & \cdots & f_3(1,n_1) & 0 & 1 & \cdots & 0 \\
f_3(2,1) & f_3(2,2) & \cdots & f_3(2,n_1) & 0 & 0 & \cdots & 0 \\
\vdots & \vdots & \cdots & \vdots & \vdots & \vdots & \cdots & \vdots \\
f_3(n_2-1,1) & f_3(n_2-1,2) & \cdots & f_3(n_2-1,n_1) & 0 & 0 & \cdots & 1 \\
f_3(n_2,1) & f_3(n_2,2) & \cdots & f_3(n_2,n_1) & f_4(n_2,1) & f_4(n_2,2) & \cdots & f_4(n_2,n_2)
\end{array}\right]$$

(3.1)

where F_1 and F_4 are $n_1 \times n_1$, $n_2 \times n_2$ companion matrices of $P_0(z_1)$ and $Q_0(z_2)$ respectively i.e.,

$$f_1(n_1, i) = -P(0, n_1 - i + 1), i = 1, \ldots, n_1$$
$$\text{and} \quad f_4(n_2, j) = -Q(0, n_2 - j + 1), j = 1, \ldots, n_2 \quad (3.2)$$

where $\det(z_1 I_{n_1} - F_1) \equiv P_0(z_1) = z_1^{n_1} + P(0,1)z_1^{n_1-1} + P(0,2)z_1^{n_1-2} + \ldots\ldots + P(0,n_1)$
and $\det(z_2 I_{n_2} - F_4) \equiv Q_0(z_2) = z_2^{n_2} + Q(0,1)z_2^{n_2-1} + Q(0,2)z_2^{n_2-2} + \ldots\ldots + Q(0,n_2)$.

The matrix F_2 is $n_1 \times n_2$ and has all its columns zero except for the first one which is given by E_{n_1} i.e., the first column of the identity matrix I_{n_1}.
The elements of F_3 are uniquely and recursively determined from the following equation :

$$f_3(i,j) = Q(0,i)P(0,n_1-j+1) - P(i,n_1-j+1) - \sum_{k=1}^{i-1} Q(0,i-k) f_3(k,j) \quad (3.3)$$
$$\text{for } i = 1,2,\ldots\ldots,n_2 \text{ and } j = 1,2,\ldots\ldots,n_1.$$

where the $P(i,j)$ and $Q(h,k)$ are defined by the following relations :

$$P_i(z_1) = \sum_{j=0}^{n_1} P(i,j) z_1^{n_1-j} \quad ,i = 0,1,2,\ldots\ldots,n_2$$

$$\text{and} \quad Q_h(z_2) = \sum_{k=0}^{n_2} Q(h,k) z_2^{n_2-k} \quad ,h = 0,1,2,\ldots\ldots,n_1 \quad (3.4)$$

Furthermore if $d(z_1, z_2)$ is separable i.e., can be written as a product of two 1-D polynomials, then the matrix F_3 is taken to be the null matrix.

The proof of the proposition follows in a simple way by expanding the determinant of the matrix $zI_{n_1+n_2} - F$ and equating the result with the polynomial $d(z_1, z_2)$. A detailed proof is set out in [1].
Obviously a similar form to F can be obtained based on the matrices A_1, A_3 and A_4 in such a way that the overall matrix obtained is the transposed matrix of F.

Example 1.

Let $d(z_1,z_2) = (z_1^2+z_1+1)z_2^2+(3z_1+2)z_2+2z_1^2-z_1+2$
$= (z_2^2+2)z_1^2+(z_2^2+3z_2-1)z_1+z_2^2+2z_2+2.$

Here we have,

$P_0(z_1)=z_1^2+z_1+1\ ,\ Q_0(z_2)=z_2^2+2\ ,$
$P_1(z_1)=3z_1+2\ ,\ Q_1(z_2)=z_2^2+3z_2-1\ ,$
$P_2(z_1)=2z_1^2-z_1+2\ ,\ Q_2(z_2)=z_2^2+2z_2+2.$

So that,

$P(0,1)=1,\ P(0,2)=1,$
$P(1,0)=0,\ P(1,1)=3,\ P(1,2)=2,$
$P(2,0)=2,\ P(2,1)=-1,\ P(2,2)=2,$
$Q(0,1)=0,\ Q(0,2)=2,$
$Q(1,0)=1,\ Q(1,1)=3,\ Q(1,2)=-1,$
$Q(2,0)=1,\ Q(2,1)=2,\ Q(2,2)=2.$

It follows that

$f_1(2,1)=-P(0,2)=-1,\ f_1(2,2)=-P(0,1)=-1,$
$f_4(2,1)=-Q(0,2)=-2,\ f_4(2,2)=-Q(0,1)=0,$
$f_3(1,1)=Q(0,1)P(0,2)-P(1,2)=0\times1-2=-2,$
$f_3(1,2)=Q(0,1)P(0,1)-P(1,1)=0\times1-3=-3,$
$f_3(2,1)=Q(0,2)P(0,2)-P(2,2)=2\times1-2=0,$
$f_3(2,2)=Q(0,2)P(0,1)-P(2,1)-Q(0,1)\times f_3(1,2)=2\times1-(-1)-0\times(-3)=3.$

Therefore,

$$F_1=\begin{bmatrix}0&1\\-1&-1\end{bmatrix},\ F_2=\begin{bmatrix}0&0\\1&0\end{bmatrix},\ F_3=\begin{bmatrix}-2&-3\\0&3\end{bmatrix}\ \text{and}\ F_4=\begin{bmatrix}0&1\\-2&0\end{bmatrix}$$

so that the overall matrix F is given by the following companion form:

$$F=\left[\begin{array}{cc|cc}0&1&0&0\\-1&-1&1&0\\\hline -2&-3&0&1\\0&3&-2&0\end{array}\right]$$

4. Algebraic Equivalence

In 1-D systems theory, two $n\times n$ matrices A and B are algebraically equivalent (similar) if and only if their corresponding characteristic matrices sI_n-A and sI_n-B are equivalent i.e., there exist unimodular $n\times n$ matrices over the ring $R[s]$, $M(s)$ and $N(s)$ such that :

$$sI_n-B=M(s)(sI_n-A)N(s) \qquad (4.1)$$

In fact, it can be shown that when it exists this transformation can be reduced to a similarity transformation i.e.,

$$sI_n-B=M_0(sI_n-A)M_0^{-1} \qquad (4.2)$$

In 2-D systems theory, however, this result is not true i.e., two matrices $zI_{n_1+n_2}-A$ and $zI_{n_1+n_2}-B$ in the form (2.3) may be equivalent over the ring $R[z_1,z_2]$ without implying that the matrices A and B being similar. In fact the similarity transformation used in the literature e.g.[5] and [7] which is a block diagonal transformation is only a special case of the general equivalence. In the following, a more general notion of algebraic equivalence is used.

Definition 1:

Two matrices A and B in the form (2.2) are algebraically equivalent if their corresponding characteristic matrices $zI_{n_1+n_2}-A$ and $zI_{n_1+n_2}-B$ are equivalent over the the ring $R[z_1,z_2]$, i.e. if there exist $(n_1+n_2)\times(n_1+n_2)$ unimodular matrices over $R[z_1,z_2]$, $M(z_1,z_2)$ and $N(z_1,z_2)$ such that :

$$zI_{n_1+n_2}-B=M(z_1,z_2)(zI_{n_1+n_2}-A)N(z_1,z_2) \qquad (4.3)$$

Using this definition of algebraic equivalence, we now present a theorem that gives necessary and sufficient conditions under which a matrix A in the form (2.2) is algebraically equivalent to the companion matrix F given by (3.1).

Theorem 1:

A matrix A in the form (2.2) is equivalent to the companion matrix F given by (3.1) if and only if its characteristic matrix $zI_{n_1+n_2} - A$ is equivalent over $R[z_1,z_2]$ to the Smith form:

$$S(z_1,z_2)=\begin{bmatrix} I_{n_1+n_2-1} & 0 \\ 0 & \det(zI_{n_1+n_2}-A) \end{bmatrix} \quad (4.4)$$

Proof:

Necessity: Suppose that the matrix A is equivalent to the companion form F, then it is clear from the form of the matrix $zI_{n_1+n_2} - F$ that it can be brought by elementary row and column operations to the Smith form $S(z_1,z_2)$ given in (4.4). It follows that the matrix $zI_{n_1+n_2} - A$ is equivalent to the Smith form $S(z_1,z_2)$.

Sufficiency: Suppose that the matrix $zI_{n_1+n_2} - A$ is equivalent to the Smith form $S(z_1,z_2)$. By Proposition 1, there exists a companion form F associated with the characteristic polynomial given by $\det(zI_{n_1+n_2} - A)$. Now since both $zI_{n_1+n_2} - A$ and $zI_{n_1+n_2} - F$ are equivalent to the same Smith form $S(z_1,z_2)$, they are equivalent to each other i.e., the matrices A and F are algebraically equivalent.

Theorem 2.

A matrix A in the form (2.2) is equivalent to the companion matrix F given by (3.1) if and only if there exists a (n_1+n_2) column vector $b(z_1,z_2)$ which has no zeros such that the matrix:

$$\begin{pmatrix} zI_{n_1+n_2} - A & b(z_1,z_2) \end{pmatrix} \text{ has no zeros.}$$

The definition of a zero of a matrix over $R[z_1,z_2]$ is the value of the complex pair (z_1,z_2) such that the matrix is rank deficient, see e.g. [4].

Proof:

Necessity: Suppose that the matrix A is equivalent to the companion form F, then there exist $(n_1+n_2)\times(n_1+n_2)$ unimodular matrices over $R[z_1,z_2]$ $M(z_1,z_2)$ and $N(z_1,z_2)$ such that:

$$zI_{n_1+n_2} - A = M(z_1,z_2)(zI_{n_1+n_2} - F)N(z_1,z_2) \quad (4.5)$$

It follows that :

$$M(z_1,z_2)(zI_{n_1+n_2} - F \quad E_{n_1+n_2})N(z_1,z_2) = (zI_{n_1+n_2} - A \quad b(z_1,z_2)) \quad (4.6)$$

It is clear that the matrix $(zI_{n_1+n_2} - F \quad E_{n_1+n_2})$ has no zeros since it has one highest order minor equal to 1. Therefore the matrix $(zI_{n_1+n_2} - A \quad b(z_1,z_2))$ has also no zeros. It remains to prove that the vector $b(z_1,z_2)$ has no zeros. This follows from the fact that $b(z_1,z_2) = M(z_1,z_2)E_{n_1+n_2}$.

Sufficiency: Suppose that there exists a (n_1+n_2) column vector $b(z_1,z_2)$ which has no zeros such that the matrix $(zI_{n_1+n_2} - A \quad b(z_1,z_2))$ has also zeros. Then, since the vector $b(z_1,z_2)$ has no zeros, there exists a $(n_1+n_2)\times(n_1+n_2)$ unimodular matrix $M_1(z_1,z_2)$ over $R[z_1,z_2]$ such that :

$$M_1(z_1,z_2)b(z_1,z_2) = E_{n_1+n_2} \quad (4.7)$$

i.e.,
$$M_1(z_1,z_2)\begin{pmatrix} zI_{n_1+n_2} - A & b(z_1,z_2) \end{pmatrix} = \left[\begin{array}{c|c} T_1(z_1,z_2) & 0 \\ \hline T_2(z_1,z_2) & 1 \end{array}\right] \quad (4.8)$$

where $T_1(z_1,z_2)$, $T_2(z_1,z_2)$ are $(n_1+n_2-1)\times(n_1+n_2)$ and $1\times(n_1+n_2)$ polynomial matrices respectively. Now since the matrix on the RHS of (4.8) has no zeros, the matrix $T_1(z_1,z_2)$ must also have no zeros. Therefore there exists a unimodular $(n_1+n_2)\times(n_1+n_2)$ matrix $N(z_1,z_2)$ such that :

$$T_1(z_1,z_2)N(z_1,z_2) = \begin{bmatrix} I_{n_1+n_2-1} & 0 \end{bmatrix}$$

i.e.,

$$M_1(z_1,z_2)\left(zI_{n_1+n_2-1}-A \qquad b(z_1,z_2)\right)\left[\begin{array}{c|c} N(z_1,z_2) & 0 \\ \hline 0 & 1 \end{array}\right]=\left[\begin{array}{cc|c} I_{n_1+n_2-1} & 0 & 0 \\ T_3(z_1,z_2) & t_4(z_1,z_2) & 1 \end{array}\right] \qquad (4.9)$$

Premultiplying the matrix on the RHS of (4.9) by the $(n_1+n_2)\times(n_1+n_2)$ unimodular matrix:

$$M_2(z_1,z_2)=\begin{bmatrix} I_{n_1+n_2-1} & 0 \\ -T_3(z_1,z_2) & 1 \end{bmatrix},$$

yields the matrix $\left[\begin{array}{cc|c} I_{n_1+n_2-1} & 0 & 0 \\ 0 & t_4(z_1,z_2) & 1 \end{array}\right]$ (4.10)

where $t_4(z_1,z_2)=\lambda.\det(zI_{n_1+n_2}-A)$, $\lambda\in R^*$. It follows that the matrices $zI_{n_1+n_2}-A$ and $S(z_1,z_2)$ are related by the following unimodular transformation:

$$M_2(z_1,z_2)M_1(z_1,z_2)\begin{bmatrix} I_{n_1+n_2-1} & 0 \\ 0 & \lambda^{-1} \end{bmatrix}\left(zI_{n_1+n_2}-A\right)N(z_1,z_2)=S(z_1,z_2) \qquad (4.11)$$

i.e., the matrix $zI_{n_1+n_2}-A$ is equivalent to its Smith form $S(z_1,z_2)$. Therefore by Theorem 1, the matrix A is algebraically equivalent to the companion form F. This completes the proof.

Example 2.

Let $A=\left[\begin{array}{cc|c} 0 & 1 & 1 \\ 6 & 1 & -2 \\ \hline 2 & 1 & 2 \end{array}\right]$, then it can be easily verified that the vector $b=\left[\begin{array}{c} 1 \\ 3 \\ \hline 0 \end{array}\right]$

satisfies the conditions in Theorem 2. Furthermore here we have $\det(zI_3-A)=(z_1^2-z_1-6)(z_2-2)$ i.e., the determinant is separable. In fact by premultiplying the matrix zI_3-A by the unimodular matrix :

$$\begin{bmatrix} -3 & 1 & -1 \\ 3z_1-6 & -z_1+2 & z_1-1 \\ -3z_1z_2+6z_1+12z_2-34 & z_1z_2-2z_1-4z_2+3 & -5z_1+15 \end{bmatrix}$$

and postmultiplying it by the unimodular matrix:

$$\frac{1}{25}\begin{bmatrix} (z_1+2)(z_1-4)(z_2-7)-15 & -5 & z_2-7 \\ (z_1+2)(z_1-4)(z_2-7)-20 & -15 & -3(z_2-7) \\ 5(z_1+2)(z_1-4) & 0 & -5 \end{bmatrix}$$

yields the charactersitic matrix:

$$zI_3-F\equiv\left[\begin{array}{cc|c} z_1 & -1 & 0 \\ -6 & z_1-1 & -1 \\ \hline 0 & 0 & z_2-2 \end{array}\right]$$

corresponding to the companion form:

$$F\equiv\left[\begin{array}{cc|c} 0 & 1 & 0 \\ 6 & 1 & 1 \\ \hline 0 & 0 & 2 \end{array}\right]$$

Notice that because the determinant of the matrix zI_3-A is separable, the matrix F_3 is zero.

5. Conclusions

In this paper, a canonical form which may be considered as a 2-D companion matrix is presented. By introducing a more general notion of equivalence, some of the conditions of equivalence to the companion form existing in 1-D systems theory are extended to the 2-D case. This work can be taken further by considering the usefulness of this 2-D companion matrix in the solution of problems such as realisation, controllability, observability, pole assignability, etc. of 2-D linear systems.

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6. References

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