APPLICATION OF NORMAL FORM METHODS TO THE ANALYSIS OF RESONANCES IN PARTICLE ACCELERATORS

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ABSTRACT

The transformation to normal form in a Lie-algebraic framework provides a very powerful method for identifying and analysing non-linear behaviour and resonances in particle accelerators. The basic ideas are presented and illustrated by a simple yet important example.

1. Introduction

In this paper, we will discuss the advantages of using state-of-the-art mathematical tools, in particular normal form methods and maps in a Lie-algebraic framework[1-3], to help us solve very complicated non-linear behaviour in particle accelerators. Although these methods were invented about 100 years ago, they would still have limited utility without the availability of modern computers and considerable mathematical development[3]. Two important developments are the advent of algebraic codes and differential algebra[4].

In general, this is a study of systems of differential equations of the form

$$\dot{x} = f(x, \lambda, t); \quad x \in \mathbf{R}^n, \ \lambda \in \mathbf{R}^m, \ t \in \mathbf{R}. \tag{1.1}$$

In accelerator physics, these equations become Hamilton's equations

$$\zeta' = -[H, \zeta] \tag{1.2}$$

where $f(x, \lambda, t) \Rightarrow H$, $' = d/d\theta$, $t \Rightarrow \theta$, the new "time"; $x \in \mathbf{R}^n \Rightarrow \zeta = (x, p_x, z, p_z, \tau, p_\tau) \in \mathbf{R}^6$, the "canonical variables" and the $[\cdot, \cdot]$ are the familiar Poisson brackets. The relativistic Hamiltonian, H, in cylindrical coordinates will be used for explicitness.

$$H = -e\rho A_\theta - \rho\left\{(\frac{p_0}{c})^2 - m^2c^2 - [p_\rho - eA_\rho]^2 - [p_z - eA_z]^2\right\}^{1/2} \tag{1.3}$$

where $\rho = x + \rho_0$; ρ_0 is the radius of curvature of the central or "design" trajectory, p_0 is the total energy of the system and e is the electronic charge. The magnetic vector potential A has the form

$$A = A(x, z, \tau; \theta); \quad A \in \mathbf{R}^3. \tag{1.4}$$

Equation (1.2) could be solved by brute-force numerical integration, which works up to a point for small systems like cyclotrons, but not for large systems. For example, the main ring of the Superconducting Supercollider has about 10 000 magnets; the Hamiltonian has 20 000 piece-wise continuous sections! Each particle trajectory must be integrated separately. Consequently, exploring phase space is extremely expensive and is accompanied by loss of numerical accuracy. In particular, the *symplectic condition* (conservation of phase space) for Hamiltonian systems is generally not preserved. Alternatively, we can use "state-of-the-art" methods to simplify the problem. This is essential for large systems. A truly enormous simplification of the problem is achieved by employing: 1) maps $\zeta_f = \mathcal{M}(\theta)\zeta_i$; 2) normal forms (reduces the problem to its "simplest" form — isolates resonances); 3) a Lie-algebraic formulation (ensures symplecticity, minimizes the number of terms, enhances understanding).

2. Lie-Algebraic Maps

In the development, we use a Lie-algebraic formulation. The Poisson bracket $[H,\zeta]$ in Hamilton's equations (1.2) satisfies all of the requirements of a Lie algebra, namely, the commutation relation

$$A * B = (A,B) = AB - BA \tag{2.1}$$

and the Jacobi identity

$$(A,(B,C)) + (B,(C,A)) + (C,(A,B)) = 0. \tag{2.2}$$

Note: This Lie algebra is *neither commutative nor associative.*

Traditionally, Taylor-series maps of the form $\zeta_f = \mathcal{M}(\theta)\zeta_i$ have been employed, which when expanded become

$$\zeta_i^{FIN} = R_{ij}\zeta_j^{IN} + T_{ijk}\zeta_j^{IN}\zeta_k^{IN} + U_{ijkm}\zeta_j^{IN}\zeta_k^{IN}\zeta_m^{IN} + \cdots \tag{2.3}$$

where, in general, R, T and U have 36, 126, and 336 terms, respectively. In a Lie-algebraic representation of $\mathcal{M}(\theta)$, we write

$$\mathcal{M} = \mathcal{M}_f; \qquad \mathcal{M}_f = e^{:f:}; \qquad :f:g = [f,g] \tag{2.4}$$

where

$$e^{:f:} \stackrel{\text{def}}{=} \sum_{n=0}^{\infty} \frac{:f:^n}{n!}; \quad :f:^0 = \mathcal{I} = \{\textit{identity map}\}. \tag{2.5}$$

The map $\mathcal{M}_f$ can be factored [1] into

$$\mathcal{M}_f = \mathcal{M}_2\mathcal{M}_3\mathcal{M}_4\ldots = e^{:f_2:}e^{:f_3:}e^{:f_4:}\ldots \tag{2.6}$$

where $\mathcal{M}$ and $\mathcal{M}_n$ are *symplectic maps* and f_n are homogeneous polynomials of degree n; f_2, f_3, and f_4 contain 21, 56 and 126, terms respectively (compare with (2.3)). The maps $\mathcal{M}_n$ are elements of the symplectic group $sp(2n,\mathbf{R})$.

3. Normal Form

The normal form is the transformation $\mathcal{N} = \mathcal{A}\mathcal{M}\mathcal{A}^{-1}$ such that $\mathcal{N}$ is in its "simplest" form; $\mathcal{M}$ must be expanded about a *"centre manifold"* or "closed orbit" (*i.e.* no H_1 term in the Hamiltonian, H), which implies no $\mathcal{M}_1$ term in the map $\mathcal{M}$. The *symplectic map* $\mathcal{A}$ is an n^{th} order canonical transformation that isolates the *tune shifts* and *resonances* to n^{th} order. Using the factored map, we can normalize $\mathcal{M}$ order-by-order to obtain

$$\mathcal{N} = \mathcal{A}_n\cdots\mathcal{A}_4\left\{\mathcal{A}_3\left\{\mathcal{A}_2\mathcal{M}\mathcal{A}_2^{-1}\right\}\mathcal{A}_3^{-1}\right\}\mathcal{A}_4^{-1}\cdots\mathcal{A}_n^{-1}. \tag{3.1}$$

We proceed as follows. From the eigenvectors of the matrix representation R, of $\mathcal{M}_2$, we get a canonical transformation that *rescales and block diagonalizes* R.

$$\mathcal{R} = \mathcal{A}_2 R \mathcal{A}_2^{-1} \Rightarrow \mathcal{R} = \begin{pmatrix} \mathcal{R}_1 & & 0 \\ & \mathcal{R}_2 & \\ 0 & & \mathcal{R}_3 \end{pmatrix}; \mathcal{R}_i = \begin{pmatrix} cos(\mu_i) & sin(\mu_i) \\ -sin(\mu_i) & cos(\mu_i) \end{pmatrix} or \begin{pmatrix} 1 & L_i \\ 0 & 1 \end{pmatrix}. \tag{3.2}$$

Next we normalize $\mathcal{M}_3$

$$\mathcal{N} = \mathcal{A}_3\mathcal{A}_2\mathcal{R}e^{:f_3:}\cdots\mathcal{A}_2^{-1}\mathcal{A}_3^{-1} = e^{:G_3:}\mathcal{R}e^{:g_3:}\cdots e^{-:G_3:} = \mathcal{R}\exp\left\{:-(\mathcal{I}-\mathcal{R}^{-1})G_3 + g_3:\right\}. \tag{3.3}$$

A further simplification results if we expand $(\mathcal{I}-\mathcal{R}^{-1})$ and g_3 in a resonance or complex basis, as follows:

$$h_\pm = x \pm ip_x; \qquad v_\pm = z \pm ip_z. \tag{3.4}$$

Hence,

$$(\mathcal{I}-\mathcal{R}^{-1})|\mathbf{n},\mathbf{m}> = \left[1 - \sum_{\mathbf{n}+\mathbf{m}=3} e^{i(\mathbf{n}-\mathbf{m})\cdot\mu}\right]|\mathbf{n},\mathbf{m}>, \tag{3.5}$$

and

$$g_3 = \sum_{\mathbf{n}+\mathbf{m}=3} A_{\mathbf{nm}}|\mathbf{n},\mathbf{m}> \qquad \text{where} \qquad |\mathbf{n},\mathbf{m}> = h_+^{n_1}h_-^{m_1}v_+^{n_2}v_-^{m_2}\cdots \tag{3.6}$$

and $\mathbf{n},\mathbf{m},\mu$ are vectors with components $(n_1,n_2,\cdots)$ *etc.* Substituting (3.5) and (3.6) into (3.3) and solving for G_3, we obtain

$$G_3 = \sum_{\mathbf{n}+\mathbf{m}=3} \frac{A_{\mathbf{nm}}}{\{1 - exp[i(\mathbf{n}-\mathbf{m})\cdot\mu]\}}. \tag{3.7}$$

All terms in g_3 can be incorporated into G_3 except when $\mathbf{n}-\mathbf{m}=0$ or $(\mathbf{n}-\mathbf{m})\cdot\mu = 2\pi k$; k = integer; *i.e.*, $(n_1-m_1)\mu_x + (n_2-m_2)\mu_z \neq 2\pi k$; $k = 0,\pm1,\pm2\cdots$. In a similar fashion, we could proceed to find G_4, *etc.* A further substitution of $I_x = h_+h_-$, $I_z = v_+v_-,\cdots$ — the *action invariants*— leads to

$$\mathcal{N} = \exp:\left\{(\mu_x + \mu_x'\rho_\tau + \mu_x''p_\tau^2)I_x + (\mu_z + \cdots)I_z + a_xI_x^2 + a_zI_z^2 + \cdots\right\}: \tag{3.8}$$

where we have assumed no acceleration and no explicit resonances; μ are the *phase advances*, μ' and μ'' are the first- and second-order *chromaticities* and all terms proportional to I^n are non-linear *tune shifts*. If explicit resonances occur with $(\mathbf{n}-\mathbf{m})\cdot\mu = 2\pi k$, k = integer, these must be added to (3.8). Finally, the *normal form* has the remarkable property that

$$\mathcal{M}^n = \mathcal{A}^{-1}\mathcal{N}^n\mathcal{A} \tag{3.9}$$

which also results in an enormous saving of work when applicable.

As an example, consider the Hamiltonian for the superposition of a uniform magnetic field with quadrupole and sextupole components. The vector potential for this Hamiltonian up to 3^{rd} order is

$$A_\theta = -\frac{x+\rho_0}{2}B_0 - \frac{1}{\rho_0}\left\{Q_xx^2 + Q_zz^2\right\} - \frac{S_x}{\rho_0}\left\{\frac{x^3}{3} - xz^2\right\}, \tag{3.10}$$

with $A_\rho = A_z = 0$; Q_x, Q_z, S_x are the quadrupole and sextupole strength, respectively. (With a proper choice of parameters, this Hamiltonian models all of the non-linearities of the TASCC superconducting cyclotron magnetic field up to third order.) Because (3.10) is independent of θ, the Hamiltonian (1.3) has the trivial solution

$$\mathcal{M}(\theta) = \exp\left\{-\int_0^\theta :H: d\theta'\right\} \quad \Longrightarrow \quad \mathcal{M}(\theta) = e^{-\theta:H:} \tag{3.11}$$

which can be factored (with some work) into the form of (2.6). Because we are neglecting acceleration, the energy of each particle is conserved and we define $p_\tau = (E_0 - E)/E_0$, where E is the particle energy and E_0 the energy of the central trajectory. If we factor (3.11), we find that f_3 contains 19 complicated terms.

If we carry out the normalization procedure described above and there are no explicit resonances, the normal form contains $g_2 = \mu_x h_+ h_- + \mu_z v_+ v_-$, the "rotation matrix R" and

$$g_3 = \frac{(1-\nu_x)}{4\nu_x\beta} h_+ h_- p_t + \frac{(1-\nu_z)}{4\nu_z\beta} v_+ v_- p_t$$
$$+\pi S_x \frac{k_x^2}{\nu_x\beta} h_+ h_- p_t - \pi S_x \frac{k_x k_z}{\nu_x\beta} v_+ v_- p_t$$
$$+\pi k_x \frac{1-\beta^2}{\nu_x\beta} p_t^3 + 2\pi S_x \frac{k_x^3}{3\nu_x^3\beta^3} p_t^3 \qquad (3.12)$$

which contains 4 *chromaticity* terms proportional to p_τ and two time-of-flight aberration terms proportional to p_τ^3, a substantial simplification. Furthermore, these are the terms that could potentially cause us trouble.

We see from (3.12) that if $\nu_x = \nu_z = 1 \Rightarrow \mu_x = \mu_z = 2\pi$ (that is, we make the system resonant!), then 2 terms disappear and we are left with the terms proportional to S_x. If the remaining *chromaticity* terms can be removed (which they can), then $\mathcal{R}_1 = \mathcal{R}_2 = \mathcal{I}$ or $\mathcal{R}_\perp = \mathcal{I}$. Hence

$$\mathcal{M} = \{\mathcal{A}_3^{-1}\mathcal{A}_2^{-1}\mathcal{N}\mathcal{A}_2\mathcal{A}_3\} = \mathcal{I}e^{-:\alpha p_\tau^3:} \;!! \qquad (3.13)$$

Eq. (3.13) tells us that under these circumstances, all of the non-linearities, except the one proportional to p_τ^3, annihilate each other up to 3^{rd} order in the map! Although one would never want to make the whole accelerator resonant in this way, parts of it can be made so to great advantage.

4. Conclusions

In the design and analysis of particle accelerators, we can achieve truly enormous savings in time and effort by:

1. using *maps*,
2. transforming to *normal form* to study *tune-shifts* and *resonances*, and
3. using the *Lie-algebraic* framework which enhances *insight and understanding.*

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References

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