FINITE ELEMENT METHOD - THEORY AND APPLICATIONS

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ABSTRACT

A brief summary of the mathematical basis of the Finite Element Method (FEM) has been presented. Attention is drawn to the natural development of the method as an engineering analysis tool into a general numerical analysis tool. A particular application to the stress analysis of rubber materials is presented. Special advantages and issues associated with the method are mentioned.

1. Mathematical Basis of FEM

The FEM is typically viewed as an approximation to the integral form of governing equations. Two distinct procedures are available to achieve such approximation; the weighted residual method and variational method.

1.1 Weighted Residual Method

The starting point here is the set of differential equations and boundary conditions which govern the behaviour inside the domain V and on the surface S,

\[ A(u) = 0 \quad \text{in} \quad V \]  
\[ B(u) = 0 \quad \text{on} \quad S \]

where the dependent variable \( u = u(x, t) \) is a scalar (or vector) function of space and time. The differential operators A and B are obtained from the physics of the problem.

The above equations can be combined in a single integral form, as follows:

\[ \int_V C^T A(u) \, dV + \int_S D^T B(u) \, dS = 0 \]  

If Eq. (1.3) is satisfied for any arbitrary choice of functions C and D, then the differential Eqs. (1.1) and (1.2) would be satisfied at all points. It is usually possible to integrate Eq. (1.3) by parts to yield another integral form known as "the weak integral form". Of course, the shape of the functions A, B, C and D will now be different. As a result, the new differential operators A and B will include a lower order of differentiation, albeit at the expense of higher orders in the arbitrary functions C and D. This results in loosening the continuity condition on the trial functions in the approximation process. In some problems, it has been found that the boundary term in the weak integral is much simpler than the corresponding term in the original form. Such a simplification of boundary conditions constitutes one of the most impressive advantages of the FEM. It is interesting to note herein that if the integration by parts is continued once more, it is possible to entirely get rid of the domain integral by careful choice of arbitrary functions, which satisfies the differential equations in the domain, and hence reduces the dimensionality of the problem by one order of magnitude. This has led to the development of a new method, called the "Boundary Element Method", in the 1970's(2). Now, if an approximation to the variable \( u \) is assumed, Eqs. (1.3) will not be identically satisfied for any arbitrary function unless these functions are chosen to minimize the errors (residuals) to zero. Hence the name of weighted residual methods.

1.2 Variational Method

A variational principle is a scalar quantity (functional), which is defined in integral form in terms of the dependent variable (e.g., \( u \ )) and/or its derivatives. The solution to the continuum problem is the function \( u \ ), which makes the functional stationary with respect to any arbitrary variation \( \delta u \). Such variational principles are called natural if the physical aspects of the problem can be stated directly in such a way as to minimize some quantities, such as the potential energy of a mechanical system, energy dissipation in viscous flow, etc. If such a principle exists, standard procedures can be immediately established to approximate the solution(1). It is easy to show that this procedure is mathematically equivalent to the Galerkin weighted residual method. More important is the fact that the procedure will result in symmetric FEM equations.
which are attractive from a computational point of view. Unfortunately, the opposite is not true, i.e., a weighted residual procedure does not necessarily produce symmetric equations. In this regard, the weighted residual method retains its advantage as a more general method, because it is not possible to find a variational principle for some problems which can otherwise be described by differential equations. Some researchers still like to construct the so-called contrived variational principles by introducing additional variables called Lagrange multipliers. In these situations, the gain in having symmetric equations can be upset by having more variables and singular equations, which require special care in solving them.

1.3. Trial (Shape or Interpolation) Functions
The FEM can now be recognized as the special case when the approximation of the variable \( u \) is expressed in terms of piecewise (i.e., defined element wise) trial functions, as follows:

\[
u = \hat{u} = \sum_{i} N_i a_i = N a\quad (1.4)
\]

where \( N_i \) are assumed trial function (usually polynomial) and \( a_i \) are the values of the variable \( u \) at the nodal points which constitute a given element in the domain.

As for the weighting functions \( C \) and \( D \) in Eq. (1.3), various methods can be used. The most common one is known as the Galerkin method, wherein \( C \) and \( D \) are chosen to be the same as the trial functions \( N \) of Eq. (1.4). It can be seen by now that the most important step in FEM is the choice of trial functions \( N \). In general, such functions should be continuously integrable (over a single element) up to the highest order of differentiation in Eq. (1.3). In addition, they should avoid the presence of infinity at the inter-element boundaries. Polynomial functions which are chosen to equal unity at the corresponding nodal point and zero elsewhere are commonly used. They are usually expressed in terms of normalized space (i.e., ranging between -1 and +1). They can then be mapped to the real geometry using standard transformation techniques. Experience has shown that simpler functions produce better results, especially in nonlinear analysis. Figure (1.1) shows some such functions for 2- and 3-node elements in 1-dimension space. Extension to 2- and 3-dimension space is straightforward.

\[
N_1 = (1 - \xi)/2 \\
N_1 = -\xi(1 - \xi)/2
\]

Figure 1.1 Shape Functions for 2-node and 3-node Finite Elements in 1-Dimension Space

If the coordinates \( x \) are interpolated using the same trial functions \( N \) of Eq. (1.4), the element is further classified as an isoparametric element. Some researchers have shown that by simple relocation of the mid-side node or by changing the trial function itself, it is possible to model such peculiarities as the \( 1/\sqrt{r} \) singularity of linear fracture mechanics, or to map a semi-infinite domain to a finite space. Except for such peculiarities, experience has shown, however, that the simpler (linear) functions do usually produce better results (for the same degree of discretization), especially in nonlinear analysis. Finally, it is interesting to note that the well-known finite difference method can be viewed as a special case of the weighted residual method, in which the weighting function happens to be the Dirac function applied at grid points.

2. Applications in Solid Mechanics

The most convenient approach in solid mechanics is to write down the potential energy \( \Pi \) of the system as follows:

\[
\Pi = \frac{1}{2} \int_{V} \epsilon_{ij} \sigma_{ij} \, dV - \int_{S} u_{i} t_{i} \, dS \quad (2.1)
\]

where \( \sigma_{ij} \) and \( \epsilon_{ij} \) are the stress and strain tensors inside the domain, while \( u_{i} \) and \( t_{i} \) are the displacement and traction vectors on the boundary, respectively. If the problem is to be reduced to one dependent variable only (e.g., \( u_{i} \)), all other variables have to be expressed in terms of \( u_{i} \) using whatever kinematic and constitutive...
material relationship applicable to the problem. For example, in the case of small displacements and linear elastic material, the following relations can be assumed:

$$\sigma_{ij} = E_{ijkl} \varepsilon_{kl}$$

$$\varepsilon_{ij} = \frac{1}{2} \left( \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right)$$

(2.2)

where $E_{ijkl}$ is the generalized Hook's law. Now, by assuming a FEM approximation (similar to Eq. 1.4), the variable $u$ and its derivatives can be expressed in terms of nodal variables $a$ as follows:

$$\{ u \} = [N] \{ a \}$$

$$\{ \varepsilon \} = [B] \{ a \}$$

(2.3)

(2.4)

The equilibrium of the mechanical system corresponds to the function $u$ which minimizes the functional $\Pi$ (i.e., $\delta \Pi = 0$). By substituting Eqs. (2.2), (2.3) and (2.4) into Eq. (2.1), the variation of functional $\Pi$ is as follows:

$$\delta \Pi = \{ \delta a \}^T \left( \int \left[ B^T [E] [B] \{ a \} \right] dV - \int_{S} [N]^T \{ t \} dS \right) = 0$$

(2.5)

Since $\delta a$ is any arbitrary variation, the expression inside the $\{}$ brackets should equal zero. This leads to the following set of algebraic equations:

$$[K] \{ a \} = \{ F \} = \{ 0 \}$$

(2.6)

where:

$$[K] = \sum_{\text{elements}} \int_{V} [B]^T [C] [B] dV$$

$$\{ F \} = \sum_{\text{elements}} \int_{S} [N]^T \{ t \} dS$$

(2.7)

It is understood that the summation in Eq. (2.7) represents the contribution at one node from all the elements connected to it. The formulation of several element matrices can be executed in parallel, making the FEM suitable for modern parallel and vector computer technology.

In the case of rubber analysis, there are many sources of nonlinearities, such as the kinematic and material relationship (Eqs. 2.2), large displacements due to moving boundaries, gap/friction and slip/stick surface conditions, as shown in Figure 2.2. The solution procedure is similar to the above one, with the coefficient matrices $K$ and $F$ now being functions of the displacement $u$ and stress $\sigma$. The resulting algebraic equations are nonlinear, and standard numerical linearization procedures (e.g., Newton-Raphson) are usually used for the solution.

Figure 2.1 Typical O-ring Problem

The general purpose FEM program MARC (3) has been used in the stress analysis of the O-ring of the solid rocket in the space shuttle. The stress, strains, displacements and reactions can be found at different loads, boundary and/or temperature conditions. These results can help the designer determine contact and/or maximum stress locations, which are crucial in operation conditions. The amount of output in a typical FEM analysis is usually very large, and interactive graphics tools(4) have to be used to investigate the results. Figure 2.2 shows a typical color (reprinted here in gray) contour of stress being superimposed on the deformed mesh.

3- Current Activities in FEM

With the mathematical foundation of FEM being established, a lot of R and D is still going on in this field. Questions such as accuracy, convergence, existence and/or uniqueness of the solution are being addressed. The development of new complicated materials and the need to push operational conditions to new heights
(e.g., high temperatures, supersonic speeds, etc.) have moved research beyond the realm of linear analysis. Automatic-adaptive mesh generation, interactive graphics and animation techniques are becoming indispensable standard tools of numerical analysis. Mixed (hybrid) formulation, wherein several dependent variables are explicitly included, have yielded very accurate results in some particular applications.

4- Conclusions

The FEM has been shown to be based on solid mathematical foundations. This explains the reason why the method has developed from being an intuitive engineering tool in its infancy, to being such a comprehensive numerical method tool now. The successful application of this method to the stress analysis of the O-ring used in the field joints of the solid rocket of the space shuttle has been demonstrated. More development is still taking place to extend FEM to new fields and applications.

References

Figure 2.2 Contour of Horizontal Stress in the O-ring under 35% Vertical Compression and Upstream Gas Pressure of 250 psi.