TOPICS IN INDUSTRIAL MATHEMATICS

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ABSTRACT

Mathematical methods are widely used to solve practical problems arising in modern industry. This article outlines some of the topics relevant to AECL programmes. This covers the applications of Transmission and Neutron Transport Tomography to determine density distributions in rocks and two phase flow situations. Another example covered is the use of variational methods to solve the problems of aerosol migration and control theory.

1. Introduction

Mathematics may have first developed out of our need to solve practical problems, but it has evolved into an independent discipline. While the aim in pure mathematics is to find all possible implications of the underlying assumptions, the relevance and acceptability of the final results impose constraints in applied mathematics. Industrial mathematics is essentially applied mathematics characterized by serving the needs of industry. The field of industrial mathematics is quite wide. A few examples are described below.

2. Nonintrusive Imaging Techniques

For some measurements, probe-based (intrusive) techniques are impractical, leaving nonintrusive ones as the only alternatives. Most nonintrusive techniques are based on the principle of tomography. Transmission tomography has been used to determine the local density of rocks(1), X-ray and γ-ray transmission tomography, to determine the density distribution of fluids(2), industrial products and for medical diagnosis. Neutron transport topographic methods are better suited for some fluid flow situations.(3) Methods developed and in use at Whiteshell are described here.

2.1 Geophysical tomography

Seismic tomography is used to determine density variations in rocks. Measured quantity is the travel time $T_i = T(L_i)$ of sound along each of many paths $L_i, i = 1, 2, \ldots n$, given by

$$T_i = \int \eta(r) d\ell(r) \tag{1}$$

where $\eta(r)$ is the slowness to be computed at point r and $d\ell(r)$ is the infinitesimal-length. All of the transmission tomographic problems reduce to an equivalent problem. Standard methods may be classified as algebraic reconstruction, and the transform methods. A new method, called the areal basis inversion technique (ABIT) reduces (1) to a Fredholm-type integral equation of the first kind and solves it by the L_2-inversion theorem.(1) A comparison of results produced from synthetic data demonstrates that ABIT defines the anomaly more accurately.

2.2 Gamma-ray densitometer

The phase-distribution is needed to understand many physical situations involving two-phase flow. Multi-beam gamma densitometers are commonly used for this purpose; they measure, essentially, the chordal void fractions. An algorithm developed recently(2) to extract the phase-distribution is outlined below.

The chordal void fraction $\alpha(\phi)$ along $L(\phi)$ is given by

$$\alpha(\phi) = \frac{1}{d(\phi)} \int_{r_0(\phi)}^{r_1(\phi)} \Gamma(s(\phi), r) dr \tag{2}$$

where $\Gamma(s(\phi), r) = 1$ for r in $s(\phi)$, and zero otherwise, $d(\phi) = (r_1(\phi) - r_0(\phi))$ and $s(\phi)$ is a subset of $[r_0(\phi), r_1(\phi)]$ defined by the phase distribution. The points $r_0(\phi)$ and $r_1(\phi)$ are the intersections of the γ-ray with the circumference of the pipe. The purpose here is to approximate Γ. The procedure used was based on the series expansion method; *i.e.*, expand Γ as a linear combination of some basis functions β_j. The coefficients are determined by (2). For the three-beam densitometers in use at Whiteshell, the problem was reduced to solving one nonlinear equation: $h(\gamma) = 0$, after some manipulations. The method was verified for several realistic flow patterns and found to be a vast improvement over the existing methods.

2.3 Neutron transport tomography

While the quantities of interest are still the distribution of the phases and the void fraction, the system of interest is the heated section of an experimental test facility. The γ-rays are attenuated almost out of existence by this type of system, rendering the multi-beam gamma densitometers inapplicable. A method based on neutron transport tomography solves the technical problem satisfactorily, but the mathematical problem encountered is quite complex.

In addition to other difficulties in solving the inverse problem of nonlinear tomography, the number of detector locations in a realistic experimental situation is quite small, two or three at the most. As was the case with gamma densitometry, the series expansion method is, therefore, better suited in this kind of situation. This procedure was used to reduce the problem to solving $\phi(\gamma) = 0$. The D vector $\phi(\gamma)$ is defined by $\phi_d(\gamma) = u_d - \mu_0(r_d), d = 1, 2, \ldots, D$, and γ is the D vector to be computed, u_d is determined by the measured count and

$$\mu_i(r) = \int \frac{d^2r'}{|r-r'|} exp\left[-\kappa_1 \sum_{i=1}^{3} \gamma_i \xi_i(r,r')\right] \left\{ \sum_{c=1}^{3} \frac{a_c}{|r_c - r'|} + \lambda \left[\sum_{i=1}^{3} \gamma_i \beta_i(r')\right] (A\mu(r'))\right\}_i \qquad i = 0, 1, 2, \cdots$$

where $\xi_i(r, r')$ is the line integral of the basis vectors β_i along the straight line joining r and r'.

The solution γ yields the void fraction and the phase distribution in a straightforward manner. The method was verified by using experimental neutron counts for a number of void fraction values in a stratified flow pattern. Agreement between the experimental and computed values of the void fraction is excellent for most of the measured values. The present method also identifies the flow pattern quite well.

3. Variational Methods

Variational calculus is one of the major tools used to formulate physical problems. Fermat's "least time" and Hamilton's "least action" principles have enjoyed widespread applications in the framework of variational calculus. On the other hand, variational methods such as the Rayleigh-Ritz method, the Bubnov-Galerkin method and the method of moments provide powerful tools to solve differential, integral and integro-differential equations. Both types of applications are indicated here.

3.1 Geophysical tomography

It was assumed in Section 2.1 that a signal travels along straight lines joining the sources and the detectors. A more accurate model may be used to describe the signal path. According to Fermat's principle, the travel time along a ray path joining the given points is minimum compared to all of the neighbouring paths. The physical path is then described by the Euler equation of the associated variational problem, which reduces to

$$\nabla \eta = \frac{d}{d\ell}(\eta \frac{dr}{d\ell}) \tag{3}$$

Equation (3) forms the basis of the ray tracing and the shooting methods.[4]

3.2 Aerosol migration

The transport of aerosols is described by a multi-component, nonlinear integrodifferential equation. Here we consider a simplified version that is adequate enough to address most of the complications.

The number density, $C(m, t)$, of particles of mass m in size range $(0, M)$ at time t is the solution of

$$C'(m,t) = F(m,t,C(m,t))$$

where the prime denotes the derivative with respect to time, and F is a nonlinear operator. The value of $C(m,0)$ is known and $C(0,t) = C(M,t) = 0$. In the direct variational method, the solution $C(m,t)$ is treated as an element of an appropriate Hilbert space H with a scalar product (,). For the present case, H may be taken to be the space of square integrable functions over the interval $(0, M)$. The solution $C(m,t)$ may be approximated by a linear combination of basis functions $\{v_j(m)\}$:

$$C(m,t) = \sum_{j=1}^{n} \alpha_j(t) v_j(m).$$

This reduces the problem to solving

$$A\frac{\partial\alpha(t)}{\partial t} = g(\alpha(t))$$

where A is the normalization matrix, and the vector function $g(\alpha(t))$ is as defined above. Standard methods are used to integrate this equation. A commonly used basis set in this case is given by

$$v_i(m) = m^{i-1}e^{-\gamma m}$$

where γ is an arbitrary positive constant. This scheme was tested for a number of cases.[5] Variational approximations converge rapidly to the exact solution as the basis size increases.

3.3 Control theory

The dynamics of a large class of linear control systems is described by

$$dx/dt = A(t)x + B(t)u, \qquad x(0) = x_0, \tag{4}$$

where x and u are vectors, and A and B are matrices. It is required to find the value u^* of the control function u that optimizes a functional termed the performance index. It can be shown that u^* is related to the solution ξ^* of an opérator equation of the form[6]

$$(1 + K)\xi^* = g \tag{5}$$

where g is a given vector in a Hilbert space and K is an operator.

While the problem is solvable by the variational methods of the type described above, convergence rate is often poor. However, the set $\{K^{i-1}g\}$ may be used as a basis in the direct variational method. The method based on such a representation is called the moment method.[7] The rate of convergence of the method, when applicable, is usually better, and it may be applied to generate monotonically convergent upper and lower bounds to $J^{(6)}$.

The moment method was compared with the standard variational methods for several realistic problems. The method of moments required only a few basis functions, compared to tens in the other types.

4. Concluding Remarks

As a subfield of applied mathematics, industrial mathematics overlaps with a number of areas of pure mathematics. The applications of the non-intrusive imaging techniques illustrated here cover a small section of the tomographic methods. Other methods worth mentioning are positron emission tomography and ultrasound and magnetic resonance imaging.

The topics and applications discussed in this paper were selected for their significance to AECL Research. Also, these are some of the areas in which we have made substantial improvements over the techniques that were otherwise available. However, some further work is needed before these developments can be fully exploited.

5. References

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