Generalized quantum groups

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ABSTRACT

The algebraic approach to quantum groups is generalized to include what may be called an anyonic symmetry, reflecting the appearance of phases more general than ± 1 under transposition.

1. Quantum Groups

The notion of quantum group[1], a special type of *bialgebra*, originally abstracted from studies of integrable or exactly solvable physical systems, has been found to be quite pervasive in many diverse fields of physics, such as inverse scattering theory, solitons, statistical spin models, lattice gauge theory, conformal field theory and gravitation.

There now exists a great deal of elegant mathematical machinery to unify the various historical approaches to quantum groups, but a straightforward algebraic approach[2] which led to one of the earliest incarnations of the concept is also very directly linked to interesting physical systems. A method of studying such solvable statistical systems[3] as the Heisenberg antiferromagnetic chain, the 2-d Ising model, the 1-d XY-model, the XYZ model, the hard-hexagon model, and the general eight-vertex model, centers on the existence of a *transfer matrix* T_{ij} satisfying the equation†

$$\hat{R}_{ab|mn}T_{mk}T_{nl} = T_{am}T_{bn}\hat{R}_{mn|kl} \tag{1.1}$$

for some invertible $\hat{R}$. A consistency condition for this structure is the Yang-Baxter relation

$$\hat{R}_{ab|mn}\hat{R}_{nc|pk}\hat{R}_{mp|ij} = \hat{R}_{bc|mn}\hat{R}_{am|ip}\hat{R}_{pn|jk}. \tag{1.2}$$

It can be shown[2] that the associative algebra generated by the non-commuting T_{ij}, subject to the relations (1.1), obtains the additional structure of a bialgebra when a *coproduct* is introduced of the form

$$\Delta(T_{ij}) = T_{ik} \otimes T_{kj}. \tag{1.3}$$

A coproduct is, in a sense, the opposite of a product: instead of combining two elements to yield a third, it distributes an element into a sum of pairs. A simple example is the Lie coproduct which may be defined for any Lie algebra:

$$\Delta(X) = X \otimes 1 + 1 \otimes X. \tag{1.4}$$

In elementary physics the essential physical role of a coproduct is easy to overlook, since in the theory of angular momentum, for example, the Lie coproduct translates into the simple additive relation between the angular momentum operator of a compound system and the operators of the constituent systems,

$$J_z = j_z^{(1)} + j_z^{(2)}, \tag{1.5}$$

which one is tempted to take as intuitively obvious.

The continuous deformation of Lie algebras away from their natural Lie product and the coproduct (1.5), while retaining a bialgebra structure and hence such important consequences as being able to generate all states from one 'highest weight' state using raising and lowering operators, is an important approach[4] to obtaining quantum groups of potential physical interest. The quantum groups so obtained are, in fact, a subset of those obtained by considering a structure dual to that defined by equations (1.1) and (1.3). In the following we describe this structure and present a generalization.

† Implicit summation on all repeated indices.

2. More Details

The index acrobatics involved in (1.1) and (1.2) may be eliminated by using the transposition operator $P_{ij|kl} = \delta_{il}\delta_{jk}$. Then (1.1) may be written as $\hat{R}T_1 P T_1 P = T_1 P T_1 P \hat{R}$, where $T_{1\,ij|kl} = (T \otimes 1)_{ij|kl} = T_{ik}\delta_{jl}$ and the implicit summations are simply of the form $\sum_{p,q} A_{ij|pq} B_{pq|kl}$.

The dual algebra (to which the term 'quantum group' is strictly applicable) is generated by non-commuting quantities $L^{\pm}_{ij}$ with a coproduct of the same form as (1.3). The duality pairing obtained from

$$\langle L^{\pm}_{ij}, T_{ab} \rangle = R^{\pm}_{ia|jb}, \tag{2.1}$$

where $R^+ = \hat{R}P$ and $R^- = \hat{R}^{-1}P$, requires the $L^{\pm}$ to satisfy the relations

$$\begin{aligned} \hat{R}PL^{\pm}PL^{\pm} &= L^{\pm}PL^{\pm}P\hat{R} \\ \hat{R}PL^{+}PL^{-} &= L^{-}PL^{+}P\hat{R} \end{aligned} \tag{2.2}$$

analogous to (1.1).

3. An Anyonic Generalization

In analogy with the supersymmetric generalization[5] of the above structure, which replaces the transposition operator P with the graded transposition operator $\overline{P}_{ij|kl} = (-1)^{\hat{i}\hat{j}} P_{ij|kl}$, we seek a generalization involving a transposition operator of the form $\tilde{P}_{ij|kl} = \mu_{ij} P_{ij|kl}$. Unlike the regular and supersymmetric transposition operators, $\tilde{P}$ is not *a priori* assumed to be symmetric, nor does $\tilde{P}^{-1} = \tilde{P}$. There are thus a great many 'natural' generalizations of the equations in §2, in which one replaces P's with some selection of $\tilde{P}$'s, or its inverses or their transposes. Elimination of inconsistent choices is best done with a symbolic manipulation program such as Mathematica[6]. A consistent generalization, so derived, requires the μ_{ij} to satisfy $\mu_{ij}\mu_{jk}\mu_{ki} = \mu_{ik}\mu_{kj}\mu_{ji}$ and replaces the corresponding equations in §2 with

$$\begin{aligned} \hat{R}T_1\tilde{P}T_1\tilde{P} &= T_1\tilde{P}T_1\tilde{P}\hat{R}, \\ R^+ = \hat{R}\tilde{P} \quad &, \quad R^- = P\tilde{P}\hat{R}^{-1}P, \\ \hat{R}\tilde{P}^{-1}L^{\pm}\tilde{P}L^{\pm} &= \tilde{P}^{-1}L^{\pm}\tilde{P}L^{\pm}\hat{R}, \\ \hat{R}\tilde{P}L^{+}\tilde{P}^{T}L^{-}\tilde{P}^{T} &= \tilde{P}^{T}L^{-}\tilde{P}^{T}L^{+}\hat{R}\tilde{P}. \end{aligned} \tag{3.1}$$

The utility of such an introduction of 'anyonic' symmetry in the context of solvable models is currently under investigation.

References

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