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INDUCED FERMION NUMBER, PHASE-SHIFT FLIP, AND THE AXIAL ANOMALY IN THE AHARONOV-BOHM POTENTIAL

ALEXANDER MOROZ*

Division de Physique Théorique†, *Institut de Physique Nucléaire, Université Paris-Sud, F-91 406 Orsay Cedex, France*
and
‡*School of Physics and Space Research, University of Birmingham, Edgbaston, Birmingham B15 2TT, U. K.*

ABSTRACT

The spectral properties of the Dirac and the Klein-Gordon equations in the the Aharonov-Bohm potential are discussed. The density of states for different self-adjoint extensions is calculated. The Aharonov-Casher and the index theorems are corrected in the sense that they give rather an upper bound on the number of threshold or zero modes in a given finite-flux background than their actual number. There are no zero (threshold) modes in the Aharonov-Bohm potential and the index of the massless two-dimensional Euclidean Dirac operator is zero. As in the nonrelativistic case, whenever a bound state is present in the spectrum it is always accompanied by a (anti)resonance at the energy proportional to the absolute value of the binding energy. The presence of the bound state manifests itself by asymmetric differential scattering cross sections. The results are applied to several physical quantities: the total energy, induced fermion-number, and the axial anomaly. Stability of the system is discussed. It is shown that the axial anomaly is related to the phase-shift flip. The predictions of a persistent current in the presence of a cosmic string and a gravitational vortex are made.

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*e-mail address : moroz@fzu.cz
†Unité de Recherche des Universités Paris XI et Paris VI associée au CNRS
‡Present address

1 Introduction

In this paper the change over all space of the density of states (DOS) $\triangle\rho_\alpha(E)$ induced by the Aharonov-Bohm (AB) potential $\mathbf{A}(r)$ [1],

$$A_r = 0, \qquad A_\varphi = \frac{\Phi}{2\pi r} = \frac{\alpha}{2\pi r}\Phi_0, \tag{1}$$

in the radial gauge, is calculated for the Dirac and the Klein-Gordon equations. We shall talk about the AB potential in a more general sense here. Usually $\Phi = \alpha\,\Phi_0$ is the total flux through the flux tube and Φ_0 is the flux quantum, $\Phi_0 = hc/|e|$. However, the same potential (1) is induced by a cosmic string provided the identification $\alpha = e/Q_{Higgs}$ is made, where e and Q_{Higgs} are respectively the charge of a test particle and the charge of the Higgs particle [2, 3]. In the case of a point spectrum [4] one finds that there are no threshold (zero) modes in the AB potential for any α. The Aharonov-Casher and the index theorems [5, 6, 7] are corrected in the sense that they give rather an upper bound on the number of threshold or zero modes in a given finite-flux background than their actual number. Similarly as in the nonrelativistic case, whenever a bound state occurs in the spectrum one finds that it always is accompanied by a (anti)resonance. The presence of a bound state manifests itself in asymmetric differential scattering cross sections (cf. [10, 11]). To calculate the contribution of scattering states to the integrated DOS (IDOS) the Krein-Friedel formula is used [8, 9]. The formula is the basic tool for calculating the change of the IDOS in solid state physics induced by potentials of a finite range. Recently we have shown in the discussion of the nonrelativistic AB scattering, that when regularized by the zeta function it gives the correct answer even for the long-ranged AB potential [10, 11]. The DOS $\rho_\alpha(E)$ provides an important link between different physical quantities. It enables one to calculate the total energy and effective action, and to discuss the stability properties of matter against the spontaneous creation of a magnetic field. This question is of principal importance in understanding the nature of the ground state both in field theories and in condensed matter physics. The knowledge of $\triangle\rho_\alpha(E)$ determines the spectral asymmetry $\sigma_\alpha(\mathcal{E})$,

$$\sigma_\alpha(\mathcal{E}) = \triangle\rho_\alpha(\mathcal{E}) - \triangle\rho_\alpha(-\mathcal{E}), \tag{2}$$

where $\mathcal{E} = |E| > 0$. The latter determines in turn the one-loop contribution E^1_{eff} to the effective energy [12], the induced fermion number Q, and the axial anomaly $\mathcal{A}$ of the massless Euclidean Dirac operator [13, 14]. One has

$$E^1_{eff} = \int_0^\infty \mathcal{E}\,\sigma_\alpha(\mathcal{E})\,d\mathcal{E}, \tag{3}$$

$$Q = -\frac{1}{2}\int_0^\infty \sigma_\alpha(\mathcal{E})\,d\mathcal{E}, \qquad \mathcal{A} = \int_0^\infty \sigma_\alpha(\mathcal{E})\,d\mathcal{E}, \tag{4}$$

and $Q = -\mathcal{A}/2$. By using the above relations one can check for the consistency of the result for either of these quantities. Moreover, thanks to the recent advances in the fabrication of microstructures, and in mesoscopic physics one can realize the AB potential and study its influence in many physical systems (see [15] for a recent review). The flux tube in the above experiments can be either penetrable [16] or impenetrable.

2 Basic properties of the Dirac Hamiltonian

Let us collect the basic properties of two (2d) and three (3d) dimensional Dirac Hamiltonians that will be used in the paper (see [7] for a recent review). The Dirac Hamiltonian $H(A)$ in the *standard* representation in a magnetic field (in units with $\hbar = c = 1$) is

$$H(A) = \begin{pmatrix} m & D^\dagger \\ D & -m \end{pmatrix}, \tag{5}$$

where

$$D = \begin{cases} \sum_{j=1}^{3} \sigma_j (p_j + A_j) & n = 3, \\ \sum_{j=1}^{2} (p_1 + A_1) + i(p_2 + A_2) & n = 2, \end{cases} \tag{6}$$

and $p_j = i\partial_j$. We have taken the standard representation of γ matrices in 2d,

$$\gamma^0 = \beta = \sigma_3, \qquad \gamma^1 = i\sigma_2, \qquad \gamma^2 = -i\sigma_1, \tag{7}$$

where σ_j are the Pauli matrices. Let us lift the coordinate z in the 3d Dirac Hamiltonian. Then

$$\boldsymbol{\sigma}\cdot\mathbf{p} = \sigma_1 p_1 + \sigma_2 p_2 = i\begin{pmatrix} 0 & \partial_1 - i\partial_2 \\ \partial_1 + i\partial_2 & 0 \end{pmatrix}. \tag{8}$$

This term is nothing but the massless 2d Euclidean Dirac operator. After some algebra one finds that the components (χ^1, χ^4) and (χ^3, χ^2) of the four-spinor $\chi = (\chi_1, \chi_2, \chi_3, \chi_4)$ combine to give the components of two spinors that satisfy 2d Dirac equations with *opposite* signs of the mass term. In two space dimensions, mass breaks the parity: one has two irreducible representation of the γ matrices since, in contrast to 3d, $\{\gamma_0, \gamma_1, \gamma_2\}$ and

$\{-\gamma_0, -\gamma_1, -\gamma_2\}$ are not unitary equivalent. The above decomposition just tells us why the 3d Dirac Hamiltonian with non-zero mass does not break parity despite the fact that the 2d Dirac Hamiltonian with non-zero mass does. Let us fix a 2d Dirac Hamiltonian $H(A)$ in the AB potential by fixing the sign of mass m and the flux $\alpha \geq 0$ to be positive, and $e = -|e|$. After separation of variables in polar coordinates it reduces to the direct sum, $H(A) = \oplus_l h_{m,l}$, of channel operators $h_{m,l}$ in $L^2[(0,\infty), rdr]$,

$$h_{m,l} = \begin{bmatrix} m & -i\left(\partial_r + \frac{\nu+1}{r}\right) \\ -i\left(\partial_r - \frac{\nu}{r}\right) & -m \end{bmatrix}, \tag{9}$$

where $\nu = l + \alpha$ [3]. One finds immediately that

$$h^*_{m,l}|_{m\to -m} = -h_{m,l}, \tag{10}$$

where '$*$' means the complex conjugation. Therefore, for $E = -\mathcal{E} < -m$, the scattering states are given by

$$\Psi_{-\mathcal{E};l}(t,r,\varphi) = \Psi^*_{\mathcal{E};l}(t,r,\varphi)|_{m\to -m}. \tag{11}$$

In what follows the parameter $\mathcal{E}$ will stand for $|E|$.

3 Impenetrable flux tube

Let us first consider the conventional set-up where wave functions are zero at the position of the flux tube and discuss the point spectrum [4]. The spectrum of the 2d Dirac Hamiltonian is in general *asymmetric*: $D \neq D^\dagger$ in contrast to three dimensions [7]. The Aharonov-Casher theorem [5] tells us that in a general finite-flux magnetic field $B(\mathbf{r})$,

$$\int_\Omega B(\mathbf{r})\, d^2\mathbf{r} = \Phi = \text{const}, \tag{12}$$

the 2d Dirac Hamiltonian has exactly either $\alpha - 1 = n - 1$ or $[\alpha] = n$ threshold states at one and only one of the thresholds $E = \pm m$ (in the present case at $E = m$) depending whether the flux α is an integer or not. In the case of the 2d massless Euclidean Dirac operator, the threshold states are actually zero modes. The reason is that the component of the threshold mode that would be multiplied by mass m is zero and so the threshold mode is the eigenmode of $\boldsymbol{\sigma}\cdot\mathbf{p}$, too [5, 7]. The proof of the theorem is an application of the Atiyah-Singer index theorem [6] and nobody seems to have checked its validity for

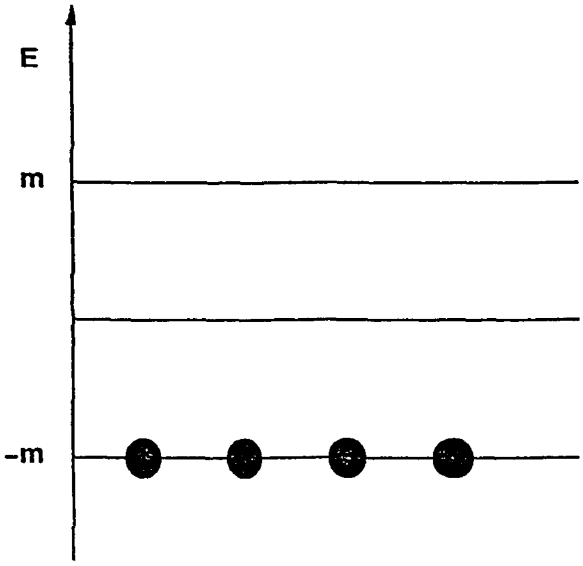

Figure 1: In the case of a 2d Dirac Hamiltonian in a generic finite-flux magnetic field the threshold states occur at one and only one threshold. In the case of the 2d Euclidean massless Dirac operator they are actually zero modes. There are, however, neither threshold nor zero modes in the AB potential.

singular field configurations. In the latter case one has to check for square integrability not only at infinity but at the position of singularities of the field, too. It is here where the theorems fail. If one takes their proof as presented, for example in Ref. [7], p. 198, and substitutes for $B(\mathbf{r})$,

$$B(\mathbf{r}) = 2\pi\alpha\,\delta(\mathbf{r}), \tag{13}$$

one finds immediately that $\phi(\mathbf{r})$, defined there as

$$\phi(\mathbf{r}) = \frac{e}{2\pi}\int_{R^2} \ln|\mathbf{r}-\mathbf{r}'|\,B(\mathbf{r}')\,d^2\mathbf{r}' = \alpha\ln|\mathbf{r}|, \tag{14}$$

can be calculated exactly. One finds

$$\phi(\mathbf{r}) = \alpha\ln|\mathbf{r}|. \tag{15}$$

The threshold (zero) modes presented there,

$$e^{-\phi(r)},\; e^{-\phi(r)}(x_1 - ix_2),\; \ldots,\; e^{-\phi(r)}(x_1 - ix_2)^{n-1}, \tag{16}$$

are then obviously singular at the origin and are not elements of $L^2[(0,\infty), rdr]$. One can show this directly by using $h_{m,l}$ as well. For a threshold state at $E = m$ to exist, the upper component χ_1 of the Dirac spinor has to obey

$$\left(\partial_r - \frac{\nu}{r}\right)\chi_1(r) = 0. \tag{17}$$

At the threshold $E = -m$ one then obtains

$$\left(\partial_r + \frac{\nu+1}{r}\right)\chi_2(r) = 0. \tag{18}$$

These two equations can be easily integrated. The solutions are

$$\chi_1(r) = r^{\nu} \qquad \text{and} \qquad \chi_2(r) = r^{-(1+\nu)}, \tag{19}$$

and they are obviously not in $L^2[(0,\infty), rdr]$ for any l. If they are square integrable at the infinity they are not so at the origin and *vice versa*.

In the case of a continuous spectrum the respective radial equation for the up (down) component of the spinor reduces to the Bessel equation with a δ-function potential,

$$H_l^{\pm}\chi(r) = k^2\chi(r) \tag{20}$$

where

$$H_l^{\pm} = -\frac{d^2}{dr^2} - \frac{1}{r}\frac{d}{dr} + \frac{\nu_{\pm}^2}{r^2} \pm g_m \frac{\alpha}{r}\delta(r). \tag{21}$$

Here $g_m = 2$ and $(\nu_+ = l + \alpha / \nu_- = l + 1 + \alpha)$ for the (up/down) component [17]. The dispersion relation is $k = \sqrt{E^2 - m^2}$. The spectrum of the Dirac Hamiltonian for $E > m$ is given in terms of the Bessel functions,

$$\Psi_{\mathcal{E},l} = \chi(r)e^{il\varphi}e^{-i\mathcal{E}t/\hbar}, \tag{22}$$

where

$$\chi(r) = \frac{1}{N}\begin{pmatrix} \sqrt{\mathcal{E}+m}\,(\varepsilon_l)^l J_{\varepsilon_l\nu}(kr) \\ i\sqrt{\mathcal{E}-m}\,(\varepsilon_l)^{l+1} J_{\varepsilon_l(\nu+1)}(kr)e^{i\varphi} \end{pmatrix}, \tag{23}$$

N is a normalization factor, and $\varepsilon_l = \pm 1$. Let us write $\alpha = n + \eta$ where n is the integer part $[\alpha]$ of α and η is the fractional part. Then square integrability at the origin fixes the sign of ε_l except for the channel $l = -n - 1$ [3]. The two-component solutions of the massive Dirac equation have only *one degree of freedom* that is reflected in the equality of up and down phase shifts [18]. For the conventional AB problem

$$\delta_l^u = \delta_l^d = \delta_l = \frac{1}{2}\pi(|l| - |l + \alpha|). \tag{24}$$

The AB potential is *long-ranged* and the phase shifts δ_l are *singular*: they do not depend on the energy and do not decay to zero for $E \to \infty$. In the channel $l = -n - 1$ *two different solutions with opposite phase shifts* are possible,

$$\chi^-(r) = \frac{1}{N} \begin{pmatrix} \sqrt{\mathcal{E}+m}\, J_{\eta-1}(kr) \\ i\sqrt{\mathcal{E}-m}\, J_\eta(kr) e^{i\varphi} \end{pmatrix}, \tag{25}$$

and

$$\chi^+(r) = \frac{1}{N} \begin{pmatrix} \sqrt{\mathcal{E}+m}\, J_{1-\eta}(kr) \\ -i\sqrt{\mathcal{E}-m}\, J_{-\eta}(kr) e^{i\varphi} \end{pmatrix}. \tag{26}$$

Note that in the limit $\eta \to 0$ the state $\chi^-(r)$ is *singular* and there is no such state in the spectrum at $\eta = 0$. By looking at the behaviour as $r \to \infty$ of these two solutions, one finds, for example that for $n = 0$, their phase shifts are

$$\delta^- = -\frac{1}{2}\pi\eta, \qquad \delta^+ = \frac{1}{2}\pi\eta. \tag{27}$$

If one takes the solution χ^+ for $l = -n - 1$ then the phase shifts for all l will be given by (24).

4 Self-adjoint extensions and the point spectrum

From the set (9) of radial Hamiltonians only that with $\nu < 1$ admits a non-trivial one-parametric family of self-adjoint extensions [3]. In this *single* channel that corresponds to $l = -n - 1$ ($l = -n$ for positive charge) the AB potential is in the so-called *limit circle case* at zero (see Ref. [19], p.152) where the boundary conditions have to be specified. Let us consider the influence of the bound state of energy E_b in this channel [3, 20],

$$B_{E_b;-n-1}(t, r, \varphi) = \frac{1}{N} \begin{pmatrix} \sqrt{m+E_b}\, K_{1-\eta}(\kappa r) \\ i\sqrt{m-E_b}\, K_\eta(\kappa r) e^{i\varphi} \end{pmatrix} e^{-i(n+1)\varphi} e^{-iE_b t/\hbar}. \tag{28}$$

Here $K_\nu(z)$ is the modified Bessel function [21] and $\kappa = \sqrt{m^2 - E_b^2}$. The wave function is in $L^2[(0,\infty), rdr]$ as it decays exponentially outside the flux tube and is square integrable at the origin. However, in the limit $E_b \to \pm m$ the square integrability is lost, in accord with our statement about the nonexistence of threshold states in the AB potential. In the presence of a bound state of energy E_b the scattering states have to be modified. The reason is that the Dirac Hamiltonian $h_{m,l}$ has to be a symmetric operator when defined on a dense subset of $L^2[(0,\infty), rdr]$, such as the set of absolutely continuous functions regular at the origin [3]. The condition implies that any two states $\chi_1(\mathbf{r})$ and $\chi_2(\mathbf{r})$ in the Hilbert space $L^2[(0,\infty), rdr]$ have to satisfy

$$r\,\phi^\dagger\sigma_1\xi = r\,\left[\chi_2^{*}\chi^1 + \chi_1^{*}\chi^2\right] \to 0 \qquad (r\to 0), \tag{29}$$

where σ_1 is the Pauli matrix. Eq. (29) is nothing but the boundary condition at zero. To satisfy (29) scattering states with energy $E > m$ in the channel where the bound state is present have to take a general form

$$\Psi_{\mathcal{E};-n-1}(t,r,\varphi) = \left[\chi^+(r)\,\sin\mu + \chi^-(r)\,\cos\mu\right] e^{-i(n+1)\varphi} e^{-i\mathcal{E}t/\hbar}, \tag{30}$$

where $\chi^+(r)$ and $\chi^-(r)$ are given respectively by (25) and (26) [22]. In contrast to [3] we shall not use the factor $(-1)^{n+1}$ for $\chi^-(r)$. Similarly, as in the non-relativistic case [10, 11], in the presence of a bound state of energy E_b the condition (29) determines the parameter μ of the self-adjoint extension to be a function of energy E,

$$\tan\mu = \frac{m-E_b}{(m^2-E_b^2)^\eta}\frac{(\mathcal{E}^2-m^2)^\eta}{\mathcal{E}-m}\,. \tag{31}$$

The formula is valid only for $E > m$. Thanks to (10) the scattering states at energies $E < -m$ are given by (11) where the substitution $m \to -m$ is made in μ as well. From now on we shall write if necessary $\mu_\pm$ for respectively positive and negative energies. A close inspection of (31) reveals that under $m \to -m$ both the sign and the value of $\tan\mu_+$ change,

$$\tan\mu_+ \to \tan\mu_- = -\frac{m+E_b}{(m^2-E_b^2)^\eta}\frac{(\mathcal{E}^2-m^2)^\eta}{\mathcal{E}+m}\,. \tag{32}$$

The change amounts to the following two changes in (31),

$$E_b \to -E_b, \qquad \mathcal{E} \to -\mathcal{E}. \tag{33}$$

One cannot simply replace $\mathcal{E}$ by $-\mathcal{E}$ in (31) to obtain the value of μ_- for the negative energy states.

The spectrum [see (30) and (31)] is different for different bound state energies E_b. Therefore, in physical terms, it is the bound state energy E_b that parametrizes different self-adjoint extensions. Now, by comparing the asymptotic behaviour of (30) for $r \to \infty$ with the behaviour when $\alpha = 0$ one finds the channel S-matrix $S_{-n-1} = e^{2i\delta_{-n-1}}$ with the phase shift given by

$$\delta_{-n-1}(E) = \frac{1}{2}\pi\alpha + \Delta_{-n-1}(E). \tag{34}$$

The change $\Delta_{-n-1} = \Delta_{-n-1}(E)$ of the conventional phase shifts is

$$\Delta_{-n-1} = -\arctan\frac{\sin(\eta\pi)}{\cos(\eta\pi) - \tan\mu}\,. \tag{35}$$

5 The density of states and the Krein-Friedel formula

As has been established earlier, the point spectrum consists at best of one point that lies in the interval $-m < E < m$. The contribution of scattering states to the change $\Delta N_\alpha(E)$ of the integrated DOS induced by the presence of a scatterer is given by the Krein-Friedel formula [8] directly as the sum over all phase shifts,

$$\Delta N_\alpha(E) = N_\alpha(E) - N_0(E) = (2\pi i)^{-1}\ln\det\mathrm{S} = \frac{1}{\pi}\sum_l \delta_l(E), \tag{36}$$

S being the total on-shell S-matrix. The integrated density of states $N_\alpha(E)$ here is defined as

$$N_\alpha(E) = \int_{-\infty}^{E} \rho_\alpha(E')\,dE', \tag{37}$$

where

$$\rho_\alpha(E) = -\frac{1}{\pi}\mathrm{Im}\mathrm{Tr}\,G_\alpha(\mathrm{x},\mathrm{x},E+i\epsilon), \tag{38}$$

and $G_\alpha(\mathrm{x},\mathrm{x},E+i\epsilon)$ is the resolvent of H in the AB potential with the flux corresponding to α. The fact that phase shifts can be rather easily calculated without any care of the proper normalization of wave functions greatly facilitates the calculation. Moreover, by means of the Krein-Friedel formula it is rather easy to calculate the change of the IDOS for all possible self-adjoint extensions of (21). In the conventional AB set-up the phase shifts are given by (24) and by using the ζ-function regularization one finds,

$$\begin{aligned}\ln\det\mathrm{S} &= \sum_{l=-\infty}^{\infty} 2i\delta_l \\ &= i\pi\sum_{l=-\infty}^{\infty}(|l| - |l+\alpha|) = i\pi\left[2\sum_{l=1}^{\infty} l^{-s} - \sum_{l=0}^{\infty}(l+\eta)^{-s} - \sum_{l=1}^{\infty}(l-\eta)^{-s}\right]\Bigg|_{s=-1} \\ &= i\pi\left[2\zeta_R(s) - \zeta_H(s,\eta) - \zeta_H(s,1-\eta)\right]\big|_{s=-1} = -i\pi\eta(1-\eta),\end{aligned} \tag{39}$$

where ζ_R and ζ_H are the Riemann and the Hurwitz ζ-functions. In the presence of the bound state the phase shift is changed only in the channel $l = -n-1$ and one obtains for the change of the IDOS

$$\triangle N_\alpha(E) = -\frac{1}{2}\eta(1-\eta) - \frac{1}{\pi}\arctan\frac{\sin(\eta\pi)}{\cos(\eta\pi)-\tan\mu}\,. \tag{40}$$

For $E > m$ one finds that $\tan\mu_+$ is positive [see (31)] and hence there is a typical *(anti)resonance* in the relativistic case for $0 < \eta < 1/2$ at the energy $\mathcal{E}_{res}$ which has the form

$$\frac{\mathcal{E}_{res}-m}{(\mathcal{E}_{res}^2-m^2)^\eta} = \frac{1}{\cos\eta\pi}\frac{m-E_b}{(m^2-E_b^2)^\eta}. \tag{41}$$

The phase shift $\delta_{-n-1}(E)$ (34) changes by $-\pi$ in the direction of increasing energy and the integrated DOS (40) has a sharp decrease by one. For $m - E_b << 1$ one finds

$$\mathcal{E}_{res} \sim m + (\cos\eta\pi)^{1-\eta}(m-E_b), \tag{42}$$

and $\mathcal{E}_{res} \downarrow m$ in the limit $E_b \uparrow m$. In the limit $E_b \uparrow m$ the antiresonance merges with the bound state. For $1/2 < \eta < 1$ the cosine in (40) changes its sign and the resonance disappears at positive energies. However, at negative energies $\tan\mu_-$ is given by (32) and has a diffferent sign, too. Therefore, the resonance will occur now at *negative* energies [23]. However, in distinction to the previous case the (anti)resonance obviously will not merge with the bound state provided $E_b \uparrow m$. $\eta = 1/2$ is a special point since the antiresonance is at infinity. Irrespective of the value of η, one finds for any fixed $E > m$ that $\tan\mu_+ \to 0$ in the limit $E_b \uparrow m$. Therefore, in this limit, $\mu_+ = 0$ for positive energies [see (31)]. For a fixed negative energy then [see (32)]

$$\tan\mu_- \to -\infty, \tag{43}$$

and hence $\mu_- = -\pi/2$. Therefore, at the upper threshold the scattering states (30) are given only in terms of χ^- while at the lower threshold only in terms of χ^+. Thus, one has the *phase-shift flip* at positive energies and the conventional phase shift (24) at negative energies. In other words, the phase-shift flip occurs only either at positive energies or only at negative energies depending on whether the bound state (28) merges with the (anti)resonance into the continuous spectrum at the upper or at the lower threshold. Hence, in the present case, the contribution of *scattering* states with energy $E \geq m$ to the change of the DOS is

$$\triangle\rho_\alpha(E) = \rho_\alpha(E) - \rho_0(E) = -\frac{1}{2}\eta(1+\eta)\,\delta(E-m). \tag{44}$$

while the contribution of *scattering* states with $E \leq -m$ to the change of the DOS is

$$\triangle\rho_\alpha(E) = -\frac{1}{2}\eta(1-\eta)\,\delta(E+m). \tag{45}$$

Note that the change of the DOS is concentrated at the thresholds where it is proportional to delta functions.

Since the phase shifts are known one can calculate differential scattering cross sections. Similarly, such as in the nonrelativistic case [10, 11] one finds for $\varphi \neq 0$ that

$$\left(\frac{d\sigma}{d\varphi}\right)(k,\varphi) = \left(\frac{d\sigma^0}{d\varphi}\right)(k,\varphi) + \frac{8\pi}{k}\sin^2\Delta_{-n-1} + \frac{4}{k}\frac{\sin(\pi\alpha)}{\sin(\varphi/2)}\sin\Delta_{-n-1}\cos\left(\Delta_{-n-1}+\pi\alpha-\varphi/2\right), \tag{46}$$

where

$$\left(\frac{d\sigma^0}{d\varphi}\right)(k,\varphi) = \frac{1}{2\pi k}\frac{\sin^2(\pi\alpha)}{\sin^2(\varphi/2)}. \tag{47}$$

is the conventional differential scattering cross section. Note that in contrast to the conventional case, in the presence of a bound state the differential scattering cross section becomes *asymmetric* with regard to $\varphi \to -\varphi$ and give rise to the Hall effect [11]. The asymmetry is easy to understand as one has only one bound state (28) which, for $\alpha \geq 0$, occurs for $l < 0$.

6 Regularization, renormalization, $R \to 0$ limit, and the interpretation of selfadjoint extensions

To identify the physics that corresponds to different self-adjoint extensions one starts with a flux tube of finite radius R and a magnetic field B satisfying (12), and considers the limit $R \to 0$ [3, 24]. The limit is curious in the following sense. Since the flux tube is not singular any more the Aharonov-Casher [5, 7] (or index [6]) theorem applies and one has, in general, $[\alpha]-1$ or $[\alpha]$ threshold (zero) modes (at the lower thershold) depending on whether the flux is an integer or not. Then, in the limit $R \to 0$ these modes merge with the continuous spectrum. To obtain the bound state in the spectrum for $R \neq 0$, the resulting interaction inside the flux tube for both up and down components of the Dirac spinor has to be an attractive potential $V(r)$ of strength at least equal to

$$V(r)|_{r\leq R} = -\frac{\hbar^2}{2m}\frac{\alpha}{R^2}c(R), \tag{48}$$

where $V(r) = 0$ for $r > R$. Here, $c(R) = 2[1 + \varepsilon(R)]$ and $\varepsilon(R) > 0$ [24, 25]. Note that in the limit $R \to 0$

$$V(r)|_{r\leq R} \to -[1 + \varepsilon(0)]\frac{\hbar^2}{m}\frac{\alpha}{r}\delta(r). \tag{49}$$

If one starts with a homogeneous field regularization [$B(x, y) = const$ inside the flux tube] the magnetic moment coupling induces an *attractive* potential $V_m(r)$ for the *down* component,

$$V_m(r)|_{r\leq R} = -g_m\frac{\alpha}{R^2}, \tag{50}$$

and a *repulsive* potential $-V_m(r)$ for the *up* component inside the flux tube. Here g_m is the (possibly anomalous) magnetic moment. Therefore, in order to obtain a bound state in the spectrum the potential $W_c(r)$,

$$W_c(r) = V(r) + V_m(r), \tag{51}$$

has to be put in by hand. Here $V_m(r)$ cancels the repulsive potential $-V_m(R)$ for the up component. An arbitrary weak attractive potential *cannot* lead to bound states (cf. [17]). For a bound state to survive as $R \to 0$ the coupling constant has to be renormalized, $\varepsilon(R) \to 0$ as $R \to 0$ [26]. The potential $W_c(r)$ is then the *critical* potential. For either weaker or for stronger potentials and without the renormalization, bound states do not survive the limit $R \to 0$ [11]. If $\varepsilon(R)$ in (48) is small but kept constant as $R \to 0$ then, in the nonrelativistic case, $E_b \to -\infty$ [10, 11, 27]. Nevertheless, in the limit $R \to 0$ bound states decouple. This is seen in the phase shifts, since $\Delta_l(E) \to 0$ when $E_b \to -\infty$ [10, 11]. In the relativistic case then the bound states moves through the gap $(-m, m)$ from one continuum into another and no trace of them remains as $R \to 0$, except when $W_c(R)$ is present. The same situation persists with the cylindrical shell regularization but the number of bound states for a given α can be *different* [10, 11].

7 Spectral asymmetry, effective energy, induced fermion number, and the axial anomaly

In the absence of the bound state, the spectral asymmetry is determined by the continuous part of the spectrum. One finds from (44) and (45) that the contribution is actually determined by the phase-shift flip,

$$\sigma_\alpha(\mathcal{E}) = -\eta\,\delta(\mathcal{E} - m). \tag{52}$$

Therefore, according to (3),

$$E^1_{eff} = -m\eta, \tag{53}$$

and consequently, according to (4),

$$Q = \frac{1}{2}\eta, \qquad \mathcal{A} = -\eta. \tag{54}$$

When the bound state (28) is present then

$$\sigma_\alpha(\mathcal{E}) = \text{sign}(E_b)\,\delta(\mathcal{E} - |E_b|) + \frac{1}{\pi}\frac{d}{d\mathcal{E}}\left\{\left[\Delta^+_{-n-1}(\mathcal{E}) - \Delta^-_{-n-1}(\mathcal{E})\right]\Theta(\mathcal{E} - m)\right\}, \tag{55}$$

where Θ is the Heaviside step function, and $\Delta^\pm_{-n-1}(\mathcal{E})$ is the change of the conventional phase shift (35) respectively for positive and negative energy states in the $l = -n-1$ channel. The superscripts '$\pm$' means that the value of $\tan\mu_\pm$ is taken in (35). The values of E^1_{eff}, Q, and $\mathcal{A}$ are then obtained by substituting (55) in (3) and (4). It should be stressed out that $\sigma_\alpha(\mathcal{E})$ can have both signs depending on the sign of the mass in a 2d Dirac Hamiltonian (6), (9). In the case of the 3d Dirac Hamiltonian then

$$\sigma_\alpha(\mathcal{E}) = E^1_{eff} = Q = 0. \tag{56}$$

8 Energy calculations

As has been shown, in the case of the Dirac Hamiltonian (in contrast to the Schrödinger Hamiltonian) the bound state is not present when the flux tube is regularized with a given distribution of magnetic field that satisfies (12). In this case the one-loop contribution E^1_{eff} to the effective energy is given by (53). From the 3d point of view the contributions then cancel. On the other hand the classical energy of the magnetic field (58) tends to infinity as $R \to 0$ (when the total flux Φ is kept fixed). By summing the two contributions one finds that the total energy changes by an infinite positive amount. Therefore, up to one-loop the matter is *stable* against a spontaneous field formation. The latter statement is in accord with the so-called *diamagnetic inequality* [28].

In the nonrelativistic case in $2+1$ dimensions we have shown, in an idealistic situation when magnetic moment $g_m > 2$ is kept constant and dynamics is ignored that a quantum-mechanical instability against a magnetic field formation arises. The reason is the formation of bound states that decouple from the Hilbert space by taking away negative energy [10, 11]. One finds that whenever the ratio of the energy at rest to the electromagnetic energy satisfies an inequality

$$\frac{mc^2}{e^2} < X(\alpha, g_m), \tag{57}$$

the total energy of the field and matter together goes to $-\infty$ as $R \to 0$, where R is the radius of the flux tube. The flux tube has been regularized by a homogeneous magnetic field $B(\mathbf{r})$. Note that the homogeneous magnetic field optimizes the energy functional

$$E = \int_\Omega B^2(\mathbf{r})\, d^2\mathbf{r} \tag{58}$$

subject to the constraint (12). The function $X(\alpha, g_m)$ is determined by the solutions x_l of the matching equation,

$$X(\alpha, g_m) = \frac{1}{4\pi\alpha^2}\sum_l x_l^2(\alpha, g_m) \geq 0. \tag{59}$$

The relativistic treatment shows that the instability is in fact a nonrelativistic artefact.

9 Gravitational vortex

It is illustrative to check our calculations in the AB potential for the case of a gravitational vortex [29]. A localized massive particle of mass M in a 2+1 dimensional space-time induces a conical geometry with a deficit angle $\delta\varphi = 2\pi(1-\gamma) \equiv 8\pi GM$, where G is Newton's constant. Solutions of the Dirac equation in this case are given essentially by (22-23) provided $\varepsilon_l \nu = \varepsilon_l(l+\alpha)$ is replaced by $\nu = |l|/\gamma$ [29]. Since

$$0 < \gamma \leq 1, \tag{60}$$

it is impossible to find a nontrivial self-adjoint extension because one of the components of the Dirac spinor is always given by a Bessel function of order $\nu \geq 1$. Hence, the phase shift for all l has the conventional form,

$$\delta_l^u = \delta_l^d = \delta_l = \frac{1}{2}\pi|l|\left(1-\gamma^{-1}\right) \tag{61}$$

(see (24) or [29]) and no phase-shift flip occurs. By using the Krein-Friedel formula, the change of the DOS is given by

$$\Delta\rho_\gamma(E) = -\frac{1}{12}\left(1-\gamma^{-1}\right)\left[\delta(E-m)+\delta(E+m)\right], \tag{62}$$

and the spectral asymmetry $\sigma_\gamma(E) \equiv 0$. Hence, there is no induced charge in this case in accord with [30].

10 The Klein-Gordon equation

Let us consider the Klein-Gordon Hamiltonian with the minimal and an additional *non-minimal* Pauli coupling,

$$J^\mu \left[eA_\mu - (1/2)g\lambda^{-2}\varepsilon_{\mu\nu\sigma}F^{\nu\sigma}\right], \tag{63}$$

with λ an arbitrary scale parameter. It is peculiar to $2+1$ dimensions that the Pauli coupling exits without any reference to the spin [31]. The first term couples the current to the gauge potential while the second, nonminimal terms, couples the current directly to the magnetic field and as such it is an analogue of the magnetic moment coupling in the Pauli Hamiltonian [31]. By using separation of variables in polar coordinates one then arrives exactly at H_l^+ [see (21)]. The latter is nothing but the radial part of the Pauli Hamiltonian [10, 11, 17] with only one change: the dispersion has the relativistic form, $k = \sqrt{E^2 - m^2}$. In contrast to the Dirac equation the wave function of a scattering state has a single component which is given in terms of Bessel functions of order $|l+\alpha|$. Therefore, the equivalent results for the Klein-Gordon equation are obtained by substituting the relativistic dispersion into nonrelativistic results given in [10, 11]. Bound states can occur in two channels, $l = -n$ and $l = -n-1$. The parameter μ_l in these channel is then determined by [10, 11, 22]

$$\begin{aligned} \cot\mu_{-n} &= -A_{-n} = -(k/\kappa_{-n})^{2\eta}, \\ \cot\mu_{-n-1} &= -A_{-n-1} = -(k/\kappa_{-n-1})^{2(1-\eta)}, \end{aligned} \tag{64}$$

where $\kappa_l = \sqrt{m^2 - E_l^2}$ is as above, with E_l, $0 \le E_l \le m$, being the binding energy either in the $l = -n$ or in the $l = -n-1$ channel. The corresponding change of the conventional phase shift (24) (the latter is the same as in the case of the Schrödinger and the Dirac equations) is then

$$\Delta_l = \arctan\left(\frac{\sin(|l+\alpha|\pi)}{\cos(|l+\alpha|\pi) - A_l^{-1}}\right). \tag{65}$$

For $0 < \eta < 1/2$ the *resonance* appears in the $l = -n$th channel,

$$E_{res}^2 - m^2 = \frac{m^2 - E_{-n}^2}{[\cos(\eta\pi)]^{1/\eta}} > 0. \tag{66}$$

The phase shift $\delta_{-n}(E)$ changes by π [see (65)] in the direction of increasing energy and the change of the integrated density of states,

$$\triangle N_\alpha(E) = -\frac{1}{2}\eta(1-\eta) + \frac{1}{\pi}\arctan\left(\frac{\sin(\eta\pi)}{\cos(\eta\pi) - A_{-n}^{-1}}\right) - \frac{1}{\pi}\arctan\left(\frac{\sin(\eta\pi)}{\cos(\eta\pi) + A_{-n-1}^{-1}}\right), \tag{67}$$

has a sharp increase by one. For $1/2 < \eta < 1$ the resonance is then shifted to the $l = -(n+1)^{\text{th}}$ channel.

11 Discussion of the results

Closed analytical results have been obtained for the density of states $\rho_\alpha(E)$ and the spectral asymmetry $\sigma_\alpha(\mathcal{E})$ of the 2d Dirac Hamiltonian induced by the Aharonov-Bohm potential, and consequently for the one-loop contribution E^1_{eff} to the effective energy, the induced charge Q, and the axial anomaly $\mathcal{A}$ of the Dirac operator. These quantities have been calculated for different self-adjoint extensions when generically, a bound state is present in the spectrum. Physically this corresponds to the situation when an attractive δ-function potential $W_c(r)|_{R\to 0}$ (51) is put on top of the AB potential. In the case of the "magnetic" AB potential, a repulsive interaction is induced for the spin up component of the Dirac spinor and hence, the attractive potential may be easier to realize in the AB potential of non-magnetic origin, such as in the field of a cosmic string. A nontrivial self-adjoint extension manifests itself by an asymmetric differential scattering cross section which has been calculated here. Our result (54) for Q gives another example where a transcendental charge is induced (cf. [32]). Our result for the axial anomaly $\mathcal{A}$ [see (4) and (55)] gives an answer to [33] where an attempt to find its analytical form was made. In a particular case when the bound state is absent, our result for $\triangle\rho_\alpha(E)$ are consistent with an earlier calculations of the spectral asymmetry $\sigma_\alpha(\mathcal{E})$, Q, and $\mathcal{A}$ [14].

The Aharonov-Casher and index theorems have been corrected for singular field configurations. One has to check for square integrability of solutions at the position of a singularity, too. There are neither threshold nor zero modes in the AB potential and the index of massless 2d Euclidean operator is zero. The index is *discontinuous* in the limit $R \to 0$ since for any $R \neq 0$ the Aharonov-Casher and index theorems hold. The relevance of the phase-shift flip for the axial anomaly in a general finite-flux field has been discussed in [34]. Their treatment, however, differs from ours since they considered a complementary situation of a regular field configuration and the limit $k \to 0$. We do not find any instability of minimally-coupled relativistic matter in $2+1$ dimensions (cf. [35] when the Abelian Chern-Simons term is present). Due to the symmetry of the spectrum of the 3d Dirac Hamiltonian any question about the instability is pointless (cf. the suggestion of [36] for a 'flux spaghetti' vacuum in the spirit of [37] as a mechanism for avoiding the divergence of perturbative QED).

The formal technical similarity between scattering in the AB potential and scattering in the field of a cosmic string [2, 3] enables us to make the conclusion that as in the former case a *persistent current* [38] will also occur in the field of a cosmic string. A persistent current is essentially due to the momentum that electrons acquire in the AB potential. According to the formulae in [11, 39, 40] the scattering-state contribution to the persistent current is directly related to the change $\triangle N_\alpha(E)$ of the integrated DOS (IDOS),

$$dI(E,\alpha) = \partial_\alpha[\triangle N_\alpha(E)]dE, \tag{68}$$

where $dI(E,\alpha)$ is the differential contribution to the persistent current at energy E. Because (68) involves only the IDOS, in an ideal situation measurements of a persistent current can test the above theoretical results. The persistent current is defined with respect to a point. It is given by the total current through a line that extends from that point to infinity, in the absence of currents through the external leads [39]. A similar scenario can also occur in the field of a gravitational vortex. A self-adjoint extension is actually the $R \to 0$ limit, where R is the radius of a flux tube. Experimentally, infinitely thin means nothing but that the radius of the flux tube is negligibly small when compared to any other length, such as a wavelength of particles, in the system. Therefore, this is the regime in which our results can be applied. The parameter $\triangle_{-n}$ of a particular self-adjoint extension is then determined by a bound state energy in the $l = -n$ channel.

Note that one has the unitary equivalence between a spin 1/2 charged particle in a 2d magnetic field and a spin 1/2 neutral particle with an anomalous magnetic moment in a 2d electric field [41] and our results apply to the this case as well.

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