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ABSTRACT

The linear stability problem of the rotational motion of a top around a fixed point containing an inner cavity filled up with a viscous fluid is considered. The effect of the viscosity in the stability problem is studied.

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1 Introduction

We will consider the stability problem of the rotational motion of a top around a fixed point with a cavity filled up with a viscous fluid.The study of this problem started in the fourties with S.L.Sobolev.He considered only the case of an ideal fluid.Sobolev's paper has not been well known (it was published in a journal of physics,approximately 15 years after his research, see [14]) and only the ideas and results contained in his subsequent papers,connected with the linearized systems of the equation describing the motion a rotating fluid,have been a source of important new directions,both in the theory of Partial Differential Equations as well as in the Spectral Theory of abstract operators (see [3], [4] and [5]).In his not less remarkable research about the stability of this system (which we call from now on Sobolev's systems) Sobolev found that an ellipsoidal form of the inner cavity is preferable in order to communicate a stable movement to the top.

In the present paper the following result is proved:for any angular velocity ω in Sobolev's system,different from a certain value ω^* there is a value of the viscosity $\nu(\omega)$ such that,for every $\nu > \nu(\omega)$ the system is instable.The conditions under which this result is obtained are:the rigid body (and so the inner cavity) is symmetric with respect to a vertical axis and the inner cavity has an order $m(m > 2)$ of symmetry under rotation around this vertical axis.[3]

2 System of equations

In this section we shall introduce the equations representing the oscillation of Sobolev's system under small perturbations.
Let (x', y', z') be an inertial system of orthogonal coordinates such that its origin coincides with a fixed point of Sobolev's system placed in the base of the rigid body.Let us assume that the Sobolev's system rotates uniformly with constant angular velocity ω around the axis z' (which is also an axis of symmetry) i.e.,both the rigid body and the viscous rotate at the same velocity ω.We consider also a second system (x, y, z) in such a way that the axis z coincides with the axis z' and have also its origin in the fixed point.We shall assume the system (x, y, z) rotating uniformly with angular velocity ω around the axis z'.We remark that before the perturbation such system is fixed to the rigid body.
Initially the coordinates of the center of gravity of the rigid body $(X'_{cm}, Y'_{cm}, Z'_{cm})$ referred to the system (x', y', z') will be $X'_{cm} = Y'_{cm} = 0$ and $Z'_{cm} = l_1$,where l_1 is the distance of this point to the origin.If the Sobolev's system is slightly disturbed,the rigid body will start to oscillating and therefore its center of gravity shall describe a path represented by a vector radius (w.r.t the system at rest).Now we shall start the physical modeling of the Sobolev's system under slight perturbations.
We just take the equations for the viscous fluid from the hydrodynamics.With respect to the system (x', y', z') they have the following form:

$$\frac{\partial v'}{\partial t} - \nu \Delta v' + \frac{1}{\rho} \nabla p' = F' - g\vec{k}. \tag{2.1}$$

and

$$\text{div } v' = 0. \tag{2.2}$$

[3]If we consider the inner cavity Ω as a set of points in the space,after a rotation of angle $\frac{2\pi}{m}$,we recover Ω.

where v' denotes the field of velocity of the fluid,p' the hydrodynamical pressure,$\rho F'$ is the field of the external forces, ν the viscosity coefficient,ρ the fluid's density and g the gravitational constant.Here and in what follows F' will be a given solenoidal vector field.The equation (2.1) is the linearized equation of Navier-Stoke and (2.2) the continuity equation.In this last equation the divergence of the field v' is taken w.r.t (x', y', z').
Let us write the equations (2.1) and (2.2) in the system (x, y, z).It is easy to see that

$$\frac{\partial v'}{\partial t} = \frac{\partial v}{\partial t} - 2\omega(v \times \vec{k}) - \omega^2(x, y, 0). \tag{2.3}$$

where $\vec{k} = (0, 0, 1)$.Let

$$p' = -\rho g z + \omega^2 \rho \frac{(x^2 + y^2)}{2} + p = p_0 + p. \tag{2.4}$$

In (2.3) and (2.4) v and p are the deviations of the field of velocity and the pressure respectively written in coordinates (x, y, z).From (2.3) and (2.4) the equations (2.1) and (2.2) can be written in the following form

$$\frac{\partial v}{\partial t} - 2\omega(v \times \vec{k}) - \nu \Delta v + \frac{1}{\rho}\nabla p = F, \tag{2.5}$$

$$\mathrm{div}\, v = 0 \tag{2.6}$$

where ρF is the field of the external forces expressed in the coordinates (x, y, z).
The vectorial equation describing the small oscillations of the rigid body after the perturbation is the following

$$\frac{d\mathcal{L}'}{dt} = \mathcal{N}' \tag{2.7}$$

where $\mathcal{L}'$ is the angular momentum of the rigid body w.r.t the fixed point placed in its base and $\mathcal{N}'$ is the momentum of the forces influence on the rigid body w.r.t the same fixed point.These forces are the gravity,the pressure of the fluid and the friction.Let us introduce a third system of orthogonal coordinates,whose versors on every one of the axis are given by the vectors

$$i'_{cm} = \frac{\dot{r}'_{cm} \times r'_{cm}}{l_1^2}, \qquad j'_{cm} = \frac{\dot{r}'_{cm}}{l_1}, \qquad k'_{cm} = \frac{r'_{cm}}{l_1}.$$

Recall that we have assumed that the rigid body is symmetric w.r.t the axis z' (and therefore also w.r.t the axis defined by the vector k'_{cm}).Under this only condition we have

$$\mathcal{L}' = \omega I_{\parallel} k'_{cm} - I_{\perp} i'_{cm}. \tag{2.8}$$

here

$$I_{\perp} = \int \left(r'^2 - (x'_2)^2\right) dr', \qquad I_{\parallel} = \int \left(r'^2 - (z'_2)^2\right) dr'$$

where $r' = x'_2 i'_{cm} + y'_2 j'_{cm} + z'_2 k'_{cm}$ and the integration is in the region occupied by the rigid body .Deriving (2.8) we obtain

$$\frac{d\mathcal{L}'}{dt} = \omega I_{\parallel} \frac{\dot{r}'_{cm}}{l_1} - I_{\perp} \frac{\ddot{r}'_{cm} \times r'_{cm}}{l_1^2} \tag{2.9}$$

projecting (2.9) on the axes x' and y' one gets the following relations:

$$\frac{d\mathcal{L}'}{dt} \cdot i' = \omega I_{\parallel} \frac{\dot{X}'_{cm}}{l_1} - I_{\perp} \frac{\left(\ddot{Y}'_{cm} Z'_{cm} - Y'_{cm} \ddot{Z}'_{cm}\right)}{l_1^2} \tag{2.10}$$

and

$$\frac{d\mathcal{L}'}{dt} \cdot j' = \omega I_{\parallel} \frac{\dot{Y}'_{cm}}{l_1} - I_{\perp} \frac{\left(\ddot{Z}'_{cm} X'_{cm} - Z'_{cm} \ddot{X}'_{cm}\right)}{l_1^2} \tag{2.11}$$

linearizing (2.10) and (2.11) in a neighborhood of the values of the coordinates of the center of gravity before the perturbation,we obtain the expressions

$$\frac{d\mathcal{L}'}{dt} \cdot i' = \omega I_{\parallel} \frac{\dot{X}'_{cm}}{l_1} - I_{\perp} \frac{\ddot{Y}'_{cm}}{l_1} \tag{2.12}$$

and

$$\frac{d\mathcal{L}'}{dt} \cdot j' = \omega I_{\parallel} \frac{\dot{Y}'_{cm}}{l_1} + I_{\perp} \frac{\ddot{X}'_{cm}}{l_1} \tag{2.13}$$

Having in mind the moment of the gravity,the pressure and the friction,after a calculation we get

$$\mathcal{N}' \cdot i' = -g M_1 \frac{Y'_{cm}}{l_1} + \rho \int_{\Omega} \left(z' \frac{\partial v'_{y'}}{\partial t} - y' \frac{\partial v'_{z'}}{\partial t} \right) d\Omega \tag{2.14}$$

and

$$\mathcal{N}' \cdot j' = -g M_1 \frac{X'_{cm}}{l_1} + \rho \int_{\Omega} \left(x' \frac{\partial v'_{z'}}{\partial t} - z' \frac{\partial v'_{x'}}{\partial t} \right) d\Omega \tag{2.15}$$

where Ω is the region occupied by the fluid and M_1 is the mass of all system.
Let $\phi' = X'_{cm} + iY'_{cm}$.From (2.7) and (2.12) – (2.15) it follows

$$I_{\perp} \ddot{\phi}' + i\omega I_{\parallel} \dot{\phi}' + i(\mathcal{N}' \cdot i' + i\mathcal{N}' \cdot j') = 0.$$

In the system (x, y, z) the above the equation can be written in the form

$$I_{\perp} \ddot{\phi} + i\omega(2I_{\perp} - I_{\parallel})\dot{\phi} + \omega^2(I_{\parallel} - I_{\perp})\phi + i(\mathcal{N}_x + i\mathcal{N}_y) = 0, \tag{2.16}$$

where $\phi = e^{-i\omega t}\phi'$.Finally just remark that

$$\mathcal{N}_x + i\mathcal{N}_y = -iN(v) + igM_1 l_1 \phi - iN_0^1. \tag{2.17}$$

where N_0^1 is a constant and

$$N(v) = \rho \int_{\Omega} r \cdot \frac{\partial v}{\partial t} d\Omega - 2\omega\rho \int_{\Omega} r \cdot (v \times \vec{k}) d\Omega. \tag{2.18}$$

Here $r = (z, iz, -(x + iy))$.From (2.16) and (2.17)

$$I_{\perp} \ddot{\phi} + i\omega(2I_{\perp} - I_{\parallel})\dot{\phi} + (\omega^2(I_{\parallel} - I_{\perp}) - gM_1 l_1)\phi + N(v) + N_0^1 = 0. \tag{2.19}$$

Let as before Ω be the region occupied by the fluid.Since on $\partial\Omega$ (the boundary of Ω) both the velocity of the fluid and the rigid body must be equal,we have

$$\begin{aligned} v'_{x'} &= z'\omega\frac{Y'_{cm}}{l_1} - y'\omega + z'\frac{\dot{X}'_{cm}}{l_1}, \\ v'_{y'} &= -z'\omega\frac{X'_{cm}}{l_1} + x'\omega + z'\frac{\dot{Y}'_{cm}}{l_1}, \\ v'_{z'} &= \omega\frac{(y'X'_{cm} - x'Y'_{cm})}{l_1} - \frac{\left(y'\dot{Y}'_{cm} + x'\dot{X}'_{cm}\right)}{l_1} \end{aligned}$$

on $\partial\Omega$.In the system (x, y, z) this boundary condition is

$$v = \left(z\frac{\dot{X}_{cm}}{l_1}, z\frac{\dot{Y}_{cm}}{l_1}, -\frac{\left(y\dot{Y}_{cm} + x\dot{X}_{cm}\right)}{l_1}\right)$$

It is not difficult to see that we can write it in the following form

$$v = \frac{1}{2}\left(\bar{r}\dot{\phi} + r\dot{\bar{\phi}}\right). \tag{2.20}$$

Here the bar means complex conjugation.

3 Sobolev's form of the equations

It follows from (2.5),(2.6) and (2.18) – (2.20) that the linear equations describing the small oscillations of Sobolev's system in a neighbourhood of the initial state $v = 0, p = p_{(0)}$ and $\phi = 0$ are:

$$\frac{\partial v}{\partial t} - 2\omega(v \times \vec{k}) - \nu\Delta v - \frac{1}{\rho}\nabla p = F, \qquad (\Omega) \tag{3.1}$$

$$\text{div } v = 0, \qquad (\Omega) \tag{3.2}$$

$$v = \frac{1}{2}(\bar{r}\dot{\phi} + r\dot{\bar{\phi}}), \qquad (\partial\Omega) \tag{3.3}$$

and

$$I_\perp\ddot{\phi} + i\omega(2I_\perp - I_\parallel)\dot{\phi} + (\omega^2(I_\parallel - I_\perp) - gM_1l_1)\phi + N(v) + N_0^1 = 0. \tag{3.4}$$

where

$$N(v) = \rho\int_\Omega r\cdot\frac{\partial v}{\partial t}d\Omega - 2\omega\rho\int_\Omega r\cdot(v\times\vec{k})d\Omega. \tag{3.5}$$

We recall that in (3.3) and (3.4) the dot means derivative with respect to time while in (3.5) it means the usual scalar product. In what follows we will make use of the system (3.1) - (3.5) written in Sobolev's form.In order to get that form we represent the quantities v,F and p as finite Fourier series due to the invariance under rotations of angle $2\frac{\pi}{m}$ of the cavity Ω.More exactly,proceeding as Sobolev did in the ideal case,it is possible to obtain

complex vector fields $v_{[s]}$,$F_{[s]}$ and complex functions $p_{[s]}$,$s = 0, \pm1, \pm2, \cdots, \pm(m-1)$,such that

$$v = \frac{1}{4}\left(\sum_0^{(m-1)} v_{[s]} + \sum_{-(m-1)}^0 v_{[s]}\right), \tag{3.6}$$

and

$$p = \frac{1}{4}\left(\sum_0^{(m-1)} p_{[s]} + \sum_{-(m-1)}^0 p_{[s]}\right), \qquad F = \frac{1}{4}\left(\sum_0^{(m-1)} F_{[s]} + \sum_{-(m-1)}^0 F_{[s]}\right). \tag{3.7}$$

where the pairs $(v_{[s]}, p_{[s]})$ as will be shown later satisfy more convenient equations.Now we realize the following program:
Let φ be a function of the real variables x, y, z defined in the region occupied by the fluid.Let us introduce complex coordinates

$$x + iy = \xi, x - iy = \bar{\xi}$$

Starting with φ one can introduce m new functions.Let $s = 0, \cdots, m-1$.We define

$$\varphi_{(s)}(\xi, \bar{\xi}, z) = \frac{1}{m}\sum_{l=0}^{m-1} e^{\frac{s2\pi il}{m}}\varphi(\xi e^{\frac{2\pi il}{m}}, \bar{\xi}e^{-\frac{2\pi il}{m}}, z)$$

It is not difficult to prove that

$$\varphi(\xi, \bar{\xi}, z) = \sum_{s=0}^{m-1}\varphi_{(s)}(\xi, \bar{\xi}, z). \tag{3.8}$$

Note that $\varphi_{(s)}(\xi, \bar{\xi}, z)$ have a meaning even for an arbitrary integer s.The following expressions are a direct consequence of the definition of the transformation (s).

$$\left(\frac{\partial\varphi}{\partial\xi}\right)_{(s+1)} = \frac{\partial\varphi_{(s)}}{\partial\xi}, \left(\frac{\partial\varphi}{\partial\bar{\xi}}\right)_{(s-1)} = \frac{\partial\varphi_{(s)}}{\partial\bar{\xi}}. \tag{3.9}$$

We pass over to the variables ξ and $\bar{\xi}$ in the equations (3.1) and (3.2).First of all,we remark that

$$\frac{\partial}{\partial x} = \frac{\partial}{\partial\xi} + \frac{\partial}{\partial\bar{\xi}}, \frac{\partial}{\partial y} = i\left(\frac{\partial}{\partial\xi} - \frac{\partial}{\partial\bar{\xi}}\right), \Delta = 4\frac{\partial^2}{\partial\xi\partial\bar{\xi}} + \frac{\partial^2}{\partial z^2} = \Delta_{\xi\bar{\xi}}.$$

Let us write

$$v_\xi = v_x + iv_y, v_{\bar{\xi}} = v_x - iv_y, F_\xi = F_x + iF_y, F_{\bar{\xi}} = F_x - iF_y.$$

Then from (3.1) and (3.2) we obtain the following system:

$$\begin{aligned} \frac{\partial v_\xi}{\partial t} + 2\omega iv_\xi - \nu\Delta_{\xi\bar{\xi}}v_\xi + \frac{2}{\rho}\frac{\partial p}{\partial\bar{\xi}} &= F_\xi, \\ \frac{\partial v_{\bar{\xi}}}{\partial t} + 2\omega iv_{\bar{\xi}} - \nu\Delta_{\xi\bar{\xi}}v_{\bar{\xi}} + \frac{2}{\rho}\frac{\partial p}{\partial\xi} &= F_{\bar{\xi}}, \\ \frac{\partial v_z}{\partial t} - \nu\Delta_{\xi\bar{\xi}}v_z + \frac{1}{\rho}\frac{\partial p}{\partial z} = F_z \quad , \quad & \frac{\partial v_\xi}{\partial\xi} + \frac{\partial v_{\bar{\xi}}}{\partial\bar{\xi}} + \frac{\partial v_z}{\partial z} = 0. \end{aligned} \tag{3.10}$$

¿From (3.9) we prove that

$$\left(\Delta_{\xi\bar{\xi}}\varphi\right)_{(s)} = \Delta_{\xi\bar{\xi}}\varphi_{(s)}. \tag{3.11}$$

Applying conveniently the operators (s) to each one of the equations of the system (3.10) and using (3.9) and (3.11) we have:

$$\begin{aligned} \frac{\partial v_{\xi,(s-1)}}{\partial t} + 2\omega i v_{\xi,(s-1)} - \nu\Delta_{\xi\bar{\xi}}v_{\xi,(s-1)} + \frac{2}{\rho}\frac{\partial p_{(s)}}{\partial\bar{\xi}} &= F_{\xi,(s-1)}, \\ \frac{\partial v_{\bar{\xi},(s+1)}}{\partial t} + 2\omega i v_{\xi,(s+1)} - \nu\Delta_{\xi\bar{\xi}}v_{\xi,(s+1)} + \frac{2}{\rho}\frac{\partial p_{(s)}}{\partial\xi} &= F_{\bar{\xi},(s+1)}, \\ \frac{\partial v_{z,(s)}}{\partial t} - \nu\Delta_{\xi\bar{\xi}}v_{z,(s)} + \frac{1}{\rho}\frac{\partial p_{(s)}}{\partial z} &= F_{z,(s)}, \\ \frac{\partial v_{\xi,(s-1)}}{\partial\xi} + \frac{\partial v_{\bar{\xi},(s+1)}}{\partial\bar{\xi}} + \frac{\partial v_{z,(s)}}{\partial z} &= 0. \end{aligned} \tag{3.12}$$

On the other hand the conditions (3.3) can be written in the following form:

$$\left(v_{\xi}, v_{\bar{\xi}}, v_z\right) = \left(z\dot{\phi}, z\dot{\bar{\phi}}, -\frac{1}{2}\left(\bar{\xi}\dot{\phi} + \xi\dot{\bar{\phi}}\right)\right). \tag{3.13}$$

Using the result,given in ([14])

$$\sum_{s=0}^{(m-1)}\left(e^{\frac{2\pi i l}{m}}\right)^s = \begin{cases} 0 & \text{if } l \not\equiv 0 \pmod{m} \\ m & \text{if } l \equiv 0 \pmod{m} \end{cases}$$

the action of the operator (s) on (3.13) lead us to the following relations

$$\left(v_{\xi,(s-1)}, v_{\bar{\xi},(s+1)}, v_{z,(s)}\right) = \begin{cases} \left(z, 0, -\frac{\bar{\xi}}{2}\right)\dot{\phi} & \text{if } s \equiv 1 \pmod{m} \\ \left(z, 0, -\frac{\xi}{2}\right)\dot{\bar{\phi}} & \text{if } s \equiv -1 \pmod{m} \\ \vec{0} & \text{in other cases} \end{cases} \tag{3.14}$$

¿From (3.5) and (3.8) we have that

$$N(v) = N\left(\sum_{s=0}^{m-1} v_{(s)}\right) = \sum_{s=0}^{m-1} N\left(v_{(s)}\right). \tag{3.15}$$

where $v_{(s)} = \left(v_{\xi,(s-1)}, v_{\bar{\xi},(s+1)}, v_{z,(s)}\right)$ and

$$N\left(v_{(s)}\right) = \rho\int_{\Omega} z\left(\frac{\partial v_{\xi,(s-1)}}{\partial t} + 2\omega i v_{\xi,(s-1)}\right)d\Omega - \rho\int_{\Omega}\xi\frac{\partial v_{z,(s)}}{\partial t}d\Omega. \tag{3.16}$$

It follows from (3.16) that

$$N\left(v_{(s)}\right) = e^{\frac{2\pi i(1-s)}{m}} N\left(v_{(s)}\right)$$

hence the only term different from zero in (3.15) is the one corresponding to $s = 1$,that is

$$N(v) = N\left(v_{(1)}\right).$$

Similarly

$$N(\bar{v}) = N\left(v_{(-1)}\right).$$

In the boundary conditions (3.14) for $s = 1$,the derivative of ϕ is related to $v_{(1)}$ and at the same time ϕ satisfies (*see*(3.4)) the equation

$$I_{\perp}\ddot{\phi} + i\omega(2I_{\perp} - I_{\parallel})\dot{\phi} + (\omega^2(I_{\parallel} - I_{\perp}) - gM_1l_1)\phi + N\left(v_{(1)}\right) + N_0^1 = 0 \tag{3.17}$$

On the other hand,for $s = -1$ the boundary conditions set up constrains between $v_{(-1)}$ and $\bar{\phi}$,this last magnitude satisfying the conjugate equation to (3.17)

$$I_{\perp}\ddot{\bar{\phi}} - i\omega(2I_{\perp} - I_{\parallel})\dot{\bar{\phi}} + (\omega^2(I_{\parallel} - I_{\perp}) - gM_1l_1)\bar{\phi} + \bar{N}\left(v_{(-1)}\right) + \bar{N}_0^1 = 0. \tag{3.18}$$

Now we again pass to cartesian coordinates,but before let us remark that

$$\begin{array}{lll} \bar{v}_{\xi,(s-1)} = v_{\bar{\xi},(-s+1)} & , \quad \bar{v}_{\bar{\xi},(s+1)} = v_{\xi,(-s-1)} & , \quad \bar{v}_{z,(s)} = v_{z,(-s)} \\ \bar{F}_{\xi,(s-1)} = F_{\bar{\xi},(-s+1)} & , \quad \bar{F}_{\bar{\xi},(s+1)} = F_{\xi,(-s-1)} & , \quad \bar{F}_{z,(s)} = F_{z,(-s)} \\ \bar{p}_{(s)} = p_{(-s)}. & & \end{array} \tag{3.19}$$

Let

$$\begin{array}{lll} v_{x,[s]} = v_{\bar{\xi},(s+1)} + v_{\xi,(s-1)}, & v_{y,[s]} = i\left(v_{\bar{\xi},(s+1)} - v_{\xi,(s-1)}\right), & v_{z,[s]} = 2v_{z,(s)} \\ F_{x,[s]} = F_{\bar{\xi},(s+1)} + F_{\xi,(s-1)}, & F_{y,[s]} = i\left(F_{\bar{\xi},(s+1)} - F_{\xi,(s-1)}\right), & F_{z,[s]} = 2F_{z,(s)} \\ p_{[s]} = 2p_{(s)}. & & \end{array}$$

Then obviously from (3.19) the following relations are true

$$\begin{array}{lll} \bar{v}_{x,[s]} = v_{x,[-s]} & , \quad \bar{v}_{y,[s]} = v_{y,[-s]} & , \quad \bar{v}_{z,[s]} = v_{z,[-s]} \\ \bar{F}_{x,[s]} = F_{x,[-s]} & , \quad \bar{F}_{y,[s]} = F_{y,[-s]} & , \quad \bar{F}_{z,[s]} = F_{z,[-s]} \\ \bar{p}_{[s]} = p_{[-s]}. & & \end{array}$$

The expressions allowing to recover the initial magnitudes are of the type:

$$v_x = \frac{1}{4}\sum_{s=0}^{m-1}\left(v_{x,[s]} + \bar{v}_{x,[s]}\right) = \frac{1}{4}\sum_{s=0}^{m-1}\left(v_{x,[s]} + v_{x,[-s]}\right) = \frac{1}{4}\sum_{s=0}^{m-1} v_{x,[s]} + \frac{1}{4}\sum_{s=-(m-1)}^{0} v_{x,[s]}.$$

Similar expressions are also true for v_y, v_z, F_x, F_y, F_z and p,but are omitted for shortness sake.It is clear now that (3.6) and (3.7) have been proved,where $v_{[s]} = \left(v_{x,[s]}, v_{y,[s]}, v_{z,[s]}\right)$ and $F_{[s]} = \left(F_{x,[s]}, F_{y,[s]}, F_{z,[s]}\right)$.From (3.12) and (3.14) it follows that $v_{[s]}$ and $p_{[s]}$ satisfy the following system of equations:

$$\frac{\partial v_{[s]}}{\partial t} - 2\omega(v_{[s]} \times \vec{k}) - \nu\Delta v_{[s]} - \frac{1}{\rho}\nabla p_{[s]} = F_{[s]}, \qquad (\Omega) \tag{3.20}$$

$$\operatorname{div} v_{[s]} = 0, \qquad (\Omega) \tag{3.21}$$

and the following boundary conditions

$$v_{[s]} = \begin{cases} \vec{r}\dot{\phi} & \text{if } s = 1, \\ r\dot{\bar{\phi}} & \text{if } s = -1, \\ 0 & \text{in other cases.} \end{cases} \qquad (\partial\Omega) \tag{3.22}$$

On the other hand it is easy to see that

$$N(v) = N(v_{[1]}), \bar{N}(\bar{v}) = \bar{N}(v_{[-1]}) \tag{3.23}$$

Hence we can write equation (3.17) as follows

$$I_\perp \ddot{\phi} + i\omega(2I_\perp - I_\parallel)\dot{\phi} + (\omega^2(I_\parallel - I_\perp) - gM_1 l_1)\phi + N(v_{[1]}) + N_0^1 = 0 \tag{3.24}$$

and its conjugate (3.18) as

$$I_\perp \ddot{\bar{\phi}} - i\omega(2I_\perp - I_\parallel)\dot{\bar{\phi}} + (\omega^2(I_\parallel - I_\perp) - gM_1 l_1)\bar{\phi} + \bar{N}(v_{[-1]}) + \bar{N}_0^1 = 0 \tag{3.25}$$

The method that we use to obtain the equations (3.20)–(3.25) in the viscous case is due to Sobolev in the ideal case,it involves tedious calculations but leads to a suitable form of the equations.First of all,the boundary conditions (3.22) are simpler than (3.3) and second,the system of equations satisfied by $(v_{[1]}, p_{[1]}, \phi)$ is exactly the conjugate system satisfied by $(v_{[-1]}, p_{[-1]}, \bar{\phi})$,hence in our case it is enough to consider one system of equations to study the stability of Sobolev's systems.

4 Operational setting of the system of equations

Let Ω be a bounded domain of the space $\mathbb{R}^3$ and let $\tilde{L}_2(\Omega)$ be the Hilbert space of the vector functions $u(x) = (u_1(x), u_2(x), u_3(x)), x = (x_1, x_2, x_3) \in \Omega$ and $u_k \in L_2(\Omega)$.The scalar product in $\tilde{L}_2(\Omega)$ is defined by the relation

$$< u, v >= \int_\Omega u \cdot \bar{v} d\Omega = \int_\Omega (\sum_{k=1}^{3} u_k \cdot \bar{v}_k) d\Omega.$$

Let M denote the set of infinitely differentiable solenoidal vectors with compact support in Ω and let S_2 denote its closure in the $\tilde{L}_2(\Omega)$ norm.The set of the elements of $\tilde{L}_2(\Omega)$ which are orthogonal to S_2 forms a subspace which we shall denote by G_2,so that

$$\tilde{L}_2(\Omega) = S_2 \oplus G_2.$$

It is well known that G_2 consists of elements ∇g,where g is a function on Ω,which is locally square summable and with first weak derivatives belong to $L_2(\Omega)$.Let P_0 be the orthogonal projection operator from $\tilde{L}_2(\Omega)$ onto S_2.We introduce the linear operator B in S_2 making $D(B) = S_2$ and $Bu = 2iP_0(u \times \vec{k})$.$B$ is a bounded and selfadjoint operator and $\sigma(B) \subset [-2, 2]$ (see [13]).Now we can define a second operator.We denote by H the Hilbert space of vector functions,obtained by completing M in the norm corresponding to the scalar product

$$[u, v] = \int_\Omega (u \cdot \bar{v} + \sum_{k=1}^{3} \frac{\partial u}{\partial x_k} \cdot \frac{\partial \bar{v}}{\partial x_k}) d\Omega.$$

Obviously,$H \subset S_2$ is dense in S_2 (see [11]).For any $\psi \in S_2$ there is a unique weak solution (v, p),$v \in H \cap \tilde{W}_2^2$ ($\tilde{W}_2^2$ is the set of all vector functions with components in W_2^2) and $p \in W_2^1$,of the linear problem:

$$(W_0) \quad \begin{aligned} \nu\Delta v + \nabla p &= \psi \qquad (\Omega) \\ \text{div } v &= 0 \qquad (\Omega) \\ v &= 0 \qquad (\partial\Omega) \end{aligned}$$

Let A_0 be the linear operator which establishes a correspondence between the solution v of the problem (W_0) and the corresponding $\psi \in S_2$,that is $A_0 v = \psi$.$D(A_0)$ is the set of all solutions v of the problem (W_0) when ψ runs through S_2.The operator A_0 is selfadjoint and negative definite on $D(A_0)$.Its inverse A_0^{-1} is compact.In what follows we shall write the system of equations (3.20) – (3.25) for $s = 0, \pm1, \ldots, \pm(m-1)$ as evolution equations in suitable Hilbert spaces.

Suppose that $s \neq \pm1$,let $\tilde{A} = \nu^{-1} A_0$,then $\tilde{A}$ can be regarded as an extension of the operator $P_0\Delta$ and $\tilde{A}$ has the same properties as A_0.If we apply the orthogonal projection P_0 to (3.20) we obtain

$$\frac{dv}{dt} = i(-\omega B + i\nu A)v + G \tag{4.1}$$

where $A = -\tilde{A} > 0, v \in C^1([0, +\infty], D(A))$ and $G = P_0 F \in S_2$.If v and G are smooth enough, (4.1) is equivalent to (3.20).We have dropped the index s in (4.1) for more simplicity of the notations.Let us now suppose that $s = 1$,next we shall change the first boundary condition in (3.22) into a homogeneous boundary condition. In order to do this we make the following transformation $v_{[1]} = v + \bar{r}\dot{\phi}$.If we substitute $v_{[1]}$ in (3.20), (3.21) and the first boundary condition in (3.22) then

$$\begin{aligned} \frac{\partial v}{\partial t} - 2\omega(v \times \vec{k}) - \nu\Delta v - \frac{1}{\rho}\nabla p_{[1]} + \bar{r}\ddot{\phi} - 2\omega(\bar{r} \times \vec{k})\dot{\phi} &= F_{[1]}, \ (\Omega) && (4.2) \\ \text{div } v &= 0, \quad (\Omega) && (4.3) \\ v &= 0, \quad (\partial\Omega) && (4.4) \end{aligned}$$

¿From (3.5) it follows that

$$N(v_{[1]}) = N(v) + N(\bar{r}\dot{\phi})$$

It is easy to prove that

$$N(\bar{r}\dot{\phi}) = \rho\kappa^2\ddot{\phi} - 2\omega\rho iE\dot{\phi}$$

where

$$iE = \int_\Omega r \cdot (\bar{r} \times \vec{k}) d\Omega, \qquad \kappa^2 = \int_\Omega | r |^2 d\Omega.$$

Substituting $v_{[1]}$ in (3.24) we have

$$(2I_\perp + \rho\kappa^2)\ddot{\phi} - 2\omega i(A_1 + \rho E)\dot{\phi} + t_1\phi + N(v) + 2N_0^1 = 0. \tag{4.5}$$

Here

$$t_1 = 2(\omega^2(I_\parallel - I_\perp) - gM_1 l_1), A_1 = (I_\parallel - 2I_\perp).$$

One can apply P_0 to the equation (4.2) and obtain

$$\frac{dv}{dt} + P_0(\bar{r})\ddot{\phi} = i(-\omega B + i\nu A)v + 2\omega P_0(\bar{r} \times \vec{k})\dot{\phi} + G, \tag{4.6}$$

where $G = P_0(F_{[1]})$.It follows from (3.5) that

$$N(v) =< \frac{dv}{dt}, P_0(\bar{r}) > +2\omega\rho < v, P_0(\bar{r} \times \vec{k}) > . \tag{4.7}$$

Set

$$\dot{\phi} = i\omega\gamma. \tag{4.8}$$

Then,from (4.5) we have

$$\dot{\gamma} = \frac{i2\omega(A_1+\rho E)}{(2I_\perp+\rho\kappa^2)}\gamma + i\frac{\omega L}{(2I_\perp+\rho\kappa^2)}\phi + i\frac{N(v)}{\omega(2I_\perp+\rho\kappa^2)} + i\frac{2N_0^1}{\omega(2I_\perp+\rho\kappa^2)}, \tag{4.9}$$

where $L = \frac{L_1}{\omega^2}$.Finally,from (4.6) - (4.9) we obtain the following evolution equation on $S_2 \times \mathbb{C}^2$:

$$\mathcal{L}_1\frac{dR}{dt} = i\mathcal{L}_2 R + R_0, \qquad R = (v,\phi,\gamma) \in C^1([0,+\infty), D(A)\times\mathbb{C}^2) \tag{4.10}$$

where $R_0 = (P_0F_{[1]}, 0, \frac{(2iN_0^1)}{\omega(2I_\perp+\rho\kappa^2)})$,$\mathcal{L}_1$ and $\mathcal{L}_2$ are linear operators,$D(\mathcal{L}_1) = S_2\times\mathbb{C}^2$ and $\mathcal{L}_1 R = L_1 R$ for every $R \in D(\mathcal{L}_1), D(\mathcal{L}_2) = D(A)\times\mathbb{C}^2$ and $\mathcal{L}_2 R = L_2 R$ for every $R \in D(\mathcal{L}_2)$.L_1 and L_2 are operator matrices that can be defined in the following way

$$L_1 = \begin{pmatrix} I & \vec{o} & i\omega P_0(\bar{r}) \\ \mathcal{O} & 1 & 0 \\ \frac{-i\rho < \cdot, P_0(\bar{r}) >}{\omega(2I_\perp+\rho\kappa^2)} & 0 & 0 \end{pmatrix}$$

and

$$L_2 = \begin{pmatrix} -\omega B + i\nu A & \vec{o} & 2\omega^2 P_0(\bar{r}\times\vec{k}) \\ \mathcal{O} & 0 & \omega \\ \frac{2\rho < \cdot, P_0(\bar{r}\times\vec{k}) >}{(2I_\perp+\rho\kappa^2)} & \frac{\omega L}{(2I_\perp+\rho\kappa^2)} & \frac{2\omega(A_1+\rho E)}{(2I_\perp+\rho\kappa^2)} \end{pmatrix}$$

Above we denote by I the identity operator on S_2,by $\mathcal{O}$ the null functional and by $\vec{o}$ the zero of this space.
Since

$$\omega^2 \| P_0(\bar{r}) \|_{S_2} < \omega^2\kappa^2 < \frac{(2I_\perp+\rho\kappa^2)}{\frac{\rho}{\omega^2}}$$

$\mathcal{L}_1$ has an inverse $\mathcal{L}_1^{-1}$.It is easy to verify that $\mathcal{L}_1^{-1} = \tilde{I} + F$,where $\tilde{I}$ is the identity on $S_2\times\mathbb{C}^2$ and F is a finite dimensional operator.

5 Spectral properties of the operators related to the main problem

The purpose of this section is to prove some properties of the operators that appear in the evolution equations obtained in the preceding section.In the sequel,we will denote by B and A the linear operators which are introduced in the section 3.

Lemma 5.1 *The operator $D = A^{-1}B$ is a Hilbert-Schmidt operator (that is $D \in \sigma_2$,see [6] for the definition).*

Proof:We note that B may be continuously extended to $\tilde{L}_2(\Omega)$,to this purpose we define B on $\tilde{C}_0^\infty(\Omega)$ (the totality of vector fields with components in $C_0^\infty(\Omega)$).Let a be any element of $\tilde{C}_0^\infty(\Omega)$ and p_a the solution of the following Neumann problem

$$\begin{aligned} \Delta p_a &= \text{div } (a\times\vec{k}) \qquad (\Omega) \\ \frac{\partial p_a}{\partial n} &= 0 \qquad\qquad (\partial\Omega) \end{aligned}$$

According to theorem 1.7 in [15] we have

$$\| \nabla p_a \|_{\tilde{L}_2(\Omega)} \leq \| a \|_{\tilde{L}_2(\Omega)}.$$

Then $\nabla p_a \in G_2$.Now,we introduce the vector field b by setting $a\times\vec{k} = b + \nabla p_a$ and define $Ba = 2ib$.Obviously $\tilde{C}_0^\infty(\Omega)$ is dense in $\tilde{L}_2(\Omega)$ hence,B may be extended by continuity to $\tilde{L}_2(\Omega)$.The operator D is bounded on $\tilde{L}_2(\Omega)$ and its range belongs to $\tilde{W}_2^2(\Omega)$.The Lemma follows from a well known result due to Agmon (see [1],[2]). ∎
Let F be a linear operator defined in a Hilbert space.We recall that a nonzero vector φ is called a root vector of the operator F, corresponding to the eigenvalue λ,if $(F-\lambda I)^n\varphi = 0$ for some positive integer n.
Let $C_{\alpha,\beta} = \alpha B + i\beta A$ where α and β are real values,$\beta > 0$.We have

Lemma 5.2 *The operator $iC_{\alpha,\beta}$ is the generator of a C_0 semigroup of contractions.*

Proof:Note that $C_{\alpha,\beta}^{-1}$ is a compact operator.In fact

$$C_{\alpha,\beta} = (\alpha B + i\beta A) = A^{\frac{1}{2}}(\alpha A^{-\frac{1}{2}}BA^{-\frac{1}{2}} + i\beta I)A^{\frac{1}{2}}$$

Since $A^{-\frac{1}{2}}BA^{-\frac{1}{2}}$ is a selfadjoint operator $i\beta$ a is a regular point for it. But this means that the operator $(\alpha A^{-\frac{1}{2}}BA^{-\frac{1}{2}} + i\beta I)^{-1}$ is a continuous operator defined in all S_2.It follows that

$$C_{\alpha,\beta}^{-1} = A^{-\frac{1}{2}}(\alpha A^{-\frac{1}{2}}BA^{-\frac{1}{2}} + i\beta I)^{-1}A^{-\frac{1}{2}}$$

is compact.On the other hand

$$Re < iC_{\alpha,\beta}u, u > = -\beta < Au, u > < 0$$

if $u \neq 0 \in S_2$.Hence,we have that the spectrum of $iC_{\alpha,\beta}$ consists of eigenvalues,which can have the infinite as a limit point.These eigenvalues are situated in the left semi-plane.Therefore,for all $\lambda > 0$,$(iC_{\alpha,\beta} - \lambda I)D(A) = S_2$.Since,$C_{\alpha,\beta}$ is a closed operator whose domain is dense in S_2 and it is accretive,i.e.,$Re(iC_{\alpha,\beta}u, u) < 0$ we have

$$\| (iC_{\alpha,\beta} - \lambda I)^{-1} \| < \frac{1}{\lambda} \tag{5.1}$$

(see [10]).It is well-known that (5.1) is a sufficient condition on the operator $iC_{\alpha,\beta}$ to be a generator of a C_0 semigroup of contractions. ∎
In the course of the investigation it was obtained the following Lemma that has independent interest.

Lemma 5.3 *The system of all root vectors of the operator $C_{\alpha,\beta}$ is complete in S_2.*

Proof:Consider the equation

$$(\alpha B + i\beta A - \lambda I)u = v$$

where $u \in D(A)$ and $v \in S_2$.This equation is equivalent to the following one

$$(\mu A^{-1} - I - K)u = v_1$$

where $\mu = -i\frac{\lambda}{\beta}$, $K = i\frac{\alpha}{\beta}A^{-1}B$ and $v_1 = i\frac{1}{\beta}A^{-1}v$.Therefore the completeness of the system of root vectors of $C_{\alpha,\beta}$ holds if and only if the system of all proper and associate vectors of the pencil $\mu A^{-1} - I - K$ is complete in S_2.Keeping in mind that $A^{-1}K = -i\frac{\alpha}{\beta}A^{-2}B \in \sigma_2$,then $(A^{-1}K)^* \in \sigma_2$.Hence $(A^{-1}K)_{Im}$(the imaginary part of the operator $A^{-1}K$) belongs to σ_2.The Lemma follows now from Keldysh's Theorem (see [6],page 262). ∎

Now,we will show a similar result to Lemma 5.3,more exactly the system of all root vectors of operator $\mathcal{L}_2\mathcal{L}_1^{-1}$ is complete in $S_2 \times \mathbb{C}^2$.In order to see this,we note that

$$\mathcal{L}_2 = -i\begin{pmatrix} -\nu A & \vec{o} & \vec{o} \\ \mathcal{O} & -1 & 0 \\ \mathcal{O} & 0 & -1 \end{pmatrix} + \begin{pmatrix} -\omega B & \vec{o} & \vec{o} \\ \mathcal{O} & -i & 0 \\ \mathcal{O} & 0 & -i \end{pmatrix} + \mathcal{L}_4$$

where $\mathcal{L}_4$ is a finite dimensional operator.From this decomposition it follows that the spectral problem

$$(\lambda\mathcal{L}_1 - i\mathcal{L}_2)R = 0$$

is equivalent to the following

$$(\lambda\tilde{S} - \tilde{I} - \tilde{H})\tilde{R} = 0$$

where

$$\tilde{S} = \begin{pmatrix} -\frac{A^{-1}}{\nu} & \vec{o} & \vec{o} \\ \mathcal{O} & -1 & 0 \\ \mathcal{O} & 0 & -1 \end{pmatrix}$$

and

$$\tilde{H} = \begin{pmatrix} -i\omega\frac{A^{-1}B}{\nu} & \vec{o} & \vec{o} \\ \mathcal{O} & 1 & 0 \\ \mathcal{O} & 0 & 1 \end{pmatrix} + \textit{finite dimensional operator}$$

Note that $\tilde{S}$ is a negative compact operator.From Lemma 5.1 it follows that $\tilde{H}$ is a Hilbert-Schmidt operator.Hence,according to Keldysh's Theorem $(see[6], page262)$,the system of the root vectors of $\mathcal{L}_2\mathcal{L}_1^{-1}$ is complete in $S_2 \times \mathbb{C}^2$.

6 General study of the stability of Sobolev's system

In this section we consider the stability problem for the evolution equations (4.1) and (4.10).
Consider in a Banach space E the differential equation

$$\frac{dx}{dt} = Cx \tag{6.1}$$

where C is a linear operator having domain $D(C)$ dense in E. The solution $x = 0$ of (6.1) is called stable is every solution of this equation is bounded.It is called unstable if is not stable.
Now,we consider the stability problem for the homogeneous equation of (4.1)

$$\frac{du}{dt} = i(-\omega B + i\nu A)u \tag{6.2}$$

¿From Lemma 5.2 the Cauchy problem for (6.2) is correctly posed and moreover it follows that all solution of (6.2) is bounded $(see[9])$.Hence,the zero solution of this equation is stable.
The remaining part of the paper is devoted to the investigation of the influence of the rigid body in the stability problem.
In what follows we shall need some definitions and results from the theory of operators in spaces with an indefinite metric.
Let E be a separable Hilbert space with inner product $(.,.)$.Suppose that we have a second inner product $[.,.]$ and that it is indefinite.It is possible to prove that there is an operator $P, D(P) = E$,such that $[x, y] = (Px, y)$.We can say that P is the operator associated to the inner product $[.,.]$.In this sense it is the most general way of defining an indefinite inner product.
An operator J with $D(J) = E$ will be called a fundamental symmetry (f.s) if $J = J^*$ and $J^2 = I$ where I is the identity on E.It is evident that all (f.s) define an indefinite inner product.Let $[.,.]$ denote an indefinite inner product in E and let P be the operator associated with it.Then,if $0 \in \rho(P)$ (that is,zero is a regular point of P) there exists a (f.s) J_1 corresponding to a definite inner product $(.,.)_1$ equivalent to the initial one,such that $[.,.] = (J.,.)_1$.
As before let $(E, (.,.))$ be a separable Hilbert space and J some (f.s) on E.Let C be an operator having a domain $D(C)$ dense in E.The operator C is said to be $J-$dissipative if $Im[Cx, x] = Im(JCx, x) \geq 0$ and $J-$selfadjoint if $(JC)^* = (JC)$.
Assume that

$$E = E_+[\dot{+}]E_-$$

where E_+ and E_- are linear manifolds $J-$positive and $J-$negative respectively and $[\dot{+}]$ denote the $J-$orthogonal direct sum.If E_+ and E_- are Hilbert spaces corresponding to the inner products $[.,.]$ and $-[.,.]$ respectively and

$$\iota = min(dimE_+, dimE_-) < \infty$$

then,E will be called a Pontriaguin space of order ι.An $J-$dissipative operator C is called maximal if it has no proper extensions.It is well known that,if there is $\lambda \in \mathbb{C}_+$,such that $\lambda \in \rho(C)$ then C is maximal.We recall the following result:a maximal $J-$dissipative

operator on a Pontriaguin space of order ι has at most ι eigenvalues in $\mathbb{C}_+$ (we assume that $\iota = dim E_+$) moreover,the sum of the algebraic multiplicities of all these eigenvalues is less or equal than ι. [4]
Now we turn to study the homogeneous equation of (4.10),that is

$$\mathcal{L}_1 \frac{dR}{dt} = i\mathcal{L}_2 R. \tag{6.3}$$

¿From the form of the L_2 it is clear that

$$L_2 = L_3 + \textit{finite dimensional operator}$$

where

$$L_3 = \begin{pmatrix} -\omega B + i\nu A & \vec{o} & \vec{o} \\ \mathcal{O} & 0 & 0 \\ \mathcal{O} & 0 & 0 \end{pmatrix}$$

Then,according to the previous argument about the operator $\mathcal{L}_1^{-1}$,(6.3) is equivalent to

$$\frac{d\tilde{R}}{dt} = i\,(\mathcal{L}_3 + \textit{finite dimensional operator})\,\tilde{R}$$

Where $\mathcal{L}_3 R = L_3 R$, $D(\mathcal{L}_3) = D(A) \times \mathbb{C}^2$ and $R = \mathcal{L}_1^{-1}\tilde{R}$.It follows from Theorem 7.5 in [9] that the Cauchy problem for the last equation is correct.Hence,the same will hold for (6.3).
We remark that the study of the stability of the zero solution of (6.3) is equivalent to the study of the stability of the zero solution of

$$\frac{dR}{dt} = i\mathcal{L}_1^{-1}\mathcal{L}_2 R. \tag{6.4}$$

Our aim in the rest of the section is to study the boundedness of the elementary solutions of equation (6.4),this is,the study of the spectrum of $\mathcal{L}_1^{-1}\mathcal{L}_2$.It is clear that if $\lambda \in \mathbb{C}_+$ belongs to the spectrum of this operator,the zero solution of (6.4) is instable.
It is easily seen that if $L > 0$ or $L = 0$,the Sobolev's system is essentially stable.Hence below we only consider the case $L < 0$.In order to study this,we introduce a indefinite inner product

$$\tilde{Q}(R_1, R_2) = \frac{\rho}{\omega^2} < U_1, U_2 > + L\Phi_1\bar{\Phi}_2 + (2I_\perp + \rho\kappa^2)\gamma_1\bar{\gamma}_2$$

where $R_1, R_2 \in S_2 \times \mathbb{C}^2$.It is evident that this is generated by the following operator

$$G = \begin{pmatrix} \frac{\rho}{\omega^2} I & \vec{o} & \vec{o} \\ \mathcal{O} & L & 0 \\ \mathcal{O} & 0 & (2I_\perp + \rho\kappa^2) \end{pmatrix}$$

[4]this result is due to Azizov (see [4])

We introduce a second indefinite inner product

$$Q(R_1, R_2) = -\tilde{Q}(\mathcal{L}_1 R_1, R_2) \qquad R_1, R_2 \in S_2 \times \mathbb{C}^2.$$

Let us remark that $\mathcal{L}_2$ is $\tilde{Q}-$dissipative.Hence $-\mathcal{L}_1^{-1}\mathcal{L}_2$ is $Q-$dissipative.On the other hand the space $S_2 \times \mathbb{C}^2$ admits a decomposition in the following way

$$S_2 \times \mathbb{C}^2 = H_+[\dot{Q}]H_-$$

where H_+ and H_- are subspaces $Q-$positive and $Q-$negative respectively and $dim H_+ = 1$.In fact,if $R = (U, \Phi, \gamma)$ is an arbitrary element of $S_2 \times \mathbb{C}^2$ then it is easily verified that

$$Q(R, R) = -\frac{\rho}{\omega^2}\| U + i\omega\gamma P_0(\bar{r}) \|_{S_2}^2 - ((2I_\perp + \rho\kappa^2) - \frac{\rho}{\omega^2}\| \omega P_0(\bar{r}) \|^2) | \gamma |^2 - L | \Phi |^2.$$

Now,we take (recall that $L < 0$)

$$H_+ = \{(\vec{o}, \Phi, 0) \mid \Phi \in \mathbb{C}\}$$

and

$$H_- = \{(U, 0, \gamma) \mid U \in S_2, \gamma \in \mathbb{C}\}.$$

Since,the $Q-$metric is generated by $-G\mathcal{L}_1$ and $0 \in \rho(-G\mathcal{L}_1)$,the operator $-\mathcal{L}_1^{-1}\mathcal{L}_2$ will be $\tilde{J}-$dissipative in a Pontriaguin space of order one ($\tilde{J}$ is a (f.s)).The zero solution of equation (6.4) will be unstable if the operator $-\mathcal{L}_1^{-1}\mathcal{L}_2$ has at least an eigenvalue in $\mathbb{C}_+$.It follows by the Azizov's Theorem enunciated before (see [4]) that $-\mathcal{L}_1^{-1}\mathcal{L}_2$ has at most one eigenvalue in $\mathbb{C}_+$.
In the next section we will prove that this bad eigenvalue really appears for greater value of the viscosity.
¿From now follows we suppose that the cavity Ω satisfies the general conditions imposed in the introduction.
As it is observed in the previous section,the unstability of the Sobolev's system is equivalent to the existence of an eigenvalue of the linear pencil $\lambda\mathcal{L}_1 + \mathcal{L}_2$ in $\mathbb{C}_+$.The spectral problem

$$(\lambda\mathcal{L}_1 + \mathcal{L}_2)R = 0 \qquad R \in D(A) \times \mathbb{C}^2$$

is the same as

$$(-\omega B + i\nu A)U + 2\omega^2\gamma P_0(\bar{r} \times \vec{k}) + \lambda U + i\omega\lambda\gamma P_0(\bar{r}) = 0, \tag{6.5}$$
$$\omega\gamma + \lambda\Phi = 0, \tag{6.6}$$
$$\tilde{h}\gamma + \omega L\Phi + 2\rho < U, P_0(\bar{r} \times \vec{k}) > + \frac{\rho\lambda}{\omega} < U, iP_0(\bar{r}) > = 0, \tag{6.7}$$

where

$$\tilde{h} = \left[(2I_\perp + \rho\kappa^2)\lambda + 2\omega(A_1 + \rho E)\right]$$

As it was mentioned before,we are interested in the case $L < 0$ for any value of the angular velocity.Then it should be $I_\parallel \le I_\perp$.
Assume that $-\lambda \notin \sigma_p(-\omega B + i\nu A)$ then λ is an eigenvalue of the pencil $\lambda\mathcal{L}_1 + \mathcal{L}_2$ if and only if λ is a root of the following equation:

$$Q_2(\omega, \lambda) = (2I_\perp + \rho\kappa^2)\lambda^2 + 2\omega(A_1 + \rho E)\lambda - \omega^2 L = \rho\lambda < \Gamma_\lambda U_0(\lambda), U_0(\bar{\lambda}) > = f(\lambda) \tag{6.8}$$

where $\Gamma_\lambda = (-\omega B + i\nu A + \lambda I)^{-1}$ and $U_0(\mu) = 2\omega P_0(\bar{r} \times \vec{k}) + i\mu P_0(\bar{r})$ for $\mu \in \mathbb{C}$.It is possible to verify that (6.8) follows from (6.1) – (6.3) if $\lambda \notin \sigma(-\omega B + i\nu A)$,in particular if $\lambda \in \mathbb{C}_+$.
It is clear that $\lambda = 0$ is not a solution of (6.8) and keeping in mind that

$$Im < \Gamma_\lambda U, U > < 0 \qquad U \neq 0 \in S_2.$$

In a similar way no real value λ with $U(\lambda) \neq \vec{o}$ will be a solution of (6.8).Note that $U(\lambda) = \vec{o}$ for some $\lambda \in \mathbb{R}$ is a very restrictive condition over Ω.When Ω is an ellipsoid then $P_0(\bar{r} \times \vec{k}) = -iP_0(\bar{r})$ holds,hence if $\lambda = 2\omega, U(\omega) = \vec{o}$[5].In what follows we can assume that Ω is such that $U(\lambda) \neq \vec{o}$ for any real value,thus (6.8) has not real zeros.
We next specify the behavior of the roots of the polynomial function $Q_2(\omega, \lambda)$.Two case can arise
I) If $(I_\parallel + \rho E)^2 - 4\rho E I_\perp \leq 2\rho\kappa^2(I_\perp - I_\parallel)$,then for every $\omega \geq 0, Q_2(\omega, \lambda)$ has two conjugated complex roots.In this case, the pencil $\lambda\mathcal{L}_1 + \mathcal{L}_2$ has not real eigenvalues.
II) If $(I_\parallel + \rho E)^2 - 4\rho E I_\perp > 2\rho\kappa^2(I_\perp - I_\parallel)$,let

$$\omega^* = \sqrt{\frac{2gM_1 l_1(2I_\perp + \rho\kappa^2)}{(I_\parallel + \rho E)^2 - 4\rho E I_\perp + 2\rho\kappa^2(I_\parallel - I_\perp)}}$$

then if $\omega < \omega^*, Q_2(\omega, \lambda)$ has two conjugate complex roots.If $\omega > \omega^*$ this polynomial will have real zeros.In this case the zeros are

$$\lambda_\pm(\omega) = \alpha\omega \pm \sqrt{\alpha^2\omega^2 - 2((\omega^2 \frac{(I_\perp - I_\parallel)}{(2I_\perp + \rho\kappa^2)}) + \beta)}$$

where

$$\alpha = -\frac{(A_1 + \rho E)}{(2I_\perp + \rho\kappa^2)}, \qquad \beta = \frac{gM_1 l_1}{(2I_\perp + \rho\kappa^2)}$$

Lemma 6.1 *The function* $F(\lambda) =< \Gamma_\lambda U_0(\lambda), U_0(\lambda) >$ *is analytic in the halfplane* G_ν *defined by the inequality*

$$-\nu \| A^{-1} \|^{-1} < Im\lambda$$

Proof.As it is proved in section 5,$(-\omega B + i\nu A)^{-1}$ is compact.Hence the spectrum of $(-\omega B + i\nu A)$ consists of at most a countable number of eigenvalues which can have infinite as a limit point.We can prove now that the numerical rank $\hat{\theta}$ of $(-\omega B + i\nu A)$ is contained in $\nu \| A^{-1} \|^{-1} \leq Im\lambda$.
Note that $\Gamma_\lambda = R_{-\lambda}$ (here $R_\mu = (-\omega B + i\nu A - \mu I)^{-1}$).It is well known that the eigenvalues of an operator belong to its numerical rank.Hence, R_λ will be analytic in the region $Im\lambda < \nu \| A^{-1} \|^{-1}$.
We continue the proof of the Lemma.Suppose that $U \neq o \in D(A)$,by definition of numerical rank

$$\frac{< (-\omega B + i\nu A)U, U >}{\| U \|^2} = \hat{\theta}(U) \tag{6.9}$$

belongs to $\hat{\theta}$.Let $V = A^{\frac{1}{2}}U$,from (6.9) it follows that

$$\frac{< (-\omega A^{-\frac{1}{2}} B A^{-\frac{1}{2}} + i\nu I)V, V >}{< A^{-1}V, V >} = \hat{\theta}(U) \tag{6.10}$$

[5]Even in this case,we can give conditions under which $\lambda = \omega$ is not a zero of (6.8) for every value of ω

If we take the imaginary part in (6.10),we have

$$\frac{\nu \| V \|^2}{< A^{-1}V, V >} = Im\hat{\theta}(U)$$

from this last equality it is easy to obtain that

$$\nu \| A^{-1} \|^{-1} < Im\hat{\theta}(U)$$

Being U arbitrary,the numerical rank of the operator $(-\omega B + i\nu A)$ is contained in the semiplane

$$\nu \| A^{-1} \|^{-1} < Im\lambda$$

Hence,R_λ is analytic in the complement of this region.To complete the proof of Lemma it remains to recall the previous observation that $\Gamma_\lambda = R_{-\lambda}$. ∎

Lemma 6.2 *We have the following estimation*

$$\| \Gamma_\lambda \| < \frac{1}{\nu \| A^{-1} \|^{-1} + Im\lambda} \qquad \lambda \in G_\nu \tag{6.11}$$

Proof.All the points of the set $\mathbb{C} \setminus \hat{\theta}$ are regular points of $(-\omega B + i\nu A)$.From Theorem 3.2 of [7] it follows that

$$\| R_\lambda \| \leq \frac{1}{d(\lambda, \hat{\theta})} \leq \frac{1}{d(\lambda, \partial G_\nu)} \tag{6.12}$$

for every λ,such that $Im\lambda < \nu \| A^{-1} \|^{-1}$,where

$$\partial G_\nu = \left\{ \lambda \mid Im\lambda = \nu \| A^{-1} \|^{-1} \right\},$$

Now,(6.11) follows from (6.12) and $\Gamma_\lambda = R_{-\lambda}$. ∎
We recall that it was assumed that equation (6.8) has not real zeros.Let $\tilde{C}$ be an arbitrary rectificable curve in $G_\nu(\nu \gg 1)$,such that $Q_2(\omega, \lambda)$ is not zero on $\tilde{C}$ for a value ω,then keeping in mind (6.11) by Rouche's Theorem there exists a $\nu(\omega)$,such that,if $\nu > \nu(\omega)$ the function $Q_2(\omega, \lambda) - f(\lambda)$ has inside $\tilde{C}$ the same number of zeros as $Q_2(\omega, \lambda)$.Note that,if $\tilde{C}$ contains in its interior only one zero of $Q_2(\omega, \lambda)$ and $\nu \to \infty$ then,this zero of $Q_2(\omega, \lambda) - f(\lambda)$ (necessarily an eigenvalue of the pencil $\lambda\mathcal{L}_1 + \mathcal{L}_2$) converges to the zero of $Q_2(\omega, \lambda)$.This follows from (6.8) and (6.11).
In what follows $\lambda_\pm(\omega)$ will denote the zeros of the polynomial $Q_2(\omega, \lambda)$ such that $| \lambda_-(\omega) | < | \lambda_+(\omega) |$ if $\lambda_\pm(\omega)$ are real and $Im\lambda_-(\omega) < 0, Im\lambda_+(\omega) > 0$ if they are complex conjugated.It is easy to see that if $\lambda_\pm(\omega)$ are real,then $\lambda_\pm(\omega) > 0$.
Suppose that II) is true (see the specification of the behavior of the roots of $Q_2(\omega, \lambda)$) and $\omega > \omega^*$.Let $B(\lambda_+(\omega), \epsilon)$ (respectively $B(\lambda_-(\omega), \epsilon)$) be the disc $| \lambda - \lambda_+(\omega) | < \epsilon$ (respectively
$| \lambda - \lambda_-(\omega) | < \epsilon$) where $\epsilon > 0$ is sufficiently small to be sure that $\lambda_- \notin B(\lambda_+(\omega), \epsilon)$ (respectively $\lambda_+ \notin B(\lambda_-(\omega), \epsilon)$) then as before,by Rouche's Theorem,starting with a sufficiently large $\nu_+(\omega)$ (respectively $\nu_-(\omega)$) the function $Q_2(\omega, \lambda) - f(\lambda)$ has one zero sufficiently near $\lambda_+(\omega)$ (respectively $\lambda_-(\omega)$) inside $B(\lambda_+(\omega), \epsilon)$ (respectively $B(\lambda_-(\omega), \epsilon)$).Let us denote this zero by $\lambda_+(\omega, \nu)$ (respectively $\lambda_-(\omega, \nu)$).We have the following.

Lemma 6.3 *(Fundamental lemma) Suppose that II) is true,then for every value ω,such that,$\omega > \omega^*$,there exists a $\nu_0(\omega)$ sufficiently large,for which*

$$Im\lambda_-(\omega,\nu) > 0$$

and

$$Im\lambda_+(\omega,\nu) < 0$$

for $\nu > \nu_0(\omega)$.

Proof.Evidently,$\nu_0(\omega) > max(\nu_+(\omega), \nu_-(\omega))$.Let $U_0(\lambda) = 2\omega U_1 + \lambda U_2$ where $U_1 = P_0(\bar{r}\times\bar{k})$ and $U_2 = iP_0(\bar{r})$ then (6.8) can be written in the following way

$$\begin{aligned} Q_2(\omega,\lambda) &= 4\omega^2\rho\lambda < \Gamma_\lambda U_1, U_1 > +\rho\lambda^3 < \Gamma_\lambda U_2, U_2 > \\ &+2\omega\rho\lambda^2[< \Gamma_\lambda U_1, U_2 > + < \Gamma_\lambda U_2, U_1 >] \end{aligned} \tag{6.13}$$

If we multiply (6.13) by $(\bar{\lambda}^2)$ we have

$$\begin{aligned} Q_2(\omega,\lambda)\bar{\lambda}^2 &= 4\omega^2\rho\bar{\lambda}|\lambda|^2 < \Gamma_\lambda U_1, U_1 > +\rho\lambda|\lambda|^4 < \Gamma_\lambda U_2, U_2 > \\ &+2\omega\rho|\lambda|^4[< \Gamma_\lambda U_1, U_2 > + < \Gamma_\lambda U_2, U_1 >]. \end{aligned} \tag{6.14}$$

Hence,any zero of the equation (6.8) satisfies (6.14).Let $V_1 = \Gamma_\lambda U_1$ and $V_2 = \Gamma_\lambda U_2$.Notice that

$$< \Gamma_\lambda U_1, U_1 >= -\omega < BV_1, V_1 > -i\nu < AV_1, V_1 > +\bar{\lambda}\| V_1 \|^2,$$
$$< \Gamma_\lambda U_2, U_2 >= -\omega < BV_2, V_2 > -i\nu < AV_2, V_2 > +\bar{\lambda}\| V_2 \|^2$$

and

$$\begin{aligned} < \Gamma_\lambda U_1, U_2 > + < \Gamma_\lambda U_2, U_1 > &= -\omega Re < BV_1, V_2 > \\ &-i\nu Re < AV_1, V_2 > +\bar{\lambda} Re < V_1, V_2 > . \end{aligned}$$

If we substitute these expressions in (6.14) we obtain

$$\begin{aligned} Q_2(\omega,\lambda)\bar{\lambda}^2 &= -4\omega^3\rho\bar{\lambda}|\lambda|^2 < BV_1, V_1 > -i4\omega^2\nu\rho\bar{\lambda}|\lambda|^2 < AV_1, V_1 > \\ &+4\omega^2\rho\bar{\lambda}^2|\lambda|^2\| V_1 \|^2 - \omega\rho\lambda|\lambda|^4 < BV_2, V_2 > \\ &+i\nu\rho\lambda|\lambda|^4 < AV_2, V_2 > +\rho|\lambda|^6\| V_2 \|^2 \\ &+2\omega^2\rho|\lambda|^4 Re < BV_1, V_2 > -2\omega\rho\nu|\lambda|^4 Re < AV_1, V_2 > \\ &+2\omega\rho\bar{\lambda}|\lambda|^4 Re < V_1, V_2 > . \end{aligned} \tag{6.15}$$

Dividing both sides of (6.15) by $(2I_\perp + \rho\kappa^2)$ and taking the imaginary part,we have an equation of the form

$$\mathcal{A}_1(Im\lambda) = \mathcal{A}_2 \tag{6.16}$$

where $\mathcal{A}_1$ and $\mathcal{A}_2$ are two real functions of λ,more precisely

$$\begin{aligned} \mathcal{A}_1 &= 2\omega\alpha|\lambda|^2 - 4\left(\frac{\omega^2(I_\perp - I_\parallel)}{(2I_\perp + \rho\kappa^2)} + \beta\right)(Re\lambda) \\ &-4\omega^3\rho|\lambda|^2 < BV_1, V_1 > +8\omega^2\rho|\lambda|^2\| V_1 \|^2(Re\lambda) \\ &+\omega\rho|\lambda|^4 < BV_2, V_2 > +2\omega\rho|\lambda|^4 Re < V_1, V_2 > \end{aligned}$$

and

$$\begin{aligned} \mathcal{A}_2 &= (-Re\lambda)(4\omega^2\nu\rho|\lambda|^2 < AV_1, V_1 > +\nu\rho|\lambda|^4 < AV_2, V_2 >) \\ &-2\omega\nu\rho|\lambda|^4 Re < AV_1, V_2 > \end{aligned}$$

then,if λ satisfies (6.8) it also satisfies (6.16) .In particular we have

$$\mathcal{A}_1(\lambda_\pm(\omega,\nu))(Im\lambda_\pm(\omega,\nu)) = \mathcal{A}_2(\lambda_\pm(\omega,\nu)). \tag{6.17}$$

We recall that,for $\nu > max(\nu(\omega)_+, \nu(\omega)_-)$ sufficiently large,$(\lambda_\pm(\omega,\nu))$ are very close to $(\lambda_\pm(\omega))$.Hence,there exists a $\nu_0(\omega)$ such that,for $\nu > \nu_0(\omega)$ the signs of $\mathcal{A}_1(\lambda_\pm(\omega,\nu))$ and $\mathcal{A}_2(\lambda_\pm(\omega,\nu))$ coincide with the signs of $\mathcal{A}_1(\lambda_\pm(\omega))$ and $\mathcal{A}_2(\lambda_\pm(\omega))$*respectively*.It is easy to see that,as ν is sufficiently large we can take

$$\mathcal{A}_1(\lambda_\pm(\omega)) = 2\omega\alpha\lambda_\pm(\omega) - 4\left[\frac{\omega^2(I_\perp - I_\parallel)}{(2I_\perp + \rho\kappa^2)} + \beta\right]$$

and

$$\mathcal{A}_2(\lambda_\pm(\omega)) = -\lambda_\pm^3(\omega)\rho\nu\mathcal{A}_3(\lambda_\pm(\omega))$$

where

$$\begin{aligned} \mathcal{A}_3(\lambda_\pm(\omega)) &= 4\omega^2 < AV_1, V_1 > +\lambda_\pm^2(\omega) < AV_2, V_2 > \\ &+2\omega\lambda_\pm(\omega)Re < AV_1, V_2 > . \end{aligned}$$

We recall that $\lambda_\pm(\omega) > 0$ and on the other hand

$$\begin{aligned} \mathcal{A}_3(\lambda_\pm(\omega)) &= < AV_2, V_2 > \left[\lambda_\pm(\omega) + \frac{\omega Re < AV_1, V_2 >}{< AV_2, V_2 >}\right]^2 \\ &+\frac{\omega^2}{< AV_2, V_2 >}\left[4 < AV_1, V_1 >< AV_2, V_2 > -(Re < AV_1, V_2 >)^2\right]. \end{aligned}$$

¿From the trivial inequality

$$(Re < AV_1, V_2 >)^2 \leq |< AV_1, V_2 >|^2 \leq < AV_1, V_1 >< AV_2, V_2 >$$

we have that $\mathcal{A}_3(\lambda_\pm(\omega)) > 0$.Hence,$\mathcal{A}_2(\lambda_\pm(\omega)) < 0$.Finally,from $\nu > \nu_0(\omega)$ we conclude that $\mathcal{A}_2(\lambda_\pm(\omega,\nu)) < 0$.We examine now the sign of $\mathcal{A}_1(\lambda_\pm(\omega))$.

$$\begin{aligned} \mathcal{A}_1(\lambda_\pm(\omega)) &= 2\omega\alpha\left[\omega\alpha \pm \sqrt{\omega^2\alpha^2 - 2\left[\frac{\omega^2(I_\perp - I_\parallel)}{(2I_\perp + \rho\kappa^2)} + \beta\right]}\right] \\ &-4\frac{\omega^2(I_\perp - I_\parallel)}{(2I_\perp + \rho\kappa^2)} \\ &= 2(\pm\lambda_\pm(\omega))\sqrt{\omega^2\alpha^2 - 2\left[\frac{\omega^2(I_\perp - I_\parallel)}{(2I_\perp + \rho\kappa^2)} + \beta\right]}. \end{aligned}$$

¿From this expression it follows that $\mathcal{A}_1(\lambda_+(\omega)) > 0$ and $\mathcal{A}_1(\lambda_-(\omega)) < 0$.Hence,from $\nu > \nu_0(\omega)$, $\mathcal{A}_1(\lambda_-(\omega,\nu)) < 0$ and $\mathcal{A}_1(\lambda_+(\omega,\nu)) > 0$.Now the Lemma follows from (6.17). ∎

Theorem 6.1 *For every value of the angular velocity ω different from ω^*,there exists a $\nu(\omega)$ such that if $\nu > \nu(\omega)$,Sobolev's system is instable.*

Proof.If I) is true or II) holds with $\omega < \omega^*$,the result follows from a simple application of the Rouche's Theorem to the functions $Q_2(\omega,\lambda)$ and $f(\lambda)$.Now if II) holds and $\omega > \omega^*$ the Theorem follows from Lemma 6.3. ∎

7 Conclusions

The Theorem 6.1 shows the important effect of the viscosity on the stability problem considered in our paper with regard to the ideal case studied by Sobolev [14].It follows from [14] that for ellipsoidal cavities there exist at most two unstability zones whose upper limit is determined by the geometry of the elipsoid and the distribution of mass in the rigid body.As follows from the assertion of Theorem 6.1 when the fluid is viscous this upper limit increase indefinitely with the viscosity.

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References

[1] Agmon S. On Kernel, Eigenvalues and Eigenfunctions of operators related to Elliptic problems. *Comm. Math. Appl. Math.* 18 (1965) 627-663.

[2] Agmon S. *Lectures on elliptic boundary value problems.* Van Nostrand Mathematical Studies, 1965.

[3] Alexandrian R. A. Spectral properties of the operators corresponding to systems of the equations of Sobolev's type.*Memories of the Mathematical Society of Moscu.* T9 (1960) 455-505 (in Russian).

[4] Azizov T. Ya. and Iokhvido I.S *Linear operators in space with indefinite metric.* Pure and Applied mathematics series.Jhon Wiley and Sons,1989.

[5] Bognar J. *Indefinite inner product spaces.* Springer. Berlin, 1974

[6] Gohber I.C and Krein M.G. *Introduction to the theory of linear nonselfadjoint operators* Translations of Mathematical Monographs v 18,American Mathematical Society.Providence,Rodhe Island,1969

[7] Kato T. *Perturbation theory for linear operators* Springer-Verlag ,1966

[8] Krein M.G. and Langer H. On some mathematical principles in linear theory of damped oscillations of continua I. *Integral Equations and Operator Theory* . V 1/3 ,1978.

[9] Krein S.G. *Linear Differential Equations in Banach Spaces.* Translations of Mathematical Monographs v 29,American Mathematical Society.Providence,Rhode Island,1971.

[10] Lions J.L. *Problemex aux limitex daux les equations aux derivees partielles.* Les presses de L'Univers. Montreal, 1965.

[11] Ladyzhenskaya O.A. *The mathematical theory of viscous incompressible fluid.* Mathematics and its applications series.Gordon and Breach Science Publishers,1969

[12] Pontriaguin L.S. Hermitic operators in spaces with indefinite metric *Izv. Akad. Nauk. SSSR ser math* No 6 (1944) 243-280 (in Russian)

[13] Ralston J. V. On stationary modes in Inviscid rotating fluids. *Journ. Math. Anal. Appl.* 44 (1973) 366-383

[14] Sobolev S. L. About the movement of a symmetric top with a cavity filled of a fluid.*Prikl. Mekh. Tekh. Fiz.* No. 3 (1960) 20-25 (in Russian)

[15] Yudovich V. I. *The linearization method in Hydrodynamical Stability Theory* Translations of Mathematical Monographs,V 79.American Mathematical Society.Providence,Rodhe Island,1989

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