NIKHEF
NATIONAAL INSTITUUT VOOR KERNFYSICA EN HOGE-ENERGIEFYSICA

NIKHEF-H 94-15
ULB-TH-94/07
hep-th/9405194

Local BRST cohomology in the antifield formalism: II. Application to Yang–Mills theory

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Abstract

Yang-Mills models with compact gauge group coupled to matter fields are considered. The general tools developed in a companion paper are applied to compute the local cohomology of the BRST differential s modulo the exterior spacetime derivative d for all values of the ghost number, in the space of polynomials in the fields, the ghosts, the antifields (=sources for the BRST variations) and their derivatives. New solutions to the consistency conditions $sa+db=0$ depending non trivially on the antifields are exhibited. For a semi-simple gauge group, however, these new solutions arise only at ghost number two or higher. Thus at ghost number zero or one, the inclusion of the antifields does not bring in new solutions to the consistency condition $sa+db=0$ besides the already known ones. The analysis does not use power counting and is purely cohomological. It can be easily extended to more general actions containing higher derivatives of the curvature, or Chern-Simons terms.

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1 Introduction

In a previous paper [1], referred to as I, we have derived general theorems on the local cohomology of the BRST differential s for a generic gauge theory. We have discussed in particular how it is related to the local cohomology of the Koszul-Tate differential δ and have demonstrated vanishing theorems for the cohomology $H_k(\delta|d)$ under various conditions. In the present paper, we apply the general results of I to Yang-Mills models with compact gauge group and provide the explicit list of all the non-vanishing BRST groups $H^k(s|d)$ for those models.

It has been established on general grounds that the groups $H^k(s)$ and $H^k(s|d)$ are respectively given by

$$H^k(s) \simeq \begin{cases} H^k(\gamma, H_0(\delta)) & k \geq 0 \\ 0 & k < 0 \end{cases} \tag{1.1}$$

and

$$H^k(s|d) \simeq \begin{cases} H^k(\gamma|d, H_0(\delta)) & k \geq 0 \\ H_{-k}(\delta|d) & k < 0 \end{cases} \tag{1.2}$$

(see [2] and I where this is recalled). Here, γ is the longitudinal exterior derivative along the gauge orbits, denoted by d (or D) in [2]. The isomorphisms (1.1) and (1.2) are valid for arbitrary gauge theories and hold when the "cochains" (local q-forms) upon which s acts are allowed to contain terms of arbitrarily high antighost number.

Now, in the case of Yang-Mills models, the BRST differential is just the sum of δ and γ,

$$s = \delta + \gamma \tag{1.3}$$

and so, is *not* an infinite formal series of derivations with arbitrarily high antighost number (as it can a priori occur for an arbitrary gauge system). It is thus natural to consider local q-forms that have bounded antighost number, and to wonder whether the equalities (1.1)-(1.2) still hold under this restriction. Our first result, derived in section 3, establishes precisely the validity of (1.1)-(1.2) in the space of local q-forms with bounded antighost number.

The isomorphisms (1.1)-(1.2) are useful in that they indicate how BRST invariance is equivalent to - and can be used as a substitute for - gauge invariance. However, they are not very explicit and a more precise characterization of $H^k(s)$ or $H^k(s|d)$ is desired.

It has been shown in [3] that in each cohomological class of s, one can find a representative that does not involve the antifields and which is thus annihiliated by γ. It then easily follows that

$$H^k(s) \simeq H^k(\gamma, \mathcal{E})/\mathcal{N} \qquad (k > 0) \tag{1.4}$$

where (i) $\mathcal{E}$ is the algebra generated by the vector potential A^a_μ, the ghosts C^a, the matter fields y^i and their derivatives (no antifields) ; and (ii) $\mathcal{N}$ is the ideal of elements of $\mathcal{E}$ that vanish on-shell. Since the cohomology of γ in $\mathcal{E}$ is well understood in terms of Lie algebra cohomology, the equation (1.4) provides a more precise characterization of $H^k(s)$ than (1.1) does. The representatives of (1.4) are polynomials in the "primitive forms" on the Lie algebra with coefficients that are invariant polynomials in the field strengths, the matter fields and their covariant derivatives [4, 5, 6, 7, 8, 9]. Furthermore, two such objects are in the same class if they coincide on-shell. To get a non redundant list, one may split the field strengths, the matter fields and their covariant derivatives into "independent" components, which are not constrained by the equations of motion, and "dependent components", which may be expressed on-shell in terms of the independent components. The cocyles may then be chosen to depend only on the independent components. The isomorphism (1.4) is a cohomological reformulation of a theorem proved long ago by Joglekar and Lee [10]. It plays a crucial role in renormalization theory [11, 12].

We derive in this paper an analogous, more precise characterization of the local cohomology $H^k(s|d)$ of s modulo d. For each value of the ghost degree, and in arbitrary spacetime dimension, we provide a constructive procedure for building representatives of each cohomological class. We then list all the solutions, some of which are expressed in terms of non trivial conserved currents which we assume to have been determined. We find that contrary to what happens for the cohomology of s, there exists cocycles in the cohomology of s modulo d from which the antifields cannot be eliminated by redefinitions. Thus, there are new solutions to the consistency conditions $sa + db = 0$ besides the antifield independent ones, as pointed out in [13] for a Yang-Mills group with two abelian factors.

However, if the gauge group is semi-simple, these additional solutions do not arise at ghost number zero or one but only at higher ghost number. Accordingly, the conjecture of Kluberg-Stern and Zuber on the renormalization of (local and integrated) gauge invariant operators [14, 15] is valid in that case (in even dimension). Differently put, there is no consistent perturbation of the Yang-Mills Lagrangian of ghost number zero, besides the perturbations by gauge invariant operators (or Chern-Simons terms in odd dimensions). Also, in four dimensions, there is no new candidate gauge anomaly besides the well known Adler-Bardeen one. Our results were partly announced in [16] and do not use power counting. They are purely cohomological.

The BRST differential contains information about the dynamics of the theory through the Koszul-Tate differential δ. Therefore, if one replaces the Yang-Mills Lagrangian $-1/8\ trF^2$ by a different Lagrangian containing higher order derivatives of the curvature, or Chern-Simons terms in odd dimensions, the local BRST cohomology generically changes even though the gauge transformations remain the same. We show, however, that the procedure for dealing with the Yang-Mills action works also for these more general actions.

2 BRST differential

We assume throughout that the gauge group G is compact and is thus the direct product of a semi-simple compact group by abelian $U(1)$ factors. As in I, we take all differentials to act from the right.

The BRST differential [17, 18] for Yang-Mills models is a sum of two pieces,

$$s = \delta + \gamma,\ antigh\ \delta = -1,\ antigh\ \gamma = 0 \tag{2.1}$$

where δ is explicitly given by

$$\begin{aligned} \delta A^a_\mu = 0,\ \delta C^a = 0,\ \delta y^i = 0 \\ \delta A^{*\mu}_a = -\frac{\delta^L \mathcal{L}_0}{\delta A^a_\mu},\ \delta C^*_a = -D_\mu A^{*\mu}_a + gT^j_{ai} y^*_j y^i,\ \delta y^*_i = -\frac{\delta^L \mathcal{L}_0}{\delta y^i} \end{aligned} \tag{2.2}$$

Here, $\mathcal{L}_0 = \mathcal{L}^y_0(y^i, D^y_\mu y^i) + \frac{1}{8} trF^{\mu\nu}F_{\mu\nu}$, $D^y_\mu y^i = \partial_\mu y^i - gA^a_\mu T^i_{aj} y^j$, and $\mathcal{L}^y_0(y^i, \partial_\mu y^i)$ is the free matter field Lagrangian. We assume for simplicity that the matter fields do not carry a gauge invariance of their own and belong to a linear representation of G. The differential γ is given by

$$\begin{aligned} \gamma A^a_\mu = D_\mu C^a,\ \gamma C^a = -\frac{1}{2} g C^a_{bc} C^b C^c,\ \gamma y^i = gT^i_{aj} y^j C^a \\ \gamma A^{*\mu}_a = gA^{*\mu}_c C^c_{ab} C^b,\ \gamma C^*_a = gC^*_c C^c_{ab} C^b, \\ \gamma y^*_i = -gT^j_{ai} y^*_j C^a, \end{aligned} \tag{2.3}$$

There is no term of higher antighost number in s because the gauge algebra closes off-shell. One has

$$\delta^2 = 0,\ \gamma^2 = 0,\ \gamma\delta + \delta\gamma = 0. \tag{2.4}$$

As explained in I, section 4, we shall consider local q-forms that are polynomials in all the variables (Yang-Mills potential A^a_μ, matter fields y^i, ghosts C^a, antifields $A^{*\mu}_a$, y^*_i and C^*_a) and their derivatives. This is natural from the point of view of quantum field theory and implies in particular that the local q-forms under consideration have bounded antighost number.

Now, the general isomorphism theorems (1.1)-(1.2) have been established under the assumption that the local q-forms may contain terms of arbitrarily high antighost number. Our first task is to refine the theorems to the case where the allowed q-forms are constrained to have bounded antighost number. This is done in the next section.

3 Homological perturbation theory and bounded antighost number

Theorem 3.1 : *for Yang-Mills models, the isomorphisms*

$$H^k(s) \simeq \begin{cases} H^k(\gamma, H_0(\delta)) & k \geq 0 \\ 0 & k < 0 \end{cases} \tag{3.1}$$

and

$$H^k(s|d) \simeq \begin{cases} H^k(\gamma|d, H_0(\delta)) & k \geq 0 \\ H_{-k}(\delta|d) & k < 0 \end{cases} \tag{3.2}$$

also hold in the space of q-forms that are polynomials in all the variables and their derivatives.

Proof. We extend the action of the even derivation K of section 10 of I on the ghosts as follows,

$$K = N_\partial + A \tag{3.3}$$

where N_∂ is the operator counting the derivatives of all the variables,

$$\begin{aligned} N = \sum_{(k)} |k| \Big[& \frac{\partial^R}{\partial\left(\partial_{(k)} A^a_\mu\right)} \partial_{(k)} A^a_\mu + \frac{\partial^R}{\partial\left(\partial_{(k)} C^a\right)} \partial_{(k)} C^a \\ & + \frac{\partial^R}{\partial\left(\partial_{(k)} A^{*\mu}_a\right)} \partial_{(k)} A^{*\mu}_a + \frac{\partial^R}{\partial\left(\partial_{(k)} C^*_a\right)} \partial_{(k)} C^*_a \\ & + \frac{\partial^R}{\partial\left(\partial_{(k)} y^i\right)} \partial_{(k)} y^i + \frac{\partial^R}{\partial\left(\partial_{(k)} y^*_i\right)} \partial_{(k)} y^*_i \Big] \end{aligned} \tag{3.4}$$

and where A is defined by

$$\begin{aligned} A = \sum_{(k)} \Big[& 2 \frac{\partial^R}{\partial\left(\partial_{(k)} A^{*\mu}_a\right)} \partial_{(k)} A^{*\mu}_a + 3 \frac{\partial^R}{\partial\left(\partial_{(k)} C^*_a\right)} \partial_{(k)} C^*_a \\ & + 2 \frac{\partial^R}{\partial\left(\partial_{(k)} \tilde{y}^*_i\right)} \partial_{(k)} \tilde{y}^*_i + \frac{\partial^R}{\partial\left(\partial_{(k)} \bar{y}^*_i\right)} \partial_{(k)} \bar{y}^*_i - \frac{\partial^R}{\partial\left(\partial_{(k)} C^a\right)} \partial_{(k)} C^a \Big]. \end{aligned} \tag{3.5}$$

The antifields $\tilde{y}^*_i$ are associated with second order differential equations, while the antifields $\bar{y}^*_i$ are associated with first order differential equations. We give A-weigth -1 to the ghosts so that γ has only components of non positive K-degree,

$$\gamma = \gamma^0 + \gamma^{-1}, \tag{3.6}$$

just as δ,

$$\delta = \delta^0 + \delta^{-1} + \delta^{-2}. \tag{3.7}$$

As shown in I, one has $[K, \partial_\mu] = \partial_\mu$ so that the exterior derivative d increases the eigenvalue of N_∂ and K by one unit.

The ghosts are the only variables with negative K-degree ($\partial_\mu C^a$ has degree 0, $\partial_{\mu\nu} C^a$ has degree 1, etc...). Furthermore, because the antifields carry all a strictly positive degree, a form with bounded K-degree k cannot contain terms of antighost number greater than $k+g$, where g is the dimension of the Lie algebra (=number of ghosts). It is thus polynomial in the antifields.

We have indicated in section 10 of I that if a is δ-closed, has positive antighost number and has K-degree bounded by k, then $a = \delta b$ where b has also K-degree bounded by k. Similarily, if a is δ-closed modulo d, has both positive antighost and pure ghost numbers, and has K-degree bounded by k, then $a = \delta b + dc$ where b has K-degree bounded by k and c has K-degree bounded by $k-1$. Indeed, one knows from [19] that $a = \delta b + dc$. The bound on the k-degree is then easily derived by expanding the equality according to the K-degree, and using the acyclicity of δ_0, of δ_0 mod d and of d. These properties are crucial in the proof of the theorem.

Let a be a s-cocycle which is polynomial in all the variables and their derivatives. Let us expand a according to the antighost number,

$$a = a_0 + a_1 + \ldots + a_m. \tag{3.8}$$

One has

$$\delta a_{i+1} + \gamma a_i = 0, \qquad i = 0, 1, 2, \ldots, m-1 \tag{3.9}$$

and

$$\gamma a_m = 0. \tag{3.10}$$

The isomorphism between $H^k(s)$ and $H^k(\gamma, H_0(\delta))$ is defined by $[a] \mapsto [a_0]$. To prove the theorem, one must verify that this map is injective and surjective. This is done as in [2], by controlling further polynomiality through the K-degree in a manner analogous to what is done in I, section 10. For instance, let us prove surjectivity. Let a_0 be a representative of $H^k(\gamma, H_0(\delta))$, i.e., be an antifield independent solution of $\delta a_1 + \gamma a_0 = 0$. Since a_0 and a_1 are polynomials, they have bounded K-degree. We denote this bound by k. To show that a_0 is the image of a polynomial cocycle a of s, one constructs recursively a_2, a_3 etc by means of (3.9). Because both δ and γ have components of non-negative K-degree, the higher order terms a_2, a_3 etc... may be chosen to have also K-degree bounded by k. Thus, the recursive construction stops at antighost number $k+g$ (at the latest) and $a = a_0 + a_1 + \ldots + a_{k+g}$ is polynomial. Injectivity, as well as (3.2) are proved along the same lines. □

To conclude, we note that theorem 3.1 holds for all "normal" theories in the sense of section 10 of I, and, in particular, for Einstein gravity. Moreover, the reader may check that that there is some flexibility in the proof of the theorem, in that one may assign different weights to the variables and nevertheless reach the same conclusion.

4 Cohomology of γ

In order to characterize completely $H^*(s|d)$, one needs a few preliminary results. Some of them have been developed already in the literature, while some of them are new. These results are: cohomology $H^*(\gamma)$, invariant cohomology of d and invariant cohomology of δ modulo d. They are considered in this section and the next two.

The cohomology $H^*(\gamma)$ of γ has been computed completely in [4, 5, 6, 7, 8, 9, 3]. The easiest way to describe it is to redefine the generators of the algebra. The new generators adapted to γ are on the one hand A^a_μ, its symmetrized derivatives $\partial_{(\mu_1 \cdots \mu_k} A^a_{\mu_{k+1})}$, ($k = 1, 2, \ldots$) and their γ-variations ; and on the other hand χ^u_Δ and the undifferentiated ghosts C^a, where the χ^u_Δ stand for the field strengths, the matter fields, the antifields and all their covariant derivatives. (u stands for representation indices ; while Δ stands for spacetime or spinorial indices unrelated to the gauge group). The χ^u_Δ belong to a representation of the Lie algebra $\mathcal{G}$ of the gauge group. Indeed, the field strengths belong to the adjoint representation, the antifields $A^{*\mu}_a$ and C^*_a belong to the co-adjoint representation, while the antifields y^*_i belong to the representation dual to that of the y^i. As a result, the polynomials in the χ's also form a representation of the Lie algebra $\mathcal{G}$ of the gauge group: to any $x \in \mathcal{G}$, there is a linear operator $\rho(x)$ acting in the space of polynomials in the χ's as an even derivation and such that $\rho([x_1, x_2]) = [\rho(x_1), \rho(x_2)]$. The representation ρ is completely reducible. The polynomials belonging to the trivial representation are the invariant polynomials.

The crucial feature in the calculation of $H^*(\gamma)$ is that A^a_μ, its symmetrized derivatives and their γ-variations disappear from $H^*(\gamma)$ since they belong to the "contractible" part of the algebra. More precisely, one has

Theorem 4.1 : *(i) The general solution of $\gamma a = 0$ reads*

$$a = \bar{a} + \gamma b \tag{4.1}$$

where $\bar{a}$ is of the form

$$\bar{a} = \sum \alpha_J(\chi^u_\Delta)\omega^J(C^a). \tag{4.2}$$

Here, the α_J are invariant polynomials in the χ's, while the $\omega^J(C^a)$ belong to a basis of the Lie algebra cohomology of the Lie algebra of the gauge group.

(ii) $\bar{a}$ is γ-exact if and only if $\alpha_J(\chi^u_\Delta) = 0$ for all J.

Proof. the proof may be found in [4, 5, 6, 7, 9, 3] and will not be repeated here. □

Note that the α_J involve also the spacetime forms dx^μ. This will always be assumed in the sequel, where the word "polynomial" will systematically mean "spacetime form with coefficients that are polynomial in the variables and their derivatives".

5 Invariant cohomology of d

Let $\alpha(\chi^u_\Delta)$ be an invariant polynomial in the χ's. Assume that α is d-closed, $d\alpha = 0$. Then one knows from the theorem on the cohomology of d that $\alpha = d\beta$ for some β. Can one assume that β is also an invariant polynomial? If α does not contain the antifields, this may not be the case: invariant polynomials in the 2-form $F^a \equiv (1/2)F^a_{\mu\nu}dx^\mu dx^\nu$ are counterexamples (and the only ones) [7, 9]. However, if *antigh* $\alpha > 0$, one has:

Theorem 5.1 : *the cohomology of d in form degree $< n$ is trivial in the space of invariant polynomials in the χ's with strictly positive antighost number. That is, the conditions*

$$\gamma\alpha = 0, \; d\alpha = 0, \; antigh\ \alpha > 0, \; deg\ \alpha < n, \; \alpha = \alpha(\chi^u_\Delta) \tag{5.1}$$

imply

$$\alpha = d\beta \tag{5.2}$$

for some invariant $\beta(\chi)$,

$$\gamma\beta = 0, \tag{5.3}$$

Proof. the proof proceeds as the proof of the proposition on page 363 in [9]. We shall thus only sketch the salient points.

(i) First, one verifies the theorem in the abelian case with uncharged matter fields. In that case, any polynomial in the χ^u_Δ is invariant since the χ's themselves are invariant. To prove the theorem in the abelian case, one splits the differential d as $d = d_0 + d_1$, where d_1 acts on the antifields only and d_0 on the other fields. Let α be a polynomial in the field strengths, the antifields, the matter fields and their ordinary (= covariant) derivatives. If $d\alpha = 0$, then $d_1\alpha^N = 0$, where α^N is the piece in α containing the maximum number of derivatives of the antifields. But then, $\alpha^N = d_1\beta^{N-1}$, where β^{N-1} is a polynomial in the χ^u_Δ. This implies that $\alpha - d\beta^{N-1}$ ends at order $N-1$ rather than N. Going on in the same fashion, one removes successively $\alpha^{N-1}, \alpha^{N-2}, \ldots$ until one reaches the desired result.

(ii) Second, one observes that if α is invariant under a global compact symmetry group, then β can be chosen to be also invariant since the action of the group commutes with d.

(iii) Finally, one extends the result to the non-abelian case with coloured matter fields by expanding α according to the number of derivatives of all the fields (see [9] page 364 for the details). □

What replaces theorem 5.1 in form degree n is: let $\alpha = \rho dx^0 \ldots dx^{n-1}$ be exact, $\alpha = d\beta$, where ρ is an invariant polynomial of antighost number > 0. [Equivalently, ρ has vanishing variational derivatives with respect to all the fields and antifields]. Then, one may take the coefficients of the $(n-1)$-form β to be also invariant polynomials.

Theorem 5.1 can be generalized as follows. Let α be a representative of $H^*(\gamma)$, i.e.,

$$\alpha = \Sigma \alpha_J(\chi^u_\Delta)\omega^J(C^a) \tag{5.4}$$

where the $\alpha(\chi)$ are invariant polynomials. Because $d\gamma + \gamma d = 0$, d induces a well defined differential in $H^*(\gamma)$. This may be seen directly as follows. The derivative $d\alpha_J = D\alpha_J$ is an invariant polynomial in the χ's since D commutes with the representation, while $d\omega^J = \gamma\hat{\omega}^J(A, C)$ for some $\hat{\omega}^J$. Thus $d\alpha = \pm\Sigma(D\alpha_J)\omega^J + \gamma(\Sigma\alpha_J\hat{\omega}^J)$ defines an element of $H^*(\gamma)$ $(\gamma\alpha_J = 0)$, namely the class of $\Sigma D\alpha_J\omega^J \equiv \Sigma d\alpha_J\omega^J$. What is the cohomology of d in $H^*(\gamma)$? Again, we shall only need the cohomology in form degree $< n$ and antighost number > 0.

Theorem 5.2 : *$H^{g,l}_k(d, H^*(\gamma)) = 0$ for $k \geq 1$ and $l < n$. Here g is the ghost number, l is the form degree and k is the antighost number.*

Proof. let $\alpha = \Sigma\alpha_J\omega^J$ be such that $d\alpha = 0$ in $H^*(\gamma)$, i.e., $d\alpha = \gamma\mu$. From the above calculation, it follows that $\Sigma(D\alpha_J)\omega^J = \gamma\mu'$. But $\Sigma(D\alpha_J)\omega^J$ is of the form (4.2). This implies that $D\alpha_J = d\alpha_J = 0$ by (ii) of theorem 4.1. Thus, by theorem 5.1, $\alpha_J = d\beta_J$ where β_J are invariant polynomials in the χ's. It follows that $\alpha = \Sigma d\beta_J\omega^J = \pm d(\Sigma\beta_J\omega^J) \mp \gamma(\Sigma\beta_J\hat{\omega}^J)$ is indeed d-trivial in $H^*(\gamma)$. □

Theorem 5.2 is one of the main tools needed for the calculation of $H^*(s|d)$ in Yang-Mills theory. It implies that there is no nontrivial descent [20, 21, 22] for $H(\gamma|d)$ in positive antighost number. Namely, if $\gamma a + db = 0$, *antigh* $a > 0$, one may redefine $a \to a + \gamma\mu + d\nu = a'$ so that $\gamma a' = 0$. Indeed, the descent $\gamma a + db = 0, \gamma b + dc = 0, \ldots$ ends with e so that $\gamma e = 0$ and $de + \gamma(\textit{something}) = 0$. Thus e is trivial and by the redefinition $e \to e + \gamma f + dm$, may be taken to vanish, etc....

6 Invariant cohomology of δ modulo d

The final tool needed in the calculation of $H^*(s|d)$ is the invariant cohomology of δ modulo d. We have seen that $H_k(\delta|d)$ vanishes for $k > 2$. Now, let α be a δ-boundary modulo d, $\alpha = \delta\beta + d\gamma$, and let us assume that α is an invariant polynomial in the χ's (no ghosts). Can one also take β and γ to be invariant polynomials? The answer is affirmative as the next theorem shows.

Theorem 6.1 : *if the invariant polynomial α is a δ-boundary modulo d and has nonvanishing antighost number,*

$$\alpha = \delta\beta + d\gamma, \qquad antigh\ \alpha > 0, \tag{6.1}$$

then one may assume that β and γ are also invariant polynomials. In particular, $H_k(\delta|d) = 0$ for $k \geq 3$ in the space of invariant polynomials.

Proof. Let a_p^k be a k-form of antighost number p such that

$$a_p^k = \delta\mu_{p+1}^k + d\mu_p^{k-1}, \qquad p \geq 1. \tag{6.2}$$

We must show that both μ_{p+1}^k and μ_p^{k-1} may be taken to be invariant polynomials if a_p^k is an invariant polynomial. To the equation (6.2), we can associate a tower of equations that starts at form degree n and ends at form degree $k-p+1$ if $k \geq p$ or 0 if $k < p$,

$$a_{p+n-k}^n = \delta\mu_{p+n-k+1}^n + d\mu_{p+n-k}^{n-1} \tag{6.3}$$

$$\vdots$$

$$a_p^k = \delta\mu_{p+1}^k + d\mu_p^{k-1} \tag{6.4}$$

$$\vdots$$

$$\begin{cases} a_1^{k-p+1} = \delta\mu_2^{k-p+1} + d\mu_1^{k-p} \\ or \\ a_{p-k}^0 = \delta\mu_{p-k+1}^0, \end{cases}$$

where the a's are all invariant polynomials. One goes up the ladder by acting with d and using the fact that if an invariant polynomial is δ-exact in the space of all polynomials, then it is also δ-exact in the space of invariant polynomials (theorem 2 of [3]). One goes down the ladder by applying δ and using theorem 5.1.

It is easy to see, using again theorem 2 of [3] and theorem 5.1 that if any one of the μ_i^j is equal to an invariant polynomial modulo δ or d exact terms, then all of them fulfill that property. That is, if $\mu_i^j = M_i^j + \delta\rho_{i+1}^j + d\rho_i^{j-1}$ for one pair (i,j) $(j-i = k-p-1)$, then $\mu_l^m = M_l^m + \delta\rho_{l+1}^m + d\rho_l^{m-1}$ for all (l,m). Here, the M_l^m are invariant polynomials. Thus it suffices to verify the theorem for the top of the ladder, i.e., the n-forms. Furthermore, one has

Lemma 6.1 : *Theorem 6.1 is obvious for n-forms of antighost number $p > n$.*

Proof. The proof is direct. If $a_p^n = \delta\mu_{p+1}^n + d\mu_p^{n-1}$ with $p > n$, one gets at the bottom of the ladder $a_{p-n}^0 = \delta\mu_{p-n+1}^0$. But then, by theorem 2 of [3], one finds $\mu_{p-n+1}^0 = M_{p-n+1}^0 + \delta\rho_{p-n+2}^0$ where M_{p-n+1}^0 is an invariant polynomial. This implies that all the μ's are of the required form, and in particular that μ_{p+1}^n and μ_p^{n-1} may be taken to be invariant polynomials. □

We can now prove theorem 6.1. The proof proceeds as the proof of theorem 5.1. Namely, one verifies first the theorem in the abelian case with a single gauge field and uncharged matter fields. One then extends it to the case of many abelian fields with a global symmetry. One finally considers the full non-Abelian case.

Since the last two steps are very similar to those of theorem 5.1, we shall only verify explicitly here that theorem 6.1 holds for a single abelian gauge field with uncharged matter fields. So, let us start with a n-form a_p solution of (6.2) and turn to dual notations,

$$a_p = \delta b'_{p+1} + \partial_\mu j_p^\mu \qquad (p \geq 1). \tag{6.5}$$

We shall first prove that if the theorem holds for antighost number $p+2$, then it also holds for antighost number p. A direct calculation yields

$$\frac{\delta a_p}{\delta C^*} = \delta Z'_{(p-1)} \tag{6.6}$$

$$\frac{\delta a_p}{\delta A^{*\mu}} = \delta X'_{(p)\mu} + \partial_\mu Z'_{(p-1)} \tag{6.7}$$

$$\frac{\delta a_p}{\delta A_\mu} = \delta Y'^{\mu}_{(p+1)} - \partial_\nu(\partial^\mu X'^{\nu}_{(p)} - \partial^\nu X'^{\mu}_{(p)}) \tag{6.8}$$

$$\frac{\delta a_p}{\delta y^i} = D^+_{ji} X'^{i}_{(p)} + \delta Y'_{(p+1)i} \tag{6.9}$$

$$\frac{\delta a_p}{\delta y^*_i} = \delta X'^{i}_{(p)} \tag{6.10}$$

where Z'_{p-1}, $X'_{(p)\mu}$, Y'^{μ}_{p+1}, $X'^{i}_{(p)}$ and $Y'_{(p+1)i}$ are obtained by differentiating b'_{p+1} $[Z' = 0$ if $p = 1]$. The explicit expression of these polynomials will not be needed in the sequel. In (6.9), D^+_{ji} is the differential operator appearing in the linearized matter equations of motion. Because $\delta^R a_p/\delta C^*$, $\delta^R a_p/\delta A^{*\mu}$, $\delta^R a_p/\delta A_\mu$, $\delta^R a_p/\delta y^i$ and $\delta^R a_p/\delta y^*_i$ are invariant polynomials, i.e., involve only the χ's, one may replace in (6.6)-(6.10) the polynomials $Z'_{(p-1)}$, $X'_{(p)\mu}$, $Y'^{\mu}_{(p+1)}$, $X'^{i}_{(p)}$ and $Y'_{(p+1)i}$ which may a priori involve symmetrized derivatives of A_μ, by invariant polynomials $Z_{(p-1)}$, $X_{(p)\mu}$, $Y^\mu_{(p+1)}$, $X^i_{(p)}$ and $Y_{(p+1)i}$ depending only on the χ's,

$$\frac{\delta a_p}{\delta C^*} = \delta Z_{(p-1)} \tag{6.11}$$

$$\frac{\delta a_p}{\delta A^{*\mu}} = \delta X_{(p)\mu} - \partial_\mu Z_{(p-1)} \tag{6.12}$$

$$\frac{\delta a_p}{\delta A_\mu} = \delta Y^\mu_{(p+1)} - \partial_\nu(\partial^\mu X^\nu_{(p)} - \partial^\nu X^\mu_{(p)}) \tag{6.13}$$

$$\frac{\delta a_p}{\delta y^i} = D^+_{ji} X^{i}_{(p)} + \delta Y_{(p+1)i} \tag{6.14}$$

$$\frac{\delta a_p}{\delta y^*_i} = \delta X^i_{(p)}. \tag{6.15}$$

This is obvious for $Z_{(p-1)}$ and $X^i_{(p)}$ (simply set A_μ and its symmetrized derivatives equal to zero in $Z'_{(p-1)}$ and $X'_{(p)}$; this commutes with the action of δ). The assertion is then verified easily for $X^\nu_{(p)}$, $Y_{(p+1)i}$ and $Y^\mu_{(p+1)}$.

Now, the invariant polynomial Y^μ_{p+1} is δ-closed modulo d by (6.13) since $\delta a_p/\delta A_\mu = \partial_\nu(\delta a_p/\delta F_{\mu\nu})$. Thus, it is δ-exact modulo d because $H^{n-1}_{p+1}(\delta|d) \simeq H^n_{p+2}(\delta|d)$ is empty $(p+2 \geq 3)$. This means that Y^μ_{p+1} can be written as $\delta T^\mu_{p+2} + \partial_\nu S^{\mu\nu}_{p+1}$ where T^μ_{p+2} and $S^{\mu\nu}_{p+1}$ are both invariant polynomials since we assume that the theorems holds for antighost number $p+2$ in form degree n, or, what is the same, by our general discussion above, for antighost number $p+1$ in form degree $n-1$.

If one injects relations (6.11) - (6.15) in the identity

$$a_p = \int dt[\frac{\delta^R a_p}{\delta C^*}C^* + \frac{\delta^R a_p}{\delta A^{*\mu}}A^{*\mu} + \frac{\delta^R a_p}{\delta A_\mu}A_\mu + \frac{\delta^R a_p}{\delta y^i}y^i + \frac{\delta^R a_p}{\delta y^*_i}y^*_i] + \partial_\mu \rho'^\mu \tag{6.16}$$

one gets, using $Y^\mu_{p+1} = \delta T^\mu_{p+2} + \partial_\nu S^{\mu\nu}_{p+1}$ and making integrations by parts, that

$$a_p = \delta b_{p+1} + \partial_\mu \rho^\mu \tag{6.17}$$

where b_{p+1} is *manifestly invariant*. This proves that the theorem holds in antighost number p if it holds in antighost number $p+2$ (ρ^μ may also be chosen to be invariant by theorem 5.1). But we know by lemma 6.1 that the theorem is true for antighost number $> n$. Accordingly, the theorem is true for all (strictly) positive values of the antighost number. □

7 Calculation of $H^*(s|d)$ - General method

We can now turn to the calculation of $H^*(s|d)$ itself. The strategy for computing $H^*(s|d)$ adopted here [16] is to relate as much as possible elements of $H^*(s|d)$ to the known elements of $H^*(\gamma|d)$ [23, 4, 5, 6, 7, 8, 9, 3]. To that end, one controls the antifield dependence through theorems 5.2 and

6.1. This is done by expanding the cocycle condition $sa + db = 0$ according to the antighost number. At maximum antighost number k, one gets $\gamma a_k + db_k = 0$. Theorem 5.2 and its consequences for the descent equations for γ in the presence of antifields then implies, for $k \geq 1$, that one can choose b_k equal to zero. Thus $\gamma a_k = 0$, and by theorem 4.1, $a_k = \Sigma \alpha_J(\chi^u_\Delta)\omega^J(C)$ up to γ-exact terms. [The redefinition $a_k \rightarrow a_k + \gamma m_k + dn_k$ can be implemented through $a \rightarrow a + sm_k + dn_k$, which does not change the class of a in $H(s|d)$]. The equation at antighost number $k-1$ reads $\delta a_k + \gamma a_{k-1} + db_{k-1} = 0$. Acting with γ, we get $d\gamma b_{k-1} = 0$, which implies $\gamma b_{k-1} + dc_{k-1} = 0$.

If $k - 1 \geq 1$, theorem 5.2 implies again that one can choose $\gamma b_{k-1} = 0$ with $b_{k-1} = \Sigma\beta_J(\chi^u_\Delta)\omega^J(C)$. Inserting the forms of a_k and b_{k-1} into the equation at antighost number $k-1$ gives $\Sigma(\delta\alpha_J + d\beta_J)\omega^J(C) = \gamma(something)$ which implies $\delta\alpha_J + d\beta_J = 0$ by part (ii) of theorem 4.1, i.e. α_J is a δ-cycle modulo d. Suppose that α_J is trivial, $\alpha_J = \delta\mu_J + d\nu_J$. Theorem 6.1 then implies that μ_J and ν_J can be choosen to be invariant polynomials. The redefinition $a \rightarrow a \pm s(\Sigma\mu_J\omega^J - \Sigma\nu_J\hat{\omega}_J) - d(\Sigma\nu_J\omega^J)$ allows one to absorb a_k. [Recall that $\gamma\hat{\omega}^J = d\omega^J$. The corresponding redefinition of b is $b \rightarrow b - s(\Sigma\nu_J\omega^J)$, which leaves b_k equal to zero since $\gamma\nu_J = 0$]. Consequently, we have learned (i) that for $k \geq 1$, the last term a_k in any s-cocycle modulo d may be chosen to be of the form $\Sigma\alpha_J\omega^J(C)$ where the α_J are invariant polynomials ; and (ii) that for $k \geq 2$, α_J define δ-cycles modulo d which must be nontrivial since otherwise, a_k can be removed from a by adding to a a s- coboundary modulo d.

We can classify the elements of $H^*(s|d)$ according to their last non trivial term in the antighost number expansion. The results on the cohomology of $H_*(\delta|d)$ show that only three cases are possible.

Class I: a stops at antighost number 2,

$$a = a_0 + a_1 + a_2 \tag{7.1}$$

(with $a_0 = 0$ if $gh\ a = -1$, or $a_0 = a_1 = 0$ if $gh\ a = -2$). The last term a_2 is invariant,

$$a_2 = \sum \alpha_J(\chi^u_\Delta)\omega^J(C) \tag{7.2}$$

and the $\alpha_J(\chi^u_\Delta)$ define non trivial elements of $H_2(\delta|d)$.

Class II: a stops at antighost number one,

$$a = a_0 + a_1 \tag{7.3}$$

(with $a_0 = 0$ if $gh\ a = -1$). The last term a_1 is invariant,

$$a_1 = \sum \alpha_J(\chi^u_\Delta)\omega^J(C) \tag{7.4}$$

We shall see in section 9 below that the $\alpha_J(\chi^u_\Delta)$ must also be non-trivial δ-cycles modulo d.

Class III: a does not contain the antifields,

$$a = a_0. \tag{7.5}$$

Then, of course, $gh\ a \geq 0$,

8 Solutions of class I

The solutions of class I arise only when $H_2(\delta|d)$ is non trivial, i.e., when there are free abelian gauge fields. This is a rather academical context from the point of view of realistic Lagrangians, but the question turns out to be of interest in the construction of consistent couplings among free, massless vector particles [24].

One can divide the solutions of class I into three different types, according to whether they have total ghost number equal to -2 (type I_a), -1 (type I_b) or ≥ 0 (type I_c).

Type I_a: if $gh\ a = -2$, then a reduces to a_2 and cannot involve the ghosts. The solutions of type I_a have form degree n and are exhausted by theorem 13.1 of I, in agreement with the isomorphism $H^{-2}(s|d) \simeq H_2(\delta|d)$. They read explicitly

$$a \equiv a_2 = f^\alpha C^*_\alpha, \quad f^\alpha = constant \tag{8.1}$$

where C^*_α are the antifields conjugate to the ghosts of the abelian, free, gauge fields. We switch back and forth between the form notations and their dual notations. The C^*_α should thus be viewed alternatively as n-forms or as densities.

Type I_b: if $gh\ a = -1$, then a_2 must involve one ghost C^A. This ghost must be abelian since one must have $\gamma C^A = 0$. Thus,

$$a_2 = f_{A\alpha} C^{*\alpha} C^A, \quad f_{A\alpha} = const., \tag{8.2}$$

where the sum over A runs a priori over *all* abelian ghosts. The equation in antighost number one yields for a_1

$$a_1 = f_{A\alpha} A^A_\mu A^{*\alpha\mu}. \tag{8.3}$$

The next equation $\delta a_1 + db_0 = 0$ is then equivalent to

$$f_{A\alpha} F^A_{\mu\nu} F^{\alpha\mu\nu} = \partial_\rho k^\rho \tag{8.4}$$

for some k^ρ. This equality can hold only if the variational derivatives of the left hand side identically vanishes, which implies $f_{A\alpha} = 0$ for $A \neq \beta$ and $f_{\alpha\beta} = -f_{\beta\alpha}$. Thus, one gets finally

$$a = f_{\alpha\beta}(A^\alpha_\mu A^{*\beta\mu} + C^\alpha C^{*\beta}), \quad f_{\alpha\beta} = -f_{\beta\alpha}. \tag{8.5}$$

Type I_c: if $gh\ a \geq 0$, then all three terms a_0, a_1, and a_2 are in principle present. The term a_2 reads

$$a_2 = f_{\alpha J} C^{*\alpha} \omega^J(C) \tag{8.6}$$

where $\omega^J(C)$ form a basis of the Lie algebra cohomology. The $\omega^J(C)$ can be written as polynomials in the so-called "primitive forms". The primitive forms are of degree one (C^A) for the abelian factors and of degree ≥ 3 (trC^3, $trC^5, \ldots$) for each simple factor [25].

It will be useful in the sequel to isolate explicitly the abelian ghosts in (8.6). Thus, we write

$$a_2 = \sum_k \frac{1}{k!} f_{\alpha\Gamma A_1 \ldots A_k} \omega^\Gamma(C) C^{A_1} \ldots C^{A_k} C^{*\alpha} \tag{8.7}$$

where $\omega^\Gamma(C)$ involve only the ghosts of the simple factors. The pure ghost numbers of the terms appearing in (8.7) must of course add up to $2 + q$, where q is the total ghost number of a. The factors $\omega^\Gamma(C)$ have the useful property of belonging to a chain of descent equations [20, 21, 22] involving at least two steps

$$\partial_\mu \omega^\Gamma(C) = \gamma \hat{\omega}^\Gamma_\mu(C) \tag{8.8}$$

$$\partial_{[\mu} \omega^\Gamma_{\nu]}(C) = \gamma \hat{\hat{\omega}}^\Gamma_{[\mu\nu]}(C) \tag{8.9}$$

For instance,

$$\hat{\omega}^\Gamma_\mu = \frac{\partial^R \omega^\Gamma}{\partial C^a} A^a_\mu \tag{8.10}$$

(see [23, 7]). By contrast, the abelian ghosts belong to a chain that stops after the first step. One has $\partial_\mu C^A = \gamma A^A_\mu$, but there is clearly no $f_{\mu\nu}$ such that $\partial_{[\mu} A_{\nu]} = \gamma f_{\mu\nu}$. Since it will be necessary below to "lift" twice the elements $\omega^J(C)$ of the basis through equations of the form (8.8) and (8.9), the abelian factors play a distinguished role.

A direct calculation shows that

$$\delta a_2 = \gamma\Big[\Big(\sum \frac{1}{(k-1)!}\omega^\Gamma f_{\alpha\Gamma A_1\ldots A_k} C^{A_1}\ldots C^{A_{k-1}} A_\mu^{A_k} + \sum \frac{1}{k!}(-)^k \hat{\omega}_\mu^\Gamma f_{\alpha\Gamma A_1\ldots A_k} C^{A_1}\ldots C^{A_k}\Big) A^{*\alpha\mu}\Big] + \partial_\mu V^\mu \qquad (8.11)$$

for some V^μ. This fixes a_1 to be

$$a_1 = -\Big[\sum f_{\alpha\Gamma A_1\ldots A_k}\Big(\frac{1}{(k-1)!}\omega^\Gamma C^{A_1}\ldots C^{A_{k-1}} A_\mu^{A_k} + \frac{1}{k!}(-)^k \hat{\omega}_\mu^\Gamma C^{A_1}\ldots C^{A_k}\Big)\Big] A^{*\alpha\mu} \qquad (8.12)$$

up to a solution m_1 of $\gamma m_1 + dn_1 = 0$. Using again the absence of non trivial descent in positive antighost number, we may assume $n_1 = 0$ and $m_1 = \sum_J \mu_J(\chi^u_\Delta)\omega^J(C)$ by a redefinition $m_1 \to m_1 + d\alpha + \gamma\beta$ that would only affect a_0 as $a_0 \to a_0 + \delta\beta$ (if it exists). That is, a_1 takes the form (8.12) modulo an invariant object of antighost number one.

Compute now δa_1. One finds

$$\delta a_1 = -\frac{1}{2}\sum \frac{1}{(k-1)!}\omega^\Gamma f_{\alpha\Gamma A_1\ldots A_k} C^{A_1}\ldots C^{A_{k-1}} F_{\mu\nu}^{A_k} F^{\alpha\mu\nu} + \delta m_1 + \gamma(M_{\mu\nu\alpha}F^{\alpha\mu\nu}) + \partial_\mu \tilde{V}^\mu \qquad (8.13)$$

for some $\tilde{V}^\mu$. Here, $M_{\mu\nu\alpha}$ is explicitly given by

$$M_{\mu\nu\alpha} = \sum\Big[\frac{1}{2(k-2)!} f_{\alpha\Gamma A_1\ldots A_k}\omega^\Gamma C^{A_1}\ldots C^{A_{k-2}} A_\mu^{A_{k-1}} A_\nu^{A_k} + \frac{2}{(k-1)!}(-)^k \hat{\omega}^\Gamma_{[\mu} f_{\alpha\Gamma A_1\ldots A_k} C^{A_1}\ldots C^{A_{k-1}} A^{A_k}_{\nu]} - \frac{1}{k!}\hat{\hat{\omega}}^\Gamma_{[\mu\nu]} f_{\alpha\Gamma A_1\ldots A_k} C^{A_1}\ldots C^{A_k}\Big]. \qquad (8.14)$$

Thus, δa_1 is γ-closed modulo d and a_0 exists if and only if the first term on the right hand side of (8.13) is weakly γ-exact modulo d, i.e.,

$$-\frac{1}{2}\sum \frac{1}{(k-1)!}\omega^\Gamma f_{\alpha\Gamma A_1\ldots A_k} C^{A_1}\ldots C^{A_{k-1}} F_{\mu\nu}^{A_k} F^{\alpha\mu\nu} + \delta m_1 = \gamma m_0 + \partial_\mu n_0^\mu \qquad (8.15)$$

for some m_0 and n_0^μ of antighost number zero. This forces this first term to vanish, as we now show.

By acting with γ on (8.15), one gets $d\gamma n_0 = 0$ and thus $\gamma n_0 + dn_0' = 0$. Accordingly, n_0 is an antifield independent solution of the γ-cocycle condition modulo d. This equation has been completely solved in the literature [4, 7, 8, 9] and the solutions fall into two classes: those that are annihilated by γ and are therefore invariant objects (up to redefinitions) ; and those that lead to a non trivial descent, for which no redefinition can make n_0' equal to zero. This second class involves only the forms $A^a = A^a_\mu dx^\mu$, $F^a = dA^a + A^2$, their exterior products, and the ghosts. Thus, $n_0 = \bar{n}_0 + \bar{\bar{n}}_0$, where $\bar{n}_0$ belongs to the first class and $\bar{\bar{n}}_0$ belongs to the second class.

The solutions of the second class are obtained by considering the descent $\gamma\bar{\bar{n}}_0 + d\bar{\bar{n}}_0' = 0$, $\gamma\bar{\bar{n}}_0' + d\bar{\bar{n}}_0'' = 0$ etc .... One successively lifts the last term of the descent, which is annihilated by γ all the way to $\bar{\bar{n}}_0$. The term $d\bar{\bar{n}}_0$ itself can be written as a γ-exact term, unless there is an "obstruction". This obstruction is an invariant polynomial which involves $\omega^J(C)$ and the components $F^a_{\mu\nu}$ but only through the forms F^a and their exterior products, but no other combination [23]. In particular, the dual of F^a cannot occur. Accordingly, the obstruction cannot be written as a term involving $F^A_{\mu\nu}F^{\alpha\mu\nu}$ plus a term involving the equations of motion, plus a term of the form $d\bar{n}_0$, with $\bar{n}_0$ invariant. This means that the obstruction must be zero if a_0 is to exist, so that $d\bar{n}_0 = \gamma\mu_0$ by itself. By adding to a_0 a solution of type III if necessary, we may assume $\bar{n}_0$ to be absent.

If n_0 reduces to the invariant piece $\bar{n}_0$, the equation (8.15) and theorem 4.1 imply that

$$-\frac{1}{2}\sum\frac{1}{(k-1)!}\omega^\Gamma f_{\alpha\Gamma A_1\dots A_k}C^{A_1}\dots C^{A_{k-1}}F^{A_k}_{\mu\nu}F^{\alpha\mu\nu}+\delta m_1$$
$$-\sum(D_\mu n^\mu_J)\omega^J=0 \tag{8.16}$$

with $\bar{n}_0=\sum n^\mu_J\omega^J$. If we set in this equality the covariant derivatives of $F^a_{\mu\nu}$ equal to zero, one gets the desired result that $f_{\alpha\Gamma A_1\dots A_k}C^{A_1}\dots C^{A_{k-1}}F^{A_k}_{\mu\nu}F^{\alpha\mu\nu}$ should vanish. This implies that $f_{\alpha\Gamma A_1\dots A_k}$ (i) has as non vanishing components only $f_{\alpha\Gamma\alpha_1\dots\alpha_k}$; and (ii) is completely antisymmetric in $(\alpha,\alpha_1,\dots,\alpha_k)$. The solutions of class I_c are consequently exhausted by

$$a=\sum f_{\alpha\Gamma\alpha_1\dots\alpha_k}\Big[\Big(-\frac{1}{2(k-2)!}\omega^\Gamma C^{\alpha_1}\dots C^{\alpha_{k-2}}A^{\alpha_{k-1}}_\mu A^{\alpha_k}_\nu$$
$$-\frac{2}{(k-1)!}(-)^k\hat{\omega}^\Gamma_{[\mu}C^{\alpha_1}\dots C^{\alpha_{k-1}}A^{\alpha_k}_{\nu]}+\frac{1}{k!}\hat{\hat{\omega}}^\Gamma_{[\mu\nu]}C^{\alpha_1}\dots C^{\alpha_k}\Big)F^{\alpha\mu\nu}$$

$$-\Big(\frac{1}{(k-1)!}\omega^\Gamma C^{\alpha_1}\dots C^{\alpha_{k-1}}A^{A_k}+\frac{1}{k!}(-)^k\hat{\omega}^\Gamma_\mu C^{\alpha_1}\dots C^{\alpha_k}\Big)A^{*\alpha\mu}$$
$$+\frac{1}{k!}\omega^\Gamma C^{\alpha_1}\dots C^{\alpha_k}C^{*\alpha}\Big] \tag{8.17}$$

(modulo solutions of class II). This ends our discussion of the solutions of class I, corresponding to elements of $H_2(\delta|d)$.

[The analysis has been performed explicitly for spacetime dimensions greater than or equal to three. In two spacetime dimensions, there are further solutions. The solutions of ghost number -2 read $(\partial f/\partial F^a_{01})C^*_a+(1/2)(\partial^2 f/\partial F^b_{01}\partial F^a_{01})\epsilon_{\mu\nu}A^{*\mu}_a A^{*\nu}_b$, where f is an invariant polynomial in those field strengths $F^a_{\mu\nu}$ that obey $D_\mu F^a_{01}=0$ on-shell. The solutions of ghost number -1 and higher are constructed as above, by multiplying the solutions of ghost number -2 with the γ-invariant polynomials $\omega^J(C)$, and then solving successively for a_1 and a_0. There are possible obstructions in the presence of abelian factors which restrict the coefficients of ω^J. We leave the details to the reader.]

9 Solutions of class II

The next case to consider is given by a cocycle a whose expansion stops at antighost number 1. Again, we may consider two subcases: type II_a, with $gh\ a=-1$; and type II_b, with $gh\ a\geq 0$.

Type II_a: if $gh\ a=-1$, then a reduces to a_1 and does not involve the ghosts. It is clearly an element of $H_1(\delta|d)$, by the equation $\delta a_1+db_0=0$. The groups $H^k_1(\delta|d)$ are non empty in form degree n (conserved currents) and $n-1$ (if there are uncoupled abelian fields). Let j^μ_Δ be a complete set of inequivalent non trivial conserved currents and let $X^a_{\mu\Delta}$, X^i_Δ be the corresponding global symmetries of the fields,

$$\delta(X^a_{\mu\Delta}A^{*\mu}_a+X^i_\Delta y^*_i)=\partial_\mu j^\mu_\Delta \tag{9.1}$$

We impose to $X^a_{\mu\Delta}A^{*\mu}_a+X^i_\Delta y^*_i$ to be annihilated by γ i.e., to be invariant. Because the equations of motion involve derivatives of the field strengths, and are not invariant polynomials in the forms F^a, there is no obstruction to taking j^μ_Δ annihilated by γ as well.

One gets for the BRST cohomology $H^{-1}(s|d)$.

In form degree $n-1$:

$$a = f^{\alpha} A^{*\mu}_{\alpha}, \quad f^{\alpha} = constant. \tag{9.2}$$

In form degree n:

$$a = f^{\Delta}(X^{a}_{\mu\Delta} A^{*\mu}_{a} + X^{i}_{\Delta} y^{*}_{i}), \quad f^{\Delta} = constant. \tag{9.3}$$

Turn now to the solutions of type II_b.

Type II_b: We must solve $\gamma a_0 + \delta a_1 + db_0 = 0$ with $a_1 = \sum \alpha_J \omega^J$. The derivation above does not imply that b_0 is annihilated by γ and thus, it is not clear at this stage that the α_J belong to $H(\delta|d)$. However, by acting with γ on $\delta a_1 + \gamma a_0 + db_0 = 0$, one gets again that $\gamma b_0 + dc_0 = 0$. The analysis proceeds then in a manner similar to that of type II_c. As mentioned above, the general solution to $\gamma b_0 + dc_0 = 0$ is known [4, 7, 8, 9] and takes the form $b_0 = \bar{b}_0 + \bar{\bar{b}}_0$, where (i) $\bar{b}_0$ is annihilated by γ and thus given by $\bar{b}_0 = \Sigma\beta_{0J}(\chi)\omega^J(C)$ (up to irrelevant γ-exact terms) with β_{0J} invariant polynomials in the χ's ; and (ii) $\bar{\bar{b}}_0$ is γ closed only modulo a non-trivial d exact term and involves the forms $A^a = A^a_\mu dx^\mu$, $F^a = dA^a + A^2$, and C^a. The obstruction [23] to writing $d\bar{\bar{b}}_0$ as a γ exact term involves the forms F^a and $\omega^J(C)$. It cannot be written as the sum of a term proportional to the equations of motion and a term of the form $d\bar{b}_0$ with $\bar{b}_0$ invariant since such terms involve unavoidably the covariant derivatives of the field strengths. Thus, the obstruction must be absent and $d\bar{\bar{b}}_0 = -\gamma\bar{\bar{a}}_0$, for some $\bar{\bar{a}}_0$. The equation $\delta a_1 + \gamma a_0 + db_0 = 0$ splits therefore into two separate equations $\gamma\bar{\bar{a}}_0 + d\bar{\bar{b}}_0 = 0$ and $\gamma\bar{a}_0 + d\bar{b}_0 + \delta a_1 = 0$.

The first equation defines a solution of class III. We need only consider in this section the second equation. Because $\bar{b}_0$ is annihilated by γ, we may follow the procedure of section 7 to find again that the invariant polynomials α_J in a_1 define elements of $H_1(\delta|d)$. One gets explicitly.

In form degree $n-1$:

$$a = f^{\alpha}_{J}(\hat{\omega}^{J}_{\nu}(C)F^{\mu\nu}_{\alpha} + \omega^{J}(C)A^{*\mu}_{\alpha}), \quad f^{\alpha}_{J} = constant. \tag{9.4}$$

In form degree n:

$$a = f^{\Delta}_{J}[\hat{\omega}^{J}_{\mu} j^{\mu}_{\Delta} + \omega^{J}(C)(X^{a}_{\mu\Delta}A^{*\mu}_{a} + X^{i}_{\Delta}y^{*}_{i})], \quad f^{\Delta}_{J} = constant. \tag{9.5}$$

(with $\gamma\hat{\omega}^J_\mu = \partial_\mu \omega^J$).

[In two dimensions, there are further solutions obtained by taking $f^a_J = \partial f_J/\partial F^a_{01}$, where f_J are arbitrary invariant polynomials in the F^a_{01}. We leave the details to the reader.]

The solutions of class I exist only if there are free, abelian gauge fields. For a semi-simple gauge group, class I is empty. By contrast, the solutions of class II in form degree n exist whenever there are non trivial conserved currents, or, equivalently, non trivial global symmetries. They occur at ghost number -1, or $-1 + l_J$, where l_J is the ghost number of the element ω^J of the chosen basis of the Lie algebra cohomology. For a semi-simple gauge group, l_J is greater than or equal to three. Thus, the solutions of class II occur at ghost number -1, 2, and higher, but not at ghost number 0 or 1. The solutions at ghost number 2 are given by (9.5) with $\omega^J = trC^3$ and $\hat{\omega}^J_\mu = 3trC^2A_\mu$.

We close this section by pointing out that one may regroup the conserved currents j_Δ (viewed as $(n-1)$-forms) and the coefficients X^i_Δ into a single object

$$\tilde{G}_\Delta = d^n x(X^a_{\mu\Delta}A^{*\mu}_a + X^i_\Delta y^*_i) + j_\Delta, \tag{9.6}$$

which has the remarkable property of being annihilated by the sum $\tilde{s} = s + d$,

$$\tilde{s}\tilde{G}_\Delta = 0. \tag{9.7}$$

This equation is the analog of a similar equation holding for $\tilde{q}^*_\alpha$,

$$\tilde{q}^*_\alpha = C^*_\alpha + A^*_\alpha + *F_\alpha \tag{9.8}$$

where the C^*_α are viewed as n-forms, the A^*_α are viewed as $(n-1)$-forms and the dual $*F_\alpha$ to the uncoupled free abelian field strength are $(n-2)$-forms. One has also

$$\bar{s}q^*_\alpha = 0. \tag{9.9}$$

In veryfing these relations, one must use explicitly the fact that the spacetime dimension is n through $d(n\text{-form}) = 0$.

10 Non-triviality of solutions of classes I and II

We verify in this section that the solutions of types I and II are all non trivial.

Theorem 10.1 : *any BRST cocycle a modulo d belonging to the class I or to the class II is necessarily non trivial, $a \neq sc + de$.*

Proof. the idea of the proof is to show that if $a = sc + de$, then, the $\alpha_J(\chi^u_\Delta)$ all define trivial elements of $H_2(\delta|d)$ or $H_1(\delta|d)$. So, assume that $a = sc + de$. Expand this equation according to the antighost number. One gets

$$a_0 = \gamma c_0 + \delta c_1 + de_0, \quad a_1 = \gamma c_1 + \delta c_2 + de_1 \tag{10.1}$$

and

$$0 = \gamma c_i + \delta c_{i+1} + de_i \qquad (i \geq 2) \tag{10.2}$$

(we assume a to belong to the class II for definiteness ; the argument proceeds in the same way for the class I). Let c stop at antighost number M, $c = c_0 + c_1 + \ldots + c_M$. Then, one may assume that e stops also at antighost number M. Indeed, the higher order terms can be removed from e by adding a d-exact term since $H^k(d) = 0$ for $k < n-1$. Now, the equation (10.2) for $i = M$ reads $\gamma c_M + de_M = 0$ and is precisely of the form analysed above. Since $M \geq 2$, one may assume $e_M = 0$ and then by adding to c_M a s-exact modulo d-term (which does not modify a), that c_M is of the form $c_M = \sum \gamma_J(\chi^u_\Delta)\omega^J(C)$. Next, the equation at order $M-1$ shows that c_M can actually be removed, unless $M = 2$. Thus, we may assume without loss of generality that $c = c_0 + c_1 + c_2$, $c_2 = \sum \gamma_J(\chi^u_\Delta)\omega^J(C)$ and $e = e_0 + e_1$. It follows that the equation for a_1 reads

$$\sum \alpha_J(\chi^u_\Delta)\omega^J(C) = \gamma c_1 + \sum \delta\gamma_J(\chi^u_\Delta)\omega^J(C) + de_1. \tag{10.3}$$

By acting with γ on this equation, we obtain as above that e_1 may also be chosen to be invariant, $e_1 = \sum \epsilon_J(\chi^u_\Delta)\omega^J(C)$. Accordingly, (10.3) reads

$$\sum (\alpha_J(\chi^u_\Delta) - \delta\gamma_J(\chi^u_\Delta) - d\epsilon_J(\chi^u_\Delta))\,\omega^J(C) = \gamma c'_1 \tag{10.4}$$

from which one infers, using theorem 4.1, that

$$\alpha_J(\chi^u_\Delta) - \delta\gamma_J(\chi^u_\Delta) - d\epsilon_J(\chi^u_\Delta) = 0. \tag{10.5}$$

This shows that all the α_J are δ-exact modulo d, in contradiction to the fact that they define non trivial elements of $H_*(\delta|d)$. Therefore, the cocycle a cannot be s-exact modulo d. □

11 Solutions of class III

The solutions of class III do not depend on the antifields and fulfill $\gamma a_0 + db_0 = 0$. As we have recalled, these equations have been extensively studied previously and their general solution is known [4, 23, 7, 8, 9]. For this reason, we refer the reader to the existing literature for their explicit construction.

The solutions are classified according to whether b_0 can be removed by redefinitions or not.

Type III_a : $\gamma a_0 = 0$. The abelian anomaly $CF_{\mu\nu}dx^\mu dx^\nu$ in two dimensions belongs to this class.

Type III_b : $\gamma a_0 + db_0 = 0$, with b_0 non trivial. In that case, a_0 and b_0 may be assumed to depend only on the forms A^a, F^a, C^a and their exterior products.

The elements of $H(\gamma|d)$ not involving the antifields are non trivial as elements of $H(s|d)$ if and only if they do not vanish on-shell modulo d. Thus, the non trivial elements of $H(\gamma|d)$ of type III_b remain non trivial as elements of $H(s|d)$ since the forms A^a and F^a are unrestricted by the equations of motion. However, the solutions of type III_a may become trivial even if they are non trivial as elements of $H(\gamma|d)$.

The solutions of direct interest are those of ghost number zero and one. At ghost number zero, class III_a contains the invariant polynomials in the field strengths, the matter fields and their covariant derivatives. The Yang-Mills Lagrangian belongs to class III_a. Class III_b contains non trivial solutions at ghost number zero only in odd spacetime dimensions $2k+1$. These non trivial solutions are the Chern-Simons terms, given by

$$\mathcal{L}_{CS} = tr(AF^k + \ldots) \tag{11.1}$$

where the dots denote polynomials in A^a and F^a whose degree in F is smaller than k and whose form degree equals $2k+1$.

At ghost number one, type III_a contains solutions of the form "abelian ghost times invariant polynomial". It contains no solution if the group is semi-simple. Type III_b contains the famous Adler-Bardeen anomaly.

12 More general Lagrangians

In the previous discussion, we have assumed that the Lagrangian was the standard Yang-Mills Lagrangian. This assumption was explicitly used in the calculation since the dynamics enters the BRST differential through the Koszul-Tate differential.

It turns out, however, that for a large class of Lagrangians, one can repeat the analysis and get similar conclusions. These Lagrangians are gauge invariant up to a total derivative and thus read

$$\mathcal{L} = \mathcal{L}_0(y, F_{\mu\nu}, D_\mu y, D_\rho F_{\mu\nu}, \ldots) + \mathcal{L}_{CS} \tag{12.1}$$

where $\mathcal{L}_0$ is an invariant polynomial in the matter fields, the fields strengths and their covariant derivatives, and where the Chern-Simons term $\mathcal{L}_{CS}$ is available only in odd dimensions. We shall assume that the Yang-Mills gauge symmetry exhausts all the gauge symmetries. We shall also impose that the Lagrangian $\mathcal{L}$ defines a normal theory in the sense of section 10 of I. The calculation of $H(s|d)$ can then be performed along the lines of this paper.

(i) First, one verifies that the γ-invariant cohomology $H_k(\delta|d)$ is described as before : $H_k(\delta|d)$ vanishes for k strictly greater than 2; for $k = 2$, it is non-empty only if there are uncoupled abelian gauge fields, in which case it is spanned by C^*_α; and for $k = 1$, it is isomorphic to the set of non trivial global symmetries with invariant a_1. Thus, the dynamics enters explicitly $H_k(\delta|d)$ only at $k = 1$, through the conserved currents.

(ii) The solutions of class I makes a further use of the dynamics through the study of the obstructions to the existence of a_0. A case by case analysis, which proceeds as in section 8, is in principle required. Recall, however, that class I exists only in the academic situation where there are uncoupled abelian gauge fields.

(iii) Class II also uses the equations of motion in the proof that a_1 should define elements of $H(\delta|d)$. It must be verified whether the equations of motion can or cannot remove obstructions given by polynomials in the forms F^a. Again, the analysis proceeds straightforwardly as in section 9.

(iv) Class III is obviously unchanged since it does not involve the antifields (only the coboundary condition is modified, since the concept of "on-shell trivial" is changed).

The analysis is particularly simple for the pure Chern-Simons theory in three dimensions, without the Yang-Mills part. We take a semi-simple gauge group. Class I is then empty. Class II is empty as well since there is no non-trivial a_1 annihilated by γ. Only class III is present. Among the solutions of class III, those that are of the subtype III_a turn out to be trivial since the field strengths and their covariant derivatives vanish on-shell. Thus, we are left with class III_b. These solutions are obtained from the standard descent, with bottom given by the elements ω^J of the basis of the Lie algebra cohomology (trC^3, trC^5 etc), with constant coefficients (no F since $F = 0$ on-shell). This agrees with the analysis of [26].

13 Conclusion

In this paper, we have explicitly computed the cohomological groups $H^k(s|d)$ for Yang-Mills theory. Our work goes beyond previous analyses on the subject [17, 20, 4, 27, 28, 29, 30, 23, 22, 5, 7, 9], in that (i) we do not use power counting; and (ii) we explicitly include the antifields (= sources for the BRST variations). We have shown that new cohomological classes depending on the antifields appear whenever there are conserved currents, but only at antighost number ≥ 2 for a semi-simple gauge group. Our results confirm previous conjectures in the field. [The existence of antifield-dependent solutions of the consistency equation at ghost number one for a theory with abelian factors was anticipated in [27]. The structure of these solutions was partly elucidated and an argument was given that they cannot occur as anomalies].

The central feature behind our analysis is a key property of the antifield formalism, namely, that the antifields provide a resolution of the stationary surface through the Koszul-Tate differential [2]. It is by attacking the problem from that angle that we have been able to carry out the calculation to completion, while previous attempts following different approaches turned out to be unsuccessful. Thus, even in the familiar Yang-Mills context, the formal ideas of the antifield formalism prove to be extremely fruitful.

Our results can be extended in various directions. First, one can repeat the Yang-Mills calculation for Einstein gravity with or without matter. This will be done explicitly in [31]. Second, at a more theoretical level, one can analyze further the connection between the local BRST cohomology, the characteristic cohomology and the variational bicomplex [32]. This will be pursued in [33].

14 Acknowledgements

We are grateful to J. Collins, M. Dubois-Violette, O. Piguet, A. Slavnov, J. Stasheff, R. Stora, M. Talon, C. Teitelboim and C. Viallet for fruitful discussions. This work has been supported in part by research funds from the Belgian "F.N.R.S." as well as by research contracts with the Commission of the European Communities. M.H. is grateful to the CERN theory division for its kind hospitality while this work was being carried out and completed.

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