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MODULAR INVARIANTS AND FUSION RULE AUTOMORPHISMS FROM GALOIS THEORY

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Abstract. We show that Galois theory of cyclotomic number fields provides a powerful tool to construct systematically integer-valued matrices commuting with the modular matrix S, as well as automorphisms of the fusion rules. Both of these prescriptions allow the construction of modular invariants and offer new insight in the structure of known exceptional invariants.

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1. The Galois group and the modular matrix S

The classification of modular invariants and of fusion rule automorphisms are among the most challenging problems in conformal field theory. In [1,2] it was observed that Galois theory applied to elements of the modular matrix S can shed some light on these issues. In this note we point out that these connections can be exploited further so that they can actually be used to construct fusion rule automorphisms and modular invariants.

Given a rational fusion ring with generators ϕ_i, $i \in I$ (I some finite index set), and relations $\phi_i \star \phi_j = \sum_{k\in I} \mathcal{N}_{ij}^{\ k} \phi_k$ with $\mathcal{N}_{ij}^{\ k} \in \mathbb{Z}_{\geq 0}$, there exists a unitary and symmetric matrix S that diagonalizes the fusion matrices, i.e. the matrices $\mathcal{N}_i$ with entries $(\mathcal{N}_i)_j^k := \mathcal{N}_{ij}^{\ k}$. Together with the matrix T with entries $T_{ij} = T_i\,\delta_{ij} := e^{2\pi i(\Delta_i - c/24)}\delta_{ij}$, S generates a finite-dimensional representation of the modular group $SL_2(\mathbb{Z})$. In particular, $S^2 = C$, $(ST)^3 = C$, $C^2 = \mathbb{1}$. The charge conjugation matrix C, a permutation of order two, will be written as $C_{ij} = \delta_{i,j^+}$. By the Verlinde formula [3]

$$\mathcal{N}_{ij}^{\ k} = \sum_{\ell\in I} \frac{S_{i\ell} S_{j\ell} S^*_{k\ell}}{S_{0\ell}}\,, \tag{1}$$

the eigenvalues of the fusion matrices $\mathcal{N}_i$ are the *generalized quantum dimensions* S_{ij}/S_{0j}; here the label $0 = 0^+ \in I$ corresponds to the unit of the fusion ring (or in terms of conformal field theory, to the identity primary field, i.e. to the vacuum of the theory). They realize the irreducible representations of the fusion ring, i.e. we have

$$\frac{S_{i\ell}}{S_{0\ell}}\,\frac{S_{j\ell}}{S_{0\ell}} = \sum_{k\in I} \mathcal{N}_{ij}^{\ k}\,\frac{S_{k\ell}}{S_{0\ell}} \tag{2}$$

for all $\ell \in I$. The generalized quantum dimensions $S_{i\ell}/S_{0\ell}$ are the roots of the characteristic polynomial $\det(\lambda\mathbf{1} - \mathcal{N}_i)$, which is a normalized polynomial with integral coefficients, and hence they are algebraically integer numbers in some algebraic number field L over the rational numbers $\mathbb{Q}$. The extension $L/\mathbb{Q}$ is normal [1], and hence (using also the fact that $\mathbb{Q}$ has characteristic zero) a *Galois extension*; its *Galois group* $\mathcal{G}al(L/\mathbb{Q})$ is abelian. It follows [1] that L is contained in some cyclotomic field $\mathbb{Q}(\zeta_n)$, where ζ_n is a primitive nth root of unity.

Applying an element $\sigma_L \in \mathcal{G}al(L/\mathbb{Q})$ on equation (2) and using the fact that the fusion coefficients $\mathcal{N}_{ij}^{\ k}$ are integers and hence invariant under σ_L, we learn that the numbers $\sigma_L(S_{ij}/S_{0j})$, $i \in I$, again realize a one-dimensional representation of the fusion ring. As the generalized quantum dimensions exhaust all inequivalent one-dimensional representations of the fusion ring [4,5], there must exist some permutation of the labels j which we denote by $\dot\sigma$, such that

$$\sigma_L\Big(\frac{S_{ij}}{S_{0j}}\Big) = \frac{S_{i\dot\sigma(j)}}{S_{0\dot\sigma(j)}}\,. \tag{3}$$

The field M defined as the extension of $\mathbb{Q}$ that is generated by all S-matrix elements extends L. The extension $M/\mathbb{Q}$ is again normal and has abelian Galois group [2], so that $\mathcal{G}al(M/L)$ is a normal subgroup of $\mathcal{G}al(M/\mathbb{Q})$. Elementary Galois theory then shows that

$$0 \to \mathcal{G}al(M/L) \xrightarrow{\imath} \mathcal{G}al(M/\mathbb{Q}) \xrightarrow{r} \mathcal{G}al(L/\mathbb{Q}) \to 0\,, \tag{4}$$

with $\imath$ the canonical inclusion and r the restriction map, is an exact sequence, and hence

$$\mathcal{G}al(L/\mathbb{Q}) \cong \mathcal{G}al(M/\mathbb{Q})\,/\,\mathcal{G}al(M/L)\,. \tag{5}$$

In particular any $\sigma_M \in \mathcal{G}al(M/\mathbb{Q})$, when restricted to L, maps L onto itself and equals some element $\sigma_L \in \mathcal{G}al(L/\mathbb{Q})$. Conversely, any $\sigma_L \in \mathcal{G}al(L/\mathbb{Q})$ can be obtained this way. Therefore by a slight abuse of notation we will frequently use the abbreviation σ for both σ_M and its restriction σ_L.

Working in the field M, it follows from (3) that for any $\sigma_L \in \mathcal{G}al(L/\mathbb{Q})$ there exist signs $\epsilon_\sigma(i) \in \{\pm 1\}$ such that the relation

$$\sigma_M(S_{ij}) = \epsilon_\sigma(i) \cdot S_{\dot\sigma(i)j} \tag{6}$$

is fulfilled for all $i, j \in I$ [2]. We note that the Galois group element σ and the permutation $\dot\sigma$ of the labels that is induced by σ need not necessarily have the same order. However, it is easily seen (see the remarks around (21) below) that an extra factor of 2 is the only difference that can appear.

In this letter we describe how these observations can be extended in two directions. First, we show that Galois theory can be used to construct automorphisms of the fusion rules. Second, we derive from Galois theory a prescription for the systematic construction of integral-valued matrices in the commutant of the modular matrix S, and hence of candidate modular invariants. We describe how this method is implemented for WZW theories. As it turns out, our general prescription is able to explain many of the modular invariants that are usually referred to as 'exceptional'.

2. Fusion rule automorphisms

We first show that, if the permutation $\dot\sigma$ induced by the Galois group element σ leaves the identity fixed,

$$\dot\sigma(0) = 0\,, \tag{7}$$

then $\dot\sigma$ is an automorphism of the fusion rules. To prove this, we first calculate

$$\frac{S_{0i}}{S_{00}} = \sigma_L\Big(\frac{S_{0i}}{S_{00}}\Big) = \frac{\sigma_M(S_{0i})}{\sigma_M(S_{00})} = \frac{\epsilon_\sigma(i)\, S_{0\dot\sigma(i)}}{\epsilon_\sigma(0)\, S_{00}}\,. \tag{8}$$

Since S_{0j}/S_{00}, the main (i.e., zeroth) quantum dimensions, are positive, we learn that the sign $\epsilon_\sigma(i)$ is the same for all $i \in I$,

$$\epsilon_\sigma(i) = \epsilon_\sigma(0) =: \epsilon_\sigma = \text{const}\,. \tag{9}$$

Applying σ on the Verlinde formula (1), we then find

$$\mathcal{N}_{ij}^{\;\;k} = \sigma(\mathcal{N}_{ij}^{\;\;k}) = \sum_{l\in I} \frac{\epsilon_\sigma^3\, S_{\dot\sigma(i)l} S_{\dot\sigma(j)l} S^*_{\dot\sigma(k)l}}{\epsilon_\sigma\, S_{0l}} = \mathcal{N}_{\dot\sigma(i)\dot\sigma(j)}^{\quad\;\;\dot\sigma(k)}\,. \tag{10}$$

Next we note that in terms of the cyclotomic field $\mathbb{Q}(\zeta_n) \supseteq M \supseteq L$, the elements $\sigma_{(\ell)} \in \mathcal{G}al(L/\mathbb{Q})$ are simply the restrictions of elements $\tilde\sigma_{(\ell)} \in \mathcal{G}al(\mathbb{Q}(\zeta_n)/\mathbb{Q})$; the latter act as $\zeta_n \mapsto (\zeta_n)^\ell$, and $\mathcal{G}al(\mathbb{Q}(\zeta_n)/\mathbb{Q}) \cong \mathbb{Z}_n^*$ is the set of all such maps with ℓ coprime to n. In particular, $\ell = -1$ corresponds to complex conjugation; the associated permutation of the generators of the fusion ring is the charge conjugation C. As the Galois group is abelian, it follows that $\dot\sigma$ is compatible with charge conjugation,

$$\dot\sigma(i^+) = (\dot\sigma(i))^+\,. \tag{11}$$

Together with (7), the results (10) and (11) show that, as claimed, $\dot\sigma$ is an automorphism of the fusion rules.

Our result can be interpreted as follows. Let $G := \{\sigma \in \mathcal{G}al(L/\mathbb{Q}) \mid \dot\sigma(0) = 0\}$, and let L^G be the subfield in L that is left fixed under G. The elements of the subgroup G of $\mathcal{G}al(L/\mathbb{Q})$ leave the main quantum dimensions invariant, and hence the main quantum dimensions are already contained in L^G. The automorphisms of the fusion rules that are obtained from the Galois group as described above are thus a manifestation of the fact that the main quantum dimensions do *not* exhaust the field spanned by all generalized quantum dimensions.

The general result is nicely illustrated by the example of complex conjugation. Suppose that the fusion ring is non-selfconjugate, i.e. there is at least one $i \in I$ such that $i^+ \neq i$. Then the modular matrix S is complex, and as already mentioned the charge conjugation C which acts as $i \mapsto i^+$ is induced by $\sigma_C = \sigma_{(-1)} \in \mathcal{G}al(L/\mathbb{Q})$, i.e. $i^+ = \dot\sigma_C(i)$. As the main quantum dimensions are real (which is equivalent to $(0)^+ = 0$), G contains at least σ_C as a nontrivial element, and charge conjugation is the corresponding non-trivial automorphism.

As a second illustration, consider the extremal case $G = \mathcal{G}al(L/Q)$. This means that all main quantum dimensions are rationals (and, since they are algebraic integers, in fact even ordinary integers). This situation is realized e.g. for $c = 1$ conformal field theories, both for compactification of the free boson on a circle and for compactification on those $\mathbb{Z}_2$ orbifolds for which the number of fields is $m^2 + 7$ for some $m \in \mathbb{Z}$, as well as for the $(\mathrm{so}(N^2))_2$ and $(\mathrm{su}(3))_3$ WZW theories. Consider e.g. the theory of a free boson on the circle, with $N \in 2\mathbb{Z}$ primary fields. The fusion rules read $p \star q = p + q \bmod N$, and the modular matrix S has entries $S_{pq} = \mathrm{e}^{-2\pi i pq/N}$. The permutations induced by the Galois group are parametrized by l, with l and N coprime, and act like $p \mapsto lp \bmod N$. This is invertible just because l and N are coprime, and clearly an automorphism. Thus G is the full Galois group, $G \cong \mathbb{Z}_N^*$. Analogous considerations hold for the orbifolds and for the WZW theories just mentioned.

Note that a permutation automorphism of generic order N does not directly lead to a modular invariant since the corresponding permutation matrix Π_σ generically does not commute with S, but rather obeys $S^{-1}\Pi_\sigma S = \Pi_\sigma^{-1}$. For $N = 2$ (such as e.g. charge conjugation), Π_σ does commute with S, and hence provides a candidate modular invariant. For being indeed a modular invariant, Π_σ also has to commute with the modular matrix T; it is not difficult to establish (see the remarks around (24) below) that any automorphism of the fusion rules that fulfills (6) and commutes with the T-matrix has order two.

Sometimes there also exist automorphisms of the fusion rules that *cannot* be obtained from elements of the Galois group. This happens for instance if the S-matrix elements of all fields that are permuted are rational numbers; in this situation, any element of the Galois group necessarily leaves these fields fixed, and hence cannot induce the fusion rule automorphism.

3. The construction of S-matrix invariants

As an easy consequence of the relation (6) between $S_{\dot\sigma(i)j}$ and S_{ij}, it follows that for any matrix Z which satisfies

$$[Z,S] = 0\,, \qquad Z_{ij} \in \mathbb{Z} \ \ \forall\, i,j \in i\,, \tag{12}$$

the relation $Z_{\dot\sigma(i)\dot\sigma(j)} = \epsilon_\sigma(i)\epsilon_\sigma(j)\, Z_{ij}$ holds [2]. This leads to a selection rule for those matrices Z which obey $Z_{ij} \geq 0$ in addition to (12), and which hence provide a candidate modular invariant $\mathcal{Z}(\tau,\bar\tau) = \sum_{i,j} \chi_i^*(\bar\tau) Z_{ij} \chi_j(\tau)$ for the associated conformal field theory (this restriction is a generalization of the 'parity rule' of [6] and the 'arithmetical symmetry' of [7]).

Here we will go beyond the level of mere selection rules and show that Galois theory can be used to *construct* modular invariants. Let us apply σ^{-1} to the relation (6); then we have

$$S_{ij} = \sigma^{-1}\sigma(S_{ij}) = \sigma^{-1}(\epsilon_\sigma(i) S_{\dot\sigma(i)j}) = \epsilon_\sigma(i)\epsilon_{\sigma^{-1}}(j)\, S_{\dot\sigma(i)\,\dot\sigma^{-1}(j)}\,, \tag{13}$$

where in the last equality one uses the fact that $\epsilon_\sigma(i) = \pm 1$ is rational and hence fixed under σ. Using (13) l times, we obtain

$$S_{ij} = \epsilon_l(i)\epsilon_{-l}(j)\, S_{\dot\sigma^l(i)\,\dot\sigma^{-l}(j)}\,, \tag{14}$$

where the signs $\epsilon_l(i) \equiv \epsilon_{\sigma^l}(i) \in \{\pm 1\}$ are determined by $\epsilon_1 \equiv \epsilon_\sigma$ through $\epsilon_l(i) = \prod_{m=0}^{l-1} \epsilon_1(\dot\sigma^m(i))$. We will employ the simple result (13), respectively (14), to show that to any element of the Galois group one can associate a matrix Z which obeys (12).

Before proceeding, we should point out that a relation of the form (14) need not necessarily stem from Galois theory. In the proof we actually use only this relation, but not the information whether it is derived from Galois theory or not. [1] In particular, we need not assume that the signs ϵ_l are prescribed by some Galois group element σ, but only use that they are determined by the permutation $\dot\sigma$. However, Galois theory constitutes the only systematic tool that is known so far to derive such relations, even though it does not provide an exhaustive list. (A situation where the symmetry property (14) of the modular matrix S is satisfied in the absence of Galois symmetries is provided by mutually local simple currents [8] of order two.) [2]

Thus assume that $\dot\sigma$ is a permutation, of order $\dot N$, of the index set I of a fusion ring and satisfies a relation of the type (14), and define the integer N to be the order of the associated map $S_{ij} \mapsto \epsilon_\sigma(i)\, S_{\dot\sigma(i)j}$. We can then show that for any set $\{f_l \mid l = 1,2,\ldots,N\} \subset \mathbb{Z}$ of integers that satisfy

$$f_l = f_{-l} \equiv f_{N-l}\,, \tag{15}$$

the matrix Z with integral entries

$$Z_{jk} := \sum_{l=0}^{N-1} f_l\, \epsilon_l(k)\, \delta_{j,\dot\sigma^l(k)} \tag{16}$$

commutes with the modular matrix S. Namely, by direct calculation we have

$$(SZ)_{ik} = \sum_{j\in I} \sum_{l=0}^{N-1} S_{ij} \cdot f_l\, \epsilon_l(k)\, \delta_{j,\dot\sigma^l(k)} = \sum_{l=0}^{N-1} f_l \epsilon_l(k)\, S_{i\dot\sigma^l(k)} \tag{17}$$

as well as

$$\begin{aligned}(ZS)_{ik} &= \sum_{j\in I} \sum_{l=0}^{N-1} f_l \epsilon_l(j)\, \delta_{i,\dot\sigma^l(j)} \cdot S_{jk} = \sum_{l=0}^{N-1} f_l \epsilon_l(\dot\sigma^{-l}(i))\, S_{\dot\sigma^{-l}(i)k} \\ &= \sum_{l=0}^{N-1} f_l \epsilon_l(\dot\sigma^{-l}(i)) \cdot \epsilon_l(\dot\sigma^{-l}(i)) \epsilon_{-l}(k)\, S_{i\dot\sigma^{-l}(k)} = \sum_{l=0}^{N-1} f_l \epsilon_{-l}(k)\, S_{i\dot\sigma^{-l}(k)}\,,\end{aligned} \tag{18}$$

where in the transition to the second line we employed (14). Now one merely has to replace the sum on l in (18) by one on $-l$ and use (15) to conclude that indeed S and Z commute. The terms in the sum of (16) correspond to the elements of the cyclic group that is generated by the element σ appearing in (13); considering more generally an arbitrary abelian group G whose generators satisfy (13), one proves that the prescription (16) generalizes to

$$Z_{jk} = \sum_{\sigma\in G} f_\sigma\, \epsilon_\sigma(k)\, \delta_{j,\dot\sigma(k)}\,, \tag{19}$$

with f_σ restricted by

$$f_\sigma = f_{\sigma^{-1}} \tag{20}$$

[1] This remark applies in fact equally to the considerations about fusion rule automorphisms above.

[2] Considering simple currents of general order would amount to allow the ϵ's in (14) to be arbitrary phases instead of signs. Unfortunately there are no nontrivial cases with $N > 2$ and (16) being real-valued.

for all $\sigma \in G$.

Returning to the interpretation in terms of the Galois group, we note that according to (6) the upper limit N of the summation in equation (16) is precisely the order of the Galois group element σ (in particular, Galois theory provides a relation of the type (14) with $-l \equiv N-l$), and recall that this order need not necessarily coincide with the order of the permutation $\dot\sigma$ of the labels that σ induces. However, the following consideration shows that the distinction between N and $\dot N$ is actually not very relevant to applications. First, at most a relative factor of 2 can be present; namely, since $\dot\sigma$ is of order $\dot N$, one has in particular $\dot\sigma^{\dot N}(0)=0$, which by (9) implies that the sign $\epsilon_{\sigma^{\dot N}}$ is universal, and hence

$$\sigma^{2\dot N}(S_{ij}) = \sigma^{\dot N}(\epsilon_{\sigma^{\dot N}}\, S_{ij}) = (\epsilon_{\sigma^{\dot N}})^2\, S_{ij} = S_{ij}\,, \tag{21}$$

so that $\sigma^{2\dot N} = id$ on M; thus either $N=\dot N$ or else $N=2\dot N$. Furthermore, for $N=2\dot N$ the terms in the formula for Z are easily seen to cancel out pairwise, so that the proposed invariant is identically zero, and hence the case $N \neq \dot N$ is rather uninteresting.

We can make another statement about $\dot\sigma$ by assuming that it commutes with the T-matrix, $T_{\dot\sigma(i)} = T_i$. Applying this property together with the relation (13) to the identity

$$T_i^{-1} S_{ik} T_k^{-1} = \sum_{j\in I} S_{ij} T_j S_{jk} \tag{22}$$

which follows from $(ST)^3 = S^2 = C$, we obtain

$$\begin{aligned}\epsilon_\sigma(i)\epsilon_{\sigma^{-1}}(k)\, T_i^{-1} S_{\dot\sigma(i)\dot\sigma^{-1}(k)} T_k^{-1} &= \epsilon_{\sigma^{-1}}(i)\epsilon_{\sigma^{-1}}(k) \sum_{j\in I} S_{\dot\sigma^{-1}(i)\dot\sigma(j)} T_{\dot\sigma(j)} S_{\dot\sigma(j)\dot\sigma^{-1}(k)} \\ &= \epsilon_{\sigma^{-1}}(i)\epsilon_{\sigma^{-1}}(k)\, T^{-1}_{\dot\sigma^{-1}(i)} S_{\dot\sigma^{-1}(i)\dot\sigma^{-1}(k)} T^{-1}_{\dot\sigma^{-1}(k)}\,.\end{aligned} \tag{23}$$

Thus

$$S_{\dot\sigma^{-1}(i)j} = \epsilon_\sigma(i)\epsilon_{\sigma^{-1}}(i)\, S_{\dot\sigma(i)j} \tag{24}$$

for all $i,j \in I$. As S is unitary, its rows are linearly independent, and hence (24) implies that $\dot\sigma(i) = \dot\sigma^{-1}(i)$ for all i, i.e. that $\dot\sigma^2 = id$. Hence any σ that fulfills (6) and commutes with the T-matrix has order two. (Again, this result is just based on the property (13) of $\dot\sigma$, and therefore is valid independently of whether $\dot\sigma$ comes from a Galois group element σ or not.) As we will see below, at least for WZW theories a kind of converse statement is also true, namely that any Galois group element of order two respects the T-matrix up to possibly minus signs.

Due to the presence of the signs ϵ_σ, the invariants (16) are generically *not* positive. However, at least for order $N=2$ one sometimes gets invariants that are completely positive and moreover have a non-degenerate vacuum. The only required property of σ is that $\epsilon_\sigma(i)$ is universal for all length-two orbits, while the sign for fixed points is arbitrary. Fixed points with $\epsilon_\sigma(i) = -1$ simply get projected out; in fact, the latter are the only fields that can be directly projected out.

The kind of invariant that is defined by (16) depends on the vacuum orbit. If the identity is a fixed point, the signs $\epsilon(i)$ are all equal to the same overall sign ϵ, as shown in section 2. Then, for $N=2$, the choice $f_0=0$ and $f_1=\epsilon$ in (16) immediately gives us a positive matrix Z that commutes with S and generates a fusion rule automorphism. If the vacuum is not fixed, the choice $f_0=0$, $f_1=\epsilon(0)$ leads to an invariant with an extended chiral algebra in which at least the identity block is positive. It follows from unitarity of S that in such an invariant not all coefficients $f_l\epsilon_l(i)$ can be positive (otherwise $Z_{ij} \geq \delta_{ij}$, and hence $Z_{00} = \sum_{i,j\in I} S_{0i} Z_{ij} S_{0j} \geq \sum_{i\in I} S_{0i} S_{0i} = 1$, with equality only if $Z_{ij} = \delta_{ij}$; this is clearly a contradiction). The only way to get a positive invariant is then that the negative signs occur precisely for the fixed point orbits, which are then projected out. If $N = 2$ this is indeed possible. Note that T-invariance still remains to be checked in both cases.

For $N > 2$ it is much harder to get a physical invariant. First of all there must exist orbits that violate T-invariance, although such orbits might be projected out by the summation in (16). It is in fact easy to see that no positive integer invariant can be obtained from from (16) if N is odd, for any choice of f_l. If N is odd, all coefficients except f_0 come in pairs f_l, f_{-l}. It follows that $Z_{jj} = f_0 \bmod 2$ for all $j \in I$, and since $Z_{00} = 1$ this means that none of the fields is projected out. Then the unitarity argument given above shows that a non-trivial positive invariant cannot exist. If $N > 2$ and even, hence not a prime, one has to distinguish various kinds of fixed point orbits. Positive modular invariants may then well exist, but we will not consider this more complicated case in this paper.

Let us stress that even if the matrix (16) contains negative entries, or does not commute with T, it can still be relevant for the construction of physical invariants, because the prescription may be combined with other procedures in such a manner that the negative contributions cancel out. For example one may use simple currents to extend the chiral algebra before employing the Galois transformation, or it may happen that a certain linear combination with other known [illegible]s of the integer commutant of S is a physical invariant.

4. WZW theories

In the special case where the fusion ring describes the fusion rules of a WZW theory based on an affine Lie algebra $\hat{g}$ at level k, the Galois group is a subgroup of $\mathbb{Z}^*_{M(k+g^\vee)}$, where $g^\vee$ is the dual Coxeter number of the horizontal subalgebra g of $\hat{g}$ and M is the denominator of the metric on the weight space of g. [3]

Let us label the primary fields by their g-weight Λ which corresponds to an integrable highest weight of $\hat{g}$ at level k, and denote by ρ the Weyl vector of g. Then a Galois transformation labelled by $\ell \in \mathbb{Z}^*_{M(k+g^\vee)}$ acts as the permutation [2]

$$\dot{\sigma}_{(\ell)}(\Lambda) = w(\ell \cdot (\Lambda + \rho)) - \rho \,. \tag{25}$$

That is, one first performs a dilatation of the shifted weight $\Lambda + \rho$ by a factor of ℓ. The weight $\tilde{\Lambda} = \ell(\Lambda + \rho) - \rho$ so obtained is not necessarily an integrable highest weight at level k. If it is not integrable, then one has to supplement the dilatation by (the horizontal projection of) an affine Weyl transformation $w \equiv w_{\ell;\Lambda}$. Note that $\Lambda + \rho$ is an integrable weight at level $k + g^\vee$. Using affine Weyl transformations w at this level we can rotate $\ell(\Lambda + \rho)$ back to another integrable weight at level $k + g^\vee$, which is in fact unique. In general there is no guarantee that after subtraction of ρ one gets an integrable weight at level k, but it is not hard to see that this does indeed work simultaneously for all integrable weights if the integer ℓ is coprime with $(k + g^\vee)$, which indeed follows from the requirement for (25) to correspond to an element of the Galois group. Finally, there is a general formula for the sign $\epsilon_{\sigma_{(\ell)}}$, namely

$$\epsilon_{\sigma_{(\ell)}}(\Lambda) = \eta_\ell \, \mathrm{sign}(w_{\ell;\Lambda}) \,, \tag{26}$$

i.e. the sign is just given by that of the Weyl transformation w, up to an overall sign η that only depends on $\sigma_{(\ell)}$ [2], but not on the individual highest weight Λ.

[3] In [2] a larger cyclotomic field is used in order to take care of the overall normalization factor of S. But the permutation of the primary fields can be read off the generalized quantum dimensions, which do not depend on the normalization of S. The correct Galois treatment of the normalization of S leads to an overall sign, which however is irrelevant for our purpose.

That it is the shifted weight $\Lambda + \rho$ rather than Λ that is scaled is immediately clear from the Kac–Peterson formula for the modular matrix S. In fact, it is possible to derive formula (14) directly by scaling the row and column labels of S by ℓ and ℓ^{-1}, respectively, using (25). Galois symmetry is thus not required to derive this formula, nor is it required to show that (16) commutes with S. Galois symmetry has however a general validity and is not restricted to WZW models.

Substituting (25) into the formula for WZW conformal weights one easily obtains a condition for T-invariance, namely $(\ell^2 - 1) = 0 \bmod 2M(k + g^\vee)$ (or mod $M(k + g^\vee)$ if all integers $M(\Lambda + \rho, \Lambda + \rho)$ are even). Since ℓ has an inverse mod $M(k + g^\vee)$, it follows that $\ell = \ell^{-1} \bmod M(k + g^\vee)$, i.e. the order of the transformation must be 2, as we already saw in the general case.

It is straightforward to find solutions to these conditions, and a little bit more work to check if the resulting modular invariants are indeed positive. Without any claim to generality we list here some examples of known invariants that come out in this way:

- First of all we get some (though not all) simple current invariants. The D-type invariants of A_1 at level $4m$ appear for $\ell = 4m + 1$. In general integer spin invariants of order 2 simple currents and automorphism invariants generated by fractional spin simple currents of odd prime order seem to come out as Galois invariants, but except for A_1 we do not have a general proof.
- Several chiral algebra extensions corresponding to conformal embeddings [9–11] are obtained, for example for $(A_2)_5$, $(A_4)_3$, $(G_2)_3$, $(G_2)_4$ and $(F_4)_3$.
- We also found four extensions by currents of spin higher than 1, namely for $(A_9)_2$, $(D_7)_3$, $(E_6)_4$ and $(E_7)_3$. The first two are expected on the basis of rank-level duality [12], and all four appeared already in [13].
- Finally we constructed two pure automorphisms for $(G_2)_4$ and $(F_4)_3$, which were first found in [14].
- In other cases we could obtain positive invariants after extending the chiral algebra by simple currents, for example for $(A_1)_{28}$, $(A_2)_9$, $(A_3)_8$, $(C_3)_4$, $(C_4)_3$, $(C_{10})_1$.

In some other cases expected invariants appeared as linear combinations. A detailed description of these and other examples will be presented elsewhere.

However, there also exist invariants that cannot be explained by Galois symmetry. One such example is the E_7-type invariant of $(A_1)_{16}$, an automorphism built on top of the D invariant. This automorphism is of the form (14), but it relates S-elements that are rational numbers, and hence transform trivially under Galois transformations.

5. Discussion

There are several striking similarities between Galois symmetries and simple current symmetries. First of all both are related to general properties of fusion rings, and not to particular (e.g. WZW) models. Both imply equalities among certain matrix elements of S up to signs or phases. Both symmetries organize the fields of the theory into orbits, whose length is a divisor of the order N of the symmetry. In both cases one can give very simple generic formulas for S-invariants, and in both cases the phenomenon of 'fixed points', i.e. of orbits whose length is less than N, occurs. In both cases such fixed points can appear with multiplicities larger than 1 in certain modular invariants in which the chiral algebra is extended. Note that this kind of structure is empirically observed in nearly all exceptional (not simple current generated) invariants found thus far. However, we believe this is the first time that at least in some cases the apparent 'orbits' and 'fixed points' of exceptional invariants are actually related to an underlying discrete symmetry. This might in fact be of some help in the still open problem of resolving fixed points of exceptional invariants.

There is also an important difference between Galois and simple current symmetries. In the latter case on can give a general construction of invariants that are positive and are also T-invariant. For Galois invariants it may well be possible to find a general criterion for T-invariance (as we have done for WZW models), but positivity appears to be a much more difficult requirement. Experience with WZW theories suggests that positive invariants only occur for low levels. At higher levels the signs $\epsilon(i)$ are distributed without any obvious pattern, and since the number of representations increases rapidly, positive invariants become less and less likely. This explains the 'exceptionality' of these invariants. Unfortunately it is far from obvious how to make this statement precise (except perhaps for $g = A_1$), and furthermore we know already one counter-example: the D invariants of A_1 at levels $4m$ form an infinite series of positive Galois invariants.

There is, however, one set of S-invariants that is always positive, namely those due to a Z_2 Galois symmetry that fixes the vacuum. In WZW models such invariants (that also commute with T) are abundant: this includes all charge conjugation invariants and also at least some of the simple current automorphism invariants that were first constructed in [15]. Remarkably, very few exceptional ones are known.

Let us also note that formula (10) can be generalized to automorphisms $\dot\sigma$ which change the vacuum, i.e. obey $\dot\sigma(0) \neq 0$ (and hence are not automorphisms of the fusion ring as a *unital* ring). In this situation, (10) gets replaced by

$$\begin{aligned} \mathcal{N}_{ij}{}^k &= \sigma(\mathcal{N}_{ij}{}^k) = \sigma(\sum_{l\in I} \frac{S_{il}S_{jl}S^*_{kl}}{S_{0l}}) \\ &= \sum_{l\in I} \frac{\epsilon_\sigma(i)\epsilon_\sigma(j)\epsilon_\sigma(k)\, S_{\dot\sigma(i)l}S_{\dot\sigma(j)l}S^*_{\dot\sigma(k)l}}{\epsilon_\sigma(0)S_{\dot\sigma(0)l}} = \epsilon_\sigma(0)\epsilon_\sigma(i)\epsilon_\sigma(j)\epsilon_\sigma(k)\; {}_{\dot\sigma(0)}\mathcal{N}_{\dot\sigma(i)\dot\sigma(j)}{}^{\dot\sigma(k)}, \end{aligned} \tag{27}$$

where ${}_l\mathcal{N}_{ij}{}^k \equiv \sum_{m\in I} S_{im}S_{jm}S^*_{km}/S_{lm}$. Note that the numbers ${}_l\mathcal{N}_{ij}{}^k$ are well-defined only if $S_{lm} \neq 0$ for all $m \in I$, in which case according to (27) they are actually integers; in the present situation this condition is met because $S_{\dot\sigma(0)i} = \epsilon_\sigma(0)\epsilon_\sigma(i)\, S_{0\,\dot\sigma(i)} \neq 0$ for all $i \in I$. This result can be interpreted as follows. Allowing also for negative structure constants, we can introduce a second fusion product $\star_\sigma$, with structure constants ${}_{\dot\sigma(0)}\mathcal{N}_{ij}{}^k$, on the same ring $Z^{|I|}$. Defining $\tilde\phi_i := \epsilon_\sigma(0)\epsilon_\sigma(i)\,\phi_{\dot\sigma(i)}$, it follows that $\tilde\phi_i \star_\sigma \tilde\phi_j = \sum_{k\in I} \mathcal{N}_{ij}{}^k\, \tilde\phi_k$, i.e. both fusion structures are isomorphic. Some special cases of this phenomenon have already been noticed in [16]. While our argument uses symmetries of number fields, in [16] the representation theory of the modular group is employed; thus our observation suggests a relation between number fields and modular forms.

In this paper we have presented a procedure for constructing modular invariant partition functions directly from symmetries of the matrix S, without any explicit knowledge of its matrix elements. This method is valid for all rational conformal field theories, and not a priori restricted to WZW models and coset theories, unlike conformal embeddings or rank-level duality. Previously only two such methods were known, namely charge conjugation (actually an example of Galois symmetry) and simple currents, and usually the term 'exceptional invariant' was used to refer to anything else. By providing a third general procedure, the results of this paper define a new degree of 'exceptionality' for modular invariant partition functions. Invariants satisfying this new definition of exceptionality do exist; this may be taken as an indication that still more interesting structure remains to be discovered.

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