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The next-next-to-leading QCD approximation for non-singlet moments of deep inelastic structure functions

S.A. Larin, [1] T. van Ritbergen, J.A.M.Vermaseren

NIKHEF-H
P.O. Box 41882
1009 DB Amsterdam.

Abstract

We obtain the analytic next-next-to-leading perturbative QCD corrections in the leading twist approximation for the moments $N = 2, 4, 6, 8$ of the non-singlet deep inelastic structure functions F_2 and F_L. We calculate the three-loop anomalous dimensions of the corresponding non-singlet operators and the three-loop coefficient functions of the structure function F_L.

[1] on leave from the Institute for Nuclear Research (INR) of the Russian Academy of Sciences, 60^{th} October Anniversary Prospect 7^a, Moscow 117312, Russia.

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1 Introduction

Deep inelastic lepton-nucleon scattering is an important tool for quantitative tests of perturbative QCD. This subject is now of special interest due to the operation of the HERA accelerator where the structure functions of electron-proton scattering are measured at high energies. Calculation of the next-next-to-leading (NNL) QCD approximation for the structure functions F_2 and F_L of deep inelastic electron-nucleon scattering is of interest for an accurate comparison of perturbative QCD with experiment. To obtain the NNL approximation for these structure functions one needs anomalous dimensions of the operators in the 3-loop order and Wilson coefficient functions in the 2-loop order for F_2 and in the 3-loop order for F_L. At present, these structure functions were calculated in the next-to-leading approximation only. The 2-loop anomalous dimensions were obtained in refs. [1]-[4]. The 2-loop approximation of the anomalous dimensions can be considered as well established after confirming [4] the validity of the anomalous dimensions of gluon operators obtained in ref. [3]. The 2-loop Wilson coefficient functions were calculated in refs. [5]-[9]. They can be also considered as well established after finding the agreement between ref. [8] and ref. [9]. Before the present paper the deep inelastic calculations in the NNL order were performed only for the sum rules [10, 11] and for the lowest moment of F_L [12].

In this paper we present the three-loop α_s^3 perturbative QCD corrections to the coefficient functions of the operator product expansion for the non-singlet moments $N = 2, 4, 6, 8$ of the

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structure function F_L in the leading twist and massless quark approximation. We also calculate the α_s^3 corrections to corresponding anomalous dimensions. We combine this to obtain the NNL approximation for the non-singlet moments $N = 2, 4, 6, 8$ of both structure functions F_2 and F_L.

2 Preliminaries

We need to calculate the hadronic part of the amplitude of unpolarized deep inelastic electron-nucleon scattering which is given by the tensor

$$\begin{aligned} W_{\mu\nu}(p,q) &= \int d^4z e^{iqz} < p, nucleon|J_\mu(z)J_\nu(0)|\ nucleon, p > \\ &= e_{\mu\nu}\frac{1}{2x}F_L(x,Q^2) + d_{\mu\nu}\frac{1}{2x}F_2(x,Q^2) \\ e_{\mu\nu} &= (g_{\mu\nu} - \frac{q_\mu q_\nu}{q^2}) \\ d_{\mu\nu} &= (-g_{\mu\nu} - p_\mu p_\nu\frac{4x^2}{q^2} - (p_\mu q_\nu + p_\nu q_\mu)\frac{2x}{q^2}) \end{aligned} \tag{1}$$

where J_μ is the electromagnetic quark current, $x = \frac{Q^2}{2pq}$, $Q^2 = -q^2$, $|\ nucleon, p >$ is the nucleon state with a momentum p (spin averaging is assumed here).

The moments of the structure functions F_k are expressed[2] through the parameters of the following operator product expansion of the T-product of electromagnetic currents:

$$\begin{aligned} T_{\mu\nu} &= i\int d^4z e^{iqz}T\{J_\mu(z)J_\nu(0)\} \\ &= \sum_N((g_{\mu\nu} - \frac{q_\mu q_\nu}{q^2})q_{\mu_1}q_{\mu_2}C^\alpha_{L,N}(\frac{Q^2}{\mu^2},a_s) \\ &\quad -(g_{\mu\mu_1}g_{\nu\mu_2}q^2 - g_{\mu\mu_1}q_\nu q_{\mu_2} - g_{\nu\mu_2}q_\mu q_{\mu_1} + g_{\mu\nu}q_{\mu_1}q_{\mu_2})C^\alpha_{2,N}(\frac{Q^2}{\mu^2},a_s))\times \\ &\quad \times q_{\mu_3}...q_{\mu_N}(\frac{1}{Q^2})^N O^{\alpha,\{\mu_1,...,\mu_N\}} + \text{singlet contributions + higher twists.} \end{aligned} \tag{2}$$

Here and throughout the whole paper we use the notation

$$a_s = \frac{g^2}{16\pi^2} = \frac{\alpha_s}{4\pi}$$

for the QCD strong coupling constant. The sum in (2) runs over the standard set of the spin-N, twist-2 irreducible (i.e. symmetrical and traceless in indices $\mu_1, ..., \mu_N$) flavor nonsinglet quark operators:

$$O^{\alpha,\{\mu_1,...,\mu_N\}} = \overline{\psi}\lambda^\alpha\gamma^{\{\mu_1}D^{\mu_2}...D^{\mu_N\}}\psi, \quad \alpha = 1,2,...,8 \tag{3}$$

[2] For reviews see [13, 14, 15]

where D^{μ_i} are the covariant derivatives, λ^α are the generators of the flavor group $SU(n_f)$. $C^\alpha_{k,N}(\frac{Q^2}{\mu^2}, a_s)$ are the corresponding coefficient functions.

The non-singlet moments[3] of the structure functions F_k are expressed through the operator product expansion (2) in the following form:

$$M_{k,N} = \int_0^1 dx\ x^{N-2} F_k^{ep-en}(x, Q^2) = \sum_\alpha C^\alpha_{k,N}(\frac{Q^2}{\mu^2}, a_s) \left(A^\alpha_{N,proton}(\mu^2) - A^\alpha_{N,neutron}(\mu^2)\right) \qquad (4)$$

where $A^\alpha_{N,nucleon}$ is the spin-averaged nucleon matrix elements of the operator:

$$< p, nucleon|O^{\alpha,\{\mu_1,\dots,\mu_N\}}|nucleon, p >= p^{\{\mu_1}\dots p^{\mu_N\}} A^\alpha_{N,nucleon}(\mu^2) \qquad (5)$$

The coefficient functions $C^\alpha_{k,N}$ of non-singlet operators can be reduced to coefficient functions $C_{k,N}$ independent of α as will be shown below.

Application of the renormalization group technique gives for the coefficient functions the following standard expression:

$$C_{k,N}(\frac{Q^2}{\mu^2}, a_s(\mu^2)) = C_{k,N}(1, a_s(Q^2)) \times \exp\left(-\int_{a_s(\mu^2)}^{a_s(Q^2)} da'_s \frac{\gamma_N(a'_s)}{\beta(a'_s)}\right), \qquad (6)$$

The anomalous dimensions γ_N in eq.(6) are defined as

$$\gamma_N(a_s) = \mu^2 \frac{d \log Z_N}{d\mu^2} = \sum_{n=0}^\infty \gamma_N^{(n)} a_s^{n+1} \qquad (7)$$

and renormalized operators and bare ones are connected as follows:

$$(O^{\alpha,\{\mu_1,\dots,\mu_N\}})_R = (Z_N)^{-1} (O^{\alpha,\{\mu_1,\dots,\mu_N\}})_B. \qquad (8)$$

The three-loop approximation for the β-function in QCD in the MS-scheme was obtained in [16, 17]:

$$\beta(a_s) = \mu^2 \frac{da_s}{d\mu^2} = -\sum_{n=0}^\infty \beta_n a_s^{n+2}$$

$$= -\left(11 - \frac{2}{3} n_f\right) a_s^2 - \left(102 - \frac{38}{3} n_f\right) a_s^3 - \left(\frac{2857}{2} - \frac{5033}{18} n_f + \frac{325}{54} n_f^2\right) a_s^4. \qquad (9)$$

The perturbative expansion in a_s for the coefficient functions is

$$C_{k,N}(1, a_s) = B^{(0)}_{k,N} + B^{(1)}_{k,N} a_s + B^{(2)}_{k,N} a_s^2 + B^{(3)}_{k,N} a_s^3 + O(a_s^4), \qquad (10)$$

where the Callan-Gross relation gives $B^{(0)}_{L,N} = 0$ for all N and the standard deep inelastic normalization [18] of the coefficient functions implies $B^{(0)}_{2,N} = 1$.

[3] for the parity conserving process the dispersion technique gives non-trivial expressions through the parameters of the operator product expansion only for the *even* moments

The NNL approximations for the non-singlet moments after renormalization group improvement are

$$M_{2,N}(Q^2) = a_s^{\frac{\gamma_N^{(0)}}{\beta_0}} \left[B_{2,N}^{(0)} + B_{2,N}^{(1)} a_s + B_{2,N}^{(2)} a_s^2\right] \times E(a_s) \times A_N(\mu^2), \tag{11}$$

$$M_{L,N}(Q^2) = a_s^{(\frac{\gamma_N^{(0)}}{\beta_0}+1)} \left[B_{L,N}^{(1)} + B_{L,N}^{(2)} a_s + B_{L,N}^{(3)} a_s^2\right] \times E(a_s) \times A_N(\mu^2), \tag{12}$$

with,

$$E(a_s) = \Bigg[1 + \frac{\gamma_N^{(1)}\beta_0 - \gamma_N^{(0)}\beta_1}{\beta_0^2} a_s + \\ + \frac{\gamma_N^{(2)}\beta_0^3 - \gamma_N^{(0)}\beta_2\beta_0^2 - \gamma_N^{(1)}\beta_1\beta_0^2 + \gamma_N^{(0)}\beta_1^2\beta_0 + (\gamma_N^{(1)})^2\beta_0^2 - 2\gamma_N^{(1)}\beta_0\gamma_N^{(0)}\beta_1 + (\gamma_N^{(0)})^2\beta_1^2}{2\beta_0^4} a_s^2\Bigg]$$

where for the NNL approximation one should keep only the three leading orders in the product of the power series for the coefficient functions with the series $E(a_s)$. $A_N(\mu^2)$ absorbs all Q-independent factors: the μ-dependent part of the renormalization group exponent from eq.(6) and the operator matrix elements $A^{\alpha}_{N,nucleon}(\mu^2)$. We conclude that for the NNL approximation to the nonsinglet moments $M_{k,N}$ we need to calculate the previously unknown 3-loop coefficients $B_{L,N}^{(3)}$ for the coefficient functions $C_{L,N}(1,a_s)$ and the 3-loop coefficients $\gamma_N^{(2)}$ for the anomalous dimensions $\gamma_N(a_s)$.

3 The calculation

To obtain the NNL corrections we use the 'method of projectors' of ref. [19] which reduces the calculation of the coefficient functions and anomalous dimensions of the operators to the calculation of diagrams of the propagator type only. This method relies heavily on the use of dimensional regularization [20] and the minimal subtraction scheme [21]. The dimension of the space-time is $D = 4 - 2\epsilon$. Recent calculations of the three-loop corrections to the coefficient functions by means of this method can be found in refs.[10, 11, 12].

Let us schematically describe the application of the method of ref.[19] to our case. To calculate coefficient functions of quark operators one should take matrix elements of the operator expansion (2) between quark states:

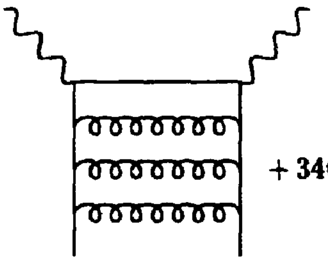
$$tr\,(\lambda^{\alpha} < p, quark|T_{\mu\nu}|quark, p >_{amputated}) \equiv e_{\mu\nu}T_L^{\alpha}(p,q) + d_{\mu\nu}T_2^{\alpha}(p,q), \tag{13}$$

where quark legs are amputated. Taking a trace with λ^{α} projects out a non-singlet operator with a given number α.

In order to calculate the coefficient function $C_{k,N}$ of the spin-N operator according to the 'method of projectors' [19] one needs to take the following derivative

$$T^{\alpha}_{k,N}(Q^2) = q^{\{\mu_1}...q^{\mu_N\}} \frac{\partial^N}{\partial p^{\mu_1}...\partial p^{\mu_N}} T_k^{\alpha}(p,q)|_{p=0} \tag{14}$$

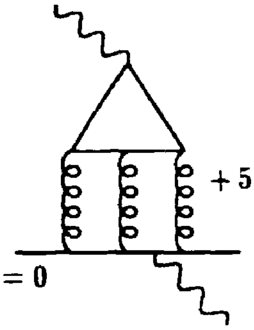
The essence of the method [19] is the nullification of the momentum p in the diagrams. Hence we need to calculate the 3-loop diagrams for the forward scattering of a photon off quarks with zero quark momentum. Typical 3-loop diagrams contributing to the eq.(14) are shown in fig.1.



+ 346 other diagrams

$p = 0$

fig.1a



+ 5 other diagrams

$p = 0$

fig.1b

Figure 1a depicts the class of diagrams which have a through going fermion line to which both external photons are connected. The diagrams in figure 1b have the photon vertices located in different fermion lines. The class of diagrams of fig.1b first appear at the three-loop level and these are the only diagrams which generate the color factor $d^{abc}d^{abc}$. In total we had to calculate 347 + 6 different diagrams. Diagrams with one or two loop gluon propagator insertions are counted as single diagrams.

After nullification of the quark momentum p the only contribution which survives in the operator expansion (2) is the tree matrix element $A^{\alpha}_{N,tree}$ of the corresponding spin-N operator $O^{\alpha,\{\mu_1\cdots\mu_N\}}$ since all massless vacuum diagrams are zero in dimensional regularization. This means that one only has to calculate the matrix elements of operators at the tree level. It is interesting to note that, after nullification of the momentum p, the diagrams that contribute to $T_{k,N}(Q^2)$ (see Fig.1) contain infrared divergences. These divergences should be canceled by the ultraviolet renormalization constants Z_N of the operators. We conclude that calculation of the diagrams in Fig.1 gives us simultaneously the necessary 3-loop coefficient functions and the 3-loop renormalization constants of the operators. This is a feature of the method that we use [19].

Finally we obtain the equation:

$$T^{\alpha}_{k,N}(Q^2) \quad = \quad C_{k,N}(\frac{Q^2}{\mu^2}, a_s)(Z_N)^{-1} A^{\alpha}_{N,tree}(\mu^2) \tag{15}$$

The ultraviolet renormalizations at the 3-loop level of the diagrams contributing to the l.h.s. of the eq.(15) can be done in the standard way by a substitution of the bare coupling constant in terms of a renormalized one $(a_s)_B = a_s - a_s^2\beta_0/\epsilon + \mathcal{O}(a_s^3)$.

The diagrams of fig.1a receive a $SU(n_f)$ flavor factor $tr(\hat{Q_f}^2\lambda^{\alpha})$, where $\hat{Q_f}$ is the quark charge matrix and λ^{α} are the (diagonal) generators of $SU(n_f)$. The diagrams of fig.1b receive a flavor factor $tr(\hat{Q_f})tr(\hat{Q_f}\lambda^{\alpha})$. If the ratio of these flavor factors would depend on the number α of a generator λ^{α}, it would be impossible to factorize the α-independent coefficient functions $C_{k,N}$. But it can be proven, fortunately, that this ratio is α-independent: $\frac{tr(\hat{Q_f})tr(\hat{Q_f}\lambda^{\alpha})}{tr(\hat{Q_f}^2\lambda^{\alpha})} = 3\sum_{f=1}^{n_f} q_f$ where q_f

is an electromagnetic quark charge. This means that factorization is possible:

$$\sum_a C^a_{k,N} A^a_{N,nucleon} =$$

$$= \left[C_{k,N}(fig.1a) + \frac{tr(\hat{Q}_f)tr(\hat{Q}_f\lambda^\beta)}{tr(\hat{Q}_f^{\,2}\lambda^\beta)} C_{k,N}(fig.1b)\right] \times \left[(\sum_a tr(\hat{Q}_f^{\,2}\lambda^a)A^a_{N,nucleon}\right]$$

$$= C_{k,N}(\frac{Q^2}{\mu^2}, a_s) \times \tilde{A}_N(\mu^2), \qquad (16)$$

where the non-singlet $C_{k,N}$ is independent of a.

The analytic calculation of the diagrams has been done with the symbolic manipulation program FORM [22] by means of the package MINCER [23]. This package implements the algorithms of [24]. MINCER was thoroughly optimized specially for these calculations and the optimized version will be published elsewhere. For the lower moments the diagrams were run with a gauge parameter ξ in the gluon propagator: $(g_{\mu\nu} - \xi\frac{q_\mu q_\nu}{q^2})/q^2$. For the higher moments (n=6,8) this turned out to be impossible because this exceeds our present computer resources. The cancellation of ξ in the coefficient functions and renormalization constants of the (gauge invariant) operators is a strong check of the calculations.

We have done the calculations for the moments $M_{k,N}$ from the second till the 8^{th} moments only: $N = 2, 4, 6, 8$. This is because the computer time increases drastically with the number of the moment and each moment takes about five times as much time as the moment one lower. The calculation of the 8^{th} non-singlet moment took the equivalent of more than 600 CPU hours on a SGI Challenge workstation with a 100 MHZ MIPS 4400 chip.

All following results are presented in the standard modification of the minimal subtraction scheme, the $\overline{MS}$-scheme [18].

Our results for $C_{L,N}$ read

$$\begin{aligned}
C_{L,2}(1,a_s) = &+a_s\left[C_F\frac{4}{3}\right] \\
&+a_s^2\left[C_FC_A(\frac{2878}{135} - \frac{32}{5}\zeta_3) + C_Fn_f(-\frac{92}{27}) + C_F^2(-\frac{1906}{135} + \frac{64}{5}\zeta_3)\right] \\
&+a_s^3C_FC_An_f(-\frac{204548}{1215} - \frac{1568}{45}\zeta_3 + \frac{320}{3}\zeta_5) \\
&+a_s^3C_FC_A^2(\frac{548668}{1215} - \frac{3680}{9}\zeta_3 + 224\zeta_5) \\
&+a_s^3C_Fn_f^2(\frac{2168}{243}) \\
&+a_s^3C_F^2C_A(-\frac{41536}{405} + \frac{73504}{45}\zeta_3 - \frac{5248}{3}\zeta_5) \\
&+a_s^3C_F^2n_f(\frac{25534}{405} - \frac{2848}{45}\zeta_3) \\
&+a_s^3C_F^3(-\frac{232798}{1215} - \frac{39424}{45}\zeta_3 + \frac{3584}{3}\zeta_5) \\
&+a_s^33(\sum_{f=1}^{n_f} q_f)\frac{d^{abc}d^{abc}}{n_c}(\frac{82}{15} - \frac{168}{5}\zeta_3 + 32\zeta_5)
\end{aligned}$$

$$= a_s\,(1.777777778) + a_s^2\,(56.75530152 - 4.543209877 n_f) +$$

$$+a_s^3 \left(2544.598081 - 421.6908887 n_f + 11.89574760 n_f^2 - 23.21009479 \sum_{f=1}^{n_f} q_f \right)$$

$$\begin{aligned}
C_{L,4}(1,a_s) \;=\; & +a_s \left[C_F \frac{4}{5} \right] \\
& +a_s^2 \left[C_F C_A (\frac{4763}{225} - \frac{48}{5}\zeta_3) + C_F n_f (-\frac{64}{25}) + C_F^2 (-\frac{19967}{1125} + \frac{96}{5}\zeta_3) \right] \\
& +a_s^3 C_F C_A n_f (-\frac{14259893}{94500} + \frac{55904}{945}\zeta_3) \\
& +a_s^3 C_F C_A^2 (\frac{171354151}{283500} - \frac{1889656}{4725}\zeta_3 + 32\zeta_5) \\
& +a_s^3 C_F n_f^2 (\frac{82688}{10125}) \\
& +a_s^3 C_F^2 C_A (-\frac{263016326}{354375} + \frac{1070306}{1575}\zeta_3 + 32\zeta_5) \\
& +a_s^3 C_F^2 n_f (\frac{258828431}{2835000} - \frac{33344}{315}\zeta_3) \\
& +a_s^3 C_F^3 (\frac{582157141}{3780000} + \frac{57356}{945}\zeta_3 - 192\zeta_5) \\
& +a_s^3 3 (\sum_{f=1}^{n_f} q_f) \frac{d^{abc}d^{abc}}{n_c} (-\frac{12359}{540} - \frac{428}{45}\zeta_3 + 32\zeta_5)
\end{aligned}$$

$$= a_s(1.066666667) + a_s^2\,(47.99398933 - 3.413333333 n_f) +$$

$$+a_s^3 \left(2523.739021 - 383.0520013 n_f + 10.88895473 n_f^2 - 15.17660866 \sum_{f=1}^{n_f} q_f \right)$$

$$\begin{aligned}
C_{L,6}(1,a_s) \;=\; & +a_s \left[C_F \frac{4}{7} \right] \\
& +a_s^2 \left[C_F C_A (\frac{172106}{11025} - \frac{48}{7}\zeta_3) + C_F n_f (-\frac{1486}{735}) + C_F^2 (-\frac{257318}{25725} + \frac{96}{7}\zeta_3) \right] \\
& +a_s^3 C_F C_A n_f (-\frac{927531079}{6945750} + \frac{88916}{1575}\zeta_3) \\
& +a_s^3 C_F C_A^2 (\frac{3186722339}{6174000} - \frac{4977821}{11025}\zeta_3 + \frac{1040}{7}\zeta_5) \\
& +a_s^3 C_F n_f^2 (\frac{986872}{138915}) \\
& +a_s^3 C_F^2 C_A (-\frac{4333877446411}{8751645000} + \frac{2071508}{2205}\zeta_3 - 480\zeta_5) \\
& +a_s^3 C_F^2 n_f (\frac{66489992539}{875164500} - \frac{31312}{315}\zeta_3)
\end{aligned}$$

$$+a_s^3 C_F^3(-\frac{90265366481}{2268945000} - \frac{536548}{2205}\zeta_3 + \frac{2560}{7}\zeta_5)$$
$$+a_s^3 3(\sum_{f=1}^{n_f} q_f)\frac{d^{abc}d^{abc}}{n_c}(-\frac{5869993}{378000} - \frac{472}{63}\zeta_3 + \frac{160}{7}\zeta_5)$$

$$= a_s(0.7619047619) + a_s^2(40.99619760 - 2.695691610 n_f) +$$
$$+a_s^3\left(2368.193774 - 340.0691070 n_f + 9.472190428 n_f^2 - 11.11683760\sum_{f=1}^{n_f} q_f\right)$$

$$\begin{aligned}
C_{L,8}(1,a_s) \;=\; & +a_s\left[C_F\frac{4}{9}\right] \\
& +a_s^2\left[C_F C_A(\frac{14741729}{1190700} - \frac{16}{3}\zeta_3) + C_F n_f(-\frac{14234}{8505}) + C_F^2(-\frac{21694349}{3572100} + \frac{32}{3}\zeta_3)\right] \\
& +a_s^3 C_F C_A n_f(-\frac{23153083641529}{198037224000} + \frac{18459136}{363825}\zeta_3) \\
& +a_s^3 C_F C_A^2(\frac{7653142193467}{18003384000} - \frac{9508318}{19845}\zeta_3 + \frac{2240}{9}\zeta_5) \\
& +a_s^3 C_F n_f^2(\frac{1435876}{229635}) \\
& +a_s^3 C_F^2 C_A(-\frac{7667007621800089}{27725211360000} + \frac{2476882549}{2182950}\zeta_3 - \frac{2720}{3}\zeta_5) \\
& +a_s^3 C_F^2 n_f(\frac{11844644404289}{198037224000} - \frac{914992}{10395}\zeta_3) \\
& +a_s^3 C_F^3(-\frac{86167166469418457}{499053804480000} - \frac{111668693}{218295}\zeta_3 + \frac{7360}{9}\zeta_5) \\
& +a_s^3 3(\sum_{f=1}^{n_f} q_f)\frac{d^{abc}d^{abc}}{n_c}(-\frac{35555777437}{3000564000} - \frac{85378}{14175}\zeta_3 + \frac{160}{9}\zeta_5) \qquad (17)
\end{aligned}$$

$$= a_s(0.5925925926) + a_s^2(35.87664404 - 2.231471683 n_f) +$$
$$+a_s^3\left(2215.210878 - 305.4730331 n_f + 8.337149534 n_f^2 - 8.741107731\sum_{f=1}^{n_f} q_f\right)$$

Here $C_F = \frac{4}{3}$ and $C_A = 3$ are the Casimir operators of the fundamental and adjoint representation of the color group $SU(3)$, for the symmetric structure constants of $SU(3)$ we have $d^{abc}d^{abc} = \frac{40}{3}$ and for the trace normalization of the fundamental representation $T_F = \frac{1}{2}$ is substituted, n_f is the number of active quark flavors, ζ is the Riemann zeta-function. These results for the non-singlet coefficient functions $C_{L,N}$ agree at 2-loop level with the previous results [5, 6]. The 3-loop result for $C_{L,2}$ also agrees with a previous calculation [12].

Our results for γ_N read

$$\begin{aligned}
\gamma_2(a_s) &= +a_s\left[C_F\frac{8}{3}\right] \\
&+a_s^2\left[C_FC_A(\frac{376}{27})+C_Fn_f(-\frac{64}{27})+C_F^2(-\frac{112}{27})\right] \\
&+a_s^3C_FC_An_f(-\frac{3128}{243}-\frac{64}{3}\zeta_3) \\
&+a_s^3C_FC_A^2(\frac{20920}{243}+\frac{64}{3}\zeta_3) \\
&+a_s^3C_Fn_f^2(-\frac{224}{243}) \\
&+a_s^3C_F^2C_A(-\frac{8528}{243}-64\zeta_3) \\
&+a_s^3C_F^2n_f(-\frac{3412}{243}+\frac{64}{3}\zeta_3) \\
&+a_s^3C_F^3(-\frac{560}{243}+\frac{128}{3}\zeta_3);
\end{aligned}$$

$$= a_s(3.555555556)+a_s^2(48.32921811-3.160493827n_f)+$$
$$+a_s^3(859.4478377-133.4381617n_f-1.229080933n_f^2)$$

$$\begin{aligned}
\gamma_4(a_s) &= +a_s\left[C_F\frac{157}{30}\right] \\
&+a_s^2\left[C_FC_A(\frac{16157}{675})+C_Fn_f(-\frac{13271}{2700})+C_F^2(-\frac{287303}{54000})\right] \\
&+a_s^3C_FC_An_f(-\frac{8802581}{486000}-\frac{628}{15}\zeta_3) \\
&+a_s^3C_FC_A^2(\frac{136066373}{972000}+\frac{1439}{75}\zeta_3) \\
&+a_s^3C_Fn_f^2(-\frac{384277}{243000}) \\
&+a_s^3C_F^2C_A(-\frac{267028157}{9720000}-\frac{1439}{25}\zeta_3) \\
&+a_s^3C_F^2n_f(-\frac{165237563}{4860000}+\frac{628}{15}\zeta_3) \\
&+a_s^3C_F^3(-\frac{714245693}{48600000}+\frac{2878}{75}\zeta_3)
\end{aligned}$$

$$= a_s(6.977777778)+a_s^2(86.28665021-6.553580247n_f)+$$
$$+a_s^3(1515.562364-244.7285919n_f-2.108515775n_f^2)$$

$$
\begin{aligned}
\gamma_6(a_s) = &+a_s\left[C_F\frac{709}{105}\right] \\
&+a_s^2\left[C_FC_A(\frac{157415}{5292})+C_Fn_f(-\frac{428119}{66150})+C_F^2(-\frac{3173311}{514500})\right] \\
&+a_s^3C_FC_An_f(-\frac{13978373}{686000}-\frac{5672}{105}\zeta_3) \\
&+a_s^3C_FC_A^2(\frac{115237918583}{666792000}+\frac{69862}{3675}\zeta_3) \\
&+a_s^3C_Fn_f^2(-\frac{80347571}{41674500}) \\
&+a_s^3C_F^2C_A(-\frac{34855421369}{1166886000}-\frac{69862}{1225}\zeta_3) \\
&+a_s^3C_F^2n_f(-\frac{44644018231}{972405000}+\frac{5672}{105}\zeta_3) \\
&+a_s^3C_F^3(-\frac{854652999073}{51051262500}+\frac{139724}{3675}\zeta_3) \\
= &\ a_s(9.003174603)+a_s^2(108.0184697-8.629256740n_f)+ \\
&+a_s^3(1891.827779-307.4236890n_f-2.570638992n_f^2)
\end{aligned}
$$

$$
\begin{aligned}
\gamma_8(a_s) = &+a_s\left[C_F\frac{9883}{1260}\right] \\
&+a_s^2\left[C_FC_A(\frac{25870049}{762048})+C_Fn_f(-\frac{36241943}{4762800})+C_F^2(-\frac{27040578211}{4000752000})\right] \\
&+a_s^3C_FC_An_f(-\frac{1578915745223}{72013536000}-\frac{19766}{315}\zeta_3) \\
&+a_s^3C_FC_A^2(\frac{8101059985033}{41150592000}+\frac{2510407}{132300}\zeta_3) \\
&+a_s^3C_Fn_f^2(-\frac{38920977797}{18003384000}) \\
&+a_s^3C_F^2C_A(-\frac{3662576699059}{112021056000}-\frac{2510407}{44100}\zeta_3) \\
&+a_s^3C_F^2n_f(-\frac{91675209372043}{1680315840000}+\frac{19766}{315}\zeta_3) \\
&+a_s^3C_F^3(-\frac{109308710097437993}{6351593875200000}+\frac{2510407}{66150}\zeta_3) \qquad (18) \\
= &\ a_s(10.45820106)+a_s^2(123.7764525-10.14583662n_f)+ \\
&+a_s^3(2164.091836-352.3116596n_f-2.882493484n_f^2)
\end{aligned}
$$

These results for the non-singlet operator anomalous dimensions γ_N, N=2,4,6,8, agree at the 2-loop level with the known results [1, 2]. The 3-loop result for γ_2 agrees with the previous calculation [12]. Recently, the leading n_f-terms were calculated for the non-singlet anomalous dimensions in all orders of a_s using the large n_f expansion [25]. Our $a_s^3n_f^2$ terms agree with these results.

The NNL approximation for moments $M_{L,N}$, N=2,4,6,8 follows from eq.(11) after substitution of the 3-loop anomalous dimension and the 3-loop coefficient functions $C_{L,N}$ from this article. We present here only the result for $M_{L,2}$.

$$M_{L,2} = a_s^{(\frac{\gamma_2^{(0)}}{\beta_0}+1)}\Big[1.777777778 + \frac{a_s}{\beta_0^2}(7167.755453 - 1421.157926n_f + 95.60409380n_f^2$$

$$-2.019204390n_f^3) + \frac{a_s^2}{\beta_0^4}\Big(38743144.32 - 15713015.60n_f + 2567100.512n_f^2$$

$$-217290.0177n_f^3 + 10075.25730n_f^4 - 242.3130145n_f^5 + 2.349777304n_f^6$$

$$+(\sum_{f=1}^{n_f} q_f)(-339818.9980 + 82380.36316n_f - 7489.123925n_f^2$$

$$+302.5908656n_f^3 - 4.584710084n_f^4)\Big)\Big] \times A_2(\mu^2)$$

For five active quark flavors the result is:

$$M_{L,2}(n_f = 5) = a_s^{\frac{101}{69}}\left[1.777777778 + 37.42345665a_s + 794.2860056a_s^2\right] \times A_2(\mu^2)$$

This result differs from the analogous result presented in [12] which contained some errors although the 3-loop γ_2 and $C_{L,2}$ were presented correctly in [12].

The NNL approximation for moments $M_{2,N}$, N=2,4,6,8 follows from eq.(11) after substitution of the 3-loop anomalous dimensions from this article and the 2-loop coefficient functions $C_{2,N}$ from [9].

$$M_{2,2} = a_s^{\frac{\gamma_2^{(0)}}{\beta_0}}\Big[1 + \frac{a_s}{\beta_0^2}(222.7325103 - 28.46639231n_f + 2.304526749n_f^2) + \frac{a_s^2}{\beta_0^4}(452305.9955$$

$$-212450.5282n_f + 33271.64596n_f^2 - 2284.081821n_f^3 + 72.76562050n_f^4 - .8714118778n_f^5)\Big] \times A_2(\mu^2)$$

$$M_{2,4} = a_s^{\frac{\gamma_4^{(0)}}{\beta_0}}\Big[1 + \frac{a_s}{\beta_0^2}(971.4864856 - 130.2064088n_f + 7.065349795n_f^2) + \frac{a_s^2}{\beta_0^4}(2558881.856$$

$$-922494.5008n_f + 128160.1658n_f^2 - 8412.746331n_f^3 + 263.9279697n_f^4 - 3.043264543n_f^5)\Big] \times A_4(\mu^2)$$

$$M_{2,6} = a_s^{\frac{\gamma_6^{(0)}}{\beta_0}}\Big[1 + \frac{a_s}{\beta_0^2}(1622.262427 - 216.8191461n_f + 10.72026875n_f^2) + \frac{a_s^2}{\beta_0^4}(5158334.044$$

$$-1742735.274n_f + 232658.0476n_f^2 - 14938.50804n_f^3 + 460.4265103n_f^4 - 5.152606648n_f^5)\Big] \times A_6(\mu^2)$$

$$M_{2,8} = a_s^{\frac{\gamma_8^{(0)}}{\beta_0}} \left[1 + \frac{a_s}{\beta_0^2}(2173.921666 - 289.4230724 n_f + 13.66606627 n_f^2) + \frac{a_s^2}{\beta_0^4}(7861494.037 \right.$$

$$\left. -2572131.269 n_f + 335782.1971 n_f^2 - 21240.70209 n_f^3 + 645.4970657 n_f^4 - 7.064286950 n_f^5)\right] \times A_8(\mu^2)$$

To see more clearly how the coefficients depend on the number N we present also results for $M_{2,N}$ for $n_f = 5$:

$$M_{2,2}(n_f = 5) = a_s^{\frac{32}{69}} \left[1 + 2.348059464 a_s - 6.052509330 a_s^2\right] \times A_2(\mu^2)$$

$$M_{2,4}(n_f = 5) = a_s^{\frac{314}{345}} \left[1 + 8.457076895 a_s + 73.59702078 a_s^2\right] \times A_4(\mu^2)$$

$$M_{2,6}(n_f = 5) = a_s^{\frac{2836}{2415}} \left[1 + 13.71561575 a_s + 192.6174600 a_s^2\right] \times A_6(\mu^2)$$

$$M_{2,8}(n_f = 5) = a_s^{\frac{9883}{7245}} \left[1 + 18.17792372 a_s + 324.5935524 a_s^2\right] \times A_8(\mu^2)$$

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