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OPTIMAL CONTROL OF ARRIVAL AND SERVICE RATES IN TANDEM QUEUES

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ABSTRACT

We consider n M/M/1 queues in series. At queue one the arrival and service rates are chosen in pair from a finite set whenever there are arrivals or service completions at any queue. Customers arriving to queue L (L=1,2,...,n-1) must go on to queue L+1 after finishing service at server L. Customers arriving to queue n leave the system after finishing service at the last server. At queues 2 to n arrival and service rates are fixed. The objective is to minimize the expected discounted cost of the sytem over finite and infinite horizons. We show that the optimal policy is of threshold type. In order to establish the result, we formulate the optimal control problem as a Linear Programming.

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1. Introduction

In last decade, many authors considered the optimal control of systems with more than one server. Rosberg, Varaiya and Walrand [1] have considered two M/M/1 service stations in tandem with uncontrollable arrival process and control of service rate $\mu \varepsilon$ [0,a] at first station only. They have showed that the optimal policy is characterized by switching curves. A generalization of the model given by [1] for controlling the service rates in a cycle of m queues has been considered by Weber and Stidham [2]. They have also studied the control of arrivals to the first queue only and service rates at each queue of m queues in series.

Hajek [3] has considered a general two-node model. Nodes 1 and 2 have Poisson arrivals at rates λ_1 and λ_2 respectively. A third stream of Poisson arrivals at rate λ can be routed to either queue. The nodes have fixed exponential service rates μ_1 and μ_2 and a third exponential service with rate μ that can be assigned to either queue. There are two additional exponential servers, with rates ν_{12} and ν_{21}. The first of which serves queue 1 and sends jobs to queue 2. The second of which serves queue 2 and sends jobs to queue 1. Service completions by these servers can be "accepted" or "rejected". The jobs arriving at rate λ must be routed to one of the two nodes. All decisions are made dynamically as a function of the number of jobs in the two ques. Hajek has used an inductive proof to establish the existence of a monotonic switching curve.

Ghoneim and Stidham [4] have studied two exponential servers in series (with mean service rates μ_1 and μ_2), Each with an infinite capacity queue. Arrivals to queue i are from a Poisson process with mean rate λ_i , i = 1, 2. Jobs arriving to queue 1 must go on to queue 2 after finishing service at server 1. Jobs arriving to queue 2 leave the system after finishing service at server 2. They have showed that λ_i are nonincreasing in the number of customers in either queue.

Moustafa [5] has considered two M/M/1 queues in series. At queue 1, the arrival and service rates are chosen in pair from a finite set. He showed numerically that the optimal policy is characterized by a switching curve, but he could not apply the induction proof to construct this structure of the optimal policy. The model considered by [5] is two node version of the model studied by Lu and Serfozo [6]. They have used an inductive proof to show that there is a monotone hysteric optimal policy in which the arrival and service rates are decreasing and decreasing respectively in queue length.

Typically, the control problem is formulated as a Markovian decision process and the tool of Dynamic Programming is used to establish the structure of the optimal policy. However, arguments based on Linear Programming (see [7,8]) may be used in models where Dynamic Programming technique fails.

In this paper, we use Linear Programming arguments to generalize the model considered by [5] and establish the structure of the optimal policy. The paper is organized as follows: In section 2 we describe the queueing model of the system. In section 3 we provide the Linear Programming formulation of the optimal control problem. In section 4 we discuss the structure of the optimal policy. Finally, conclusions are given in section 5.

2. The Systemm Model

We consider n M/M/1 queues in series. Customers arriving to queue L (L = 1,2,...,n-1) must go on to queue L+1 after finishing service at server L. Customers arriving to queue n leave the system after finishing service at the last server. At the the first queue, the arrival and service rates are chosen in pair (λ_i, μ_i)

from a finite set $A = \{(\lambda_1, \mu_1), (\lambda_2, \mu_2), \ldots, (\lambda_m, \mu_m)\}$ with $\lambda_1 \geq \lambda_2 \ldots \geq \lambda_m$ and $\mu_1 \leq \mu_2 \leq \ldots \leq \mu_m$, provided that $\lambda_i < \mu_i$. We assume that $\lambda_i + \mu_i$ are positive and not identical. At queues 2 to n, the arrival and service rates are fixed and equal to τ_j and ν_j respectively provided $\tau_j < \nu_j$ that $j=2,3,\ldots,n$.

Our objective is to minimize the expected holding cost over finite and infinite horizons. Let x_k^L be the number of customers at queue L, L=1,2,...,n at time t, when k-th transition (i.e., arrival or service completion) occurs. For finite horizon N, we consider the discounting factor $0<\beta<1$, the objective is given by

$$G = \min_{A} E \left\{ \sum_{k=1}^{N} \beta^{k} \sum_{L=1}^{n} (c_L x_k^L) \right\} \tag{2.1}$$

where c_L (L=1,2,...,n) is the holding cost per customer at queue L. We assume that:

$$c_1 < c_2 \tag{2.2}$$

For infinite horizon, we consider

$$G = \lim_{\beta \to 0} (1-\beta) \min_{A} E \left\{ \sum_{k=1}^{\infty} \beta^{k} \sum_{L=1}^{n} (c_L x_k^L) \right\} \tag{2.3}$$

3. LP Formulation

Now, we describe the LP formulation. We need the the following definitions:

Let

$$\Omega = \{A_1, A_2, \ldots, A_m, D_1, D_2, \ldots D_m, V_2, V_3, \ldots, V_n, U_2, U_3, \ldots, U_n\} \tag{3.1}$$

be the set of all transitions. Here A_i and D_i (i=1,2,...,m) represent the arrival and departure respectively at queue 1. For queue j

$(j=2,3,\ldots,n)$, V_j and U_j represent the arrival and departure respectively.

Let $\omega_k \varepsilon \Omega_k$ represent the k-th transition of the queueing system. Denote by $\Omega^k = \Omega \times \Omega \times \ldots \times \Omega$ the sample space of the system having k transitions, $1 \le k \le N$. Define $\omega^k = \omega_1 \omega_2 \cdots \omega_k \varepsilon \Omega^k$ as k transitions and $Pr(\omega^k)$ as the probability distribution over Ω^k.

Let $\xi_k^L(\omega^k)$ represent the change in the system state at queue L $(L=1,2,\ldots,n)$, incurred by transition ω^k. The function $\xi_k^L(\omega^k)$ is given by

For L= 1 (the first queue)

$$\xi_k^1(\omega^k) = \begin{cases} 1 & \omega_k = A_i \\ -1 & \omega_k = D_i \end{cases} \qquad i=1,2,\ldots,m \tag{3.2}$$

For $2 \le L \le n$

$$\xi_k^2(\omega^k) = \begin{cases} 1 & \omega_k = V_2, \text{ or } D_i \quad i=1,2,\ldots,m \\ -1 & \omega_k = U_2 \end{cases} \tag{3.3}$$

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$$\xi_k^{n-1}(\omega^k) = \begin{cases} 1 & \omega_k = V_{n-1}, \text{ or } U_{n-2} \\ -1 & \omega_k = U_{n-1} \end{cases} \tag{3.4}$$

$$\xi_k^n(\omega^k) = \begin{cases} 1 & \omega_k = V_n, \text{ or } U_{n-1} \\ -1 & \omega_k = U_n \end{cases} \tag{3.5}$$

Define the transitions process

$$\xi = (\zeta_1, \zeta_2, \ldots, \zeta_N)$$

Let $Z_k(\omega^k)$ denote control variables that represent actions taken at k-th transition instant, we define

$$Z_k(\omega^k)\varepsilon\ \{0,1\} \qquad \omega_k = A_i \ \text{ or } \ \omega_k = D_i\ ,\ i=1,2,\ldots,m$$

$$\text{and} \qquad Z_k(\omega^{k-1}A_i) = Z_k(\omega^{k-1}D_i) \tag{3.6}$$

since we control the arrival and service rates in pair (λ_i, μ_i), and

$$Z_k(\omega^k)=1, \quad \omega_k = V_j \ \text{ or } U_j\ , \quad j=2,3,\ldots,n \tag{3.7}$$

The evolution of the system is described by the following equation:

$$x^L_{k+1}(\omega^{k+1}) = x^L_k(\omega^k) + \xi^L_{k+1}(\omega^{k+1})Z_{k+1}(\omega^{k+1}), \qquad L=1,2,\ldots,n. \tag{3.8}$$

Suppose that the initial queue length is x^L_0 at queue L. Then

$$x^L_k = x^L_0 + \sum_{j=1}^{k} \xi^L_j(\omega^j)Z_j(\omega^j) \tag{3.9}$$

Using (2.1) the objective function becomes

$$G = \min \{ \sum_{k=1}^{N} \beta^k [E(c_1 x^1_0 + c_2 x^2_0 + \ldots + c_n x^n_0) + E(c_1 \sum_{j=1}^{k} \xi^1_j(\omega^j)Z_j(\omega^j)$$

$$+ \ldots + c_n \sum_{j=1}^{k} \xi^n_j(\omega^j)Z_j(\omega^j))]\}$$

$$G = \text{constant} + \min \{ \sum_{k=1}^{N} \beta^k \sum_{j=1}^{k} E(c_1 \xi^1(\omega^j)Z_j(\omega^j) + \ldots +$$

$$+ c_n \xi^n_j(\omega^j)Z_j(\omega^j))\}$$

$$G = \text{constant} + \min \{ \sum_{k=1}^{N} \beta^k \sum_{j=1}^{k} \sum_{\omega^j \varepsilon\, \Omega^j} \Pr(\omega^j)[c_1 \xi^1(\omega^j)Z_j(\omega^j) + \ldots +$$

$$+ c_n \xi^n_j(\omega^j)Z_j(\omega^j)]\}$$

$$G = \text{constant} + \min \{ \sum_{k=1}^{N} \beta^k \sum_{j=1}^{k} \sum_{\omega^j \varepsilon \Omega^j} \sum_{i=1}^{m} c_1 \quad Pr(\omega^{j-1} A_i) \; Z_j(\omega^{j-1} A_i) + \ldots$$

$$+ (-c_1 + c_2) pr(\omega^{j-1} D_i) Z_j(\omega^{j-1} D_i)\} \qquad (3.10)$$

Using (3.9) , the objective function is reduced to the following LP:

$$G = \min \{ \sum_{k=1}^{N} \beta^k \sum_{j=1}^{k} \sum_{\omega^j \varepsilon \Omega^j} \sum_{i=1}^{m} c_1 \quad pr(\omega^{j-1} A_i) \; Z_j \; (\omega^{j-1} A_i) + \ldots$$

$$+ (-c_1 + c_2) pr(\omega^{j-1} D_i) Z_j (\omega^{j-1} D_i)\} \qquad (3.11)$$

subject to the following constraints:

(a) $Z_j (\omega^{j-1} D_i) \varepsilon \{0,1\}$ (3.12)

(b) $x_j^L (\omega^j) \geq 0, \quad L=1,2,\ldots,n$ (3.13)

(c) $\sum_{i=1}^{m} Z_j (\omega^{j-1} D_i) = 1$ (3.14)

Constraints (b) are the non-negative of queue size for each queue l. Constraints (a) and (c) required as we choose a single pair of the set A.

4.The Structure of the Optimal Policy

The linear program (3.10), with constraints (a), (b), and (c) is the basis for the results we present in this section. To simplify the discussion , we first start to show the redundant constraints among (b). Since equation (3.9) is the state trajectory that has to satisfy (b) , we have

$${}_0x^L + \sum_{j=1}^{k} \xi_j^L \quad (\omega^j) Z_j \; (\omega^j) \geq 0 \qquad (3.15)$$

For L=1 and any $\omega^{k+1} \varepsilon \Omega^{k+1}$, (b) becomes

$$x_0^1 + \sum_{j=1}^{k} \xi_j^1 (\omega^j) Z_j (\omega^j) + Z_{k+1} (\omega^k A_i) \geq 0 \quad \text{where } \omega_{k+1} = A_i$$

and

$$x_0^1 + \sum_{j=1}^{k} \xi_j^1 (\omega^j) Z_j (\omega^j) - Z_{k+1} (\omega^k D_i) \geq 0 \quad \text{where } \omega_{k+1} = D_i$$

That is

$$x_{k+1}^1 (\omega^{k+1}) = x_k^1 (\omega^k) + Z_{k+1} (\omega^k A_i) \geq 0 \qquad (3.16)$$

$$x_{k+1}^1 (\omega^{k+1}) = x_k^1 (\omega^k) - Z_{k+1} (\omega^k D_i) \geq 0 \qquad (3.17)$$

Hence, if

$x_{k+1}^1 (\omega^{k+1}) \geq 0$ for all $\omega^{k+1} \varepsilon \Omega^{k+1}$, then

$x_k^1 (\omega^k) \geq 0$ for all $\omega^k \varepsilon \Omega^k$ because of $Z_j (\omega^k D_i) \geq 0$

By induction from k=1 to k= N, we know that constraint (b) with k=1,2,...,N-1 is redundant, if k=N is satisfied by (b). Among the constraints (b) with k=N , there some other redundant constraints. Consider for any $\omega^N \varepsilon \Omega^N$, we have:

$$x_N^1 (\omega^N) = x_{N-1}^1 (\omega^{N-1}) + \xi_N^1 (\omega^N) Z_N (\omega^N) \qquad (3.18)$$

If (3.18) is satisfied, we have:

$x_{N-1}^1 (\omega^{N-1}) - Z_N (\omega^{N-1} D_i) \geq 0$ for any $1 \leq i \leq m$, and

immediately we have:

$x_{N-1}^1 (\omega^{N-1}) + Z_N (\omega^{N-1} A_i) \geq 0$ since

$Z_N(\omega^{N-1} D_i) \geq 0$. Therefore , instead of considering $x^1(\omega^{N-1}) \geq 0$ for every $\omega^N \varepsilon\ \Omega^N$, we consider only $x_N(\omega^{N-1} D_i) \geq 0$ for every $\omega^{N-1} \varepsilon\ \Omega^{N-1}$, and $1 \leq i \leq m$.

(3.19)

By the same fashion of arguments, we check constraint (b) for $2 \leq L \leq n$. Consider for any $\omega^{k+1} \varepsilon\ \Omega^{k+1}$

$$x_o^2 + \sum_{j=1}^{k} \xi_j^2 (\omega^j) Z_j(\omega^j) + Z_{k+1}(\omega^k \omega_{k+1}) \geq 0 \quad \text{where } \omega_{k+1} = A_2, \text{or } D_i$$

$$x_o^2 + \sum_{j=1}^{k} \xi_j^2 (\omega^j) Z_j(\omega^j) - 1 \geq 0 \quad \text{where } \omega_{k+1} = U_2 \qquad (3.20)$$

or for $L \geq 3$

$$x_o^L + \sum_{j=1}^{k} \xi_j^L (\omega^j) Z_j(\omega^j) + 1 \geq 0 \quad \text{where } \omega_{k+1} = V_L, U_{L-1}$$

$$x_o^L + \sum_{j=1}^{k} \xi_j^L (\omega^j) Z_j(\omega^j) - 1 \geq 0 \quad \text{where } \omega_{k+1} = U_L \qquad (3.21)$$

Since having $Z_k(\omega^{k-1} U_L)=1$ constraint (b) should satisfy every transition $\omega^k \varepsilon\ \Omega^k$, $1 \leq k \leq N$, it is easy to see x_o^L should be greater than $\sum_{j=1}^{N} Z_j(\omega^j)$ where $\omega^N = U_L U_L U_L \ldots U_L$.

By our definition, $\sum_{j=1}^{N} Z_j(\omega^j)$ equals to N. That if we start with a set of initial queue size $x_o^L \geq N$, for $2 \leq L \leq n$, the the remaining constraints are (a), (c), and (3,19). As the initial queue lengths x_o^L (L=1,2,3,...,n) appear only in constraint (b), we will first start by the following lemma:

Lemma 1

The optimal solution will remain the same if the initial queue length is greater than a constant Q_L, L=1,2,...,n.

Proof:

Suppose $\{ Z_j^*(\omega^j), 1\le j \le N \}$ is an optimal solution. We have already seen that x_o^L, $2\le L \le n$, have to be greater than N in order to have the system of inequality feasible. Now, we show x_o^1 shall also be greater than a constant Q_o to satisfy the constraint (3.19). Consider constraint (3.19), i.e; for every $\omega^{N-1} \varepsilon \Omega^{N-1}$

$$x_o^1 + \sum_{j=1}^{N-1} \xi_j^1 (\omega^j) Z_j^*(\omega^j) - Z_N^* (\omega^{N-1} D_i) \ge 0 . \qquad (3.22)$$

Suppose $\omega^{*N} = \omega_1^* \omega_2^* \ldots \omega_{N-1}^* D_i$ is given as specific transition path such that $Z_j^*(\omega^j)=0$, for some j, $1 \le j \le N$ this is due to constraint (c) since the sum of some $Z_j(\omega^{j-1}D_i)$ is equal to 1. Thus, we have

$$x_o^1 \ge - \sum_{j=1}^{N-1} \xi_j^1 (\omega^{*j}) Z_j^* (\omega^{*j}) + Z_N^* (\omega^{*N-1} D_i) . \qquad (3.23)$$

If x_o^1 is greater than N, constraint (3.19) is satisfied for all ω^{N-1} and every i. Further more, if N is greater than 1, constraint (c) is a subset of (3.19). That is the optimal solution is not affected by constraint (3.19) when the initial queue sizes are increased.

Theorem 1

The optimal policy for finite horizon has threshold type after the queue lengths exceed Q_L , $L = 1,2,\ldots,n$.

Proof:

Immediate from Lemma 1.

Theorem 2

The optimal policy for infinite horizon has threshold type after the queue lengths exceed M_L , $L = 1,2,\ldots,n$.

Proof:

Applying [9], the result of finite horizon is extended to the infinite horizon.

5.Conclusions

We have shown that the optimal policy of arrival and service rates at the first queue of n queues in series, is of threshold type. We have applied the linear programming arguments to establish this structure of the optimal policy. If our assumption (2.2) is changed to $c_1 > c_2$, it is clear that the pair (λ_m , μ_m) is the optimal action no matter what the queue lengths are.

We believe that many optimal control queueing problems, in which the dynamic programming formulation fails, can be treated successfully via Linear Programming techniques.

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