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WAVELETS AND QUANTUM ALGEBRAS[1]

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ABSTRACT

A non-linear associative algebra is realised in terms of translation and dilation operators, and a wavelet structure generating algebra is obtained. We show that this algebra is a q-deformation of the Fourier series generating algebra, and reduces to this for certain value of the deformation parameter. This algebra is also homeomorphic with the q-deformed $su_q(2)$ algebra and some of its extensions. Through this algebraic approach new methods for obtaining the wavelets are introduced.

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1 Introduction

The wavelet transformation, as a method of multiresolution analysis (MRA), has gained widespread application in the last decades, in signal processing, data compression, patern recognition, statistical physics (description of cascade processes in high-energy multiparticle physics), fully developed turbulence, electromagnetic scattering and acoustics [1-3].

The central property of the wavelet transformation is the possibility of expansion of functions with respect to a selfsimilar set of (not-necessary) orthogonal basis functions, so-called wavelets. One of the main reasons of the efficiency of the wavelets is the fact that the domains of analysis are divided exponentialy, instead of linearly, by bands of equal widths. This is acomplished by using scalings rather than modulations, unlike the situation of the Fourier transform. The wavelets are able to reconstruct a signal by regular sampling, i.e. by analysing the signal at different scales (which increase/decrease exponentialy) the step between each two scales being the same.

In the analysis of real functions of real argument, the construction of wavelets is based on the action of two operators: translation of the argument with integers (T^k) and dilations with powers of 2^j (D^j). A pair of operators is constructed: the scaling operator which has an eigenfunction called the "scaling function" Φ, and the "difference operator" whose action on Φ gives the "mother wavelet" function Ψ. The translated and dilated versions of Φ generate a chain of subspaces of $L^2(R)$ and the translated and dilated versions of Ψ generate an orthonormal basis of $L^2(R)$, spliting it into a direct sum of orthogonal (ordered with respect to inclusion) subspaces. Consequently, any $L^2(R)$ function can be written as a sum of succesive approximations of it, each of them belonging to such a subspace, and describing the original function up to a certain degree of accuracy. More than this, the wavelets method is completely recursive and therefore ideal for computations, [1].

An interesting and usefull task is to construct a pure algebraic frame for the foundation of the wavelet analysis. Such trials of algebraic realisations were succesfully used in many fields of physics: Algebraic Scattering Theory, Interacting Boson Model, supersymmetric quantum theory (SUSSY), dynamical symmetries, etc. In order to construct an algebraic framework for the MRA with wavelets, one needs to modify the traditional techniques of the Fourier analysis, like differentiation or Lie algebras structures, in a suitable way. The basic functions of the multiresolution analysis, i.e. the scaling and wavelet functions, are not differentiable, in general step functions. Consequently, a possible approach consists in using finite-difference operators. In our approach, we introduce such objects through the help of q-deformed derivatives. Such q-differential operators are involved in q-deformed algebras.

Quantized universal enveloping algebras, also called q-deformed algebras, refer to some specific deformations of (the universal enveloping algebra U of) Lie algebras, to which they reduce when the deformation parameter q is set equal to one [4]. The simplest example of q-algebra, $\mathrm{su}_q(2) \equiv \mathrm{U}_q(\mathrm{su}(2))$, was first introduced by Sklyanin [5], and independently by Kulish and Reshetikhin [6] in their work on Yang-Baxter equations. A Jordan-Schwinger realization of $\mathrm{su}_q(2)$ in terms of q-bosonic operators was then derived by Biedenharn [7] and Macfarlane [8] and a q-deformation of the Euclidian and Heisenberg Lie algebras by using the contraction procedure was introduced by Celeghini *et al* [9]. Since then, the q-deformed algebras have been applied in various branches of physics (spin-chain models [10], non-commutative spaces [11], rotational spectra of deformed nuclei [12], Hamiltonian quantization [13], etc.). For a recent review see [14], for example. In addition to the usual version of the q-deformed algebras, frequently $\mathrm{su}_q(2)$, several generalized forms (of interest in applications to particle physics) have been introduced. Deformations involving one arbitrary function of J_0 were independently proposed by Polychronakos [15], Roček [16] and Ludu [17], the last one having a Hopf algebra structure. Also, deformations involving all commutator relations, with two functions of J_0 were introduced by Delbecq and Quesne [18]. Contrary to the former deformations, for which the spectrum of J_0 is linear as in the case of $su(2)$, the latter are characterised by an exponential spectrum of J_0.

In the present paper, we shall address the problem of obtaining classes of q-deformed algebras, realised in terms of q-differential operators (i.e. translation and dilation), which should carry out the algebraic frame for the wavelets theory. The basic ideas follow from the observation that several similarities between the wavelet theory and some features of non-linear (q-deformed) algebras exist, e.g.:

- The scale of the resolution of the wavelets modifies exponentialy. This fact could be related with exponential spectra of some operators arising only in non-linear algebras;

- The scaling functions and the wavelets are in general not-differentiable functions, which forbids their analysis with differential operators. The q-differential realisations of some q-deformed algebras act like finite-difference operators (the "differencing operators" used in wavelets theory [1]), and could be suitable substitutes for the normal derivative. Thus, q-deformed algebraic approach can restore the advantage of differential manipulation of functions involved in the Fourier transform, allowing the action of the q-deformed derivatives on non-differentiable functions. In adition, the q-derivative reduces to the normal derivative when the deformation parameter tends to certain values. Such procedure was shown to be usefull in studies concerning differential-finite-difference equations, [19];

- Through the continuous variation of the parameter of deformation one can move from one wavelet to another and, for some special limiting values of this parameter one can reduce the wavelet analysis to Fourier analysis;

- The "scaling operator" and the "differencing operator" can be regarded as generators of some closed non-linear algebras, tractable in the frame of q-algebra formalism.

Following these ideas, we have obtained some classes of non-linear algebras having two main properties: on one side they can be deformed towards the Fourier series generating algebra (and in this way we can regard wavelets as q-deformed generalisations of the Fourier series), and, on the other side, they can reduce to the $su_q(2)$ algebra, which gives the advantage of using the properties of this well investigated q-deformed algebras (unirreps and Casimir operator). For such purposes, we shall use a variant and extension of the deforming functional technique [4], wherein we obtained functionals mapping the generators of nonlinear algebras both to those of the Lie algebra $M(2)$ of translations and dilations of the R^2 plane (which generates the Fourier complex exponential) and to the $su_q(2)$ one.

Different from the ordinary Fourier transform, which reproduces a function as a superposition of complex exponentials, or from the windowed Fourier transform, which introduces a scale into the analysis of signals (the width of the window), MRA processes the signal locally, without prejudice the scale. Wavelet analysis is precisely a scale-independent method. We begin our investigation of the algebraic realisation with a simple example for the Fourier transform, by using a Lie algebra generated by the three generators: $J_0 = -i\partial$, $J_+ = e^{ix}$, $J_- = (J_+)^\dagger = e^{-ix}$ described by the commutators: $[J_0, J\pm] = \pm J_\pm$ and $[J_+, J_-] = 0$, which is isomorphic with $M(2)$. The space of the representations is generated by the complex exponentials $|n >= e^{inx}$, $J_0|n >= n|n >$, $J_\pm|n >= |n \pm 1 >$. The generators $J_\pm$ act like ladder operators on the $|n >$ states by increasing/decreasing the scale (i.e. n) with one integer. In order to obtain a scale-invariant algebra we have to introduce other generators and representations, which fulfill non-linear commutator relations, allowing the obtaining of scale-invariant equations. Consequently, some of the generators of these new algebras should satisfy some invariance equation, of the form $J_{\pm}|\phi >= |\phi >$. The corresponding q-deformed ladder operators will not change the scale anymore, but will keep the scale-invariance, through this equation.

2 The wavelet algebra

In the following we shall work in the space of $L^2(R)$ fuuctions. We introduce in this space the action of the operators of translation and dilation, defined by their action: $T^\alpha f(x) = f(x+\alpha)$, $D^\beta f(x) = f(2^\beta x)$, with α and β arbitrary real numbers. These operators are invertible and have the properties: $(T^\alpha)^\dagger = T^{-\alpha}$, $(D^\beta)^\dagger = D^{-\beta}$. They could be re-scaled in order to become unitary operators. These simple relations lead to some particular non-linear commutator relations which introduce exponential spectra of the generators, i.e. the scaling invariance. We have also the relations: $T^\alpha D^\beta = D^\beta T^{2^\beta\alpha}$, $T^\alpha T^\beta = T^\beta T^\alpha$, $D^\alpha D^\beta = D^\beta D^\alpha$. We stress that we define the operators T and D only from their action on $L^2(R)$ functions. If these operators act on $C^\infty(R)$ functions, we can write explicitely their exponential form, as formal series of the derivative operator: $T^\alpha = e^{\alpha\partial}$, and $D^\beta = e^{\beta ln 2 x\partial}$, where we have denoted by ∂ the differentiation with respect to x. More general, we can construct formal integer functions of T, $f(T)$. If f is an integer holomorphic function, and its action is taken on compact supported function subspace of $L^2(R)$ and the action of $f(T) = \sum_{k\in Z} C_k T^k$ reduces to the coresponding Laurent polynomial, i.e. if supp $\Phi(x) = [-M, M]$ then $f(T)\Phi(x) = \sum_{k_1(x)}^{k_2(x)} C_k\Phi(x+k)$, where $[k_1(x), k_2(x)]$ is the largest interval with integer limits, included in $[-x, M-x]$. In the following we shall denote simply $T^1 = T$, $D^1 = D$.

A scaling function is a function $\Phi(x)$ having three properties:

1. It exists an analytic function $h(x)$, such that the scaling function is a solution (eigenvalue or fixed point) of the "dilation equation": $Dh(T)\Phi = \Phi$. This condition makes a patern of $L^2(R)$ in a chain of subspaces V_m, $m \in Z$, $V_m \subset V_{m+1}$, each of them generated by all the translations of $D_m\Phi$, with integers.

2. $\int_R \Phi(x)dx = 1$, (averaging property).

3. $< \Phi_n, \Phi >\equiv \int_R \Phi(x+n)\Phi(x)dx = \delta_{n,0}$, for any $n \in Z$ (orthogonality condition).

In the following we shall use the notation $D^j T^n\Phi(x) = \Phi(2^j x+n) \equiv \Phi_{j,n}$. The action of the "differencing operator" on Φ is defined like $Dg(T)\Phi = \Psi$ with $g(T) = -T^{-\lambda}h(-T^{-1})$, λ being any odd integer. The "mother wavelet" Ψ, has the property that $\{\Psi_{j,n}\}_{j,n\in Z}$ is an orthonormal basis of $L^2(R)$, [1,2]. Consequently, $L^2(R)$ is a direct sum of the orthogonal subspaces W_m (the orthogonal complements of V_m), each of them generated by all possible translations of $\Psi_{m,0}$ with integers, [1-3]. From the above properties it results the restriction: $h(1) = g(-1) = 2$, $h(-1) = g(1) = 0$. Our aim is to construct a q-deformed algebra generated by operators (functions of D and T) related with the scaling operator $Dh(T)$ and the differencing operator $Dg(T)$, (wavelets algebra).

We introduce the q-deformation of an object x (a number, a function or an operator) in the standard form [4-9], for a real deformation parameter $q = e^s$, $s \in R$:

$$[x]_s = [x] = \frac{q^x - q^{-x}}{q - q^{-1}} = \frac{\sinh(sx)}{\sinh s}, \tag{1}$$

which reduces to x when $s \to 0$. Consequently, we introduce the q- deformation of the derivative, according to the definition of the coordinate description of the q-deformed oscillator, introduced by Macfarlane in [8], in the form:

$$[\partial]f(x) = \frac{e^{s\partial} - e^{-s\partial}}{sinh(s)}f(x) = \frac{f(x+s) - f(x-s)}{sinh(s)} = \frac{T^s - T^{-s}}{sinh(s)}f(x), \tag{2}$$

and, analogously,:

$$[x\partial]f(x) = \frac{f(2^s x) - f(2^{-s}x)}{sinh(s)} = \frac{D^s - D^{-s}}{sinh(s)}f(x). \tag{3}$$

When $s \to 0$, $[\partial] \to \partial$ and $[x\partial] \to x\partial$. The action of the operators T^k, D^j and of those given in eqs.(2,3) can be extended on $L^2(R)$, in the sens of distributions, through their differential realisation (infinite-order differential operators). Eqs.(2,3) can be inverted and we have a general connection with the q-deformation of the derivative, i.e. $[\partial]$: and the translation and dilation operators:

$$T^{\pm k} = \frac{1}{2}\Big(\pm\frac{[\partial]_k}{\eta(k)} + \frac{[\partial]^2_{k/2}}{\eta^2(k/2)} + 2\Big) \tag{4}$$

$$D^{\pm k} = \frac{1}{2}\Big(\pm\frac{[x\partial]_{k'}}{\eta(k')} + \frac{[x\partial]^2_{k'/2}}{\eta^2(k'/2)} + 2\Big), \tag{5}$$

where $k \in N$, $k' = k\ln 2$ and $\eta(k) = \frac{1}{2}cosech(k)$

In order to realise a non-linear algebra for the generation of wavelets, we first introduce the s, α parameter-depended operator:

$$J_0(s,\alpha) = \frac{e^{s\alpha\partial} - e^{-s\alpha\partial}\cos s\pi}{2\sinh s\xi(\alpha)} \tag{6}$$

where $s = ln(q)$ is the real q-deformation parameter, $\xi(\alpha)$ is a function of class $C^1(R)$, having the property: $\xi(0) = 0$, $\xi'(0) < \infty$, and $\alpha \in R$ is a free parameter which fixes the final form of the operator J_0 in different limiting cases (different fixed values for s). In the limits $s \to 0, \frac{1}{2}, 1$ and 2 we have:

$$J_0(0, \alpha) = \frac{\alpha}{\xi(\alpha)} \partial, \tag{7}$$

$$J_0\left(\frac{1}{2}, \alpha\right) = \frac{1}{2\sinh(1/2)\xi(\alpha)} T^{\frac{\alpha}{2}}, \tag{8}$$

$$J_0(1, \alpha) = \frac{T^{\alpha} + T^{-\alpha}}{2\sinh(1)\xi(\alpha)}, \tag{9}$$

$$J_0(2, \alpha) = \frac{T^{2\alpha} - T^{-2\alpha}}{2\sinh(2)\xi(\alpha)}. \tag{10}$$

We can see that the general definition (6) allows different posibilities for the deformation of the operator $J_0(s, \alpha)$. In the limit $s \to 0$, J_0 reduces to the normal derivative with respect to x, eq.(7). In the limit $s = \frac{1}{2}$, we obtain the simplest form for J_0, i.e. a power of the (unitary) translation operator T, eq.(8). This expresion, together with the expresion given in eq.(9) are usefull when in the construction of morphism of the wavelet algebra with other q-deformed algebras, like $su_q(2)$ for example. Eq.(9) defines a self-adjoint operator, $J_0(1, \alpha) = J_0(1, \alpha)^{\dagger}$, which is a sort of q-deformation of the unity, i.e. $J_0(1, 0) = \frac{1}{\sinh(1)} I$. The last limit, eq.(10), gives an anti-unitary operator, which represents the q-derivative with respect to the α-deformation for $\xi(\alpha) = \frac{\sinh(\alpha)}{\sinh(2)}$:

$$J_0(2, \alpha) = [2\partial]_{\alpha}, \tag{11}$$

and we get $J_0(2, \alpha) \to 2\partial$ in the limit $\alpha \to 0$. For $\alpha = 2k$ in eq.(8), $\alpha = k$ in eq.(9) and $\alpha = k$ or $k/2$ in eq.(10) the corresponding operator $J_0(s, \alpha)$ is a function of T only, providing the necessary connection with the wavelets operators. In order to introduce the dilation operator D, to make the connections with the dilation equation and to construct a closed (non-linear) algebra, we introduce other two operators:

$$J_{\pm}(s) = \frac{1}{2} e^{\mp ix\frac{s-1}{2}} D^{\mp s} e^{\mp ix\frac{s-1}{2}} (1 + T^{\pm s}), \tag{12}$$

having the property $(J_+)^{\dagger} = J_-$. We note that in the limit $s \to 0$ we have:

$$J_{\pm}(0) = e^{\pm ix}. \tag{13}$$

By choosing $\xi(\alpha) = i\alpha$, $J_0(0,\alpha)$ and $J_\pm(0)$ generate a Lie algebra homeomorphic with the Lie algebra $M(2)$ of translations and dilations in the R^2 plane, i.e. the Fourier series generating algebra, defined by the relations:

$$[J_0, J_\pm] = \pm J_\pm \ , \ [J_+, J_-] = 0. \tag{14}$$

The unitary irreducible representations (unirreps) of this Lie algebra are based, like in the case of $su(2)$, on the eigenvectors of the self-adjoint operator J_0, $J_0|a\rangle = a|a\rangle$. For two distinct eigenstates, $|a\rangle$ and $|a\rangle'$, by using eq.(14), we obtain $a' = a - 1$ and hence all the eigenvalues of J_0 are integer multiples of the smallest, non-zero one, and are equidistant. The Casimir operator for this algebra is J_+J_-. A basis of unirreps is the Fourier basis $\{e^{inx}\}_{n\in Z}$. In other words, the operators $J_\pm$ change, by their action on the basis functions, the scale $n \to n \pm 1$.

By denoting $J_0(s,\alpha) = J_0(T)$, the operators $J_0(T)$ and $J_\pm(s)$ from eqs.(6,12) fulfil the commutator relations:

$$[J_0(T), J_+(s)] = G_s(T) J_+(s), \tag{15}$$

$$[J_0(T), J_-(s)] = -J_-(s) G_s(T), \tag{16}$$

$$[J_+(s), J_-(s)] = F_s(T), \tag{17}$$

with:

$$G_s(T) = J_0(T) - \frac{T^{2^s s\alpha} e^{is\alpha(1+2^s)\frac{s-1}{2}} - T^{-2^s s\alpha} e^{-is\alpha(1+2^s)\frac{s-1}{2}} \cos s\pi}{2sinh(s)\xi(\alpha)}, \tag{18}$$

$$F_s(T) = \frac{1}{4}\Big((1 + e^{is(1+2^s)\frac{s-1}{2}} T^{2^s s})(1 + T^{-s}) - (1 + e^{is(1+2^{-s})\frac{s-1}{2}} T^{-2^{-s} s})(1 + T^s)\Big). \tag{19}$$

The relations (15-19) define a non-linear associative algebra denoted $\mathcal{A}_{G,F}$, depending on the functions $G_s(T)$ and $F_s(T)$, similar with the algebras introduced in [15-18]. As was shown above, it always exists an algebra morphism between the algebra $\mathcal{A}_{G,F}$ and the Fourier series generating algebra, obtained when $s \to 0$.

We can generalise the algebra $\mathcal{A}_{G,F}$ by choosing the new generators in the form:

$$J_0(T) = g(T, s), \tag{20}$$

and

$$J_\pm(s) = e^{\mp ix\frac{s-1}{2}} D^{\mp s} e^{\mp ix\frac{s-1}{2}} f(T^{\pm 1}, s), \tag{21}$$

where the operator functions $g(T, s)$ and $f(T^{\pm 1}, s)$ are integer functions of T, holomorphic in a neighborghood of 1, $f^\dagger(T^{\pm 1}, s) = f(T^{\mp 1}, s)$ and their dependence on s is such that in the limit $s \to 0$ we have $g(T,0) \to -i\partial$, $f(T^{\pm 1},0) \to 1$. We have then the commutator relations between these generators in the form given by eqs.(15-17) with:

$$G_s(T) = g(T,s) - g\left(T^{2^s} e^{i(1+2^s)\frac{s-1}{2}}, s\right), \tag{22}$$

$$F_s(T) = f(e^{i\frac{s-1}{2}(1+2^s)}T^{2^s}, s) f(T^{-1}, s) - f(T^{-2^{-s}} e^{i\frac{s-1}{2}(1+2^{-s})}, s) f(T,s). \tag{23}$$

If the function $g(T,s)$ is invertible with respect to T, $T = g^{-1}(J_0, s)$, we can further express the non-linear contribution to the algebra generated by eqs.(20-23) (denoted $\mathcal{A}_{g,f}$) in the form:

$$G_s(T) = \tilde{G}(J_0, s) \quad , \quad F_s(T) = \tilde{F}(J_0, s). \tag{24}$$

We can construct now the algebra morphism $\mathcal{A}_{g,f} \to \mathcal{A}_{G,F}$ through the mapping $g(T,s) \to J_0(s,\alpha)$ and $f(T^{\pm 1}, s) \to 1 + T^{\pm s}$.

We note that the first two commutators of the algebra $\mathcal{A}_{g,f}$ do not depend on the function $f(T^{\pm}, s)$, and that the third commutator does not depend on the function $g(T,s)$. Consequently,the closure condition for $\mathcal{A}_{g,f}$ is independent of the functions g and h and they could be choosen in a convenient way for our further aims.

The Casimir operator of the algebra $\mathcal{A}_{g,f}$ is given by [18,20]:

$$C = J_- J_+ + h(J_0) = J_+ J_- + h(J_0 - F_s(J_0, s)), \tag{25}$$

with $h(x)$ a real function, holomorphic in a neighborhood of $g(1,s)$ which must satisfy the functional equation:

$$h(x) - h(x - \tilde{G}(x,s)) = \tilde{F}(x,s). \tag{26}$$

In order to use the algebra $\mathcal{A}_{G,F}$ as a wavelet generation algebra we take the limit $s \to 1$ and the restrictions $\alpha = 1$ and $\xi(1) = 1/sinh(1)$. We obtain $J_0 = \frac{T+T^{-1}}{2}$, $T^{\pm 1}(J_0) = J_0 \pm \sqrt{J_0^2 - 1} = g^{-1}(J_0, 1)$ and $J_\pm = \frac{1}{2} D^{\mp 1}(1 + T^{\pm 1})$. This special algebra, $(\mathcal{A}^{(1)}_{G,F})$, has the commutator relations in the form:

$$[J_0, J_+] = (J_0 - 2J_0^2 + 1) J_+, \tag{27}$$

$$[J_0, J_-] = -J_-(J_0 - 2J_0^2 + 1), \tag{28}$$

$$[J_+, J_-] = \frac{1}{4}\left(T^2(J_0) - T^{1/2}(J_0) - T^{-1/2}(J_0) + T^{-1}(J_0)\right) = \tilde{F}(J_0, 1). \tag{29}$$

The defining equation for the Casimir operator, eq.(26), becomes:

$$h(J_0) - h(2J_0^2 - 1) = \tilde{F}_1(J_0), \tag{30}$$

or, in terms of the $T = T(J_0)$ operator:

$$\chi(T) - \chi(T^2) = \frac{1}{4}\Big(T^2 - T^{1/2} - T^{-1/2} + T^{-1}\Big), \tag{31}$$

where $\chi(T) = h(\frac{T+T^{-1}}{2})$. It is easy to check that the unique analytical solution for eq.(31) has the form:

$$\chi(T) = \frac{1}{4}\Big(const. - T - T^{1/2} - T^{-1/2}\Big), \tag{32}$$

which gives for the Casimir operator a constant.

The unirreps of the algebra $\mathcal{A}_{G,F}$, together with the Casimir operator, eq.(25), are corresponding to the eigenvalues of J_0, $J_0|\alpha >= \alpha|\alpha >$. The commutator (15) can be written in the form:

$$J_+ J_0 = (J_0 - \bar{G}(J_0, 1))J_+. \tag{33}$$

In the limiting case when $J_0^\dagger = \pm J_0$, (eqs.(9) and (10)), if $|a>$ and $|a'>$ are eigenvectors of J_0, with the corresponding eigenvalues a, a', we have from eq.(33) a recurrence relation in the form:

$$a' = \tilde{a} - \tilde{G}(\tilde{a}, 1), \tag{34}$$

where $\tilde{a} = a$ if J_0 is hermitian and $\tilde{a} = -a$ if J_0 is anti-hermitian. As an example, in the case of the algebra $\mathcal{A}^{(1)}_{F,G}$, eq.(34) gives $a' = 2a^2 - 1$. In this case the unirreps are infinit-dimensional and the eigenvalues are described by:

$$a_n = cosh(2^n\alpha), \tag{35}$$

with $\alpha = cosh^{-1}(a_0)$, for any arbitrary a_0. The corresponding eigenfunctions have the form:

$$|a_n >= e^{\pm 2^n \alpha x}, \tag{36}$$

which gives an exponential behaviour of the spectrum, similar with the behaviour of the scales in the wavelets theory. The action of $J_\pm$ is given by:

$$J_\pm|a_n >= \frac{1 + e^{\mp(\pm)2^n\alpha}}{2}|a_{n\pm 1} > . \tag{37}$$

Another example could be given if we choose the algebra $\mathcal{A}^{(2)}_{G,F}$ with J_0 defined in eq.(10). We choose $\alpha = 1/2$ and $2sinh(2)\xi(1/2) = 1$ and we obtain the spectrum of J_0 described by the recurrence relation:

$$a' = a\Big(a + \sqrt{a^2+1} + \frac{1}{a + \sqrt{a^2+1}}\Big), \tag{38}$$

which gives also a non-linear, infinite-dimensional, unbounded, discrete unirrep.

The obtaining of the wavelet and scaling functions in the algebraic frame is essentialy based on deformations, of the above described non-linear algebras, with $s = 1$. Different values for s are used to obtain the connection of these algebras with the Fourier series generating algebra or with other q-deformed algebras. Contrary to the case of the Fourier series, where the eigenequation for J_0 generates the representation basis, in the case of wavelets the scaling function is defined by the equation $J_-\Phi = \Phi$. For the algebra $\mathcal{A}^{(1)}_{G,F}$ this equation gives exactly the Haar scaling function. The action of the operator J_0 carries Φ into an infinite sequence of functions $J_0^n\Phi$. For n odd, these functions are mutual orthogonal and generate the space V_0. The action of $J_\pm$ on Φ constructs a subsequence $J_0^{2^n}\Phi$, so, all the generators of the algebra $\mathcal{A}^{(1)}_{G,F}$ moves Φ only within V_0.

More general, the algebra $\mathcal{A}_{g,f}|_{s=1}$ allows the choosing of different dilation equations. For different functions $f(T^{\pm 1}, 1)$ one can obtain different scaling functions. The function $g(T, 1)$ should be corespondingly choosen, in order to give the correct wavelet function through the action of J_0J_-. There are two posibilities to obtain the mother wavelet Ψ. One of them is to obtain Ψ directly from the generators of the algebra, as explained above: $J_0J- = Dg(T^2, 1)f(T^{-1}, 1)$ and $\Psi = J_0J_-\Phi$, [1]. However, in this situation $J_0(T)$ could be no more hermitic and consequently its spectrum is more difficult to be investigated. The other posibility represents a more direct way to introduce another wavelet algebra, which contains both the scaling $(Dh(T^{-1}))$ and the differencing $(D(-T^{-\lambda}h(-T))$ operators. We define:

$$J_0 = -T^{-\lambda}h(-T) \quad , \quad J_- = Dh(T^{-1}) \quad , \quad J_+ = J_-^\dagger, \tag{39}$$

with λ an arbitrary integer [1,2]. The eigenvector $J_-\Phi = \Phi$ represents in this way the scaling operator. We have:

$$J_-J_0\Phi = h(T^{-1/2})\Psi \tag{40}$$

It is easy to check that the function $\bar{\Psi} = h(T^{-1/2})\Psi$ is also an wavelet function, because satisfies the orthogonality condition (3):

$$< T^n h(T^{-1/2})\Psi, h(T^{-1/2})\Psi >= \sum_k C_k C_{2n-k} = \delta_{2n,0} \tag{41}$$

Evidently the generators in eq.(39) are closed against commutator relations. The connection with quantum groups is more transparent if we choose for $J_0 = [\partial]_{s=1/2} = J_0(2, 1/2)$ (from eq.(10)) and for Φ the Haar scaling function, i.e. $J_- = D(1 + T^{-1})$. The action of the q-derivative $[\partial]_{1/2}$ on Φ gives exactly the Haar wavelet. In this example the operator J_0 is anti-hermitic, and we can identify its spectrum (imaginary eigenvalues) from the commutator relation:

$$[J_0, J_-(1)] = -J_-(1)\tilde{G}(J_0, 1). \tag{42}$$

If $J_0|a>= a|a>$ then $a' = -a - \tilde{G}(-a, 1)$ gives the structure of the spectrum. In addition we have the relation

$$\Psi = (J_0 - \tilde{G}(J_0, 1)\Phi. \tag{43}$$

We can express the eigenvectors of J_0 in the wavelet basis of the algebra. If $|a>= \sum_{j,n} C_{j,n}\Phi_{j,n}$, by using eqs.(40, 42-43) we obtain a recurrence relation for the coefficients of $|a>$:

$$\sum_k C_{j,p-2^j k} C_k = a C_{j,p} \tag{44}$$

for any j and p integer, where C_k are the Taylor coefficients of the function $-T^{-\lambda}h(-T)$. We note that by using the q-derivative instead of a function of T (like in the case of the wavelet formalism), we obtain exactly the same mother wavelet (Haar wavelet).

3 Further extensions

In order to construct a finite-dimensional generated (non-linear) algebra for wavelets, we give a general form for the operator $f(T, s)$ as a polynomial of order M, whose coefficients are smooth functions of the q-deformation parameter s:

$$f(T, s) = \sum_{k=0}^{M} C_k(s) T^{k\alpha(s)}, \tag{45}$$

where the coefficients in eq.(45) must satisfy the constrains:

$$\sum_{k=0}^{M} C_k(s) = 2 \tag{46}$$

$$\sum_{n=0}^{M-2k} C_n(s) C_{n+2k}(s) = \frac{1}{2}\delta_{k,0} \quad , \quad k = 0, 1, ..., N-1, \tag{47}$$

$$C_k(-s) = (-1)^k C_k(s) \quad , \quad \alpha(-s) = -\alpha(s) + \lambda, \tag{48}$$

with λ an odd integer, $M = 2N-1$, for any s. The condition in eq.(45) provides the necessary restrictions for the scaling function $\Phi(x)$. Eq.(47) fulfils the orthogonality condition, (3). This is easy to check, because we have from eqs.(3,45)

$$\delta_{k,0} = < \Phi_k, \Phi > = 2 < D^{-1}\Phi_k, D^{-1}\Phi > +2 < f(T,s)\Phi_{2k}, f(t,s)\Phi >$$

$$= 2 \sum_{j,k=0}^{M} C_j C_k < \Phi_{2k+j}, \Phi_n > = 2 \sum_{j,k=0}^{M} C_j C_{2k+j} \tag{49}$$

These conditions, eqs.(47,49), are valid only for odd M (otherwise it results in $M = 0$ which contradicts the definition eq.(46) and consequently consists in $k = 0, 1, ..., [M/2] = N - 1$ quadratic algebraic equations for the functions $C_k(s)$. If we choose s_1 such that $\alpha(s_1) = 1$, the operator $f(T, s_1)$ generates the scaling equation. More than this, for $s = -s_1$ we have, through eq.(48):

$$f(T, -s_1) = T^{\lambda} f(-T^{-1}, s) \equiv \gamma(T, s) \tag{50}$$

which is the differencing operator which generates the wavelet function, i.e. $D\gamma(T, s_1)\Phi = \Psi$.

Due to the unitarity of the operator $\sqrt{2}D$ occuring in $J_{\pm}(s)$, the Casimir operator is allways a function of T only. *This interdicts from the very begining the possibility of searching the scaling operator between the Casimir or functions of it.* More than this, the functional equation for the function H, involved in C, has, in the case of the wavelet algebras $\mathcal{A}_{g,f}$, $\mathcal{A}_{G,F}$, only constant, trivial solutions. So, in the wavelet algebras the principal role is played by the scale operator, J_-. It is useful to look for other operators in $U(\mathcal{A}^{(1)}_{G,F})$ which commute with J_-. If they exist, then the scaling function, and the corresponding wavelet, is not unique in the frame of a given nonlinear algebra. We have the following:

Proposition: In $U(\mathcal{A}^{(1)}_{G,F})$ exists and it is unique an operator in the form $X = Dx(T^{-1})$ such that $[J_-, X] = 0$. The function $x(T^{-1})$ is integer, analytical in a neighborhood of 1 and and it is not a polynomial.

Proof: We take for X the form of a Laurent series $x(T^{-1}) = \sum_{k\in Z} C_k T^{-k}$ and the condition of commutation $[J_-, X] = 0$ gives the recurrence equations for the coefficients C_k:

$$C_{2k+1} = C_k - C_{2k-1},$$

$$C_{2k} = C_k - C_{2k-2}, \tag{51}$$

for any $k \in Z$. Eq.(51) has one trivial solution which reproduces J_-, $x(T^{-1}) = C_{-1}(1+T^{-1})$ and only one other solution, with $C_k = \pm C_{-1}$ or 0, uniquely defined, e.g.: $C_2 = C_{\pm 3} =$

$C_{-4} = C_5 = C_{-6} = C_7 = C_{-8} = C_{-11} ... = 0$, $C_{-2} = C_{-5} = C_6 = C_{-9} = C_{-10} = C_{10} = C_{11} = ... = C_{-1}$ and $C_1 = C_4 = C_{-7} = C_8 = C_9 = C_{-12} = C_{12} = ... - C_{-1}$, etc. The sequence of non-zero coefficients is infinite, q.e.d.

Concerning this Proposition we want to point out that in the case of a general form of the scaling operator, $J_- = Dy(T) = D\sum_{k\in Z} y_k T^k$, we have to solve the commutator equation $[Dy(T), X)] = 0$ for $x(T) = \sum_{k\in Z} X_k T^k$, $X = Dx(T)$. This implies that the arbitrary functions $x(z)$ and $y(z)$ must fulfil the functional condition:

$$x(z^2)y(z) = x(z)y(z^2). \tag{52}$$

Eq.(52) does not carry out any restriction with respect to the values of the functions in 0 and 1. Since $Dy(T)$ represents a scaling operator we have, according to the scaling equation: $y(1) = 1$, $y(-1) = 0$ which implies $x(-1) = 0$. If Φ is the corresponding scaling function for J_- then $X\Phi$ is also an eigenfunction of J_-, $J_-(X\Phi) = X\Phi$. In order to satisfy the averaging condition (2), for $Dx(T)$, we impose the supplimentar condition $x(1) = 1$. The last condition which we need for $DX(T)$, i.e. the orthogonality condition, (3), is equivalent with the condition, [1]:

$$|x(T)|^2 + |x(-T)|^2 = 1. \tag{53}$$

By introduction eq.(52) in eq.(53) we obtain $X = J_-$ which gives us a trivial indentity solution. Consequently, the translated versions of the new non-trivial scaling functions $X\Phi$ should not be any more orthogonal one to each other. In this case it is necessary to use the so-called "orthogonalization trick" [2] to generate the new scaling function $\Phi^{\#}$ that satisfies the orthogonality conditions.

4 Connection with $su_q(2)$

The quantum group $su_q(2)$ or its extensions, [4-9,15-17] are associative algebras over C generated by three operators $j_0 = (j_0)^\dagger$, j_+, and $j_- = (j_+)^\dagger$, satisfying the commutation relations:

$$[j_0, j_+] = j_+, \qquad [j_0, j_-] = -j_-, \qquad [j_+, j_-] = f(j_0), \tag{54}$$

where $f(j_0)$ is a real, parameter-dependent function of j_0, holomorphic in the neighbourhood of zero, and going to $2j_0$ for some values of the parameters. Let us now find a deforming functional that transforms the $\mathcal{A}_{g,f}$ generators into operators satisfying the commutation relations of eq.(54). For such purpose, eq.(15) can be written as

$$(J_0 - \tilde{G}(J_0, s))J_+ = J_+J_0. \tag{55}$$

Then, for every entire function $p(z)$, we can obtain

$$p(J_0 - \tilde{G}(J_0, s))J_+ = J_+p(J_0). \tag{56}$$

Let us consider the functional equation:

$$\Gamma(z - \tilde{G}(z, s)) = \Gamma(z) - 1 \tag{57}$$

for a given function $\Gamma(z)$. If this equation has a solution $\Gamma(z)$ that is an entire function, then eq.(56) can be written for $p(z) = \Gamma(z)$ in the form $[\Gamma(J_0), J_+] = J_+$, and consequently $[\Gamma(J_0), J_-] = -J_-$. The above equations give the correspondence between the wavelet algebras and the q-deformed algebras like $su_q(2)$ or other extensions of them. In terms of the function $\Gamma(z)$, eqs.(15-17) can be reduced to eqs.(54) through the transformations:

$$j_0 = \Gamma(J_0), \qquad j_\pm = J_\pm, \tag{58}$$

If Γ is invertible, the $\mathcal{A}_{G,F}$ algebra third defining equation (17,24) can then be reduced to the third defining equation (54) of an algebra corresponding to $f(J_0)$ with the identification $\tilde{F} \circ \Gamma^{-1} = f$ where $f \circ \Gamma$ means the composition of the two functions, i.e., $(f \circ \Gamma)(z) = f(\Gamma(z))$. In the case of $\mathrm{su}_q(2)$, i.e. $f(J_0) = [2J_0]$, the function $\tilde{F}(J_0, s)$ becomes:

$$\tilde{F}(J_0, s) = -\frac{(\phi(J_0))^2 - (\phi(J_0))^{-2}}{q - q^{-1}}, \tag{59}$$

where $\phi(z) \equiv q^{-\Gamma(z)}$ has to satisfy the equation $\phi(z - \tilde{G}(z, s)) = q\phi(z)$. For details of the above reduction of algebras one can see [20], for example.

In the case of the algebra $\mathcal{A}_{G,F}$ eqs.(55-57) give the condition:

$$p(j_0 - 1) = J_0 - \tilde{G}(J_0, 1). \tag{60}$$

having a particular solution

$$p(j_0) = cosh(2^{-j_0} ln(b)), \tag{61}$$

for any arbitrary constant b. By inversion of eq.(61) we obtain:

$$j_0 = \mp \frac{1}{ln(2)} ln(\frac{\partial}{ln(b)}). \tag{62}$$

In the revers way, starting from $su_q(2)$ we choose $f(j_0) = [2j_0]$. In this case we obtain for the operator $F_s(T)$:

$$F = f(\Gamma(J_0)) = f(p^{-1}(J_0)) = [2p^{-1}(J_0)]. \tag{63}$$

5 Construction of the scaling function

The dilation equation has the form of a fixed-point equation. In this sense, we can try to find its eigenvectors as limits of some functional sequences. In addition, these hypotetic sequences should be reducible to compact supported or rapidly decreasing functions, for two reasons: a good time-frequency localization (here time means the x coordinate), and a correct behaviour of the action of the operators $f(T)$. Let us consider the case of the Haar scaling function Φ whose generic operator is $D(1+T^{-1})$, i.e. the operator J_- of the algebra $\mathcal{A}^{(1)}_{G,F}$. We can express Φ in the form $\Phi(x) = H(x) - H(x-1)$, where $H(x)$ is the Heaviside distribution. For any sequence of functions, $\delta_n(x)$ which tends to the Dirac distribution $\delta(x)$, we can write:

$$\Phi(x) = \lim_{n\to\infty} (\Delta_n(x) - \Delta_n(x-1)), \tag{64}$$

where $\Delta_n(x) = \int_R \delta_n(x)dx$. For example, we can use the sequence $\delta^a_n = \frac{1}{\pi}\frac{2^{-n}}{x^2+2^{-2n}}$ or $\delta^b_n = \frac{2^{n-1}}{cosh^2(2^n x)}$ and consequently we get $\Delta^a_0 = \frac{arctan(x)}{\pi}$, $\Delta^b_0 = \frac{1}{2}tanh(x)$. The last example is a soliton-like shape with a very good localization. Eq.(64) can be written as:

$$\Phi(x) = \lim_{n\to\infty} (1 - T^{-1}) D^n \Delta_0(x), \tag{65}$$

or, similary,

$$\Phi(x) = 2D(1 - J_0) \lim_{n\to\infty} D^n \Delta_0. \tag{66}$$

By using eq.(15) and the properties of the operators D and T, we can write eq.(65) in the form

$$\Phi(x) = \lim_{n\to\infty} (1 - T^{-2^{-n}}) J^n_- \Delta_0(x). \tag{67}$$

As $\Delta_0(x)$ is integrable, it can be expanded in terms of the eigenvectors of the operator J_0. From eq.(67) we can now write the solution of the dilation equation in a purely algebraic form:

$$\Phi(x) = \lim_{n\to\infty}(1 - T^{-2^{-n}}(J_0))J_-^n\Delta_0(x). \tag{68}$$

We expect that eq.(68) can provide us with informations about the efficiency of the wavelet expansion of certain functions, through the algebraic picture. If the function $\Delta_0(x)$ is choosen from a class of functions to be analysed in the wavelets formalism, then $\Phi(x)$ from eq.(68), gives a good scaling function, in the sense of the rapidity of the convergece of the wavelets expansions.

6 Conclusions

In this paper we have found that the operators of dilation and translation (D,T) can be combined in such a way to generate non-linear algebras, depending on certain parameters (s, α). We have investigated these algebras from the point of view of quantum groups, discussing their unirreps, Casimir operators and their different reductions to some other q-deformed algebras, like $su_q(2)$ or the Fourier series generating algebra, for example. It has been shown that such algebras provide an appropriate frame for the foundation of the wavelets analysis and for the obtaining of the corresponding scaling functions. Direct applications of such an approach could be founded in the theory of finite-difference equations.

However, this study represents a first step in the direction of correlations between the theory of non-linear algebras (having exponential spectra) with wavelets theory and with finite-difference equations, occuring in mathematical physics.

We conjecture that such an algebraic approach (non-linear algebras) to the scale invariant structures could lead to interesting clasifications, like the posibility of identifying isolated coherent structures associated with some mother wavelets+limiting non-linear algebras (for certain values of q). The tranzition between such self-organized structures could be carried out through the modification of the deformation parameter q, in between them existing a sort of "noise" (no closed algebraic structures).

References

[1] G. Kaiser, *A Friendly Guide to Wavelets*, (Birkhäuser, Boston, 1994)

[2] I. Daubechies, *Ten Lectures on Wavelets*, (SIAM, Philadelphia, 1992), p. 139-140.

[3] M. Greiner, *Doctor Thesis* , (Arizona State University, 1992).

[4] V. G. Drinfeld, *Proc. Intern. Congress of Mathematicians* (in A. M. Gleason (ed.), Berkeley, CA, 1986), AMS, Providence, RI, 1987), p. 798;
M. Jimbo, *Lett. Math. Phys.*, **10**, 63 (1985).

[5] E. K. Sklyanin, Funct. Anal. Appl., **16**, 262 (1982).

[6] P. P. Kulish and N. Y. Reshetikhin, J. Sov. Math., **23**, 2435 (1983).

[7] L. C. Biedenharn, J. Phys. A: Math. and Gen., **22**, L873 (1989).

[8] A. J. Macfarlane, J. Phys. A: Math. and Gen., **22**, 4581 (1989).

[9] E. Celeghini, R. Giachetti, E. Sorace and M. Tarlini, J. Math. Phys., **31**, 2548 (1990).

[10] V. Pasquier and H. Saleur, Nucl. Phys. B, **330**, 523 (1990).

[11] S. L. Woronowicz, Commun. Math. Phys., **111**, 613 (1987);
Yu. Manin, *Quantum Groups and Non-commutative Geometry*, (Les Publications du centre des Recherches Mathematiques, Université de Montréal, 1988);
J. Wess and B. Zumino, Nucl. Phys. Proc. Suppl. B, **18**, 302 (1990).

[12] R. K. Gupta and A. Ludu, Phys. Rev. C, **48**, 593 (1993);
D. Bonatsos, E. N. Argyres, S. B. Drenska, P. P. Raychev, R. P. Roussev, and Yu. F. Smirnov, Phys. Lett. B, **251**, 477 (1990).

[13] D. B. Fairlie and J. Nuyts, J. Phys. A: Math. and Gen., **24**, L1001, (1991).

[14] L. Takhtajan, *Introduction to Quantum Group and Integrable Massive Models of Quantum Field Theory*, (ed. Mo-Lin Ge and Bao-heng Zhao, Sigapore, World Scientific, 1990).

[15] A. P. Polychronakos, Mod. Phys. Lett. A, **5**, 2325 (1990).

[16] M. Roček, Phys. Lett. B, **255**, 554 (1991).

[17] A. Ludu and R. K. Gupta, J. Math. Phys., **34**, 5367 (1993).

[18] C. Delbecq and C. Quesne, J. Phys. A: Math. and Gen., **26**, L127 (1993).

[19] V. Spiridonov, Phys. Rev. Lett., **69**, 398 (1992).

[20] D. Bonatsos, C. Daskaloyannis, P. Kolokotronis, A. Ludu and C. Quesne, *Nonlinear Deformation of* su(2) *Algebras with a Double Hopf Algebra Structure*, (submitted).