IC/95/260

INTERNATIONAL CENTRE FOR
THEORETICAL PHYSICS

ASYMPTOTICAL REPRESENTATION OF DISCRETE GROUPS

Alexander S. Mishchenko

and

Noor Mohammed

MIRAMARE-TRIESTE

INTERNATIONAL
ATOMIC ENERGY
AGENCY

UNESCO

UNITED NATIONS
EDUCATIONAL,
SCIENTIFIC
AND CULTURAL
ORGANIZATION

IC/95/260

International Atomic Energy Agency
and
United Nations Educational Scientific and Cultural Organization
INTERNATIONAL CENTRE FOR THEORETICAL PHYSICS

ASYMPTOTICAL REPRESENTATION OF DISCRETE GROUPS

Alexander S. Mishchenko[1] and Noor Mohammad[2]
International Centre for Theoretical Physics, Trieste, Italy.

ABSTRACT

If one has a unitary representation $\rho : \pi \longrightarrow U(H)$ of the fundamental group $\pi_1(M)$ of the manifold M then one can do many usefull things:

1. to construct a natural vector bundle over M,
2. to construct the cohomology groups with respect to the local system of coefficients,
3. to construct the signature of manifold M with respect to the local system of coefficients

and others. In particular, one can write the Hirzebruch formula which compares the signature with the characteristic classes of the manifold M, further based on this, find the homotopy invariant charcteristic classes (i.e. the Novikov conjecture). Taking into account that the family of known representations is not sufficiently large, it would be interesting to extend this family to some larger one. Using the ideas of A.Connes, M.Gromov and H.Moscovici ([1]) a proper notion of asymptotical representation is defined .

MIRAMARE – TRIESTE

August 1995

[1]Permanent address: Department of Mathematics and Mechanics, Moscow State University, 119899 Moscow, Russian Federation.

[2]Permanent address: Department of Mathematics, Quaid-i-Azam University, Islamabad, Pakistan.

1 Introduction

Our interest concentrates on the problem of developing the representation theory for groups and C^*-algebras, especially for the group C^*-algebras of discrete groups. This problem is stimulated by the following very well known phenomenon. *Let* M *be a closed* oriented non simply connected manifold with fundamental group π. Let $B\pi$ be the classifying space for the group π and let f_M be a map inducing the isomorphism of fundamental groups. Then if one has a finite dimensional unitary representation ρ of the finitely represented group π,

$$\rho : \pi \longrightarrow U(H), \tag{1.0.1}$$

then one can construct in a natural way a vector bundle ξ_ρ over the classifying space $B\pi$ with the fiber H and as a consequence, over an arbitrary manifold M with fundamental group π. The advantage of this construction lies for instance in this that one can write generalization of the Hirzebruch formula for the signature of the manifold in terms of the characteristic classes. In fact, if

$$\rho : \pi_1(M) \to U(n) \tag{1.0.2}$$

is a unitary representation, the cohomology with respect to a local system of coefficients generated by the representation ρ admits a nondegenerate quadratic form, and the signature, $\text{sign}_\rho(M)$, is defined. One can verify that

$$\text{sign}_\rho M^{4k} = 2^{2k}\langle \text{ch}\rho L(M^{4k}), [M^{4k}]\rangle = \sigma_{\text{ch}_\rho}(M^{4k}), \tag{1.0.3}$$

where $\text{ch}\rho$ is the Chern character of representation ρ. Since $\text{sign}_\rho M^{4k}$ is a homotopy invariant, the right side of the equality (1.0.3), $\sigma_{\text{ch}_\rho}(M^{4k})$, is also homotopy invariant. Unfortunately the Chern character $\text{ch}\rho$ is trivial for any finite dimensional unitary representation ρ. Therefore, the way to construct nontrivial examples of $\text{ch}\rho$ lies in considering indefinite metrics on H or infinite dimensional (Fredholm) representations. In ([1]) A.Connes, M.Gromov and H.Moscovici ([1]) defined a new concept which holds in the category of finite dimensional representations. They introduced a notion of almost flat bundles on a manifold M. Namely, let $\alpha \in K^0(M)$ be an element of the K-theory on a Riemannian manifold M, and let $\alpha = (E^+, \nabla^+) - (E^-, \nabla^-)$, where E^+, E^- are hermitian complex vector bundles with connections ∇^+, ∇^-. One can say that the element α is almost flat if for any $\epsilon > 0$ there exists a representation $\alpha = (E^+, \nabla^+) - (E^-, \nabla^-)$ such that

$$\|(E^+, \nabla^+)\| \leq \epsilon, \|(E^-, \nabla^-)\| \leq \epsilon, \tag{1.0.4}$$

where

$$\|(E, \nabla)\| = \text{Sup}_{x\in M}\{\|\theta_x(X \wedge Y)\| : \|X \wedge Y\| \leq 1\}, \tag{1.0.5}$$

and $\theta = \nabla^2$ is the curvature form.

Then for almost flat element $\alpha \in K^0(M)$ and any $\varepsilon > 0$ and a finite subset $\pi^0 \subset \pi$, there exists a representation $\alpha = (E^+, \nabla^+) - (E^-, \nabla^-)$ with

$$\|(E^+, \nabla^+)\| \leq \epsilon, \|(E^-, \nabla^-)\| \leq \epsilon, \tag{1.0.6}$$

such that the corresponding quasi-representations constructed by the holonomy on the family of paths which form elements of the fundamental group

$$\sigma^+ : \pi \longrightarrow U(N^+), \sigma^- : \pi \longrightarrow U(N^-) \tag{1.0.7}$$

have the following property:

$$\|\sigma^+\|_{\pi^0} \leq \varepsilon, \|\sigma^-\|_{\pi^0} \leq \varepsilon, \tag{1.0.8}$$

where

$$\|\sigma\|_{\pi^0} = \mathrm{Sup}\{\|\sigma(ab)\sigma(a)\sigma(b)\| : a, b \in \pi^0\}. \tag{1.0.9}$$

This notion has a shortcoming because the choice of quasi-representation depends on the property of almost flatness which in turns depends on the smooth structure of the manifold M. It would be interesting to construct a natural inverse correspondence from the family of quasi- representations of discrete group to the family of almost flat bundles.

Another problem related to quasi-representations was discussed by P.Halmos ([2]) and later by Voiculescu ([3]), Loring ([4]), Exel and Loring ([5],[6]). Namely, whether a pair of unitary matrices A and B which almost commute (in the sense that the operator norm $\|AB - BA\|$ *is small) can always be slightly perturbed in order to yield a pair of* commuting unitary matrices. The answer is known to be false and the proof depends on the second cohomology of the two-torus. We shall show that, in fact, the problem of almost commuting unitary matrices is a special case of quasi-representation of group π when $\pi = Z \oplus Z$ and $B\pi$ is two-torus.

So the general problem can be formulated as follows: whether an almost representation can be perturbed in order to yield a classical representation. Then the obstruction to the existence of such a perturbation can be expressed as a characteristic class of a vector bundle, which can be constructed in a natural way (see Loring [4]). For some reasons, described below, we prefer to consider the so called asymptotical representation due to A.Connes.

2 Definitions of asymptotical representations

2.1 Finite dimension case: Discrete version

There are at least two ways of giving the definition of asymptotical representations. The first is the following:

Definition 1 *Let $\varepsilon > 0$ and let $\pi^0 \subset \pi$ be a finite subset and*

$$\sigma : \pi \longrightarrow U(N) \tag{2.1.1}$$

a map such that

$$\sigma(g^{-1}) = \sigma(g)^{-1}, \tag{2.1.2}$$

and

$$\|\sigma\|_{\pi^0} = \sup\{\|\sigma(gh) - \sigma(g)\sigma(h)\| : g, h, gh \in \pi^0\} \leq \varepsilon. \tag{2.1.3}$$

Then σ is called ε-almost representation (with respect to the finite subset π^0).

Further we consider the so called stable class of ε–almost representations σ. This means *that the map σ can be substituted for the composition*

$$\sigma \oplus 1 : \pi \longrightarrow U(N) \longrightarrow U(N+1). \tag{2.1.4}$$

It is evident that

$$\|\sigma\|_{\pi^0} = \|\sigma \oplus 1\|_{\pi^0}. \tag{2.1.5}$$

Therefore we shall consider as ε–almost representations maps

$$\sigma : \pi \longrightarrow U(\infty), \tag{2.1.6}$$

which possess the properties (2.1.2), (2.1.3)

Definition 2 *Let*

$$\sigma = \{\sigma_n : \pi \longrightarrow U(\infty)\} \tag{2.1.7}$$

be a sequence of maps such that

$$\sigma_n(g^{-1}) = \sigma_n(g)^{-1},$$

and for any $\varepsilon > 0$ and a finite subset $\pi^0 \subset \pi$, there exists a number N_0 such that if $n > N_0$ then

$$\|\sigma_n(a) - \sigma_{n+1}(a)\| \leq \varepsilon, a \in \pi^0, \tag{2.1.8}$$

$$\|\sigma_n\|_{\pi^0} \leq \varepsilon. \tag{2.1.9}$$

σ is called asymptotical representation of the group π.

There is another approach:

Let π be a finitely rerepresented group and let $(A;S)$ be a presentation of π by generators A and relations S. Let $F(A)$ denote the free group generated by A .

Definition 3 *Let $\sigma : F(A) \longrightarrow SU(\infty)$ be a representation such that $\|\sigma(s) - E\| < \varepsilon$, where $s \in S$. Then σ is called ε-almost representation of the group π.*

Definition 4 *Let*

$$\sigma = \{\sigma_n : F(A) \longrightarrow SU(\infty)\} \tag{2.1.10}$$

be a sequence of ε_n–almost representations of the group π such that $\lim_{n\longrightarrow\infty} \varepsilon_n = 0$, and for any $a \in A$ one has

$$\lim_{n\longrightarrow\infty} \|\sigma_n(a) - \sigma_{n+1}(a)\| = 0. \tag{2.1.11}$$

Then σ is called asymptotical representation of the group π.

One can easily check that the definitions 2 and 4 are equivalent.

2.2 Finite dimension case: Continuous version

Definition 5 *Let*

$$\sigma = \{\sigma_t : \pi \longrightarrow U(\infty)\} \tag{2.2.1}$$

be a continuous family of maps $0 \leq t < \infty$ *such that*

$$\sigma_t(g^{-1}) = \sigma_t(g)^{-1}$$

and for any $\varepsilon > 0$ *and a finite subset* $\pi^0 \subset \pi$*, there exists a number* N_0 *such that if* $t > N_0$ *then*

$$\|\sigma_t\|_{\pi^0} \leq \varepsilon. \tag{2.2.2}$$

Then σ *is called asymptotical representation of the group* π*.*

There is another definition:

Let π be a finitely represented group and let $(A; S)$ be a presentation of π by generators A and relations S. Let $F(A)$ denote the free group generated by A.

Definition 6 *Let*

$$\sigma = \{\sigma_t : F(A) \longrightarrow SU(\infty)\} \tag{2.2.3}$$

be a continuous family of ε_t*–almost representations of the group* π *such that* $\lim \varepsilon_t = 0$, $t \longrightarrow \infty$. *Then* σ *is called asymptotical representation of the group* π*.*

One can easily check that the definitions 2 and 6 are equivalent. Namely, one has

Theorem 1 *Let* σ_t *be an asymptotical representation of the group* π *in the sense of definition (6) . Then there exists a sequence* t_n *such that* σ_{t_n} *is asymptotical representation in the sense of the definition (2) .*

Conversely, let σ_n *be an asymptotical representation of the group* π *in the sense of definition (2). Then there exist an asymptotical representation* σ_t*, in the sense of definition (6), and a sequence* t_n *such that*

$$\sigma_n = \sigma_{t_n}.$$

2.3 The ring of asymptotical representations

The family of asymptotical representations can be transformed into the Witt group $\mathcal{R}_a(\pi)$. Namely, the family $\{\sigma\}$ admits natural operations, i.e., direct sum and tensor product. Let us introduce two equivalence relations between two asymptotical representations. The first equivalence relation is precisely homotopy between two asymptotical representations. Let $\{\sigma_t\}$, $0 \leq t \leq 1$, be a family of asymptotical representations such that for any $g \in \pi$, the functions $\{\sigma_{t,n}(g)\}$ are continuous and the following property of uniformity holds:

$$\lim_{n \longrightarrow \infty} \mathrm{Max}_{\{t\}} \|\sigma_{t,n}\|_{\pi^0} = 0. \tag{2.3.1}$$

Then σ_0 and σ_1 are said to be homotopic.

The second relation admits change of basis in the space. Namely, the representations $\{\sigma_n\}$ and $\{U_n \circ \sigma_n \circ U_n^{-1}\}$ are called *equivalent* if U_n is any sequence of unitary operators from $U(\infty)$ satisfying the property

$$\lim_{n\longrightarrow\infty} \|U_n - U_{n+1}\| = 0.$$

One can prove that the second relation implies from the first one.

Then the Witt group generated by $\{\sigma\}$ modulo the equivalence relations, as defined above, will be denoted by $\mathcal{R}_a(\pi)$.

3 Construction of vector bundle

Now we want to construct a natural homomorphism

$$\phi : \mathcal{R}_a(\pi)\longrightarrow K(B\pi). \tag{3.0.1}$$

Let M be a finite CW–complex with fundamental group π, i.e.,

$$\pi_1(M) = \pi, \tag{3.0.2}$$

and let

$$f_M : M\longrightarrow B\pi, \tag{3.0.3}$$

be the map as defined before. Let $\tilde{M}$ be the universal covering of M, and $p : \tilde{M}\longrightarrow M$ the projection. Let $\bar{M} \subset \tilde{M}$ be a fundamental domain, that is, a finite closed subcomplex such that $p(\bar{M}) = M$. Put

$$\pi^0 = \{g \in \pi : g(\bar{M}) \cap \bar{M} \neq \emptyset\}. \tag{3.0.4}$$

One can construct elements α of the group $K(B\pi)$ using their restrictions to the spaces like M. More precisely, consider the category $B\pi$ whose objects are CW–complexes M with fundamental group π, and morphisms are the homotopy classes of maps $g : M_1\longrightarrow M_2$ such that the diagram

$$\begin{array}{ccc} M_1 & \xrightarrow{f_{M_1}} & B\pi \\ \downarrow g & & \downarrow = \\ M_2 & \xrightarrow{f_{M_2}} & B\pi \end{array} \tag{3.0.5}$$

commutes. Consider a natural correspondence α which associates any space $M \in B\pi$ to a vector bundle $\alpha(M) \in K(M)$ such that for any morphism $g : M_1\longrightarrow M_2$ one has

$$\alpha(M_1) = g^*(\alpha(M_2)). \tag{3.0.6}$$

Then it is clear that there is a one to one correspondence between the set $\{\alpha\}$ and the set of vector bundles over $B\pi$. For the sake of simplicity, one can restrict only to finite CW–complexes $M \in B\pi$.

To construct a vector bundle over M we proceed as follows. Consider a trivial vector bundle over $\bar{M}$, $\xi = \bar{M} \times C^n$, and an action of the group π which is compatible with the action on the base $\bar{M}$. This action can be described with a matrix function

$$T_g(x) : \xi_x \longrightarrow \xi_{gx},\ x \in \tilde{M},\ g \in \pi, \tag{3.0.7}$$

satisfying the condition

$$T_g(hx) = T_{gh}(x) \circ T_h^{-1}(x),\ x \in \tilde{M},\ g, h \in \pi. \tag{3.0.8}$$

The function (3.0.7) with the property (3.0.8) is called the *transition function.* FFrom (3.0.8) one has

$$T_e(x) = E,\ x \in \tilde{M}, \tag{3.0.9}$$

where $e \in \pi$ is the neutral element, and E is the identity matrix. It is clear that the condition (3.0.8) is related with separate orbits $\pi(x_0)$ of the action of π on the space $\tilde{M}$. Moreover, the function (3.0.7) determines its value only in one fixed point $x_0 \in \pi(x_0)$ by using the condition (3.0.9) . In fact, if the function (3.0.7) is defined at the point $x_0 \in \pi(x_0)$ for all $g \in \pi$ and satisfies the condition (3.0.9), then one can extend the function $T_g(x)$ to an arbitrary point $x \in \pi(x_0)$ using the formula (3.0.8):

$$T_g(x) = T_{gu}(x_0) \circ T_u^{-1}(x_0),\ x \in \tilde{M},\ g, u \in \pi,\ x = ux_0. \tag{3.0.10}$$

If $x = x_0$, then in (3.0.10) one has $u = e$, and hence $T_g(x) = T_{ge}(x_0) \circ E = T_g(x_0)$. One needs to verify the condition (3.0.8) for $x = ux_0$ and $y = hx = hux_0$.One has

$$T_g(hx) = T_g(hux_0) = T_{ghu}(x_0) \circ T_{hu}^{-1}(x_0), \tag{3.0.11}$$

$$T_{gh}(x) = T_{ghu}(x_0) \circ T_u^{-1}(x_0), \tag{3.0.12}$$

$$T_h(x) = T_{hu}(x_0) \circ T_u^{-1}(x_0). \tag{3.0.13}$$

Therefore

$$\begin{aligned} T_g(hx) &= T_{ghu}(x_0) \circ T_{hu}^{-1}(x_0) =, \\ &= T_{ghu}(x_0) \circ T_u^{-1}(x_0) \circ (T_u^{-1}(x_0))^{-1} \circ (T_{hu}(x_0))^{-1} = \\ &= T_{ghu}(x_0) \circ T_u^{-1}(x_0) \circ (T_{hu}(x_0) \circ T_u^{-1}(x_0))^{-1} = \\ &= T_{gh}(x) \circ T_h(x). \end{aligned} \tag{3.0.14}$$

Let $\bar{M} \subset \tilde{M}$ be the fundamental domain and let π^0 be as given in (3.0.4). Then one can consider the restriction

$$\tilde{T}_g(x) = T_g(x),\ g \in \pi^0,\ x \in \bar{M}, gx \in \bar{M}. \tag{3.0.15}$$

The function (3.0.15) satisfies the condition (3.0.8) for all admissible x, g, h where

$$x, hx, ghx \in \bar{M},\ g, h \in \pi^0. \tag{3.0.16}$$

The function (3.0.15) with the property (3.0.16) defines a vector bundle over M as well, and will be called also the transition function. Therefore one can define the transition function only for a fixed point $x_0 \in (\bar{M} \cap \pi(x_0))$ and for all $g \in \pi$ such that $gx_0 \in \bar{M}$. If $x, gx \in \bar{M} \cap \pi(x_0)$ then $x = hx_0$, and hence $h, g, gh \in \pi^0$. One can use the formula (3.0.8) to define the values of the transition function for all admissible x, g. Therefore for the construction of the vector bundle $\phi(\sigma)$ as a family of the transition functions, one should start from the zero dimensional skeleton. Put

$$\bar{T}_g(x_0) = \sigma(g) \tag{3.0.17}$$

for a representative x_0 of each zero dimensional orbit. To extend the transition function from zero dimensional skeleton to simplexes of higher dimension, one can use the property that the action of the group π is free and the property (2.1.3) for $\varepsilon = 1$. Indeed, in the general case one should do the following.

Let us choose representatives $\{a_\alpha\}$ in each orbits of the set $[\tilde{M}]_0$ of vertices. Then the set $[\bar{M}]_0 = [\tilde{M}]_0 \cap \bar{M}$ has the property that: $g(\bar{M}) \cap \bar{M} \neq \emptyset$ is equivalent to $g([\bar{M}]_0) \cap [\bar{M}]_0 \neq \emptyset$. Therefore there are elements $g_a \in \pi^0$ such that for any element $a \in [\tilde{M}]_0$, there is an α_a such that

$$a = g_a(a_{\alpha_a}). \tag{3.0.18}$$

Consider a simplex $\sigma = (b_0, b_1, \dots, b_n) \subset \tilde{M}$. This means that $\{b_0, b_1, \dots, b_n\} \subset [\tilde{M}]_0$. Then according to (3.0.4) one has

$$b_i = g_i(a_{\alpha_i}),\ i = 0, 1, \dots, n;\ g_i \in \pi^0, \tag{3.0.19}$$

and as before, one has

$$\begin{aligned} T_g(b_i) &= T_{gg_i}(a_{\alpha_i}) \circ T_{g_i}^{-1}(a_{\alpha_i}) = \\ &= \sigma(gg_i) \circ \sigma^{-1}(g_i), \\ & \quad gg_i \in \pi^0. \end{aligned} \tag{3.0.20}$$

Therefore for any i

$$\|T_g(b_i) - \sigma(g)\| = \|\sigma(gg_i) \circ \sigma(g_i^{-1}) - \sigma(g)\| < \varepsilon. \tag{3.0.21}$$

Let $x \in \sigma$ be a point with the barycentric coordinates $\lambda_0, \lambda_1, \dots, \lambda_n$,

$$x = \sum_i \lambda_i b_i.\ \lambda_i \geq 0,\ \sum_i \lambda_i = 1. \tag{3.0.22}$$

Then put

$$T_g(x) = \sum_{i=0}^{n} \lambda_i T_g(b_i). \tag{3.0.23}$$

According to (3.0.24) one has

$$\|F_x \circ G - 1\| < 1, \quad (\varepsilon = 1(\,!).) \tag{3.0.24}$$

The next problem is to establish the fact when two ε-almost representations give the same vector bundles over $B\pi$.

4 The first Chern class of vector bundle

The definition 4 is more suitable for calculating the first Chern class $c_1(\phi(\sigma)) \in H^2(B\pi; R)$. Namely, the first Chern class can be described as a two-dimensional cocycle, that is, as a function defined on the family of two dimensional cells of the space $B\pi$. Let X be a finite CW-complex with fundamental group π. Let the presentation (A,S) be induced by 2-skeleton of some simplicial structure of X.

Theorem 2 *The first Chern class $c_1(\phi(\sigma))$ can be described as a cocycle*

$$c_1(s) = tr(\log(\sigma(s)))/(2pi) \in Z, \tag{4.0.1}$$

where s runs through all 2-dimensional cells.

As a matter of fact, the construction of the vector bundle $f_M^*(\phi(\sigma))$ over a compact manifold with fundamental group π depends only on some sufficiently small ε_n, and coincides with the classical one for $\varepsilon = 0$. The magnitude of admissible ε can be estimated by $\exp(-\dim(M))$.

5 Asymptotical representations of $Z \oplus Z$.

Consider an example of fundamental group which seems to be the simplest nontrivial case when there exists an asymptotical representation giving nontrivial vector bundle over classifying space. Let $\pi = Z \oplus Z$, and let $a,\ b\ \in \pi$ be generators of the group such that $ab = ba$. To construct an asymptotical representation, it suffices to define two continuous matrix valued functions, $A(t),\ B(t) \in U(\infty),\quad 0 \le t <= \infty$ such that

$$\lim_{t\longrightarrow\infty} \|A(t)B(t) - B(t)A(t)\| = 0 \tag{5.0.1}$$

First of all we shall construct a discrete series of unitary matrices

$$A_n,\ B_n \in U(n). \tag{5.0.2}$$

Let A_n be a matrix of cyclic permutation of the orthonormal basis in C^n:

$$A_n = \begin{pmatrix} 0 & 0 & 0 & \cdots & 0 & 1 \\ 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & \cdots & 1 & 0 \end{pmatrix}, \tag{5.0.3}$$

and let B_n be the diagonal matrix

$$B_n = \begin{pmatrix} \lambda_1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & \lambda_2 & 0 & \cdots & 0 & 0 \\ 0 & 0 & \lambda_3 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & \lambda_{n-1} & 0 \\ 0 & 0 & 0 & \cdots & 0 & \lambda_n \end{pmatrix}. \tag{5.0.4}$$

Then the commutator has the following form

$$A_nB_n - B_nA_n =$$

$$= \begin{pmatrix} 0 & 0 & 0 & \cdots & 0 & \lambda_n \\ \lambda_1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & \lambda_2 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & \cdots & \lambda_{n-1} & 0 \end{pmatrix} -$$

$$- \begin{pmatrix} 0 & 0 & 0 & \cdots & 0 & \lambda_1 \\ \lambda_2 & 0 & 0 & \cdots & 0 & 0 \\ 0 & \lambda_3 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & \cdots & \lambda_n & 0 \end{pmatrix} =$$

$$= \begin{pmatrix} 0 & 0 & 0 & \cdots & 0 & \lambda_n - \lambda_1 \\ \lambda_1 - \lambda_2 & 0 & 0 & \cdots & 0 & 0 \\ 0 & \lambda_2 - \lambda_3 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & \cdots & \lambda_{n-1} - \lambda_n & 0 \end{pmatrix}. \tag{5.0.5}$$

The next step is to frame the matrices A_n and B_n with additional row and colomn:

$$\tilde{A}_n = \begin{pmatrix} A_n & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & \cdots & 0 & 1 & 0 \\ 1 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 1 & 0 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots & \\ 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & 0 & \cdots & 1 & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 & 1 \end{pmatrix}. \tag{5.0.6}$$

$$\tilde{B}_n = \begin{pmatrix} B_n & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} \lambda_1 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & \lambda_2 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & \lambda_3 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & \lambda_{n-1} & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & \lambda_n & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 & 1 \end{pmatrix}. \tag{5.0.7}$$

The commutator for the framed matrices has the form

$$\tilde{A}_n\tilde{B}_n - \tilde{B}_n\tilde{A}_n = \\ = \begin{pmatrix} A_nB_n - B_nA_n & 0 \\ 0 & 1 \end{pmatrix} = \\ = \begin{pmatrix} 0 & 0 & 0 & \cdots & 0 & \lambda_n - \lambda_1 & 0 \\ \lambda_1 - \lambda_2 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & \lambda_2 - \lambda_3 & 0 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & 0 & \cdots & \lambda_{n-1} - \lambda_n & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 & 1 \end{pmatrix}. \quad (5.0.8)$$

So the norms of two commutators coincide, i.e.,

$$\|A_nB_n - B_nA_n\| = \|\tilde{A}_n\tilde{B}_n - \tilde{B}_n\tilde{A}_n\| = \max\{|\lambda_1 - \lambda_2|, \dots, |\lambda_n - \lambda_1|\}. \quad (5.0.9)$$

Let

$$\lambda_k = \exp(\frac{2\pi ik}{n}),\ k = 1, \dots, n. \quad (5.0.10)$$

Then $\lambda_n = 1$ and

$$|\lambda_k - \lambda_{k+1}| \leq \frac{2\pi}{n}. \quad (5.0.11)$$

The next step is to connect the pair $(\tilde{A}_n, \tilde{B}_n)$ and the pair (A_{n+1}, B_{n+1}) by continuous path $(\bar{A}_t, \bar{B}_t)$ which satisfies the inequality

$$\|\bar{A}_t\bar{B}_t - \bar{B}_t\bar{A}_t\| \leq \frac{2\pi}{n}. \quad (5.0.12)$$

Put

$$\bar{A}_t = \begin{pmatrix} 0 & 0 & 0 & \cdots & 0 & \cos(2\pi t) & \sin(2\pi t) \\ 1 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 1 & 0 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots & \\ 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & 0 & \cdots & 1 & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & -\sin(2\pi t) & \cos(2\pi t) \end{pmatrix}, \quad (5.0.13)$$

$$\bar{B}_t = \tilde{B}_n = \begin{pmatrix} \lambda_1 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & \lambda_2 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & \lambda_3 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & \lambda_{n-1} & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 1 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 & 1 \end{pmatrix}. \quad (5.0.14)$$

Then one has

$$\tilde{A}_t\tilde{B}_t - \tilde{B}_t\tilde{A}_t =$$

$$= \begin{pmatrix} 0 & 0 & 0 & \cdots & 0 & \cos(2\pi t) & \sin(2\pi t) \\ \lambda_1 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & \lambda_2 & 0 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & 0 & \cdots & \lambda_{n-1} & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & -\sin(2\pi t) & \cos(2\pi t) \end{pmatrix} -$$

$$- \begin{pmatrix} 0 & 0 & 0 & \cdots & 0 & \lambda_1\cos(2\pi t) & \lambda_1\sin(2\pi t) \\ \lambda_2 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & \lambda_3 & 0 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & 0 & \cdots & \lambda_n & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & -\sin(2\pi t) & \cos(2\pi t) \end{pmatrix} =$$

$$= \begin{pmatrix} 0 & \cdots & 0 & \nu_n\cos(2\pi t) & \nu_n\sin(2\pi t) \\ \nu_1 & \cdots & 0 & 0 & 0 \\ 0 & \cdots & 0 & 0 & 0 \\ \vdots & & \vdots & \vdots & \vdots \\ 0 & \cdots & 0 & 0 & 0 \\ 0 & \cdots & \nu_{n-1} & 0 & 0 \\ 0 & \cdots & 0 & 0 & 0 \end{pmatrix}, \qquad (5.0.15)$$

where

$$\nu_k = \lambda_k - \lambda_{k+1},\ k = 1, \ldots, n;\ \lambda_{n+1} = \lambda_1.$$

When λ_k are given by (5.0.10) the inequality (5.0.12) holds.

The second part of path connects the pair ($\tilde{A}_1 = A_{n+1}$, $\tilde{B}_1 = \tilde{B}_n$) with the pair A_{n+1}, B_{n+1} . Let $\tilde{A}_t = A_{n+1}$ be constant and let $\tilde{B}_t$ be the diagonal matrix, i.e.,

$$\tilde{B}_t = \begin{pmatrix} \lambda_1(t) & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & \lambda_2(t) & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & \lambda_3(t) & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & \lambda_{n-1}(t) & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & \lambda_n(t) & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 & \lambda_{n+1}(t) \end{pmatrix}, \qquad (5.0.16)$$

where

$$\begin{gathered} \lambda_k(t) = \exp 2\pi i k((1-t)\tfrac{1}{n} + t\tfrac{1}{n+1}), \quad 1 \le k \le n \\ \lambda_{n+1}(t) = 1. \end{gathered} \qquad (5.0.17)$$

Therefore when $t = 1$, one has

$$\tilde{B}_1 = B_{n+1} \qquad (5.0.18)$$

Thus we have constructed a continuous family of (stable) matrices (A_t, B_t) such that

$$\|A_tB_t - B_tA_t\| \leq \frac{2\pi}{[t]}, \tag{5.0.19}$$

where A_n, B_n coincide with discrete sequence. Now let us apply the formula (4.0.1) to calculate the Chern class of the asymptotical representation constructed above. Since

$$s = aba^{-1}b^{-1}, \tag{5.0.20}$$

and hence

$$\sigma_n(s) = A_nB_nA_n^{-1}B_n^{-1}. \tag{5.0.21}$$

It is easy to verify that

$$A_nB_nA_n^{-1}B_n^{-1} =$$
$$= \begin{pmatrix} \frac{\lambda_1}{\lambda_2} & 0 & 0 & \cdots & 0 & 0 \\ 0 & \frac{\lambda_2}{\lambda_3} & 0 & \cdots & 0 & 0 \\ 0 & 0 & \frac{\lambda_3}{\lambda_4} & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & \frac{\lambda_{n-1}}{\lambda_n} & 0 \\ 0 & 0 & 0 & \cdots & 0 & \frac{\lambda_n}{\lambda_1} \end{pmatrix} =$$
$$\approx \begin{pmatrix} \exp(\frac{-2\pi i}{n}) & 0 & \cdots & 0 & 0 \\ 0 & \exp(\frac{-2\pi i}{n}) & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \cdots & \exp(\frac{-2\pi i}{n}) & 0 \\ 0 & 0 & \cdots & 0 & \exp(\frac{-2\pi i}{n}) \end{pmatrix}. \tag{5.0.22}$$

Thus

$$c_1(s) = \mathrm{tr}(\log(\sigma(s)))/(2pi) = 1. \tag{5.0.23}$$

Consequently, one has

Theorem 3 *The homomorphism*

$$\varphi : \mathcal{R}_a(Z \oplus Z) \longrightarrow K(T^2)$$

is surjective.

References

[1] A. Connes, M. Gromov et H. Moscovoci, *Conjecture de Novicov et fibrés presque plats,* C.R.Acad.Sci. Paris, t.310, Série I, (1990), 273–277.

[2] P.R.Halmos, *Some unsolved problems of unknown depth about operators on Hilbert space,* Proc. Roy. Soc. Edinburgh Sect. A 76 (1976), 67–76.

[3] D.Voiculescu, *Remarks on the singular extension in the C^*-algebra of the Heisenberg group,* J. Operator Theory 5 (1981), 147–170.

[4] T.A.Loring, *K-theory and asymptotically commuting matrices,* Canad. J. Math.,40 (1988), 197–216.

[5] R.Excl, T.Loring, *Almost commuting unitary matrices,* Proc. Amer. Math. Soc., vol. 106, No. 4, (1989), 913–915.

[6] R.Exel, T.Loring, *Invariants of Almost Commuting unitaries,* J. Funct. Analysis, *Vol. 95, No. 2, (1991), 364–376.*

[7] A.Stern, Russian Journal of Math.Phys., vol. 1, No. 1 (1993)