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ABSTRACT

The properties of particle motion in the vicinity of the separatrix in a phase plane are investigated. The change of adiabatic invariant value due to the separatrix crossing is evaluated as a function of a perturbation parameter magnitude and a phase of a particle for time dependent Hamiltonians. It is demonstrated that the change of adiabatic invariant value near the separatrix birth is much larger than that in the case of the separatrix crossing near the saddle point in a phase plane. The conditions of a stochastic regime to appear around the separatrix are found. The results are applied to study the longitudinal invariant behaviour of charged particles near singular lines of the magnetic field.

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1 Introduction

In the last chapter of the monograph by T.Northrop [1] there is a statement of the problem on the calculation of adiabatic invariant value for the particle orbits that cross a separatix in a phase plane. This problem is relevant to particle motion described by the Hamiltonian $\mathcal{H}(p,q,\epsilon t)$ which is supposed to depend on a 'slow' time variable, i.e. $\epsilon \ll 1$, and the phase plane corresponding to $\mathcal{H}$ with frozen time contains the lobes of the finite motion separated by the separatrix. It is well known that the adiabatic invariant to the lowest order in the smallness parameter ϵ is equal to the area enclosed by a contour of a constant energy $h = \mathcal{H}(p,q,\epsilon t)$ where $\mathcal{H}(p,q,\epsilon t)$ is a Hamiltonian describing nearly periodic orbit with slow time dependence $\epsilon \ll 1$ [2]. There exists an adiabatic invariant which is conserved to all orders in ϵ. The smallness parameter ϵ can be estimated as $\dot{\omega}/\omega^2$, where ω is the typical frequency value and $\dot{\omega}$ is its time derivative. This parameter is of order ϵ far from the separatrix and the conservation accuracy of adiabatic invariant is exponential. On the separatrix the frequency vanishes and ϵ tends formally to the infinity. Here the validity of adiabatic approximation breaks and the detailed analysis is needed in the vicinity of the separatrix.

This problem has been analysed previously for a non-linear oscillator [3] and it was found the change of adiabatic invariant of order $\epsilon \ln \epsilon$ for the main fraction of particles. Afterwards this approach has been developed in [4, 5, 6, 7].

The problem of existence and accuracy of the conservation of the adiabatic invariant is one of the fundamental problems in theory of dynamic systems and has a wide range of applications in modern physics. Among these applications are the theory of transport in toroidal plasma [6, 8, 9], acceleration of charged particles in linear accelerators [10, 11] and by strong wake waves in nonuniform plasmas [12], during the reconnection of magnetic field lines in plasmas [13, 14], particle motion in the Earth magnetosphere [15] and high-frequency plasma heating [16].

Change of adiabatic invariant in the couse of crossing the separatrix with one saddle point has been analyzed in detail earlier in [3, 4, 5, 6] for generic Hamiltonian. The aim of the paper is the investigation of the Hamiltonian system describing the particle motion in the vicinity of the sepaiatrix to undergo its birth or vanishing.

In particular, this dynamic system behaviour corresponds to the splitting of saddle points which lead to the formation of a trajectory which has the cusp shape in a phase plane. One of the problems releavant to the formation of such a singularity in the phase plane corresponds to the charged particle motion in the tokamak magnetic field. In [6, 9] there is discussed the structure of the magnetic field which strength has the form

$$B = B_0(1 - \epsilon_t \cos(s) - \epsilon_h \cos(\lambda) + \alpha^* \epsilon_h \lambda)\,, \tag{1}$$

where the coordinats are $s = (\theta - \iota\phi)/(1 - l\iota/M)$, $\lambda = l\theta - M\phi$; and B_0 is the magnetic field value on the axis; $\epsilon_t = r/R_0$ and ϵ_h are the toroidal and helical modulation amplitudes; l and M are the poloidal and toroidal mode numbers; ι is the rotational transform; dimensionless parameter α^* is equal to $\alpha_0 \sin(s)$, where $\alpha_0 = \iota\epsilon_t/M\epsilon_h$. For stellarator magnetic fields this parameter is small: $\alpha^* \ll 1$, and in a tokamak $\alpha^* \le 1$. As it is well known, in a drift approximation the magnetic field (1) plays a role of an effective potential and when $\alpha^* \approx 1$ the singularity in a phase plane appears with a transformation of saddle points on the separatrix into a cusp configuration [6].

Another range of problems which demand the detailed analysis of the accuracy of adiabatic invariant conservation is connected with charged particle motion near the separatrics of magnetic field. It is well known neglecting the particle drift their motion along the magnetic field lines is described by the Hamiltnian system in the approximation of zeroth Larmour radius. In this case the separatrics of magnetic field correspond to the separatrics of the Hamiltonian system in a phase plane. The account of a slow electric drift which causes the shift of Larmour circle transverse to magnetic field lines leads to the crossing the separatrix in a phase plane by some trajectories and to the change of longitudinal adiabatic invariant.

To define more exactly the problem of accuracy of adiabatic invariant conservation and to demonstrate the approach to decide the problem below we reproduce the results obtained earlier [3, 4, 5, 6] for the case of one saddle point. Next we use this approach to calculate the adiabatic invariant change in the vicinity the separatrix with a cusp point and to define the condition of appearance of stochastic layer near a cusp point in the presence of damping and periodic force. Except the analitical approach we perform the results of numerical integration of the equations of motion. At the end of the paper we apply the results to calculate the change of adiabatic invariant near the separatrix of magnetic field.

2 Accuracy of the adiabatic invariant change

Let us consider the adiabatic invariant conservation in the case of linear oscillator with the Hamiltonian

$$\mathcal{H} = (p^2 + \omega_0^2(\epsilon t)q^2)/2 = h, \tag{2}$$

for which we have

$$\ddot{q} + \omega_0^2(\epsilon t)q = 0. \tag{3}$$

Here a slowly varying frequency $\omega_0(\epsilon t)$ is supposed to be a monotonous function of time. This equation has a form of the Schrödinger equation with slowly varying energy potential. That means that we can use the WKB approximation to obtain the solution of Eq.3. The change of the adiabatic invariant in this case is proportional to the amplitude of the reflected wave [17]:

$$\Delta I \propto \exp\{-2\,Im \int_{t_1}^{t_0} \omega_0(t)dt\}, \tag{4}$$

where t_0 is a complex number for which $\omega_0(t_0) = 0$. As a result the change of adiabatic invariant is of order $\exp(-1/\epsilon_0)$, where $1/\epsilon_0$ is proportional to the distance between points t_1 and t_C.

On the separatrix the period of particle motion is infinite and the effective parameter ϵ_0 tends to infinity.

In the vicinity of the X-point (saddle point) the Hamiltonian can be approximated as

$$\mathcal{H} \approx (p^2 - q^2)/2. \tag{5}$$

The time during which the particle dwells on the separatrix depends on the proximity of the trajectory to the X-point. Let us assume that the energy on the separatrix is zero, i.e. $h = 0$, then the proximity is determined by the value of h. The motion governed by

this Hamiltonian has an exponential time dependence. Typically, the particle lingers near the X-point during the time

$$T \simeq \ln(1/|h|). \tag{6}$$

The change of the energy, ΔE, during one period of particle motion, is of order of h. It is of order ϵ, the perturbation parameter, therefore $h \approx \Delta E \simeq \epsilon$. Hence, the area enclosed by the trajectory changes as

$$\Delta I \approx \Delta(TE) \simeq E\Delta T + T\Delta E \approx |h|\ln|h| \approx \epsilon \ln \epsilon, \tag{7}$$

which is in quantitative agreement with the results obtained earlier [3, 4, 5, 6] .

If the Hamiltonian is different from that given by (5), then we cannot use estimations (7) for the invariant change. For the Hamiltonian

$$\mathcal{H} = \frac{p^2}{2} + V(q, \epsilon t), \tag{8}$$

with $V(q) \propto q^n$, $n > 2$ near $q = 0$, the period of motion is not logarithmic [18]:

$$T = \oint \frac{dq}{(2(h - V(q)))^{1/2}} \cong h^{(2-n)/2n}. \tag{9}$$

The proper estimate for the adiabatic invariant change gives

$$\Delta I \simeq \epsilon^{(2+n)/2n}. \tag{10}$$

This dependence corresponds to the case when several saddle points merge and to the rise or disappearance of the region of the finite motion in the phase plane and, in particular to the birth of the separatrix. Meanwhile, for already existing motion it is typical the presence of hyperbolic saddle point determined by (5), the case of separatrix birth is determined by the semicubic cusp point, where $p \propto q^{3/2}$. The last corresponds to $n = 3$ and the adiabatic invariant change is of order $\Delta I \simeq \epsilon^{5/6}$.

3 Motion near the separatrix

In this section we estimate the change of the adiabatic invariant for the case of one X-point. The system under consideration has the Hamiltonian of the form

$$\mathcal{H} = \frac{p^2}{2} - (1 + \epsilon t)\left(\frac{1}{2}q^2 - \frac{1}{4}q^4\right). \tag{11}$$

The potential in this case has two minima. The location of the separatrix corresponding to $h = 0$ depends on the time ϵt, that leads to changes of the potential shape and depth.

Let us introduce new function

$$\gamma^2(\epsilon t) = 1 + \epsilon t, \tag{12}$$

and new time variable

$$\eta = \frac{1}{\epsilon}\int^{\epsilon t} \gamma(x)dx. \tag{13}$$

Under these transformation the equation of motion takes the form

$$q'' - q + q^3 = -\frac{\gamma'}{\gamma} q', \tag{14}$$

where the prime denotes a derivative with respect to η. The advantage of this transformation is that the obtained equation in the zeroth order on the parameter ϵ value describes the motion in frozen potential with a stationary separatrix. The non-stationarity of the Hamiltonian is described by the effective friction term in the right hand side of Eq.(14) which is of order $\gamma'/\gamma \approx \epsilon/2 \ll 1$ for $\epsilon \ll 1$. This transformation has been introduced in [6] and used in [9]. Another remarkable circumstance is that the change of the coordinates inverse to that is given by Eq. (13) transforms the dissipative system to a Hamiltonian one. This fact may be useful for the analysis of particle motion in dissipative systems.

The solution of zeroth order in ϵ which corresponds to the motion on the separatrix for the Hamiltonian with constant ϵt for so-called frozen dependence on time, for $\epsilon t = 0$ obeys the equation

$$\left(q^{(0)}\right)'' - q^{(0)} + \left(q^{(0)}\right)^3 = 0, \tag{15}$$

and is given by the expression

$$q^{(0)} = \frac{2^{1/2}}{\operatorname{ch}\eta}, \tag{16}$$

$$\left(q^{(0)}\right)' = -\frac{2^{1/2}\operatorname{sh}\eta}{\operatorname{ch}^2\eta}. \tag{17}$$

Now we seek the first order solution in ϵ in the form of a series

$$q = q^{(0)} + \epsilon q^{(1)} + \cdots. \tag{18}$$

The problem under consideration is reduced to the solution of equations with the Hamiltonian given by Eq.(11). They correspond to the Hamiltonian $\mathcal{H}_0$ in the zeroth order in ϵ. The solution $(q^{(0)}, p^{(0)})$ governed by Eq.(16)–(17) for $p^{(0)} \equiv (q^{(0)})'$ corresponds to the energy $h = 0$ in this case. The equation of motion in zeroth order on ϵ may be written as

$$\left(q^{(0)}\right)'' = F\left(q^{(0)}\right). \tag{19}$$

In the first order in ϵ we have

$$\left(q^{(1)}\right)'' = \frac{\partial F}{\partial q^{(0)}} q^{(1)} + \epsilon f(q^{(0)}, \left(q^{(0)}\right)', \eta), \tag{20}$$

where ϵf is a perturbation. Differentiating Eq.(19) on time, we obtain

$$\left(q^{(0)}\right)''' = \frac{\partial F}{\partial q^{(0)}} \left(q^{(0)}\right)', \tag{21}$$

hence Eq.(20) can be rewritten in the form

$$\left(q^{(1)}\right)'' - \frac{\left(q^{(0)}\right)'''}{\left(q^{(0)}\right)'} q^{(1)} = \epsilon f. \tag{22}$$

Multiplying both parts of this equation on $\left(q^{(0)}\right)'$ and integrating on time we obtain

$$\left(q^{(1)}\right)'\left(q^{(0)}\right)' - \left(q^{(0)}\right)'' q^{(1)} = \epsilon \int_0^\eta f \left(q^{(0)}\right)' d\tilde{\eta} + h, \tag{23}$$

where h, the perturbation of particle energy, is assumed to be of order ϵ. As a result we have

$$q^{(1)}(\eta) = \left(q^{(0)}\right)'(\eta) \left(C + \int_0^\eta \frac{d\tilde{\eta}}{\left(\left(q^{(0)}\right)'\right)^2} \left(\epsilon \int_0^{\tilde{\eta}} f \left(q^{(0)}\right)' d\tilde{\tilde{\eta}} + h\right)\right), \tag{24}$$

here the constant C is proportional to $q^{(1)}(0)$.

If the integral $\int_0^\eta f \left(q^{(0)}\right)' d\tilde{\eta}$ converges for $\eta \to \pm\infty$ and its value is equal to $M_\pm$ respectively, then for $h = -\epsilon M$ the expression (24) is needed to be evaluated carefully. Namely, if the function f is such that

$$\epsilon \int f q_0' d\eta \approx \epsilon M - C_3 \exp(-\eta), \tag{25}$$

where C_3 is constant, then we have

$$q^{(1)} \approx C\, C_4\, exp(-\eta) + \frac{C_3}{C_4}. \tag{26}$$

Here we use the fact that the solution $q^{(0)}$ can be approximated as

$$q^{(0)} \approx C_4 \exp(\pm\eta) \tag{27}$$

for $\eta \to \pm\infty$. In particular case corresponding to (14) respectively the constant is $C_4 = 2\sqrt{2}$. From Eq.(26) it follows that for $\eta \to \pm\infty$ $q^{(1)}$ tends to the constant C_3/C_4. These solutions describe splitting of the separatrix under the perturbation into two branches with the values of energy $h_\pm = -\epsilon M_\pm$ and shift of a singular point to the value C_3/C_4^2 [20]. Below the separatrix means the unperturbed separatrix, which position is calculated for a frozen dependence of the Hamiltonian on time.

Substituting in Eq.(24) the expressions given by Eq. (16)- (17) we find the solution of the first order in ϵ for the Hamiltonian (11). For the initial conditions

$$q^{(1)}(0) = \frac{h}{2^{1/2}}, \tag{28}$$

$$\left(q^{(1)}\right)'(0) = 0, \tag{29}$$

it has the form

$$q^{(1)} = \epsilon \frac{2^{1/2}}{12} \frac{\operatorname{sh}^3\eta}{\operatorname{ch}^2\eta} + h \frac{2^{1/2}}{4} \left(\frac{3}{\operatorname{ch}\eta} - \operatorname{ch}\eta - 3\eta \frac{\operatorname{sh}\eta}{\operatorname{ch}^2\eta}\right). \tag{30}$$

The particle crosses the separatrix for the variable η value equal to η^*, such that $q^{(1)}(\eta^*) = 0$ and $\left(q^{(1)}\right)'(\eta^*) = 0$. The crossing takes place if $|h| < \epsilon/3$. This gives an estimate for the width of non-adiabatic region.

We estimate the adiabatic invariant value as a function of time which is presented as an area enclosed by the trajectory,and using the obtained solutions. The area enclosed by a perturbed trajectory between the points η_+ and η_-, for which $q' = \left(q^{(0)}\right)' + \left(q^{(1)}\right)'$ is equal to 0 divided by π, can be represented as

$$I = \frac{1}{\pi}\int p dq = I^{(0)}(\epsilon t) + I^{(1)}, \tag{31}$$

where $I^{(0)}$ stands for the area enclosed by the separatrix and $I^{(1)}$ is due to the perturbation $q^{(1)}$.

As a function of η, $I^{(0)}$ is constant, i.e. the separatrix is immobile in terms of η. But $I^{(0)}$ is dependent on t:

$$I^{(0)} \cong \frac{1}{\pi}\left(\int_{-\infty}^{+\infty}\left(\left(q^{(0)}\right)'\right)^2 dt + \frac{\epsilon}{2}\int_{-\infty}^{+\infty} t\left(\left(q^{(0)}\right)'\right)^2 dt\right). \tag{32}$$

For $I^{(1)}$ we have:

$$I^{(1)} = \frac{1}{\pi}\int (p^{(0)}dq^{(1)} + p^{(1)}dq^{(0)}) = \frac{2}{\pi}\int_{\eta-}^{\eta+}\left(q^{(1)}\right)'\left(q^{(0)}\right)' d\eta =$$

$$= \frac{1}{\pi}\left\{\frac{\epsilon}{6}\left(\frac{1}{\mathrm{ch}^4\eta} - 7\frac{1}{\mathrm{ch}^2\eta} - 10\ln(\mathrm{ch}\,\eta)\right) + h\left(\eta - \mathrm{th}\,\eta + \frac{3}{2}\mathrm{th}^3\eta\right)\right\}\Bigg|_{\eta_-}^{\eta_+}, \tag{33}$$

where

$$\eta_\pm = \pm\frac{1}{2}\ln\left(\frac{16}{\frac{\epsilon}{3} \pm h}\right). \tag{34}$$

The solution has the following asymptotes as η tends to $\pm\infty$

$$q = 2^{3/2}\exp(\eta) - \frac{1}{2^{5/2}}\left(h + \frac{\epsilon}{3}\right)\exp(-\eta),\ \eta \to -\infty,$$
$$q = 2^{3/2}\exp(-\eta) + \frac{1}{2^{5/2}}\left(h - \frac{\epsilon}{3}\right)\exp(\eta),\ \eta \to +\infty. \tag{35}$$

Matching of the solution asymptotes for each step of motion near the separatrix gives the changes of the energy and dimensionless time interval $\Delta\eta$ during which this step is accomplished

$$h_n = h_0 - \frac{2}{3}\epsilon n,$$
$$\Delta\eta = \ln\left(\frac{16}{h_n - \frac{\epsilon}{3}}\right). \tag{36}$$

For the energy magnitudes $h = \pm\epsilon/3$ the duration of one step becomes equal to infinity, as it follows from Eq.(36). From the dependences given by Eq.(35) we see that the trajectories asymptotically approach the coordinate origin on a phase plane, that is on the coming into the saddle point vicinity and outcoming from the saddle point vicinity the separatrix branches. These branches appear as a result of a perturbation and are described by Eq.(26) for $h_\pm = -\epsilon M_\pm$. In this case $\epsilon M_\pm = \mp\epsilon/3$.

Summation over n- steps leads to the following value of the invariant in the n-th step

$$I_n = I_0 + \frac{1}{\pi}\left[h_n \ln\frac{16}{(h_n^2 - \frac{\epsilon^2}{9})^{1/2}} + \frac{5}{6}\epsilon\ln\left(\frac{h_n + \frac{\epsilon}{3}}{|h_n| - \frac{\epsilon}{3}}\right) + h_n + \frac{1}{3}\epsilon\sum_{k=-n}^{k=+n}\ln\left(\frac{16}{h_k - \frac{\epsilon}{3}}\right)\right]. \tag{37}$$

Here $I_0 = 8/\pi$ is the area enclosed by the separatrix at $t = 0$.

4 Change of adiabatic invariant during the separatrix birth

Let us consider the particle motion near the critical point where the separatrix birth occurs. The time dependent Hamiltonian has a form

$$\mathcal{H}(p, q, \epsilon t) = \frac{p^2}{2} - \epsilon t\frac{q^2}{2} + \frac{q^3}{3} + \frac{q^4}{4} = h. \tag{38}$$

The separatrix birth ($h = 0$) takes place for $\epsilon t = 0$. If $t < 0$ the frozen time dependence of the Hamiltonian corresponds to one region in a phase plane containing closed trajectories, for $t > 0$ we have two regions with particle trajectories with $h < 0$. The potential energy has one minimum for $t < 0$ and two minima for $t > 0$ (see Fig.1).

The equation of motion,

$$\ddot{q} - \epsilon t q + q^2 + q^3 = 0, \tag{39}$$

in zeroth order in ϵ has a solution corresponding to the motion on the separatrix when $h = 0$:

$$q^{(0)} = -\frac{6}{t^2 + \frac{9}{2}}, \tag{40}$$

$$\dot{q}^{(0)} = \frac{12t}{(t^2 + \frac{9}{2})^2}. \tag{41}$$

Near a critical point in the phase plane $p = 0$, $q = 0$, the trajectory has the shape of a cusp with $p \propto q^{3/2}$.

In the first order in ϵ from Eq. (39) we have for $q^{(1)}(t)$ the equation

$$\ddot{q}^{(1)} + 2q^{(0)}q^{(1)} + 3\left(q^{(0)}\right)^2 q^{(1)} = \epsilon t q^{(0)} \tag{42}$$

with the solution

$$q^{(1)} = \dot{q}^{(0)}\left\{\int^t \frac{d\bar{t}}{(\dot{q}^{(0)})^2}\left[\frac{4}{9}h + \epsilon\int^{\bar{t}} \bar{\bar{t}}\dot{q}^{(0)}q^{(0)}d\bar{\bar{t}}\right]\right\}. \tag{43}$$

When $t \to \pm\infty$ the asymptotical dependence of zeroth order solution is

$$q^{(0)} \propto t^{-2},\ \dot{q}^{(0)} \propto t^3. \tag{44}$$

As a result we have

$$q = q^{(0)} + q^{(1)} \approx -\frac{6}{t^2} + \frac{1}{84}\left(h \mp \frac{\pi\epsilon}{3\sqrt{2}}\right)t^4. \tag{45}$$

The trajectory crosses the ordinate axis in a phase plane at the time values t_+ and t_- respectively. These time values are found from the requirement the solution to vanish ($q(t_\pm) = 0$). From Eq.45 we obtain

$$t_\pm = \pm\left[\frac{504}{h \mp \frac{\pi\epsilon}{3\sqrt{2}}}\right]^{1/6}. \tag{46}$$

For $h \to \pm\pi\epsilon/3\sqrt{2}$ these values tend to infinity that corresponds to the motion along the incoming and outcoming the critical point separatrix branches. As we have noted above under the action of perturbations the separatrix splitting occurs into two branches.

Substitution the expressions (43)–(46) into the expression for the change of adiabatic invariant during one step of motion,

$$\Delta I = \frac{2}{\pi}\int_{t_-}^{t_+} \dot{q}^{(1)}\dot{q}^{(0)}dt, \tag{47}$$

leads to the estimation

$$\Delta I = \frac{8}{7\pi}(504)^{1/6}\left(\left(h - \frac{\pi\epsilon}{3\sqrt{2}}\right)^{5/6} - \left(h + \frac{\pi\epsilon}{3\sqrt{2}}\right)^{5/6}\right). \tag{48}$$

This formula is in agreement with the value estimated by formula (10) for $n = 3$.

In Fig.2–3 the results of the numerical integration of the equation of motion corresponding to the Hamiltonian

$$\mathcal{H}(p, q, \epsilon t) = \frac{p^2}{2} - \alpha\,\mathrm{th}\left(\frac{t}{\tau}\right)\frac{q^2}{2} + \frac{q^3}{3} + \frac{q^4}{4} = h \tag{49}$$

are presented. Here the function $\alpha\,\mathrm{th}(t/\tau)$ describes the change with time of the potential energy shape. When t tends to $-\infty$ the potential energy has only one minimum, while t tends to $+\infty$ it has two minima (see Fig.1). For $t \approx 0$ the Hamiltonian is equivalent to that given by Eq.(38) with $\epsilon = \alpha/\tau$. In this calculations $\alpha = 1$, $\tau = 4$.

In Fig.2 (a–e) we choose the phase plane for the Hamiltonian given by Eq. (49) with frozen time dependence for the time: -16, 0, 2, 4, 8 , 80.

In Fig.3 it is shown the time evolution of those particles which for $t \to -\infty$ uniformly fill the energy level $h_0 = 0.1$. We can see that the time of the separatrix crossing and the energy gain of particle depend on the initial position on the line of the fixed energy.

The considered behaviour corresponds to a slow change of the system parameters with time. To compare it with non-adiabatic case when the perturbation is switched on sharply we show in Fig.4 the phase plane for a sharp change of the potential energy. The Hamiltonian is chosen to be

$$\mathcal{H}(p, q, \epsilon t) = \frac{p^2}{2} - \mathrm{sign}\,(t)\frac{q^2}{2} + \frac{q^3}{3} + \frac{q^4}{4}. \tag{50}$$

In Fig.4 the same dependences as for the adiabatic change of the potential energy are presented. In this case the spreading in energy is smaller. The separation of particles into two groups with positive and negative energy that fill the region near the separatrix is seen. The time evolution of individual particle motion according to the Hamiltonian given by Eq. (50), is shown in Fig.5.

5 The distribution of adiabatic invariant change

In Fig.2 we see that particles with time fill the region of a finite width near the separatrix. Their distribution in the phase plane is quite non uniform. The question what is a form of

distribution on adiabatic invariants is of much interest. We analyze the form of adiabatic invariant distribution depending on the type of a singular point, if it is a saddle or a cusp type. The distribution function $f(I)$ of particles crossing the separatrix has been calculated for a cusp and for a saddle point, respectively. We define the distribution function as a number of particles with adiabatic invariant equal I in unit interval ΔI:

$$f(I) = \frac{dN(I)}{dI}. \tag{51}$$

To find the form of a distribution function near a cusp the Caushy problem for the equation

$$\ddot{q} - \alpha(t)q + q^2 + q^3 = 0 \tag{52}$$

was solved, where $\alpha(t) = \alpha_0\,\mathrm{th}(t/\tau)$, with parameters: $\tau = 4$, $\alpha_0 = 1$. At the initial time the particles are placed along the curve with constant energy $h_0 = 0.05$. If $t < 0$ there is no any separatrix on the phase plane, when $t = 0$ it appears. At the time $t = 0$ the singular point has a cusp form, and for $t \to +\infty$ there are three regions on the phase plane similar to those discussed above. The value h_0 is choosen to have the regime when main fraction of particle trajectories crosses the separatrix around the time of the separatrix birth.

The distribution function is shown in Fig.6a. We see that it can be approximated by power dependence $f(I) = KI^{-\gamma}$.

In case of a saddle point similar to the performed in previous section the integration of the equation of motion has been done:

$$\ddot{q} - (1 + \alpha(t))(q - q^3) = 0\,, \tag{53}$$

where $\alpha(t) = \alpha_0\,\mathrm{th}(t/\tau)$, with parameters: $\tau = 4$, $\alpha_0 = 0.5$, $h_0 = 0.05$.

The distribution function for a saddle point is shown in Fig.6b in a logarithmic scale. We see that the distribition function has an exponential dependence that is $f(I) \propto \exp(-I^\kappa)$.

To elucidate the reason why the distribution function formed in the course of the crossing the separatrix with a saddle point differs from the distribution formed when trajectories cross the separatrix with a casp we discuss the regimes of crossing the separatrix for h close to h_c in these two cases. We suppose the parameter $\alpha(t)$ to be depended on time as $\alpha(t) \approx t$ if $t \gg 1/\epsilon$. We take into account the difference of the particle initial coordinates from the values calculated for the separatrix that location corresponds to dependence $\alpha(t)$ on time as a parameter. Let us take into consideration that the time that particle spends near the separatrix, t_0, is very large but finite.

Near the saddle point [6] we use the following approximation of the Hamiltonian

$$\mathcal{H} = \frac{p^2}{2} - t\left(\frac{q^2}{2} - \frac{q^4}{4}\right). \tag{54}$$

We evaluate the integral $I(t) = \int p dq$ on the separatrix as $I(t) \approx p_{max}q_{max}$, where p_{max} and q_{max} are maximal values of p and q on the phase plane. We seek for p_{max} and q_{max}. Since for $p = 0$ the value q_{max} equals $q_{max} = 2^{1/2}$, while $p_{max} = t^{1/2}/2^{1/2}$. Hence for particles with the initial coordinates on the separatrix for which $h = h_c$ the magnitude of adiabatic invariant is increased as

$$I \propto p_{max}q_{max} \approx t^{1/2}. \tag{55}$$

To find the time the particle spends near the saddle point in the limit $\epsilon t \gg 1$ we use the approximation given by Eq. (54) for the Hamiltonian. Keeping the square term in (54) in the limit $\epsilon t \gg 1$, we use the WKB approximation to solve the equation $\ddot{q} - tq = 0$. This solution is

$$q(t) \approx \frac{q(0)\exp(\pm\frac{2}{3}t^{3/2})}{t^{1/4}}, \tag{56}$$

where $q(0)$ is the initial coordinate of particle. Hence we have the estimate of time the particle spends near the separatrix:

$$t = \left(\frac{3}{2}\ln\frac{t^{1/4}}{q(0)}\right)^{2/3}. \tag{57}$$

Taking into account $I \propto t^{1/2}$ we find the magnitide of the adiabatic invariant of particle with the initial coordinate $q(0)$

$$I = \left(\frac{3}{2}\ln\frac{I^{1/2}}{q(0)}\right)^{1/3}, \tag{58}$$

and inverting this dependence we obtain

$$q(0) = I^{1/2}\exp\left(-\frac{2}{3}I^3\right). \tag{59}$$

The number of particles with the initial coordinate $q(0)$ in the interval $dq(0)$ for initially uniform particle distribution is equal to $dN = \mathrm{const}\, dq(0)$. Due to the conservation of the number of particles we have $f(I)dI = \mathrm{const}\, dq(0)$, that is $f(I) \propto dq(0)/dI$. Using the relationship between I and $q(0)$ described by Eq. (58), we have

$$f(I) \propto \frac{dq(0)}{dI} = \exp\left(-I^3\right). \tag{60}$$

This expression is in qualitative agreement with the dependence obtained before (see Fig.6b) by numerical integration of the Eq.53.

When the separatrix has a cusp singular point we analyze the trajectory behaviour using Eq.(52) in the neighbourhood of the solution $q^{(0)} = 0$, $p^{(0)} = 0$, that is near the singular point. In the first order in the perturbation amplitude we have the equation $\ddot{q}^{(1)} = 0$. Its solution is

$$q^{(1)}(t) = q^{(1)}(0) + \dot{q}^{(1)}(0)t. \tag{61}$$

From Eq.(61) it follows the estimation of characteristic time during which the particle stays near the unstable equilibrum, $q = 0, p = 0$. The time value is inversely proportional to $p^{(1)}(0) \equiv \dot{q}^{(1)}(0)$ and equals

$$t_0 \approx \frac{1}{p^{(1)}(0)}. \tag{62}$$

Now we can estimate the value of the integral $I = \int p dq$ as $p_{max}q_{max}$. Since $q_{max} \approx t^{1/2}$, and $p_{max} \approx t^{1/2}q_{max} = t$, the change of adiabatic invariant value for particles initially located on the separatrix is of order of

$$I \propto pq \approx t^{3/2}. \tag{63}$$

Similar to that has been obtained we find for the saddle point the relationship between the initial momentum value and the adiabatic invariant gain,

$$p(0) \approx I^{-2/3}, \tag{64}$$

and an expression for the distribution function

$$f(I) = \frac{dp(0)}{dI} \propto I^{-5/3}. \tag{65}$$

This dependence is in qualitative accordance with the curve shown in Fig.6a.

It is necessary to emphasize that general conclusion about exponential and power dependence of distribution function is in accordance with the results of numerical integration of the equation of motion. However the value of κ in exponential curve and the value of γ are sensitive to the specific dependence of $\alpha(t)$.

We have calculated $f_s(I) \propto \exp(-I^3)$ and $f_c(I) \propto I^{-5/3}$ for a linear dependence of $\alpha(t)$ on time. For another dependence $\alpha(t)$ the values of κ and γ in distribution functions $f \propto \exp(-I^\kappa)$ and $f \propto I^{-\gamma}$ may be another, respectively. In particular, the dependence $\alpha(t)$ in Eq.52 is not strictly linear, that is why we have the difference between the obtained numerical dependences and the theoretical dependences.

6 Condition of stochastic regime appearance in the course of the separatrix birth

Let us discuss the problem of appearance of a stochastic regime of the motion near the separatrix given by equations (40)-(41). Following to the approach developed in Ref. [19] (see the detailed discussion in Ref. [20]), we calculated the Melnikov distance which is the measure of the separatrix splitting

$$D(t, t_0) = \mathbf{N} \cdot \mathbf{d}. \tag{66}$$

Here $\mathbf{N}$ is the vector of a normal to the unperturbed separatrix and $\mathbf{d}$ is a distance between perturbed trajectories which appear under the action of perturbations (see Eq.(43) for $\epsilon M_\pm = -h_\pm$). The equation of motion corresponding to the considered structure of the separatrix is the following:

$$\ddot{q} + q^2 + q^3 = -\epsilon\delta\dot{q} + \epsilon\gamma\cos\omega t, \tag{67}$$

This equation describes the periodic motion in the vicinity of the separatrix (see Eq.(39) for $t = 0$), perturbed by small damping $\epsilon\delta$ and the action of the periodic force with the amplitude $\epsilon\gamma$. Calculation of the vectors $\mathbf{N}$ and $\mathbf{d}$ gives for the value $D(t, t_0)$ in correspondence with Eq.(43)

$$D(t, t_0) = -12\gamma\sin\omega t_0 \int_{-\infty}^{+\infty} d\tau \frac{\tau}{\left(\tau^2 + \frac{9}{2}\right)^2}\sin\omega\tau +$$
$$+144\delta \int_{-\infty}^{+\infty} \frac{\tau^2}{\left(\tau^2 + \frac{9}{2}\right)^4} d\tau = -12\pi\gamma\sin\omega t_0 \exp\left(-\frac{3}{2^{1/2}}\omega\right) + \frac{2^{5/2}}{27}\pi\delta. \tag{68}$$

The stochastic motion near the separatrix occurs when the perturbed trajectories are crossing, i.e. when the function D changes the sign. From (68) it follows that the condition of this is

$$\delta < \delta_c = \frac{81}{2^{1/2}} \gamma \exp\left(-\frac{3}{2^{1/2}}\omega\right). \tag{69}$$

The solution of a similar problem for a saddle point is reported in [20].

In Fig.7 the results of integration of the equation of motion (67) are shown for $\epsilon = 1$, $\delta = 0.005$, $\gamma = 0.1$, $\omega = 3$. The integration has been performed for the group of particles uniformly distributed on the curve corresponding to the constant energy value $h_0 = 0.01$ at $t = 0$. The integration was done in positive and negative direction for change of time. In this figure the appearance of homoclinic structure and stochastic motion is shown near the separatrix for δ, satisfying the condition (69). Similar calculations for $\delta = 0.1$ ($\delta > \delta_c$) are shown in Fig.8. In this case the separatrics do not intersect and the first branch (a) includes the second one (k). The typical trajectories of individual particles are shown in Fig.9 for both limiting cases.

7 Accuracy of the conservation of the longitudinal adiabatic invariant of particles near a singular magnetic field line

According to the terminology introduced by S.I.Syrovatskyi [21] a singular line of the magnetic field is a field line the electric field along this line is non-zero. The simplest and the common configuration describing a singular field line corresponds to magnetic and electric field of the form

$$\mathbf{B} = hx\mathbf{e}_x - hy\mathbf{e}_y + B_\parallel \mathbf{e}_z, \tag{70}$$

$$\mathbf{E} = E_\parallel \mathbf{e}_z, \tag{71}$$

where $\mathbf{e}_i$ are unit vectors along axes x, y and z, respectively. In plane (x, y) the magnetic field $\mathbf{B}_\perp = hx\mathbf{e}_x - hy\mathbf{e}_y$ has a hyperbolic (saddle) singular point $(x = 0, y = 0)$, and has two separatrics $x = 0$ and $y = 0$. A singular field line is coincident with z-axis where the magnetic field is equal to $B_\parallel$, and an electric field is parallel to z-axis.

Analyzing the particle motion it is convenient to subdivide the region of their motion into two regions. The first region is near a singular field line and the second region is supposed to be far. In the first region particles are accelerated by the electric field non-adiabatically from the singular line during the finite time. Due to instability of trajectories particles leave this region and fall into the region of drift motion. Depending on the initial conditions the particle trajectory does cross or does not cross the first region.

The process of nonadiabatic acceleration of charged particles near a singular field line is investigated in Ref. [22]. In this section we analyze the particle motion far from the nonadiabatic subregion.

Assume the smallness of the transverse component of magnetic field, $|\mathbf{B}_\perp| = h(x^2 + y^2)^{1/2} \equiv h\rho$, comparing to the longitudinal one, $B_\parallel$, and the conservation of the first (transverse) adiabatic invariant, $\mu = mv_\perp^2/2B = \text{const}$. In this approximation the strength of

magnetic field depends on coordinates as

$$B \approx B_\parallel + \frac{h^2\rho^2}{B_\parallel}. \tag{72}$$

We write the equations of motion in drift approximation [1] as

$$\dot{x} = v_\parallel \frac{h}{B_\parallel} x - c\frac{Eh}{B_\parallel^2} y, \tag{73}$$

$$\dot{y} = -v_\parallel \frac{h}{B_\parallel} y + c\frac{Eh}{B_\parallel^2} x, \tag{74}$$

$$\dot{z} \approx v_\parallel = \pm\left(\frac{2\mathcal{E} + 2eEz - \mu B_\parallel}{m} - \frac{\mu h^2}{2B_\parallel m}(x^2 + y^2)\right)^{1/2}, \tag{75}$$

where $\mathcal{E}$ is the initial particle energy. Here we neglected the gradient and centrifugal drift comparing with electrical drift. That assumes smallness of particle velocity ($v^2 < eEB_\parallel/hm$). We change the coordinate $z = z - z_0$, where $z_0 = (2\mathcal{E} - \mu B_\parallel)/2eE$.

In dimensionless units where the velocity is measured in $(8e^2E^2B_\parallel/\mu h^2)^{1/2}$, time in $(8e^2E^2B_\parallel^3/\mu m h^4)^{1/2}$, the plane scale is equal to $4eEB_\parallel/\mu h^2$, the system of equations (73)-(75) becomes:

$$\dot{x} = v_\parallel x - \epsilon y, \tag{76}$$

$$\dot{y} = -v_\parallel y + \epsilon x, \tag{77}$$

$$\frac{\dot{z}^2}{2} = z - \frac{x^2 + y^2}{2}. \tag{78}$$

Here the dimensionless parameter,

$$\epsilon = \left(\frac{8e^2E^4c^2}{\mu m h^2 B_\parallel}\right)^{1/2} \tag{79}$$

is assumed to be small. In the zeroth order on this parameter the solution of equations (76) and (77) can be presented as

$$x = x_0 \exp(z), \tag{80}$$

$$y = y_0 \exp(-z). \tag{81}$$

We introduce new variables

$$q = \exp(-z), \tag{82}$$

$$p = \dot{z}\exp(-z). \tag{83}$$

From Eq. (78) we find

$$p^2 = -q^2 \ln q - \frac{x_0^2}{2} - \frac{y_0^2 q^2}{2}. \tag{84}$$

That is the motion in the zeroth order in ϵ is described by the Hamiltonian

$$\mathcal{H} = \mathcal{H}_0(p, q), \tag{85}$$

$$\mathcal{H}_0(p, q) = \frac{p^2}{2} + q^2 \ln q + \frac{x_0^2}{2} + \frac{y_0^2 q^2}{2}. \tag{86}$$

It is seen that the unperturbed Hamiltonian $\mathcal{H}_0$ describes the dynamic system. This system has a separatrix with a saddle point at the origin of coordinates on a phase plane (p, q). One can see that the motion along the separatrix of magnetic field described by Eq. (70) corresponds to the motion along the separatrix in the plane (p, q).

Now we estimate the change of longitudinal adiabatic invariant

$$J_{\parallel} = \frac{m}{2\pi} \oint v_{\parallel} dl, \tag{87}$$

where the integration is performed along the field magnetic line in the course of trajectory crossing the separatrix. We use the solution of system (76)-(78), assuming smallness of parameter ϵ. The separatrix corresponds to $x_0 = 0$ and $\mathcal{H}_0 = 0$. When $t \to \infty$ near the saddle point we have

$$z^{(0)}(t) \approx \frac{t^2}{2}, \tag{88}$$

$$q^{(0)}(t) \approx \exp\left(-\frac{t^2}{2}\right), \tag{89}$$

$$p^{(0)}(t) \approx t \exp\left(-\frac{t^2}{2}\right). \tag{90}$$

In higher orders in ϵ we seek the solution near the separatrix in the form of expansion

$$x = \epsilon x^{(1)} + \dots, \tag{91}$$

$$y = y^{(0)} + \epsilon^2 y^{(2)} + \dots, \tag{92}$$

$$z = z^{(0)} + \epsilon^2 z^{(2)} + \dots. \tag{93}$$

Near the saddle point we have

$$\dot{x}^{(1)} - t x^{(1)} = -y^{(0)}, \tag{94}$$

$$\dot{y}^{(2)} + t y^{(2)} = -y^{(0)} \dot{z}^{(2)} - x^{(1)}, \tag{95}$$

$$\dot{z}^{(2)} - \frac{z^{(2)}}{t} = -(x^{(1)})^2 - 2 y^{(0)} y^{(2)}. \tag{96}$$

Taking into consideration main terms when $t \to \infty$ we obtain

$$x^{(1)} \approx -\frac{\pi^{1/2}}{2} y_0 \exp\left(\frac{t^2}{2}\right), \tag{97}$$

$$y^{(2)} \approx -\frac{\pi}{4} y_0^2 \exp\left(t^2\right), \tag{98}$$

$$z^{(2)} \approx -\frac{\pi}{4} y_0^2 t \int_0^t \frac{\exp(\tilde{t}^2) d\tilde{t}}{\tilde{t}}. \tag{99}$$

That is

$$y \approx y_0 \exp\left(-\frac{t^2}{2}\right) - \epsilon^2 \frac{\pi}{4} y_0^2 \exp\left(t^2\right). \tag{100}$$

In correspondence with used above approach we find the time when the trajectory crosses the x-axis. It is equal to

$$t_+ = \left(\frac{2}{3} \ln \frac{4}{\pi \epsilon^2 y_0}\right)^{1/2}. \tag{101}$$

Here y_0 is the phase of the separatrix crossing. On the separatrix the change of longitudinal adiabatic invariant is equal to

$$\Delta J_{\parallel} \approx \frac{2\epsilon^2}{\pi} \int^{t_+} \dot{y}^{(0)} \dot{y}^{(2)} dt \approx \frac{y_0^{8/3} \epsilon^{4/3}}{2^{4/3}} \left(\frac{2}{3} \ln \frac{4}{\pi \epsilon^2 y_0}\right)^{3/2}. \tag{102}$$

As the change of particle energy is proportional to the increase of coordinate z, as $\Delta\mathcal{E} = eE\Delta z$, from Eq.(102) it follows the relation between $\Delta\mathcal{E}$ and the initial coordinate y_0. Assuming a uniform particle distribution in values y_0 we derive that the energy distribution $f(\mathcal{E}) = dy_0/d\mathcal{E}$ has the form

$$f(\mathcal{E}) = \frac{4}{3\epsilon^2 \pi} \exp\left(-\frac{4\mathcal{E}}{3}\right). \tag{103}$$

It means that the energy spectrum is exponential with the characteristic energy in dimensional unities equal $3e^2 E^2 B_{\parallel}/\mu h^2$. It should be noted that the performed above calculations have been accomplished for a given magnitude of a transverse adiabatic invariant μ. For the further estimations it is necessary to execute the everaging on the possible values of this invariant.

Conclusion

Summing the results obtained in the paper it should be noted the following.

Accuracy of the conservation of the adiabatic invariant value in the course of the crossing the separatrix near the saddle point is higher comparing with the case of the separatrix birth. In the first case the characteristic value of the change of adiabatic invariant is of order $\epsilon \ln \epsilon$, while in the second case it is of order $\epsilon^{5/6}$.

The investigation of the condition of the appearance of stochastic layer near the separatrix in the regime of its birth demonstrated appearance of homoclinic structures near the singular point. When a singular point has the cusp form the layer width of the stochastic motion is larger than the width of a stochastic layer near the saddle point.

The calculation of the change of adiabatic invariant for the trajectories with different initial conditions made it possible to estimate the distribution function type for saddle and cusp. In the case of a saddle point the distribution function is described by the exponential dependence and in the case of a cusp point it has a power dependence.

The results of these investigations may be applied to different problems. In particular, we used the developed approach to the problem of acceleration of charged particles near singular lines of magnetic field. This problem is of interest in connnection with acceleration of charged particles during magnetic reconnection. We considered the limit when the first adiabatic invariant (transverse) is conserved and the second one (longitudinal) undergoes changes. The considered estimation of the value of longitudinal adiabatic invariant as the function of the initial position of particle made it possible to calculate the energy spectrum of fast particles which is proved to be exponential.

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FIGURE CAPTIONS

Figure 1: Potential $U = -\mathrm{th}(\epsilon t)\, q^2/2 + q^3/3 + q^4/4$ and frozen trajectories (p, q) with $h = 0.1,\ 0.05,\ 0.0,\ -0.05,\ -0.1,\ -0.15,\ -0.3,\ -0.6,\ -0.9$ for the time a). $t = -\infty$, b). $t = 0$, c). $t = +\infty$.

Figure 2: Phase plane (p, q) for the potential $U = -\alpha\, \mathrm{th}(t/\tau)\, q^2/2 + q^3/3 + q^4/4$, where $\alpha = 1$, $\tau = 4$, for the time a). $t = -16$, b). $t = 0$, c). $t = 2$, d). $t = 4$, e). $t = 8$, f). $t = 80$.

Figure 3: Typical trajectories of particles corresponding to Fig.2 trapped by the left (a,b) and right (c) lobes in the phase plane (p, q).

Figure 4: Phase plane (p, q) for the potential $U = -\mathbf{sign}(t)\, q^2/2 + q^3/3 + q^4/4$, for the time a). $t = -\infty$, b). $t = 2$, c). $t = 4$, d). $t = 6$, e). $t = 10$, f). $t = 80$.

Figure 5: Corresponding to Fig.4 typical trajectories of particles in the phase plane (p, q), left out of the separatrix (a) and trapped by the left (b) and right (c) lobe.

Figure 6: Distribution functions of particles after the separatrix crossing (a) for a cusp, trapped by the left (1) and right (2) lobes and for a saddle. Comparison with model functions: (a) $N(I) = KI^{-\gamma}$, $\gamma = 1$ (3) and (b) $N(I) = exp(-I^{\kappa})$ (2), $\kappa = 4$, respectively.

Figure 7: Phase plane (p, q) for the equation of motion Eq.67 for the parameters $\epsilon = 1$, $\delta = 0.005$, $\gamma = 0.1$, $\omega = 3$. The integration has been performed starting from the time $t = 0$ for $h = 0.01$ (e) in positive: f). $t = 4$, g). $t = 8$, h). $t = 12$, i). $t = 16$ and negative time direction: d). $t = -4$, c). $t = -8$, b). $t = -12$, a). $t = -16$.

Figure 8: The resuilts similar to Fig.7 for the same parameters except $\delta = 0.1$, for the time: a). $t = -16$, b). $t = -12$, c). $t = -8$, d). $t = -4$, e). $t = 0$, f). $t = 4$, g). $t = 8$, h). $t = 12$, i). $t = 16$.

Figure 9: Typical trajectories of particles in the phase plane (p, q), corresponding to Fig.7 and Fig.8.

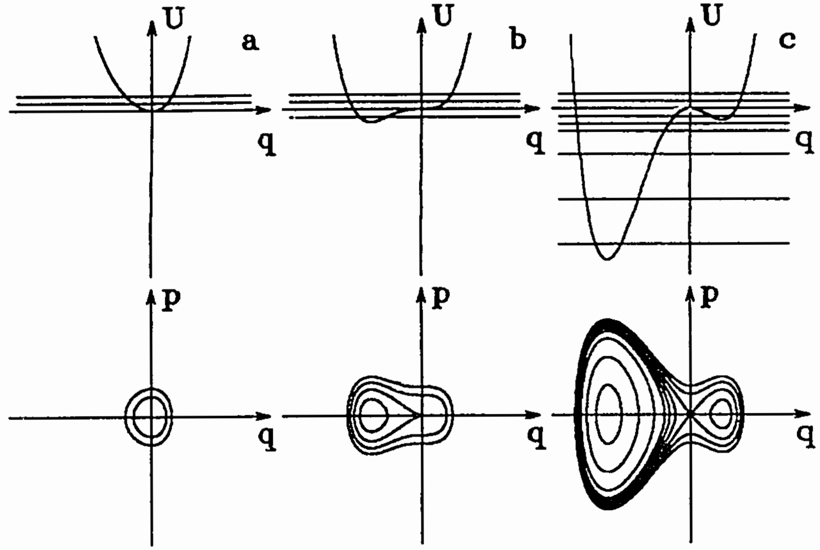


Fig.1

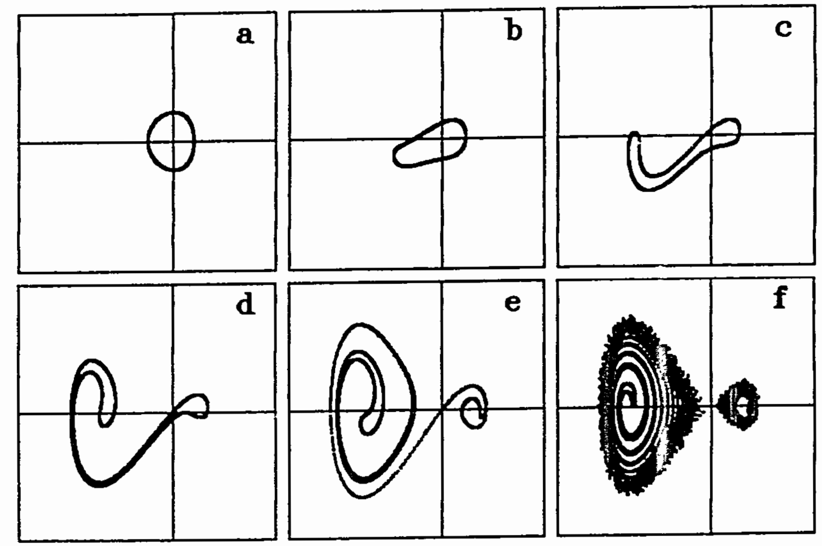


Fig.2

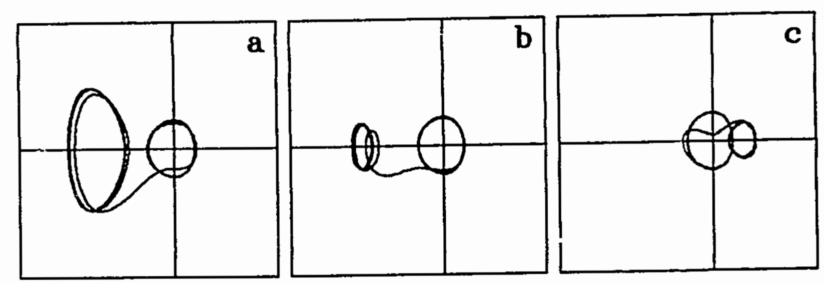
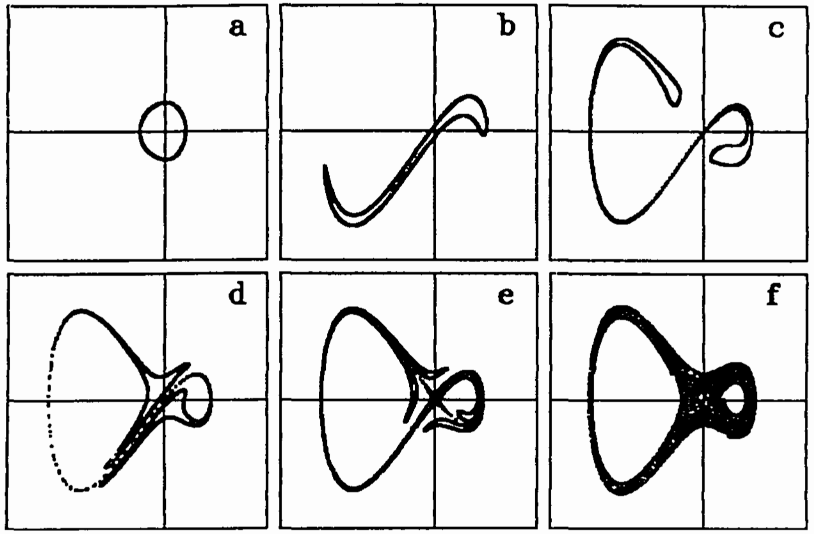


Fig.3



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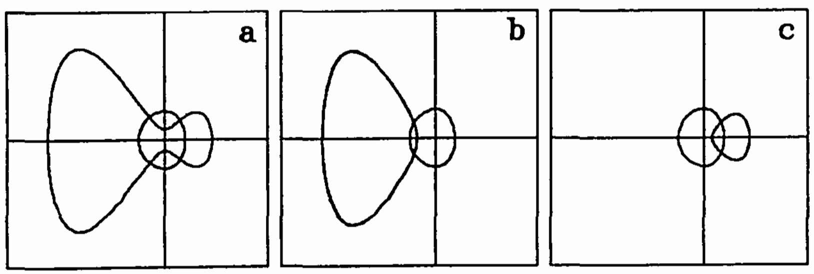

Fig.5

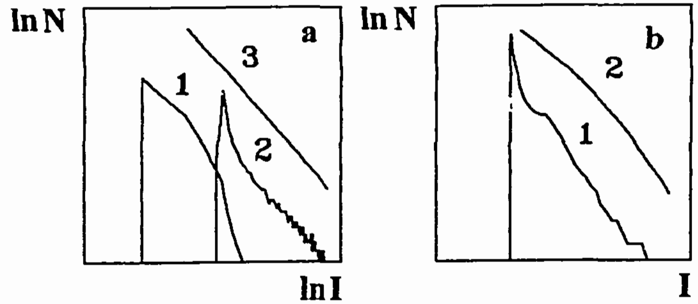

Fig.6

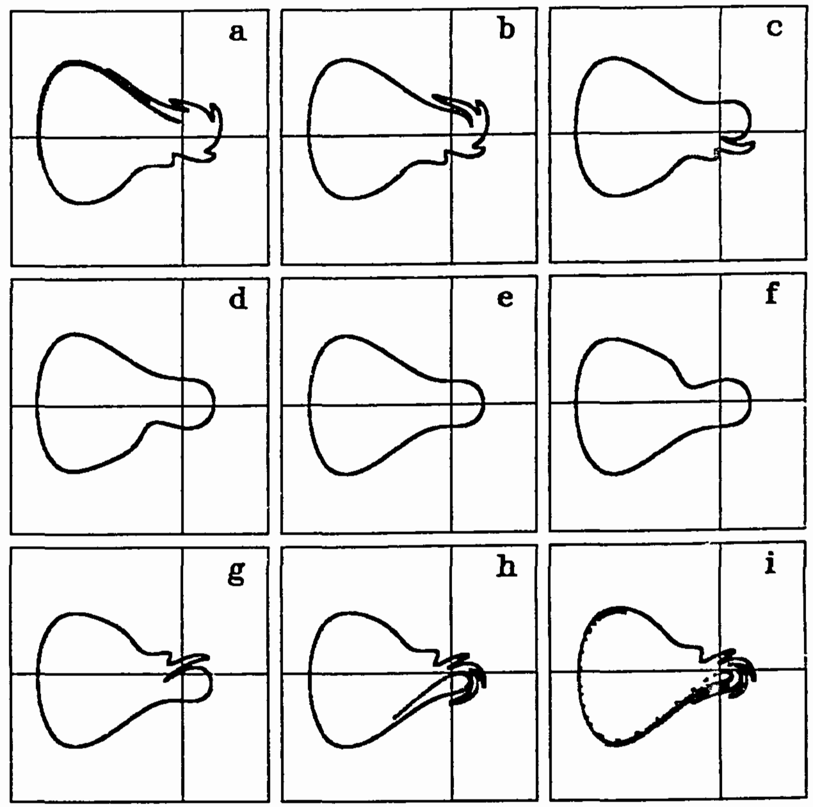

Fig.7

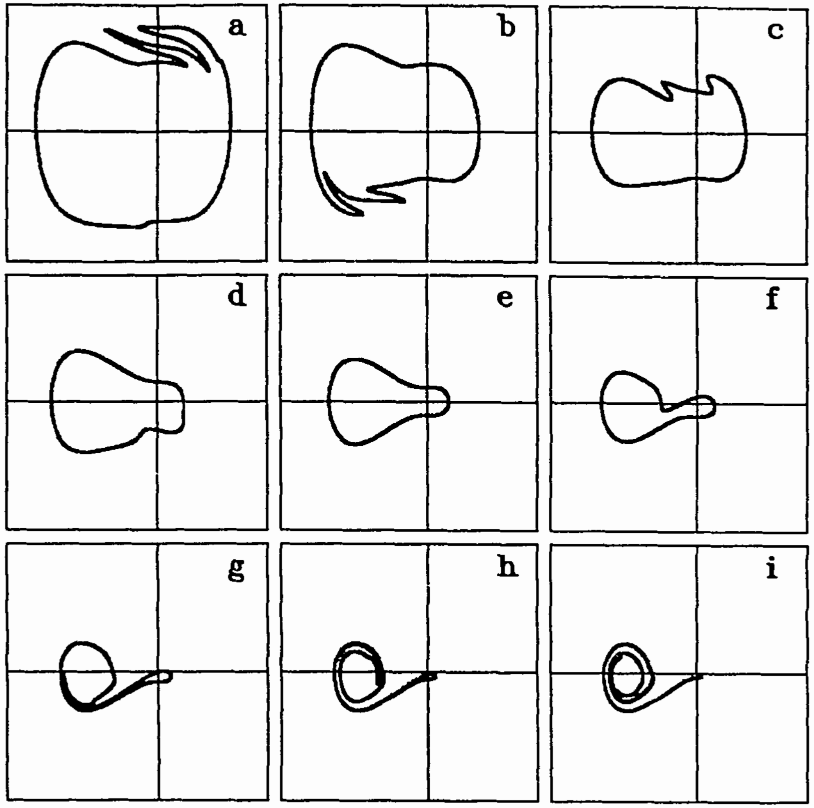

Fig.8

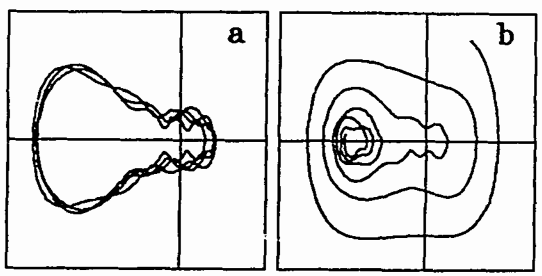

Fig.9

Fig. 6