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REPRESENTATIONS OF LOCALLY SYMMETRIC SPACES

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REPRESENTATIONS OF LOCALLY SYMMETRIC SPACES

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ABSTRACT

Locally symmetric spaces in reference to globally and Hermitian symmetric Riemannian spaces are studied. Some relations between locally and globally symmetric spaces are exhibited. A lucid account of results on relevant spaces, motivated by fundamental problems, are formulated as theorems and propositions.

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1. Introduction

There is a class of Riemannian manifolds, the symmetric spaces, which plays a very important role in Differential Geometry.

A symmetric space is a Riemannian manifold whose curvature tensor is invariant under all parallel translations.

The theory of symmetric spaces was initiated by É. Cartan in 1926, and was vigorously developed and classified completely by him by a reduction to the classification of simple Lie algebras in the late 1920s. In spirit to their definition, symmetric spaces form a special topic in Riemannian geometry.

This theory leads to the remarkable fact that symmetric spaces are locally just the Riemannian manifolds of the form

$$R^n \times \frac{G}{K}$$

where R^n is a Euclidean n-space, G is a semi-simple Lie group which has an involutive automorphism with fixed point set as the (essentially) compact group K, and $\frac{G}{K}$ is treated with a G-invariant Riemannian structure.

The theory of symmetric spaces helps to unify and explain in a general way various phenomena in classical geometries.

The sphere, the hyperbolic space, and the Euclidean space, are the best known examples of symmetric spaces.

On a symmetric space with its well-developed geometry, global function theory becomes particularly interesting. Integration theory, Fourier analysis, and partial differential operators have a place here in a canonical fashion as demanded by geometric invariance.

A general duality for symmetric spaces is looked upon as a special case for the analogy between the spherical geometries and hyperbolic geometries.

Symmetric spaces give considerable insight into an active area of recent research. There is now a large literature on them.

This article gives a rather brief survey of some results on areas of growing interest in symmetric spaces and their applications.

In particular, we present, to a certain extent, a preliminary exposition of locally, globally and Hermitian symmetric Riemannian spaces.

2. Locally symmetric spaces

We begin a study of Riemannian locally symmetric spaces. These are defined as Riemannian manifolds for which the curvature tensor is invariant under all parallel translations.

É. Cartan (1926, 27) set himself the problem of giving a complete classification of these spaces. It was based on the fact that the invariance of the curvature tensor under parallelism is equivalent to the condition that the geodesic symmetry with respect to each point be a local symmetry. This explains the term 'locally symmetric'.

Let a Riemannian manifold M have the curvature tensor R.

Note the equivalent conditions:

(i) the sectional curvature under parallel translation of tangent 2-planes of M is invariant;

(ii) R is parallel;

(iii) a linear isometry $M_x \to M_z$ of tangent spaces sending R_x to R_z extends to an isometry of normal neighbourhoods, and

(iv) the geodesic symmetry $\exp_x(X) \to \exp_x(-X)$ with $x \in M$ is an isometry on a normal neighbourhood of x.

The manifold M satisfying these conditions is called *locally symmetric*. In other words, a Riemannian manifold M for which there exists a normal neighbourhood of each point $p \in M$ containing the geodesic symmetry with respect to p as an isometry is called a *(Riemannian) locally symmetric space.*

The two definitions of locally symmetric spaces are equivalent.

Theorem 2.1 *If the sectional curvature of a Riemannian manifold M is invariant under all parallel translations then M is a Riemannian locally symmetric space.*

Proof. Suppose a Riemannian manifold M has its sectional curvature invariant under all parallel translations.

Let the points $p, q \in M$ be joined by a curve segment σ and let τ be the parallel translation from p to q along σ.

If $X, Y \in M_p$, the tangent space of M at p, then we have

$$g_p(R_p(X,Y)X,Y) = g_q(R_q(\tau X,\tau Y)\tau X,\tau Y),$$
$$g_p(R_p(X,Y)X,Y) = g_q(\tau(R_p(X,Y)X,\tau Y).$$

Let us set (Helgason 1978, p.201) the quadrilinear form B:

$$B(X,Y,Z,T) = g_q(R_q(\tau X,\tau Y)\tau Z,\tau T) - g_q(\tau(R_p(X,Y)Z,\tau T) \qquad (2.1)$$

for $X, Y, Z, T \in M_p$.

We now refer to a Lemma and a Theorem (Helgason 1978, p.68; p.198):

Lemma *Let A be a ring with identity element e such that $6a \neq 0$ for $a \neq 0$ in A. Let E be a module over A. Suppose a mapping B: $E \times E \times E \times E \to A$ is quadrilinear and satisfies the identities*

(a) $B(X,Y,Z,T) = -B(Y,X,Z,T)$

(b) $B(X,Y,Z,T) = -B(X,Y,T,Z)$

(c) $B(X,Y,Z,T) + B(Y,Z,X,T) + B(Z,X,Y,T) = 0$

Then

(d) $B(X,Y,Z,T) = B(Z,T,X,Y)$.

If, in addition to (a), (b), *and* (c), *B satisfies*

(e) *$B(X,Y,X,Y) = 0$ for all $X, Y \in E$, then $B = 0$.*

Theorem *A manifold M is affine locally symmetric if and only if $T = 0$ and $\nabla_z R = 0$ for all $Z \in D^1(M)$.*

Making use of $B = 0$ in (2.1) we find

$$R_q(\tau X,\tau Y)\tau Z = \tau(R_p(X,Y)Z).$$

It follows now

$$R_q = \tau \cdot R_p,$$

showing that $\nabla_z R = 0$ for all $Z \in D^1(M)$, the set of 1-forms vector fields. This illustrates the fact that the curvature tensor R of M is invariant under all parallel translations.

Hence M is a Riemannian locally symmetric space.

QED

The characterization of Riemannian locally symmetric spaces reflects, on using curvature conditions, as follows:

Proposition 2.2 *Let a Riemannian space (M, h) have Riemannian connection ∇ and curvature tensor R. Then (M, h) is Riemannian locally symmetric when and only when $\nabla R = 0$.*

[For a proof of Proposition 2.2 cf. Theorem 2.1].

Theorem 2.3 *If M is a connected locally symmetric space then M is locally homogeneous.*

Proof. By 'locally homogeneous' we mean that given $y, z \in M$ there exists an isometry of a neighbourhood of y onto a neighbourhood of z carrying y to z.

We denote by '$x \sim y$' an isometry of a neighbourhood of x onto a neighbourhood of y carrying x to y.

Clearly, $\sim$ is an equivalence relation. By inspection at the local transvections the equivalence classes are visibly open.

Since M is connected we must have *only one* equivalence class. This asserts that M is locally homogeneous.

QED

We are concerned with a result which imposes a condition on locally symmetric spaces to be symmetric.

A complete simply connected locally symmetric space is symmetric.

3. Globally symmetric spaces

A Riemannian space M whose every geodesic symmetry extends to a globally defined isometry of M is called *globally symmetric space*.

Two-point homogeneous spaces. Symmetric spaces of non-compact type.

Irreducible spaces

Two-point homogeneous spaces were first studied by Busemann in 1942 and by Birkhoff in 1944. Wang (1952) classified them in the compact case and Tits (1955) in the non-compact case.

A *two-point homogeneous space* is a Riemannian manifold M if for any two point pairs x_i, y_i $(i = 1, 2) \in M$ satisfying $d(x_1, y_1) = d(x_2, y_2)$ there is an isometry g of M such that $g(x_1) = x_2$ and $g(y_1) = y_2$.

The *Euclidean spaces*, the *circle* S^1 and the *symmetric spaces of rank* 1 *of the compact type and non-compact type* are two-point homogeneous spaces.

We say that a symmetric space M is of *non-compact type* if its universal Riemannian covering manifold $\widetilde{M}$ is a product of non-compact irreducible spaces.

A Riemannian manifold is called *irreducible* if it is not locally a product of lower dimensional manifolds.

Examples

A non-compact two-point homogeneous space is *globally symmetric.*

A compact semi-simple Lie group is a *Riemannian globally symmetric space* in each two-sided invariant Riemann structure.

Theorem 3.1 *If a Riemannian globally symmetric space M is connected then M is* (i) *complete,* (ii) *locally symmetric, and* (iii) *homogeneous.*

Proof. (i) We first introduce the concept of a geodesic to mean a complete space.

A *geodesic* is a smooth curve $\sigma(t)$ in a differentiable manifold M of dimension n such that $\sigma'(t)$ is parallel along σ.

The geodesic is said to be *complete* if the parameter t has the domain $-\infty < t < \infty$. If geodesics are extended by geodesic symmetries then we call M complete.

(ii) The local symmetry follows from Theorem 2.3.

(iii) Let $x, y \in M$. The completeness affords us to have a geodesic segment σ from x to y. Taking p as the middle point of σ the geodesic symmetry at p sends x to y. Hence

M is homogeneous.

Theorem 3.2 *Suppose that a Riemannian space M is symmetric. Then the fundamental group $\pi_1(M)$ is commutative.*

Proof. Let $f : \widetilde{M} \to M$ be the universal Riemannian covering, and let Γ be the group of deck transformations. Then Γ has Clifford (1873) translations of $\widetilde{M}$. Let $\tilde{x} \in \widetilde{M}$, $x = f(\tilde{x}), \nu \in \Gamma$. A geodesic σ_ν in $\widetilde{M}$ from $\tilde{x}$ to $\nu(\tilde{x})$ is chosen as a minimizing geodesic. Then $f(\sigma_\nu)$ is a minimizing geodesic for the homotopy class of curves which gives rise to the element of $\pi_1(M, x)$ corresponding to ν. Since ν is Clifford the length $L(f(\sigma_\nu))$ is solely dependent on the free homotopy class. Now that we cannot shorten $f(\sigma_\nu)$ by rounding a corner it must be smooth at x. Thus every element of $\pi_1(M, x)$ is represented by a closed geodesic through x. Each geodesic is reversed by the symmetry at x thereby inducing an automorphism $a \to a^{-1}$ of $\pi_1(M, x)$. A group A which has $a \to a^{-1}$ as an automorphism is commutative.

We formulate, in a simplified form,

Proposition 3.3 *A non-compact type Riemannian globally symmetric space is simply connected.*

4. Locally and Globally symmetric spaces

É. Cartan gave a global classification in his extensive paper (Cartan 1927). But the relation between locally and globally symmetric spaces does not seem to be altogether clear from his work.

A globally symmetric space may be constructed from a locally symmetric one.

Mention is made of some relations between locally and globally symmetric Riemannian spaces.

The universal covering manifold of a complete Riemannian locally symmetric space is *globally symmetric.*

A complete, simply connected Riemannian locally symmetric space is Riemannian *globally symmetric.*

5. Hermitian symmetric spaces

The subject of Hermitian symmetric spaces is a very classical one in the domain of Complex Differential Geometry.

Hermitian symmetric spaces were first studied by É. Cartan (1935) who classified them by means of his classification (1926, 27) of Riemannian symmetric spaces.

A Hermitian symmetric space is a complex manifold with Hermitian metric such that a Riemannian manifold consisting of the underlying real differentiable manifold and the real part of the Hermitian metric is symmetric and its symmetries are Hermitian isometries.

A Hermitian symmetric space is a Riemannian symmetric space of even dimension.

Every Hermitian symmetric space is simply connected.

Examples

All simply-connected two-dimensional Riemannian globally symmetric spaces are illustrated as examples of Hermitian symmetric spaces.

Thus *two-dimensional sphere* S^2 which is Riemannian globally symmetric space is Hermitian symmetric.

Example of a space which is not Hermitian

Two-dimensional projective space P^2 is Riemannian globally symmetric space but *not* Hermitian symmetric.

Proposition 5.1 *Let X_0 be an irreducible Hermitian symmetric space. The boundary components of X_0 then have a classical type irreducible Hermitian symmetric spaces.*

Proof. Consider that W_Γ is the subgroup of the Weyl group of G preserving ψ and acting trivially on $\psi - \Gamma$. It induces every permutation of Γ (Boothby and Weiss 1972, p.285). This speaks of the irreducibility of Hermitian symmetric spaces $X_{r,0}$.

It remains to show that X_0 is of classical type. This follows from the classification theorem (Boothby and Weiss 1972, p.292).

Irreducible Hermitian symmetric space

Let X_c and Y_c be Hermitian symmetric spaces of compact type with X_c irreducible, and let $f : X_c \to Y_c$ be a holomorphic mapping.

We can make Y_c irreducible keeping in view that f is not a holomorphic immersion. In such case Y_c are, in general, of greater dimensions than X_c.

It is important to note that if X_c is of rank ≥ 2 and Y_c irreducible then $f : X_c \to Y_c$ is necessarily a totally geodesic isometric embedding provided that Y_c is of smaller dimension than X_c.

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References

Birkhoff, G. 1944. Metric foundations of geometry I, *Trans. Amer. Math. Soc.* **55**: 465-492.

Boothby, W. M. and G. L. Weiss. 1972. *Symmetric Spaces*, Marcel Dekker, Inc., New York.

Busemann, H. 1942. *Metric Methods in Finsler Spaces and in the Foundations of Geometry*, Annals of Mathematics Studies, Princeton.

Cartan, É. 1926; 1927. Sur une classe remarquable d'espaces de Riemann, *Bull. Soc. Math. France*, **54**: 214-264; **55**: 114-134.

Cartan, É. 1927. Sur certaines formes riemanniennes remarquables des géométries a groupe fondamental simple, *Ann. Sci. École Norm. Sup.* **44**: 345-467.

Cartan, É. 1935. Sur les domaines bornes homogènes de l'espace de n variables complexes, *Abh. Math. Sem. Univ. Hamburg*, **11**: 116-162.

Clifford, W. K. 1873. On a surface of zero curvature and finite extent, *Proc. Lond. Math. Soc.* **4**: 381-395.

Helgason, S. 1978. *Differential Geometry, Lie Groups, and Symmetric Spaces*, Academic Press, Inc., Florida.

Tits, J. 1955. Sur certaines classes d'espaces homogènes de groupes de Lie, *Acad. Roy. Belg. Cl. Sci. Mém. Coll.* (3) **29**:

Wang, H.-C. 1952. Two-point homogeneous spaces, *Ann. of Math.* **55**: 177-191.