\textbf{N = 2 Super - $W_3^{(2)}$ Algebra in Superfields}

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\textbf{Abstract}

We present a manifestly $N = 2$ supersymmetric formulation of $N = 2$ super-$W_3^{(2)}$ algebra (its classical version) in terms of the spin 1 unconstrained supercurrent generating a $N = 2$ superconformal subalgebra and the spins 1/2, 2 bosonic and spins 1/2, 2 fermionic constrained supercurrents. We consider a superfield reduction of $N = 2$ super-$W_3^{(2)}$ to $N = 2$ super-$W_3$ and construct a family of evolution equations for which $N = 2$ super-$W_3^{(2)}$ provides the second hamiltonian structure.

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1 Introduction

In recent years a plenty of various superextensions of nonlinear $W$ algebras were constructed and studied from different points of view, both at the classical and quantum levels (see, e.g., [1] and references therein). An interesting class of bosonic $W$ algebras is so called quasisuperconformal algebras which include, besides the bosonic currents with the canonical integer conformal spins, those with half-integer spins [2, 3, 4]. The simplest example of such an algebra is the Polyakov-Bershadsky $W_3^{(2)}$ algebra [5, 6]. It is a bosonic analog of the linear $N = 2$ superconformal algebra (SCA) [7]: apart from two currents with the spins 2 and 1 (conformal stress-tensor and $U(1)$ Kac-Moody (KM) current), it contains two bosonic currents with spins 3/2. For the currents to form a closed set (with the relevant Jacobi identities satisfied), the OPE between the spin 3/2 currents should necessarily include a quadratic nonlinearity in the $U(1)$ KM current. So $W_3^{(2)}$, in contrast to its superconformal prototype, is a nonlinear algebra.

It is natural to seek for supersymmetric extensions of this type of $W$ algebras and to see how they can be formulated in terms of superfields. First explicit example of such an extension, $N = 2$ super-$W_3^{(2)}$ algebra, has been constructed at the classical level in [8] (its quantum version is given in [9]). It involves fermionic currents with integer spins 1 and 2 and contains both $N = 2$ SCA and $W_3^{(2)}$ as subalgebras. Actually, it can be regarded as a nonlinear closure of these two algebras.  

Curiously enough, the spin content of the currents of $N = 2$ super-$W_3^{(2)}$ algebra is such that they cannot be immediately arranged into $N = 2$ supermultiplets with respect to the $N = 2$ SCA which is manifest in the formulation given in [8]. This means that $N = 2$ super-$W_3^{(2)}$, as it stands, does not admit the standard $N = 2$ superfield description, in contrast, e.g., to $N = 2$ SCA and $W_3^{(2)}$ as subalgebras. Actually, it can be regarded as a nonlinear closure of these two algebras.  

2 Preliminaries

For the reader's convenience we review here the salient features of $N = 2$ super-$W_3^{(2)}$ algebra in terms of component currents [8].

\footnote{Zamolodchikov's $W_3$ algebra [10] can be nonlinearly embedded into $W_3^{(2)}$ [11], so it also forms a subalgebra in $N = 2 W_3^{(2)}$ (in some special basis for the generating currents of the latter).}
A powerful method of constructing conformal (super)algebras is the Hamiltonian reduction method [16, 3, 17]. In this approach one writes down a gauge potential $A$ valued in the appropriate (super)algebra $g$ and then constrain some components of $A$ to be equal to constants. From the residual gauge transformations of the remaining components of $A$ one can immediately read off the OPEs of some conformal $W$ (super)algebra, with these components as the generating currents. Since the residual gauge transformations clearly form a closed set, the Jacobi identities of the resulting $W$ algebra prove to be automatically satisfied.

A straightforward application of Hamiltonian reduction to superalgebra $sl(3|2)$ gives rise to the classical $N = 2$ super-$W_3$ algebra [12]. In [8] a different choice of constraints has been made (it corresponds to some non-principal embedding of $sl(2)$ into the bosonic $sl(3) \times sl(2)$ subalgebra of $sl(3|2)$). The residual gauge transformations of the remaining currents yield just the $N = 2$ super-$W_3(2)$ algebra we will deal with here.

More precisely, starting with the following constrained $sl(3|2)$ gauge potential $A$

$$A = \frac{1}{c} \begin{pmatrix} 2J_s - 3J_w & G^+ & T_1 & S_1 & S_2 \\ 0 & 2J_s - 6J_w & G^- & 0 & S \\ 1 & 0 & 2J_s - 3J_w & 0 & S_1 \\ \tilde{S}_1 & \tilde{S} & \tilde{S}_2 & 3J_s - 6J_w & T_2 \\ 0 & 0 & \tilde{S}_1 & 1 & 3J_s - 6J_w \end{pmatrix}, \quad (2.1)$$

where \{J_s, J_w, G^+, G^-, T_1, T_2\} and \{S_1, \tilde{S}_1, S, \tilde{S}, S_2, \tilde{S}_2\} are, respectively, bosonic and fermionic currents, one can easily find the residual gauge transformations which preserve this particular form of $A$. They correspond to the following parameters

$$l_1, l_3, a_3, a_5, a_6, b_3, b_5, (b_1 + b_6), c_4, c_5, (c_1 + c_6) \quad (2.2)$$

in the standard infinitesimal gauge transformation of $A$

$$\delta A = \partial \Lambda + [A, \Lambda], \quad (2.3)$$

with $\Lambda$ being a $sl(3|2)$-valued matrix of the parameters

$$\Lambda = \begin{pmatrix} 2l_1 + l_2 + l_3 & a_1 & a_2 & b_1 & b_2 \\ a_3 & 2l_1 - 2l_3 & a_4 & b_3 & b_4 \\ a_5 & a_6 & 2l_1 - l_2 + l_3 & b_5 & b_6 \\ c_1 & c_2 & c_3 & 3l_1 + l_4 & a_7 \\ c_4 & c_5 & c_6 & a_8 & 3l_1 - l_4 \end{pmatrix}. \quad (2.4)$$

The remaining twelve combinations of the parameters are expressed through these ones and the currents. After representing these transformations in the form

$$\delta \phi(z_1) = c \oint dz_2 \left\{ -6l_1 J_s + 18l_3 J_w + a_3 G^+ + a_6 G^- + 3a_5 T_1 - 3a_5 T_2 + b_3 S + b_5 \tilde{S}_2 + (b_1 + b_6) \tilde{S}_1 - c_4 S_2 - c_6 S - (c_1 + c_6) S_1 \right\} \phi(z_1), \quad (2.5)$$

where $\phi(z)$ is any current, a self-consistent set of OPEs for the currents can be extracted from eq.(2.5).
To understand why this superalgebra was called $N = 2$ super-$W_3^{(2)}$, it is instructive to redefine the currents in the following way

$$T_w = T_1 - \frac{1}{c} S_1 \bar{S}_1 + \frac{3}{c} J_w^2, \quad T_s = -T_2 - \frac{1}{c} S_1 \bar{S}_1 + \frac{1}{c} J_s^2.$$  \hspace{1cm} (2.6)

Then the currents $\{J_w, G^+, G^-, T_w\}$ and $\{J_s, \bar{S}, T_s\}$ can be shown to obey the following OPEs

$$J_w(z_1)J_w(z_2) = \frac{6}{z_{12}^2}, \quad J_w(z_1)T_w(z_2) = \frac{J_w}{z_{12}},$$

$$J_w(z_1)G^\pm(z_2) = \mp \frac{1}{2} \frac{G^\pm}{z_{12}}, \quad T_w(z_1)G^\pm(z_2) = \frac{3}{2} \frac{G^\pm}{z_{12}^2} + \frac{G^\pm'}{z_{12}},$$

$$T_w(z_1)T_w(z_2) = \frac{3c}{z_{12}^4} + \frac{2T_w + T_w'}{z_{12}^2},$$

$$G^+(z_1)G^-(z_2) = \frac{2c}{z_{12}^2} - \frac{6J_w}{z_{12}^2} - \left( T_w - \frac{12}{c} J_w^2 + 3J_w' \right) \frac{1}{z_{12}}, \quad (2.7)$$

$$J_s(z_1)J_s(z_2) = \frac{6}{z_{12}^2}, \quad J_s(z_1)T_s(z_2) = \frac{J_s}{z_{12}},$$

$$J_s(z_1)S(z_2) = \frac{1}{2} \frac{S}{z_{12}}, \quad J_s(z_1)\bar{S}(z_2) = -\frac{1}{2} \frac{\bar{S}}{z_{12}},$$

$$S(z_1)\bar{S}(z_2) = \frac{2c}{z_{12}^2} + \frac{2J_s + T_s}{z_{12}},$$

$$T_s(z_1)S(z_2) = \frac{3}{2} \frac{S}{z_{12}^2} + \frac{S'}{z_{12}}, \quad T_s(z_1)\bar{S}(z_2) = \frac{3}{2} \frac{\bar{S}}{z_{12}^2} + \frac{\bar{S}'}{z_{12}},$$

$$T_s(z_1)T_s(z_2) = \frac{3c}{z_{12}^4} + \frac{2T_s + T_s'}{z_{12}^2}. \quad (2.8)$$

So they form $W_3^{(2)}$ and $N = 2$ SCA with the related central charges.

Thus we are eventually left with the set of currents which includes those generating $W_3^{(2)}$ and $N = 2$ SCA, as well as four extra fermionic currents $\{S_1, \bar{S}_1, S_2, \bar{S}_2\}$ with the integer spins $\{1,1,2,2\}$. The spin-statistics content of $N = 2$ super-$W_3^{(2)}$ algebra is summarized in Table 1.

<table>
<thead>
<tr>
<th>Currents</th>
<th>$J_s$</th>
<th>$J_w$</th>
<th>$S_1$</th>
<th>$\bar{S}_1$</th>
<th>$G^+$</th>
<th>$G^-$</th>
<th>$S$</th>
<th>$\bar{S}$</th>
<th>$T_s$</th>
<th>$T_w$</th>
<th>$S_2$</th>
<th>$\bar{S}_2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Spins</td>
<td>$1^B$</td>
<td>$1^B$</td>
<td>$1^F$</td>
<td>$1^F$</td>
<td>$3^B_\frac{3}{2}$</td>
<td>$3^B_\frac{3}{2}$</td>
<td>$\bar{S}$</td>
<td>$3^F_\frac{3}{2}$</td>
<td>$3^F_\frac{3}{2}$</td>
<td>$2^B$</td>
<td>$2^B$</td>
<td>$2^F$</td>
</tr>
</tbody>
</table>

$^2$Hereafter, we explicitly write down only singular terms in OPEs. All the currents appearing in the right hand sides of the OPEs are evaluated at the point $z_2$ ($z_{12} = z_1 - z_2$).
All the currents with the aforementioned spins, except for $T_\alpha$ and $T_w$, are primary with respect to the following Virasoro stress-tensor $T$ having a zero central charge

$$T = T_s + T_w + \frac{4}{c} S_1 \bar{S}_1 - \frac{4}{c} J_s^2 + \frac{12}{c} J_w J_s - \frac{12}{c} J_w^2.$$  \hfill (2.9)

The currents $T_s$ and $T_w$ are quasiprimary with the central charges $3c$ and $-3c$, respectively. It can be checked that in this $N = 2$ super-$W_3^{(2)}$ algebra there exists no basis for the currents such that all the currents are primary with respect to some (improved) Virasoro stress-tensor.

The whole set of OPEs of $N = 2$ super-$W_3^{(2)}$ algebra in terms of these currents is given in Appendix.

3 $N = 2$ super-$W_3^{(2)}$ algebra in terms of $N = 2$ supercurrents

Despite the fact that $N = 2$ super-$W_3^{(2)}$ algebra has an equal number of bosonic and fermionic currents, it is unclear how they could be arranged into $N = 2$ supermultiplets. The main obstruction against the existence of a superfield description is the fact that the superalgebra considered the numbers of currents with integer and half-integer spins do not coincide, while any $N = 2$ superfield clearly contains the equal number of components with integer and half-integer spins.

To find a way to construct the $N = 2$ super-$W_3^{(2)}$ algebra in terms of $N = 2$ superfields, two features of its components OPEs (2.7), (2.8), (A.1) must be taken into account.

First of all, $N = 2$ super-$W_3^{(2)}$ algebra is nonlinear. This means that one may choose the basis for its generating currents in many different ways. The transformations relating different bases must be invertible but in general they are nonlinear and can include derivatives of the currents along with the currents themselves.

Secondly, we would like to stress that the OPEs (2.7), (2.8), (A.1) do not fix the scale of fermionic $S_1, \bar{S}_1, S, \bar{S}, S_2, \bar{S}_2$ and bosonic $G^+, G^-$ currents. Moreover, keeping in mind that all these currents possess definite charges with respect to the $J_w$ and $J_s, U(1)$ currents, one can introduce a new “improved” stress-tensor

$$T = T + bJ'_w + gJ'_s,$$  \hfill (3.1)

with respect to which the currents $G^+, G^-, S_1, \bar{S}_1, S, \bar{S}, S_2, \bar{S}_2$, still remaining primary, have the dimensions (spins) listed in Table 2.

<table>
<thead>
<tr>
<th>Currents</th>
<th>$G^+$</th>
<th>$G^-$</th>
<th>$S_1$</th>
<th>$\bar{S}_1$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Spins</td>
<td>$\frac{3}{2} + \frac{b}{2} + g$</td>
<td>$\frac{3}{2} - \frac{b}{2} - g$</td>
<td>$1 + \frac{b}{a} + \frac{g}{2}$</td>
<td>$1 - \frac{b}{a} - \frac{g}{2}$</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Currents</th>
<th>$S$</th>
<th>$\bar{S}$</th>
<th>$S_2$</th>
<th>$\bar{S}_2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Spins</td>
<td>$\frac{3}{2} - \frac{b}{3} - \frac{g}{2}$</td>
<td>$\frac{3}{2} + \frac{b}{3} + \frac{g}{2}$</td>
<td>$2 + \frac{b}{a} + \frac{g}{2}$</td>
<td>$2 - \frac{b}{a} - \frac{g}{2}$</td>
</tr>
</tbody>
</table>
Thus we cannot exclude the possibility that in some nonlinear basis the generating currents could have appropriate spins to be organized into supermultiplets with respect to some new $N = 2$ SCA.

Fortunately, just this situation takes place for the superalgebra under consideration. In order to demonstrate this, let us pass to the new basis $(\tilde{J}, \tilde{S}_1, \tilde{S}_2, \tilde{T}, (S_1, J), (G^+, \tilde{S}), (S, G^-), (\tilde{T}, \tilde{S}_2))$, related to the original one as

\[
\begin{align*}
\tilde{J}_s &= 4J_s - 6J_w, \quad \tilde{J} = -J_s + 3J_w, \\
\tilde{T}_s &= T - J'_s, \quad \tilde{T} = 2T_s - 2J'_s, \\
\tilde{S}_2 &= S_2 + \frac{3}{c} S_1 J_s - \frac{3}{c} S_1 J_w - S'_1, \\
\tilde{S}'_2 &= \tilde{S}_2 - \frac{1}{c} \tilde{S}_1 J_s - \frac{3}{c} \tilde{S}_1 J_w + S'_1.
\end{align*}
\tag{3.2}
\]

All the newly defined currents, except for $J$, are primary with respect to the Virasoro stress-tensor $T_s$ (it corresponds to the choice of $b = 0, g = -1$ in eq. (3.1) and Table 2) and have the following spins and statistics: $(1^B, \frac{3}{2}^F, \frac{3}{2}^F, \frac{3}{2}^B), (\frac{1}{2}^F, 1^B), (\frac{1}{2}^B, 1^F), (2^F, \frac{5}{2}^B), (2^B, \frac{5}{2}^F)$. The spin 1 current $\tilde{J}$ is not even quasiprimary because its OPE with $\tilde{T}$ contains a central term.

It can be checked that the currents $(\tilde{J}_s, \tilde{S}_2, \tilde{S}_1, \tilde{T}_s)$ form $N = 2$ SCA, while the remaining eight currents $\tilde{J}, G^+, G^-, S, \tilde{S}, S_1, \tilde{S}_2, \tilde{T}$ constitute a reducible $N = 2$ supermultiplet. Namely, the sets of the currents $(S_1, \tilde{J})$ and $(G^+, \tilde{S})$ form two anti-chiral $N = 2$ spin 1/2 supermultiplets, respectively fermionic and bosonic ones, with the standard linear transformation properties under $N = 2$ SUSY (the transformation law of fermionic current $S_1$ contains in addition a shift by the transformation parameter). However, transformation properties of the pairs $(S, G^-)$ and $(\tilde{T}, \tilde{S}_2)$ are more complicated: supersymmetry $\tilde{S}$ mixes them with the composite bosonic and fermionic currents $(B_1, B_2)$ and $(F_1, F_2)$ which have the spin content $(5/2, 3), (3, 5/2)$ and are defined by

\[
\begin{align*}
B_1 &= \frac{1}{c} S_1 S, \quad B_2 = \frac{1}{c} (\tilde{J} \tilde{T} - G^+ G^- - 2S_1 \tilde{S}_2 - SS), \\
F_1 &= \frac{1}{c} (G^- S_1 - S \tilde{J}), \quad F_2 = \frac{2}{c} (S_1 \tilde{T} - G^+ S).
\end{align*}
\tag{3.3}
\]

These transformation properties follow from the OPEs:

\[
\begin{align*}
\tilde{S}_2(z_1) S(z_2) &= \frac{2B_1}{z_{12}}, \\
\tilde{S}_2(z_1) G^-(z_2) &= \frac{2S}{z_{12}^2} + \frac{2F_1 + S'}{z_{12}}, \\
\tilde{S}_2(z_1) \tilde{T}(z_2) &= \frac{F_2}{z_{12}^3}, \\
\tilde{S}_2(z_1) \tilde{S}_2(z_2) &= \frac{\tilde{T}}{z_{12}^3} - \frac{B_2 - \frac{1}{2} \tilde{T}}{z_{12}^3}.
\end{align*}
\tag{3.4}
\]

So the eight currents $(S_1, \tilde{J}), (G^+, \tilde{S}), (S, G^-), (\tilde{T}, \tilde{S}_2)$
together form a nonlinear and actually not fully reducible representation of the $N = 2$ SCA defined above.

Crucial for putting this representation in a more transparent manifestly supersymmetric form is the observation that the nonlinearly transforming pairs of the basic currents, namely $(S, G^-)$ and $(\tilde{T}, \tilde{S}_2)$, can be combined with the composites $B_1, F_1$ and $F_2, B_2$ into two linearly transforming spin 2 $N = 2$ supermultiplets with the opposite overall Grassmann parities.

Thus, the basic currents of $N = 2$ super-$W_3^{(2)}$ together with the above composites split into the five irreducible linear multiplets of the $N = 2$ superconformal subalgebra

\begin{align}
\text{bosonic, spin } 1 : & \ (S, \tilde{S}_2, \tilde{S}_1, \tilde{T}_4) \quad (N = 2 \text{ SCA}) , \\
\text{fermionic, spin } \frac{1}{2} : & \ (S_1, \tilde{J}) , \\
\text{bosonic, spin } \frac{1}{2} : & \ (G^+, \tilde{S}) , \\
\text{fermionic, spin } 2 : & \ (S, G^-, B_1, F_1) , \\
\text{bosonic, spin } 2 : & \ (\tilde{T}, \tilde{S}_2, F_2, B_2) .
\end{align}

This extended set of currents, in accordance with their spin content, is naturally accommodated by the five $N = 2$ supercurrents: general spin 1 $J(Z)$, spin 1/2 anti-chiral fermionic $G(Z)$ and bosonic $Q(Z)$, spin 2 fermionic $F(Z)$ and bosonic $T(Z)$ \(^3\). The precise relation of the components of these superfields to the currents of $N = 2 W_3^{(2)}$ is quoted in Appendix. Below we reformulate $N = 2 W_3^{(2)}$ in terms of SOPEs of these supercurrents.

The superfield $J(Z)$ generates the $N = 2$ SCA with SOPE

\begin{equation}
J(Z_1)J(Z_2) = \frac{-2c + \theta_{12} \bar{\theta}_{12} J}{Z_{12}^2} + \frac{\bar{\theta}_{12} \bar{D} J - \theta_{12} \bar{D} J + \theta_{12} \bar{\theta}_{12} J'}{Z_{12}} ,
\end{equation}

where

\begin{equation}
\theta_{12} = \theta_1 - \theta_2 , \quad \bar{\theta}_{12} = \bar{\theta}_1 - \bar{\theta}_2 , \quad Z_{12} = z_1 - z_2 + \frac{1}{2} \left( \theta_1 \bar{\theta}_2 - \theta_2 \bar{\theta}_1 \right) ,
\end{equation}

and $D, \bar{D}$ are the spinor covariant derivatives defined by

\begin{equation}
D = \frac{\partial}{\partial \theta} - \frac{1}{2} \frac{\partial}{\partial z} , \quad \bar{D} = \frac{\partial}{\partial \bar{\theta}} - \frac{1}{2} \frac{\partial}{\partial \bar{z}} ,
\end{equation}

\begin{equation}
\{D, \bar{D}\} = -\frac{\partial}{\partial z} , \quad \{D, D\} = \{\bar{D}, \bar{D}\} = 0 .
\end{equation}

The next SOPEs express the property that the remaining four supercurrents have the aforementioned spins with respect to this $N = 2$ SCA

\begin{equation}
J(Z_1)G(Z_2) = \frac{-c\theta_{12} + \frac{1}{2} \theta_{12} \theta_{12} G}{Z_{12}^2} + \frac{\theta_{12} \bar{\theta}_{12} G' - \theta_{12} \bar{D} G - G}{Z_{12}} ,
\end{equation}

\(^3\)By $Z$ we denote the coordinates of 1D $N = 2$ superspace, $Z = (z, \theta, \bar{\theta})$.\)
Let us pay attention to the presence of a central term in (3.11). It reflects the property that the superfield \( G(Z) \) transforms inhomogeneously under \( N = 2 \) SCA. All other superfields are primary with respect to the \( N = 2 \) SCA supercurrent \( J(Z) \).

In each of the supercurrents \( F(Z) \) and \( T(Z) \), the spin 3 component and one of the spin 5/2 components are composite (see (3.3)). In the superfield language, this implies that these superfields have to satisfy some constraints. Using the formulas (A.2) of Appendix, one can check that the relations (3.3) amount to the following nonlinear constraints

\[
A_1 \equiv \overline{D}F + \frac{2}{c}(GF) = 0 ,
\]

\[
A_2 \equiv \overline{D}T + \frac{2}{c}(GT) + \frac{2}{c}(QF) = 0 .
\]

For completeness, we add also the chirality conditions for \( G, Q \)

\[
\overline{D}G = 0 , \quad \overline{D}Q = 0 .
\]

By means of eq. (3.14) one could, in principle, eliminate \( F(Z) \) in terms of \( T(Z), G(Z) \) and \( Q(Z) \). If one substitutes this expression for \( F(Z) \) in the constraint (3.13), the latter is satisfied identically. However, this expression is singular at \( Q(Z) = 0 \). We prefer to deal with two constrained supercurrents in order to have polynomial non-singular expressions in all SPOEs.

Now we are ready to construct the remaining SPOEs of \( N = 2 \) super-W\(_3\)\(^{(2)}\). Taking the most general Ansatz for these SPOEs in terms of the introduced superfields, using (3.13), (3.14), (3.3) and requiring the latter to be consistent with the OPEs for the superfield components (see Appendix) we obtain the following non-trivial relations

\[
G(Z_1)Q(Z_2) = -\frac{1}{2}\theta_{12}Q \left[ \frac{1}{Z_{12}} \right] ,
\]

\[
G(Z_1)F(Z_2) = \frac{1}{2}\theta_{12}F \left[ \frac{1}{Z_{12}} \right] ,
\]

\[
G(Z_1)T(Z_2) = -2\c\theta_{12} + \theta_{12}\theta_{12}G \left[ \frac{1}{Z_{12}} \right] - \theta_{12}\theta_{12}(G' - \frac{2}{c}GDG - \frac{1}{c}JG - \frac{1}{2}\overline{D}J)
\]

\[
+ \theta_{12}(J + 2DG) - 2G - \overline{D}J - 2G' + \frac{4}{c}GDG + \frac{2}{c}JG \left[ \frac{1}{Z_{12}} \right] ,
\]

\[
Q(Z_1)F(Z_2) = \frac{2\theta_{12} \theta_{12}G \left[ \frac{1}{Z_{12}} \right] + \frac{2G - \theta_{12}(J + 2DG) + \theta_{12}(G' - \frac{2}{c}GDG - \frac{1}{c}JG - \frac{1}{2}\overline{D}J) \left[ \frac{1}{Z_{12}} \right] + \frac{2G' - \frac{4}{c}GDG - \frac{2}{c}JG - \overline{D}J + \frac{1}{2}\theta_{12}T \left[ \frac{1}{Z_{12}} \right] ,
\]

(3.12)
The above SOPEs are self-consistent only on the shell of constraints (3.13), (3.14). These constraints are first class and the Jacobi identities are satisfied only on their shell \( A_1 = A_2 = 0 \). They are consistent with the SOPEs (3.8), (3.11), (3.12), (3.16) in the sense that the SOPEs of \( A_1, A_2 \) with all supercurrents are vanishing on the constraint shell (the compatibility of the whole set of SOPEs with the linear chirality conditions (3.15) is evident by construction). It should be pointed out that it is impossible to satisfy the Jacobi identities off the constraint shell unless one further enlarges the set of supercurrents. We have checked this by inserting the expressions \( A_1 \) and \( A_2 \) (3.13), (3.14) 4 in all appropriate places in the right hand sides of the SOPEs obtained. Thus the constraints (3.13), (3.14) are absolutely necessary for the above set of \( N = 2 \) superfields to form a closed algebra. In a forthcoming paper devoted to \( N = 2 \) superfield hamiltonian reduction [15] it will be shown that these

\(^4A_1 \text{ and } A_2 \text{ are non-zero off the constraints shell.}\)
constraints (as well as the chirality conditions (3.15)) are remnants of the Hull-Spence type constraints [18] for the supercurrents of \(N = 2\) extension of affine superalgebra \(sl(3|2)^{(1)}\).

Our final remark concerns the presence of the spin 1/2 currents \(S_1\) and \(G^+\) in the basis (3.2). At first sight, following the reasonings of ref. [19], one could think that they can be factored out to yield a smaller nonlinear algebra. However, this is not true in the present case because an important assumption of ref. [19] does not hold, namely the assumption that OPEs between the spin 1/2 currents contain singularities. Indeed, the OPEs of these currents are regular in our case. So in the algebra \(N = 2\) super-\(W_3\) the spin 1/2 currents cannot be removed.

4 Superfield reduction to \(N = 2\) super-\(W_3\) algebra

In this section we show that, if one imposes the additional first-class constraint on the supercurrent \(Q(Z)\)

\[\tilde{Q}(Z) \equiv Q(Z) - c = 0\]  \hspace{1cm} (4.1)

and the condition

\[G(Z) = 0\]  \hspace{1cm} (4.2)

which fixes a gauge with respect to gauge transformations generated by (4.1) (together (4.1) and (4.2) form a set of second-class constraints), one arrives at SOPEs of some self-consistent nonlinear algebra written in terms of unconstrained supercurrents. This algebra turns out to be none other than the well-known \(N = 2\) super-\(W_3\) algebra [12] formulated in terms of \(N = 2\) superfields in [13]. Let us define new superfields \(\tilde{J}(Z)\) and \(\tilde{T}(Z)\)

\[
\tilde{J}(Z) = J(Z) - 2DG(Z) + 2\partial Q(Z),
\]

\[
\tilde{T}(Z) = T(Z) - \frac{1}{3}[D, \bar{D}]J(Z) + \frac{4}{3}DGG(Z) - \frac{4}{9c}J(Z)^2 - \frac{20}{9c}(\partial J(Z)Q(Z)) + \frac{2}{9c}J(Z)Q(Z) - \frac{2}{c}\partial J(Z)DQ(Z) - \frac{16}{9c}DGG(Z) + \frac{20}{9c}\partial Q(Z)\partial Q(Z) + \partial J(Z) - \frac{2}{3}\partial^2 Q(Z).
\]  \hspace{1cm} (4.3)

This substitution is dictated by the requirement that SOPEs of these supercurrents with \(G(Z)\) and \(\tilde{Q}(Z)\) (4.1) be homogeneous in \(\tilde{Q}(Z)\) and \(G(Z)\). The supercurrent \(\tilde{J}(Z)\) can be checked to generate another \(N = 2\) SCA, such that the conformal weights of \(\tilde{Q}(Z), G(Z), T(Z)\) and \(F(Z)\) with respect to it equal 0, 1/2, 2 and 5/2, respectively. The constraints (4.1) and (4.2) prove to be preserved by this \(N = 2\) SCA.

Thus the superfields \(\tilde{J}(Z)\) and \(\tilde{T}(Z)\) by construction are gauge invariant with respect to the gauge transformations generated by the first-class constraints (4.1). So, according to the standard ideology of hamiltonian reduction [16, 3, 17], they have to form a closed superalgebra (with all the Jacobi identities satisfied) on the shell of constraints (4.1), (4.2) and with

\[F = -\frac{1}{2}\bar{D}T.\]  \hspace{1cm} (4.4)

\(^5\)Actually, the described procedure supplies a nice example of secondary hamiltonian reduction in \(N = 2\) superspace [15].
The last relation follows by substituting (4.1), (4.2) into eq.(3.14). Note that with this $F$ eq.(3.13) is identically satisfied.

Using the SOPEs of $N = 2$ super-$W_3^{(2)}$ we find that the resulting SOPEs for the currents $\mathcal{J}(Z)$ and $\mathcal{T}(Z)$ after substituting (4.1), (4.2), (4.4) exactly coincide with SOPEs of classical $N = 2$ super-$W_3$ algebra [13]. In the next Section we will make use of this result to construct the simplest nontrivial hamiltonian flow on $N = 2$ super-$W_3^{(2)}$.

5 Generalized $N = 2$ super Boussinesq equation

The most general hamiltonian which can be constructed out of the five superfields $\mathcal{J}(Z), G(Z), \mathcal{Q}(Z), \mathcal{F}(Z)$ of $N = 2$ super-$W_3^{(2)}$ algebra under the natural assumptions that it (i) respects rigid $N = 2$ supersymmetry and (ii) has the same scaling dimension 2 as the hamiltonian of the ordinary bosonic Boussinesq equation, is given by

$$H = \int dzd\theta d\bar{\theta} \left( T + v_1 J^2 + v_2 J \mathcal{D} G + v_3 J \partial \mathcal{Q} + v_4 \mathcal{D} G \partial \mathcal{Q} \right). \quad (5.1)$$

Note the presence of the free parameters $v_1, \ldots, v_4$ in (5.1). Now, using SOPEs of $N = 2$ super-$W_3^{(2)}$ algebra and the definition

$$\frac{\partial \phi}{\partial t} = \{ \phi, H \} \quad (5.2)$$

(here $\phi(z)$ is any supercurrent of $N = 2$ super-$W_3^{(2)}$ and the Poisson brackets in the r.h.s. of (5.2) are understood), it is straightforward to find the explicit form of the evolution equations. Due to the complexity of these equations, it is not so illuminating to write down them here. We also postpone to future publications the analysis of integrability of this system.

In ref. [13] we have constructed, in $N = 2$ superfield form, the most general one-parameter super Boussinesq equation with the second hamiltonian structure given by the classical $N = 2$ super-$W_3$ algebra. With making use of the results of Sect. 4 it is not difficult to show that the obtained system of evolution equations reproduces the one of ref. [13] upon the above truncation of $N = 2$ super-$W_3^{(2)}$ to $N = 2$ super-$W_3$ and with the following relations between the parameters in (5.1)

$$v_1 = \frac{4\alpha - \frac{8}{3\epsilon}}{6}, \quad v_2 = -\frac{4}{c} - 4v_1, \quad v_3 = 4v_1. \quad (5.3)$$

Here $\alpha$ is the parameter entering the $N = 2$ super Boussinesq equation [13].

6 Conclusion

To summarize, we have concisely rewritten classical $N = 2$ super-$W_3^{(2)}$ algebra of ref. [8] in terms of five constrained $N = 2$ superfields, found its superfield reduction to $N = 2$ super-$W_3$ algebra [12, 13], and constructed a family of $N = 2$ supersymmetric equations the second hamiltonian structure for which is given by this superalgebra and which generalize
the $N=2$ super Boussinesq equation of ref. [13]. In a forthcoming publication [20] we will extend our consideration to the case of full quantum $N=2$ super-$W_3$ algebra.

An interesting problem is to find out possible string theory implications of $N=2$ $W_3$ algebra, both in its component and superfield formulations. The fact that there exists a zero central charge stress-tensor (2.9) with respect to which almost all of the currents are primary suggests that this algebra admits an interpretation as a kind of twisted topological superconformal algebra and so has a natural realization in terms of BRST structure associated with some string (the $W_3$ one?) or superstring.

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Appendix

Here we present the component OPEs for the $N=2$ super-$W_3$ algebra [8] and give the relation between the currents of the latter and the components of $N=2$ supercurrents $J(Z)$, $G(Z)$, $Q(Z)$, $F(Z)$ and $T(Z)$.

The whole set of the OPEs contains, besides those of the subalgebras $W_3$ and $N=2$ SCA (2.7), (2.8), the following non-trivial relations:

\begin{align*}
J_w(z_1) S_1(z_2) &= -\frac{1}{6} S_1 \frac{z_1}{z_2}, \quad J_s(z_1) S_1(z_2) = -\frac{1}{2} S_1 \frac{z_1}{z_2}, \quad J_s(z_1) J_w(z_2) = \frac{3}{3} S_1 \frac{z_1}{z_2}, \\
J_s(z_1) T_w(z_2) &= \frac{2 J_w}{z_2}, \quad J_s(z_1) G^+(z_2) = -\frac{G^+}{z_1}, \quad J_w(z_1) S_1(z_2) = \frac{1}{3} S_1 \frac{z_1}{z_2}, \\
J_s(z_1) S_2(z_2) &= -\frac{1}{2} S_2 \frac{z_1}{z_2}, \quad J_w(z_1) T_s(z_2) = \frac{2 J_s}{z_2}, \quad J_w(z_1) S_2(z_2) = -\frac{1}{6} S_2, \\
T_s(z_1) T_w(z_2) &= \frac{4}{c} \left( S_1 \bar{S}_1 + J_w J_s \right) + \frac{3}{c} \left( S_1 \bar{S}_1 - S_2 \bar{S}_1 + S'_1 \bar{S}_1 + 2 J_w J'_s \right) \frac{z_1}{z_2}, \\
T_s(z_1) G^+(z_2) &= -\frac{2}{c} \left( G^+ \frac{z_1}{z_2} - S_1 \bar{S}_1 \right), \quad T_s(z_1) S_1(z_2) = -\frac{1}{2} S_1 \frac{z_1}{z_2} + \frac{S_2 - S'_1 - \frac{1}{c} (S_1 J_s + 3 J_w S_1)}{2 z_1}, \\
T_s(z_1) S_2(z_2) &= -\frac{3 S_1}{z_2} + \frac{2 S_2 - 3 S'_1 + \frac{1}{c} (3 S_1 J_s - 9 J_w S_1)}{2 z_1} + \frac{2 G^+ S_1 - \frac{4}{c} S_1 T_s + \frac{3}{c} S_1 J'^2 + \frac{1}{c} S_1 J'_s}{2 z_1},
\end{align*}

\begin{align*}
&-\frac{3}{c} S_2 J_s - 3 J_w S_2 - \frac{6}{c} J_w J_s S_1 + 6 J^2 S_1 + 6 J_s S'_1 - \frac{2}{c} S'_1 J_s + \frac{3}{c} J'_s S_1 - S'_2 + S'_{2}, \\
T_w(z_1) S_1(z_2) &= -\frac{1}{2} S_1 \frac{z_1}{z_2} - \frac{S_2 - 3 S_1 J_s + 3 J_w S_1}{2 z_1}, \quad T_w(z_1) S_2(z_2) = -\frac{2 G^- S_1 - \frac{3}{c} J_w S_1}{z_1}.
\end{align*}
\[ T_w(z_1)S_2(z_2) = \frac{3S_1}{z_{12}^2} + \frac{2S_2 - \frac{3}{c} S_1 J + \frac{9}{c} J_w S_1 + 3S_1'}{2z_{12}^2} - \frac{\frac{3}{c} G^+ S + \frac{4}{c} S_1 T_w - \frac{1}{2} S_1 J^2 + \frac{1}{c} S_1 J_w}{z_{12}^2} \]

Here we omitted the OPEs which can be obtained from (A.1) via the discrete automorphisms:

\[ J_{w,*} \rightarrow -J_{w,*}, \quad \bar{G}^* \rightarrow \pm G^*, \quad \bar{S} \rightarrow S, \quad \bar{S}_1 \rightarrow \bar{S}_1, \quad \bar{S}_2 \rightarrow -S_2, \quad \bar{S}_2 \rightarrow S_2. \]

The currents are related to the components of the \( N = 2 W_{3}^{(2)} \) supercurrents in the following way:

\[ \mathcal{J}_1 = J_1, \quad \mathcal{D}\mathcal{J}_1 = \sqrt{2} S_2, \quad \mathcal{D}J_2 = \sqrt{2} S_1, \quad [\mathcal{D}, \mathcal{D}] J_1 = -2 \bar{T}_s. \]

\[ \mathcal{G} = \sqrt{2} S_1, \quad \mathcal{D}\mathcal{G} = \bar{J}, \quad Q = \sqrt{2} G^+, \quad \mathcal{D}Q = \bar{S}, \quad F = S, \quad \mathcal{D}F = -\frac{1}{\sqrt{2}} G^-, \quad \mathcal{D}\mathcal{F} = 2\sqrt{2} B_1, \quad \mathcal{D}\mathcal{D}F = 2F_1, \]

\[ \mathcal{T} = \bar{T}, \quad \mathcal{D}\mathcal{T} = -\sqrt{2} \bar{S}_2, \quad \mathcal{D}\mathcal{T} = \sqrt{2} F_2, \quad \mathcal{D}\mathcal{D}T = -2B_2. \]

The composite currents \( B_1, B_2, F_1, F_2 \) were defined in (3.3).
References

2. THERMAL BARRIER COATING SYSTEMS

Thermal barrier coating systems represent one of the above mentioned possibilities to raise the allowable material temperature. The coating systems are generally built up by a heat insulating top coating and an oxidation resistant bond coating. The main purpose with the bond coating is to protect the substrate from oxidation by formation of an alumina scale on top of the bond coating. This alumina scale is protective since it shows a parabolic oxidation with a low growth rate. Typical bond coating thickness vary around 100 μm. A ceramic top coating with a low thermal conductivity is applied above the bond coating. The thickness of the top coating is typically 300-400 μm and this can lower the substrate temperature by some hundred degrees depending on the heat flow situation. A cross section of a thermal barrier coating system is shown in Fig. 3.

![Cross section of a typical thermal barrier coating system, consisting of an yttria-stabilised zirconia top coating and a NiCrAlY bond coating. The coatings are plasma sprayed at Volvo Aero Corporation onto Hastelloy X, a nickel-base alloy.](image)

2.1. Production of Thermal Barrier Coating systems

Both the top coating and the bond coating in this study were applied by plasma spraying, which is one of the thermal spray methods. The methods involved in the thermal spray group include flame spraying, plasma spraying and different spray techniques utilising high spray...
velocities, i.e. HVOF (High Velocity Oxy-Fuel). Characteristic for these processes is the generation of a hot flame by dissociation or combustion of gases. The coating material is then fed into the hot gas stream where it melts and accelerates towards the substrate material. A short description of the processes and their characteristics are given below.

- Flame spraying. This method uses a flame created by the combustion of oxygen and an organic gas, typically acetylene. The temperature of the flame is relatively low, reaching values around 3000 °C, and the velocities of the injected particles in the flame are also low, below 100 m/s.

- HVOF processes. This is a group of techniques utilising the combustion of gases including oxygen, hydrogen and an organic gas. This results in a gas stream with a very high velocity and the injected particles reach velocities up to 800 m/s. The temperature within the gas stream, though, is low, in the order of 3000 °C.

- Plasma spraying. The process gives intermediate particle velocities, 400 m/s, and high gas temperatures, up to 20000 °C in the core of the flame.

The above given temperature and particle velocity values are rough indications for average values connected with the processes [6]. The figures can be varied substantially within each process by the change of gases, gas mixtures or other spray parameters. Nevertheless, it can be seen that plasma spraying gives temperatures high enough for melting of the zirconia top coating particle material. The HVOF processes are not suitable for spraying ceramic materials due to the low gas temperature and the high particle velocities, which gives a short dwell time for the particles before reaching the substrate. Hence, sufficient melting of the ceramic particles is usually not achieved. The flame spraying process utilises lower particle velocities, giving enough time for melting of the particles. On the other hand, a high degree of porosity is found in the sprayed coating due to insufficient kinetic energy of the particles when they hit the substrate. High porosity levels can decrease the cohesion within the coating which should be avoided. The choice of the plasma spray process for spraying of the top coating is hence connected with the ability to completely melt the injected powder particles. This is especially delicate when the material has a low thermal conductivity since heat is slowly transported to the centre of the particles.

A schematic of the plasma gun configuration is shown in Fig. 4. The gun consists of a tungsten cathode which is surrounded by a copper anode. Both the anode and the cathode are water cooled, to avoid melting by the high temperatures created in the plasma. The plasma
gases (Ar, He, H₂, N₂ etc.) are fed into the plasma gun, passing between the anode and the cathode where an arc is generated by applying a voltage over the anode-cathode. The gases are ionised when passing the arc and a gas in plasma state emerges as a hot jet stream.

Figure 4. Schematic of the plasma gun [7].

Powder particles of coating material, carried by a carrier gas (usually Ar), are then fed into the plasma flame. The size distribution of the powder particles should be kept narrow and the typical sizes vary between 20 and 100 μm. This excludes large particles that are difficult to melt and too small particles that can be overheated during the spraying. To obtain a favourable melting of the powder particles, it is important to inject the powder in the centre of the plasma flame. The particles are then accelerated towards the substrate and during the flight they interact with the hot gases, causing the particles to melt into droplets.

Before the droplets hit the substrate material, they have a temperature distribution, a velocity distribution, a size distribution and a distribution in position relative to the gun and the substrate. These variables, together with the surrounding environment, determine the microstructure and properties of the coating. In order to quantify the droplet variables within the plasma flame, in-flight measurement techniques have been developed [8-11]. The temperature measurements are based on the collection of emitted thermal radiation from the particles within the plasma flame. This pyrometer technology can also be used to determine the velocity of the particles by recording the time between the radiation signals from the same particle at a given distance between the radiation measurement points.
The measurement techniques give almost immediate results and the particle variables can be used as an on-line characterisation of the plasma flame. Future work can and are directed towards the use of developing closed-loop systems which analyse the measured data and change the spraying parameters in order to keep a constant optimum.

2.2. Top coating materials

The top coating material in thermal barrier coating systems usually consists of zirconia with eight weight percent of yttria ($\text{ZrO}_2 + 8\text{ wt.}\%\text{ Y}_2\text{O}_3$). The purpose of the top coating is to insulate the substrate material from high temperatures and zirconia is an outstanding material for thermal insulation since the thermal conductivity is very low. Figure 5 shows the thermal conductivity for some ceramic materials, and plasma sprayed zirconia shows the lowest coefficient of thermal conductivity. A difference can be seen between the sintered, dense zirconia and the plasma sprayed zirconia. While the thermal conductivity for sintered zirconia approximates 2.5 W/mK, a lower thermal conductivity around 1 W/mK is found for the plasma sprayed zirconia coatings. This decrease is dependent on the manufacturing process where plasma spraying introduces a number of defects including pores and cracks. The high pore densities or cracks, oriented perpendicular to the heat flow, hinder the heat flow and decrease the thermal conductivity [12].

Figure 5. Coefficients of thermal conductivity for some ceramic materials [13-15].

Another important property of the top coating material is the coefficient of thermal expansion (CTE). During service, the thermal barrier coating and the substrate are subjected to temperature changes and the difference in CTEs between the substrate and the coating...
causes thermal stresses. These stresses are believed to create inelastic degradation of the coating until failure occurs. Therefore, to decrease the thermal stresses, the CTE for the top coating material should match the CTE for the substrate material which in this case is a nickel-base alloy (Hastelloy X). Figure 6 shows CTEs for Hastelloy X and some ceramic materials. It can be seen that zirconia has a CTE as close as possible to Hastelloy X.

![Figure 6. Coefficients of thermal expansion for some ceramic materials and Hastelloy X [14, 16-17].](image)

Both the low thermal conductivity and the, for being a ceramic material, high coefficient of thermal expansion are important reasons for choosing zirconia as top coating material. Other important properties for the top coating material include thermal and chemical stability. Zirconia has a high melting point (2680 °C) and is chemically resistant to jet engine exhaust products. This makes zirconia a possible candidate for a top coating material.

### 2.3. Phases in pure and stabilised ZrO₂

Pure ZrO₂ is a polymorph material and can exist with three different crystal structures, face centred cubic, tetragonal and monoclinic [18]. The face centred cubic phase is present from melting temperature to 2340 °C. At 2340 °C, this phase transforms to a tetragonal phase which is stable to 1170 °C. Below 1170 °C, the monoclinic phase is present. The latter phase transformation, from tetragonal to monoclinic, is a displacive transformation associated with a significant increase in material volume, 4.6% [7]. It is notable that this transformation is used for the toughening of structural zirconia parts because the transformation consumes energy before fracture. But, for the case of thermally sprayed coatings, the transformation is
undesirable since the volume increase creates stresses in the material which is enough to cause coating failure already during spraying.

To avoid the detrimental tetragonal to monoclinic phase transformation, stabilisation of the zirconia material is made. This can be achieved through addition of other oxides, such as $\text{Y}_2\text{O}_3$, CaO, CeO$_2$, HfO$_2$ or MgO. Figure 7 shows the phase diagram for the zirconia rich part of the ZrO$_2$-$\text{Y}_2\text{O}_3$ system and the solidified system is built up by the three described phases, cubic, tetragonal and monoclinic [19]. The horizontal lines in the bottom of the phase diagram indicate the phases that are present when quenching the material from a high temperature to room temperature.

![Phase Diagram](image)

**Figure 7.** The zirconia rich part of the zirconia-yttria system. Replotted from Scott [19].

It can be seen that a mixture with more than 16 wt.% of $\text{Y}_2\text{O}_3$ totally stabilises the ZrO$_2$ and the cubic phase is stable to room temperature. For the commonly used material ZrC$_2$, partially stabilised with 8 wt.% $\text{Y}_2\text{O}_3$, the phase diagram predicts a two-phase material with monoclinic and cubic structures. This is not the case for a sprayed coating because
equilibrium is not maintained during the cooling of the sprayed material. Instead, the material is rapidly quenched and a non-equilibrium tetragonal phase, $T'$, dominates. The amount of $T'$-phase is dependent on yttria content and maximum amount is found at a composition of 8 wt.% of yttria. At lower yttria contents, increased amount of monoclinic phase is found. A similar trend is found when increasing the yttria content above 8 wt.% where higher amounts of cubic phase are found.

Empirical thermal shock and thermal cycling tests have shown that maximum coating life is found in samples with a high degree of $T'$ phase [20-21]. No clear explanation of the correlation is proposed but the results explain the industrial use of $ZrO_2$, partially stabilised with 8 wt.% $Y_2O_3$.

Since the $T'$-phase is a non-equilibrium phase, the phase will transform to the equilibrium phases, cubic and tetragonal, when the diffusion rate is high enough. Miller et al. [22] performed annealing experiments on $ZrO_2 + 8$ wt.% $Y_2O_3$ plasma sprayed coatings to investigate the stability of the in-quenched $T'$-phase. The annealing temperatures were chosen from 1200 to 1600 °C and the ageing was done up to 100 hours. Using x-ray diffraction technique, the as-sprayed specimens were found to contain 8 mole% monoclinic phase and 12 mole% cubic phase together with the in-quenched $T'$-phase. At 1200 °C, the phase diagram predicts a low-yttria tetragonal phase and a high-yttria cubic phase. Ageing at this temperature for 100 hours caused phase content changes, and the amounts of the equilibrium phases, monoclinic and cubic, increased to 11 and 21 mole%, respectively. For samples aged at 1400 °C, a similar increase was found after only one hour at the ageing temperature. Longer ageing times continued the destabilisation of the $T'$ phase and after 100 hours, 19 mole% monoclinic phase and 55 mole% cubic phase were present.

Other studies conclude that the degeneration of the $T'$ phase to monoclinic phase is slow at ageing temperatures below 1200 °C [12, 20]. The stability of $T'$ phase in plasma sprayed $ZrO_2 + 8$ wt.% $Y_2O_3$ specimens was also investigated by Lelait [23]. The results show no destabilisation at 1200 °C ageing and only low increase of monoclinic phase content was found for an ageing temperature of 1300 °C, although the ageing time was 500 hours.

It is important to be aware of the destabilisation of the $T'$ phase into monoclinic and cubic phases because higher temperatures will be used in future jet engines.
2.4. Bond coating materials

As mentioned before, the major purpose of the bond coating is to protect the superalloy from high temperature degradation such as oxidation and corrosion. The bond coating is hence oxidised instead of the superalloy. Good protection of the underlying material is achieved if the formed oxide adheres to the bond coating and the oxide growth rate decreases with time. This is the case for $\text{Al}_2\text{O}_3$ and $\text{Cr}_2\text{O}_3$, because these oxides exhibit very low diffusion coefficients for both oxygen and metal ions. At very high temperatures (around 1000 °C), chromia reacts with oxygen and forms a gaseous reactant. The chromia layer is hence not stable at these temperatures. This limits the use of chromium as an oxide former and subsequently, the majority of all bond coatings are designed to form alumina.

Bond coatings are usually divided into two main categories; diffusion coatings and overlay coatings. Typical diffusion processes are CVD (chemical vapour deposition) and pack cementation. Overlay coatings are produced by spraying processes, such as flame spraying, plasma spraying or any HVOF process.

The diffusion processes require an inert chamber containing the part to be coated and a vapour with the coating element. Consequently, production of an aluminium-rich diffusion coating on a nickel-base alloy, demands the vapour to contain aluminium. The vapour also contains a chemical activator, usually a halide. The activator takes part in the reaction between the vapour and the nickel-base material, and facilitates the enrichment of aluminium to the nickel-base surface. Since the coating process requires diffusion of atoms, the process temperature is high, usually between 700 and 1100 °C [2]. Different phases form in the coating and for nickel-base alloys, the important phases are Ni$_3$Al, NiAl and Ni$_2$Al$_3$. A high aluminium content in the coating is preferable for the formation of the protective alumina layer. However, too high aluminium concentrations form Ni$_2$Al$_3$ in the coating, which is a brittle phase to be avoided. Therefore, diffusion coatings on nickel-base alloys are generally designed to contain NiAl-phase.

Generally, the diffusion coatings have an excellent adherence to the substrate material and the life limiting mechanism for the coating is usually due to oxidation behaviour. At high temperatures, the coating is oxidised and an alumina layer forms. Due to stresses that develop in the oxide, especially under cyclic thermal conditions, the oxide scale often spalls off the alloy. Another time dependent mechanism is the diffusion of aluminium from the coating to the superalloy, caused by the coating/superalloy aluminium gradient. The depletion of
aluminium in the coating together with the spalling of the alumina layer during thermal exposure cause difficulties to maintain a continuous alumina layer. A more rapid oxidation can then take place by formation of other oxides such as NiO. These oxides penetrate the bond coating and enter the substrate material. Consequently, the bond coating has lost its protective function.

A way to postpone the depletion process during service is to increase the amount of aluminium in the superalloy. This will decrease the aluminium gradient from bond coating surface to the superalloy, and the diffusion of aluminium to the superalloy will be less pronounced. To decrease the spalling rate of the oxide layer, small amounts (0.01-0.1 wt.%) of rare earth elements such as Sc, Y, Zr, La and Hf are included in the bond coating. The element addition substantially increases the adherence of the oxide although the exact mechanism is not yet understood [2].

The overlay coatings are applied by means of different spray techniques. Almost all materials can be applied and this is an important advantage compared to the diffusion coatings. It means that the composition of the overlay coatings easily can be optimised regarding oxidation rate. A typical overlay bond coating composition is Ni-23Co-17Cr-12Al-0.5Y wt.% [24]. In general, the overlay coatings are superior to the diffusion coatings, considering oxidation rate. However, diffusion coatings are less expensive and exhibit a better adhesion to the superalloy.

**2.5. Failure of thermal barrier coatings**

Several mechanisms are identified to play important roles for the failure of thermal barrier coating systems [25-27]. The different failure mechanisms are categorised as originating from thermal loading, bond coating oxidation, top coating corrosion and top coating phase transformation. The mechanisms can act simultaneously during service but usually, it is possible to determine the dominant mechanism. The different mechanisms are discussed below:

- **Failure due to thermal loading.** The thermal loads, in service, on a coated combustion chamber are difficult to reproduce without performing a full-scale engine rig test. However, these tests are expensive to run and lab-scale tests on coated specimens are set-up to simulate the failure mechanism. This is often done in thermal shock rigs where the specimen is cycled between two different stations, a heating station and a cooling
station. Typically, the heating station consists of a burner flame directed towards the coating while the cooling station uses pressurised air to cool the specimen. Different thermal loading situations can be chosen in the specimen with varying burner flame temperature, cooling and time spent in each station. The specimen is cycled until failure occurs and the number of cycles to failure is used to compare and rank the thermal shock lives between different types of coatings.

The rapid heating of the coating, during the shock, gives rise to a transient temperature gradient in the specimen. This, together with a mismatch in physical and elastic properties between coating and substrate, create stresses in the materials. The stresses cause cracks and delaminations to propagate in the top coating. After several cycles, the cracks may have propagated to a degree that part of the top coating spalls off from the specimen. For thick thermal barrier coating systems (top coating thicker than 1 mm), it is typical that the top coating is almost completely spalled from the bond coating, leaving only small residues in the valleys of the bond coating.

- **Failure due to bond coating oxidation.** If the specimen survives the thermal shocks, other time dependent mechanisms take place. The resistance to failure due to bond coating oxidation is simulated by cycling of the specimen in an oven. A typical cycle contains heating of the specimen to 900 °C, where it is kept for 3 hours, followed by cooling to room temperature. The heating and cooling are done slowly, not to cause temperature gradients within the specimen. The high temperatures during the cycling cause the alumina oxide to grow and the specimen is examined during the cycling until failure occurs. Investigations show failure to occur as cracking in the ceramic above the bond coating peaks and along the oxide-thermal barrier coating interface [28-29]. The failure of the coating system has been correlated to out-growth of NiO pegs into the top coating [29]. These pegs strain the surrounding material and compressive stresses are thought to develop around the pegs. Other studies indicate failure to occur when a critical thickness of the formed oxide has been reached [17, 30].

- **Failure due to top coating corrosion.** This failure type is to a large degree dependent of the fuel used in the combustion. Today's flying jet engines use relatively clean fuel. However, other fuel types are considered and especially in stationary gas turbines, different fuel types are used. This brings the ceramic coating into contact with certain levels of contaminants from the exhaust fumes. Identified corrosive species are molten
V$_2$O$_5$, Na$_2$SiF$_6$ and SO$_3$-Na$_2$SO$_4$ [31]. The elements act to destabilise the top coating by attacking the Y$_2$O$_3$, forming other yttrium-rich oxides [32]. This transforms the tetragonal phase into monoclinic and cubic phases, which should be avoided.

- **Failure due to top coating phase transformation.** This is discussed in 2.3, *Phases in pure and stabilised ZrO$_2* and is only relevant at temperatures above 1200 °C. Notice above section where destabilisation can be triggered by corrosion at temperatures above 850 °C.
3. INDUSTRIAL PROGRAMME

3.1. Thermal barrier coatings in production today

Volvo Aero Corporation (VAC) sprays thermal barrier coating systems onto several components to different jet engine producers around the world. The coated components are utilised both in military and civil engines. Similarities exist between the customer specifications of the coatings concerning thickness and material of the top coating. The thickness is usually specified to around 0.3 mm, and zirconia, stabilised with 7-8 wt.% yttria, is used as top coating material. The thermal barrier coated components are found in the rear part of the jet engine, where the material is exposed to the fumes from the combustion. Figure 1 shows the coated components including combustion chamber, flame holder, afterburner and exhaust nozzle flaps.

Thermal barrier coating systems are today part of the construction work when a jet engine is designed. This means that metallic components expected to reach the maximum design temperature are coated. A second route where thermal barrier coating systems can be introduced on jet engine components is during the development stage of the component. If the component fails due to thermal stresses during the development stage, a solution is to apply a thermal barrier coating system in order to decrease the thermal loads on the component. Finally, thermal barrier coating systems are introduced onto existing components from older engines. Here, the benefit of the coating is mainly to increase the interval between the routine maintenance inspections of the component.

3.2. Future thermal barrier coating systems

Generally, today’s thermal barrier coating systems are not the life limiting part of the coated component. Therefore, development of the coating to ensure longer coating lives is not given highest priority. Instead, considering the ongoing development of jet engines, it is clear that materials with better high temperature properties will be needed in the future. This affects the development of thermal barrier coatings and an objective for the research is to produce coatings with increased insulating abilities. An improved insulation can be used to increase the combustion temperature or decrease the cooling air flow, which, in turn, decreases the overall expenses for the engine.

Improved insulation is achieved by means of reducing the thermal conductivity of the top coating or spraying of a thicker top coating. The thermal conductivity of the top coating is
determined not only by its material but is also, to a large extent, dependent on the microstructure. Investigations have shown the thermal conductivity to decrease with increasing amount of cracks oriented normal to the heat flow direction [12, 33]. Also, the lower thermal conductivity values found in sprayed zirconia compared to sintered zirconia confirm the importance of the microstructure. Another way to reduce the thermal conductivity of the top coating is to change top coating material. At this moment, however, zirconia appears to be the best choice. Development of thicker thermal barrier coating systems, with a top coating thickness reaching 2 mm, is under progress. However, the thermal shock lives of these coatings are still insufficient.

3.3. Projects related to thermal barrier coating systems

To face expected requirements on thermal barrier coating systems, Volvo Aero Corporation takes part in several research projects. A short description of the various projects and their objectives are given below.

- Brite/Euram-project "Improved Plasma Sprayed Thermal Barriers for Relevant Combustor Geometries using Enhanced Process Control and better Test Techniques". The project is divided into two parts. The objective of the first part is to produce a burner can demonstrator with a thick thermal barrier coating system, utilising today’s materials. The burner can will be optimised regarding life time and tested in a thermal shock rig. The second part of the project concentrates towards coating materials to withstand higher temperatures than 1200 °C. This means new top coating materials will be selected.

- PhD-projects at Linköping University. The work described in this thesis, which addresses the problem of characterising the top coating microstructure after spraying of a thick thermal barrier coating system. The project also aims at increasing the life of the thick thermal barrier coating system.

  The second project related to thermal barrier coating systems involves optimisation of a non-destructive microcomputer tomography system to study the crack structure in sprayed top coatings.

- PhD-projects at Lund University. First project aims at studying the edge effects of coatings and determine the influence of edges on coating life. Second project is related to residual stress measurement techniques, developed for sprayed coatings.
• PhD-projects at Högskolan Trollhättan-Uddevalla. Two projects started, to investigate the potential for artificial intelligence as a tool to determine optimum spraying parameters. Also, expert systems will be developed to correlate spraying parameters to coating properties.

• PhD-project at University of Manchester (UMIST). The project focuses generally on the mechanical properties of the bond coating and especially on the stress relaxation of the bond coating.

Two Brite/Euram applications are also being prepared. The first is focused towards characterisation of the powder particles in the plasma flame. The project contains measurement techniques as well as modelling of the process. The objective is to develop an on-line measurement technique with a closed loop to the spraying parameters in order to control the sprayed droplets. The second application aims at reducing the thermal conductivity of the top coating by means of an improved microstructure.
4. SUMMARY OF APPENDED PAPERS

4.1. Introduction and objective

The life cycle of a thermal barrier coating system can be described by five different stages, according to Fig. 8. In stage 1, a plasma arc is generated by the plasma gun and the coating powder material is fed into the hot arc. In this stage, the process can be controlled by several parameters related to the plasma gun or to the powder feed. Stage 2 involves the interactions between the powder particles and the hot gases. The powder particles are supposed to accelerate and melt within the hot gas stream. Stage 3 comprises the coating build-up, where the molten particles hit the underlying material. Here, some parameters exist to control the process such as the substrate temperature, amount of cooling and the motion of the substrate relative to the plasma arc. In stage 4, the as-sprayed coating is exposed to service conditions. Different irreversible processes take place within the coating system and finally, in stage 5, failure of the coating system occurs.

![Schematic of the life cycle of a thermal barrier coating system.]

The objective of the work performed within this project has been to characterise the as-sprayed condition of the top coating in a thick thermal barrier coating system. The work has increased the knowledge of existing defect densities within the as-sprayed top coating as well as residual stress state within the coating system. Furthermore, the defect densities and the residual stress state can be controlled by parameters such as substrate temperature during spraying. Hence, the work has been performed in stage 3, according to Fig. 8. Connections to stage 1 also exist, because the spraying parameters have been controlled.
The long term aim with the project (together with the above mentioned B/E-project) is to reach an acceptable service life for a thick thermal barrier coating system on a real component. To fulfil this aim, correlations between stage 3 and stage 5 have to be found. This raises the question of how a thick thermal barrier coating system should be composed in stage 3 in order to reach a long life time.

4.2. Paper 1, Characterization of Microstructural Defects in Plasma Sprayed Thermal Barrier Coatings

Thermal barrier coatings with a NiCrAlY bond coating and a 1.4 mm thick zirconia top coating were air plasma sprayed onto a nickel-base substrate. The top coatings were deposited with the same spraying parameters except for the amount of external cooling, which varied from no cooling to the maximum available. This resulted in four sets of samples produced with different cooling conditions where substrate temperature varied from 100 to 830 °C. The coatings were examined by electron microscopy on polished surfaces and on fracture surfaces. The crack structure in the top coating was correlated to the substrate temperature. The density both of horizontal delaminations and of vertical microcracks was shown to decrease at higher substrate temperatures. The grain structure was columnar, and smaller grains were found at lower temperatures. Explanations for the differences in defect densities are discussed.

4.3. Paper 2, Modelled and Measured Residual Stresses in Plasma Sprayed Thermal Barrier Coatings

Thermal barrier coatings consisting of a NiCrAlY bond coating and a 1.4 mm thick partially stabilised zirconia top coating were air plasma sprayed onto grit blasted nickel-base substrates. Two samples were produced using different amount of external cooling during spraying of the top coatings. The residual stress profiles in the samples were measured after each manufacturing process step with a layer removal technique. A finite element model including a thermal analysis and a stress-strain analysis of the deposition was developed to model the origin of the thermal stresses and to verify the measured residual stresses.

The main components for the residual stresses in the sprayed coatings were identified as stresses developing during the rapid cooling of individual droplets (quenching stresses) and stresses formed during cooling from deposition temperature to room temperature. The quenching stresses were predicted to be low and tensile in the top coating, due to stress relaxation by formation of vertical microcracks. During cooling to room temperature, compressive stresses were superimposed on the top coating quenching stresses. The final
residual stresses were predicted to be compressive in the top coating, which were confirmed by measurements (-15 MPa). In the bond coating, no stress relaxation by microcracking was observed and the residual stresses were found to be tensile (=100 MPa). In the substrate, compressive residual stresses reaching -200 MPa were found in a zone to a depth of 0.3 mm into the substrate. The stresses were found to have originated during the grit blasting of the substrates prior to bond coating deposition.

A correlation between modelled in-elastic strain and measured densities of vertical microcracks in the top coating was obtained. High values were found close to the bond coating, which was correlated to a low substrate temperature during spraying of the top coating material.

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