2. 3 Nuclear Reaction Theory

2. 3. 1 Quantum and Statistical Theories of Nuclear Reactions

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Abstract Basic concepts and methods in the description of nuclear reactions are briefly reviewed in a historical perspective. Discussed are the discovery and the quantum mechanical and statistical theories of compound nucleus (CN), the discovery of direct processes at intermediate and low energies, optical model, and a theoretical treatment of direct and CN processes. in coexistence.

I. Introduction

The history of nuclear reaction studies dates back to 1909 to 1911 when Rutherford's group discovered the nuclei of Au and Pt atoms through observations of large angle elastic scattering of α particles by thin foils of the metals\(^1\). The first systematic studies of nuclear reaction mechanisms, however, were carried out in 1932 to 1933 by Fermi et al.\(^2\) They irradiated 38 nuclides ranging from Hydrogen to Uranium with then newly discovered neutrons, and observed the induced radioactivity. Two remarkable features were observed: (a)(n,γ) cross sections were very large, and (b)cross sections at very low energies were very large for some nuclides, but not so much for other nuclides. (a) suggested that the system stayed in a highly excited state in which it emits photon(s) for a very long time. Fermi et al estimated the period to be at least \(10^{16}\) sec., very much longer than the collision time, of the order of \(10^{-22}\) sec. (b) was later found to be due to resonances at the incident energies.

II. Compound nucleus model and resonance theories

The features of neutron induced reactions mentioned above were explained by N. Bohr, and G. Breit and E. P. Wigner\(^3\) with compound nucleus model. The model postulates strong interaction between nucleons that causes immediate dissipation of the energy of the incident nucleon among many nucleons in the target nucleus. As a result, none of the nucleons has enough energy to escape from the compound state and a compound nucleus is formed. It is a quasi-bound state that lasts until the energy re-concentrates onto a small number of nucleon(s) that are then emitted from it. Energy eigen-values of a compound nucleus are, as those of true bound system, discrete. They are, however, complex because of its finite life time

\[ W_\lambda = E_\lambda - iT_\lambda /2 \]  \hspace{1cm} (I.1)

where \(\Gamma_\lambda\) is related to the lifetime \(\tau_\lambda\), by \(\tau_\lambda = h / \Gamma_\lambda\). Reactions only proceed when the incident energy matches one of the resonance energies \(E_\lambda\), within the width \(\Gamma_\lambda\), hence resonance. The long life explains the large \((n,\gamma)\) cross sections. If \(\Gamma_\lambda\) is a few tens eV, as often observed, \(\tau_\lambda\) is indeed of the order of \(10^{-16}\) sec. as estimated by Fermi et al.
Breit and Wigner derived the famous Breit-Wigner formula for cross sections at energies near a resonance by analogy with the atomic resonance fluorescence using Wigner-Weisskopf perturbation theory. The analogy, however, is imperfect since nuclear force is too strong to warrant perturbation theory. Besides, the incident neutron does not disappear in the intermediate state, unlike the photon in resonance fluorescence. A complete quantum mechanical formulation free from such defects was given by Kapur and Peierls.

Characteristic features of the Kapur-Peierls formalism, and of the subsequent R-matrix formalism etc., are the use of coordinate space representation. The internal region, I, is defined to be where all nucleons are close together and interact with each other by nuclear force. A channel $\gamma$ is where system is partitioned into two particles, say c and C, whose centers of mass are separated by $r_\gamma \geq a_\gamma$, where $a_\gamma$ is the channel radius, which is beyond the range of nuclear interaction between c and C. The boundary between channel $\gamma$, and I is defined by $r_\gamma a_\gamma$. The compound nucleus state wave function $X_\lambda$, is defined to satisfy the Schrödinger equation in I,

$$HX_\lambda = HW_\lambda \quad \text{(in I),}$$

where $H$ is the total Hamiltonian and $W_\lambda$ is the energy eigen-value. The boundary condition for $X_\lambda$ is given at the boundaries of I with the channels, neglecting its amplitude in the configurations in which the system is separated into more than 3 fragments. Kapur and Peierls assumed the boundary condition of a decaying state. If we neglect all angular momenta and charges of the particles for simplicity, it has the form

$$X_\lambda = \sum_\gamma A_{\lambda\gamma} \frac{e^{ik_\gamma r}}{r_\gamma} \phi_\gamma (\xi_\gamma) \quad (r_\gamma = a_\gamma \text{for all } \gamma) \quad \text{(II.2)}$$

where the summation is over all the channels. $\phi_\gamma (\xi_\gamma)$ is the internal wave function, $k_\gamma$ is the wave number of relative motion at the experimentally given total energy, $E$, and $A_{\lambda\gamma}$ is the amplitude of $X_\lambda$ at the boundary of channel $\gamma$. A decaying state is supposed to represent a compound nucleus state long after it is formed: with no incident wave any more, it only decays from all open channels in outgoing waves as in (II.2). Eqs. (II.1) and (II.2) constitute an eigen-value problem of which $W_\lambda$ is a discrete eigen-value, $X_\lambda$ is the corresponding eigenfunction. They and their time reversed functions, $\bar{X}_\lambda$, form a complete bi-orthogonal set, \{ $X_\lambda$, $\bar{X}_\lambda$ \}, with $< X_\lambda | X_\lambda > = < \bar{X}_\lambda | \bar{X}_\lambda > = 1$, $< X_\lambda | \bar{X}_\lambda > = N^\lambda \delta_{\lambda\lambda}$, and $\sum_\lambda |X_\lambda > < \bar{X}_\lambda | N^\lambda = 1$ in I.

The total wave function, $\Psi^{(s)}_\alpha$, which describes a reaction initiated in channel $\alpha$ at energy $E$, satisfies the Schrödinger equation, $H \Psi^{(s)}_\alpha = E \Psi^{(s)}_\alpha$, and an asymptotic boundary condition:

$$\Psi^{(s)}_\alpha \sim \frac{e^{-ik_\alpha r_\alpha}}{r_\alpha} \sum_\beta \sqrt{\frac{v_\alpha S_{\alpha\beta}}{v_\beta}} \frac{e^{ik_\beta \rho}}{r_\beta} \quad \text{(II.3)}$$

where $S_{\alpha\beta} = S_{\beta\alpha}$.
where \( S_{\beta \alpha} \) is the S-matrix element of transition \( \alpha \rightarrow \beta \), and the \( v \) are the relative velocities in the channels. If one expands \( \Psi_a^{(+)} \) in \( \{ X_\lambda \} \) in I as
\[
\Psi_a^{(+)} = \sum_\lambda a_\lambda X_\lambda \quad \text{(in I)},
\]
(II.4)
one can calculate the coefficients, \( a_\lambda \), using the Schrödinger equations and the boundary conditions for \( \Psi_a^{(+)} \) and \( \{ X_\lambda \} \). One can then obtain \( S_{\beta \alpha} \), if one substitutes (II.2) and (II.3) into (II.4) and compares the amplitudes of the outgoing wave in channel \( \beta \) on both sides of the equation. The result is
\[
S_{\beta \alpha} = e^{-i k_a a_\alpha} \delta_{\beta \alpha} - i \sum_\lambda \frac{g_{\lambda \beta} g_{\lambda \alpha}}{E - W_\lambda} \quad \text{(II.5)}
\]
where \( g_{\lambda \gamma} = \sqrt{\hbar v}/(2\pi)^3 N_\lambda A_{\lambda \gamma} \). Eq. (II.5) is the dispersion formula of the S-matrix element.

As one can see, few assumptions, in particular no perturbative approximations, are made in its derivation. The second term on the right hand side of (II.5) causes resonance when \( E \) gets close to one of the \( W_\lambda \). The first term represents a scattering by a hard sphere of radius \( a_\alpha \) that occurs only in the elastic channel. It is virtually the only process that occurs if the incident energy is off resonance.

The total cross section of the reaction \( \alpha \rightarrow \beta \) is given by
\[
\sigma_{\beta \alpha} = \frac{\pi}{k_a^2} |\delta_{\beta \alpha} - S_{\beta \alpha}|^2.
\]
(II.6)
If \( E \) is within \( \Gamma_\lambda \) from \( E_\lambda \), and all other resonances are so far away that they can be neglected in the S-matrix element, one obtains the Breit-Wigner one-level formula.

\[
\sigma_{\beta \alpha} = \frac{\pi}{k_a^2} \frac{\Gamma_{\lambda \beta} \Gamma_{\lambda \alpha}}{(E - E_\lambda)^2 + \Gamma_\lambda^2 / 4} \quad \text{for } \beta \neq \alpha,
\]
(II.7)
where \( \Gamma_{\lambda \gamma} = |g_{\lambda \gamma}|^2 \). Since \( g_{\lambda \gamma} \propto A_{\lambda \gamma} \), \( \Gamma_{\lambda \gamma} \) is proportional to the probability of \( X_\lambda \) having the internal state \( \phi_\gamma \). It therefore reflects the structure of \( X_\lambda \). It can be shown that for an isolated resonance
\[
\Gamma_\lambda = \sum_\gamma \Gamma_{\lambda \gamma}.
\]
The total width, \( \Gamma_\lambda \), is the total probability of decay of state \( \lambda \) per unit time and \( \gamma \)-width, \( \Gamma_{\lambda \gamma} \), is that through channel \( \gamma \).

The definition of the resonance state depends on the boundary condition. The decaying state boundary condition is physically plausible, but not unique. In fact, one can replace (2) by
\[
X_\lambda = \sum_\gamma C_{\lambda \gamma} \frac{u_{\gamma}(r_\gamma)}{r_\lambda} \phi_\gamma(\xi_\gamma)
\]
(II.8)
where \( u_{\gamma}(r_\gamma) \) satisfies
\begin{equation}
\frac{u'(r_y)}{u(r_y)} \bigg|_{r_y=a_y} = b_y \tag{II.9}
\end{equation}

where $b_y$ is a quantity that specifies the boundary condition. For a decaying state, $b_y = ik_y$ is assumed. Since $ik_y$ is $E$-dependent, so are all resonance parameters, including $W_x$, although they can be regarded as constants in a narrow energy range around the resonance. In the R-matrix theory of Wigner, $b_y = \text{real constant}$ is assumed. The resonance parameters are then energy independent. The price for that is that the dispersion formula directly derived is for the R-matrix, and the S-matrix is only obtained by a linear fractional transformation. The result of the transformation, if carried out exactly, can be shown to be term by term identical to the Kapur-Peierls formula. This is, of course, not the case if approximations are made, as is always done in practice, either to the R-matrix or to the transformation. Another choice is $b_y = ik_y$, where $k_y$ is the wave number corresponding to $E = E_y$. Then, $W_y$ is a pole of the S-matrix. Bloch pointed out that all the boundary conditions mentioned above may be included in the Hamiltonian

$$H = \sum_P \frac{\hbar^2}{2\mu_P} \delta(r_p - a_P)\left[ \frac{d}{dr_p} - b_p \right] r_p$$

In all the formalisms mentioned above make use of the concept of channel radius the choice of which is not unique. Resonance parameters depend on the choice, except for $b_y = ik_y$. The exact S-matrix is, of course, independent of the choice, but approximate ones are not. Normally, it is adjusted so that the approximate S-matrix best reproduces experimental data.

The formalism initiated by Feshbach has no channel radius since it is based on the partition of the Hilbert space of wave functions, rather than the configuration space. The partition is rather flexible. Perhaps the most common one is that into P-space that includes all open channels and the complementary Q-space. A compound nucleus state is defined as an eigen-state of $QHQ$,

$$QHQ \Phi = E_y \Phi$$

and the resonance energy, $W_y$ is given by

$$W_y = E_y + \left( \Phi \left| QHP \frac{1}{E + i\epsilon - PH} QHP \right| \Phi \right)$$

where $P$ and $Q$ are projection operators onto P and Q spaces respectively. Feshbach's formalism has no ambiguity related to channel radius. It, however, has its own non-uniqueness related to the partition of the Hilbert space. Q-space is often defined as a space in which all nucleons are bound in a single particle potential. This definition naturally depends on the single particle potential.
III. Statistical compound nucleus model

The spacing of compound nucleus levels become narrower and the widths of individual levels wider as the incident energy gets higher. The levels eventually overlap with each other. At such energies, the energy spread of the incident beam, \( I \), contains many resonance levels, and what is measured is an energy average over \( I \), hereafter denoted by \(< >\). The observed total cross section of a reaction \( \alpha \rightarrow \beta \neq \alpha \), e.g., is given by

\[
< \sigma_{\beta\alpha} > = \frac{\pi}{k_\alpha^2} \left| \sum_{\lambda} \frac{g_{\lambda \beta}g_{\lambda \alpha}}{E - E_\lambda + i\Gamma_\lambda / 2} \right|^2 . \quad \text{(III.1)}
\]

If one neglects all the resonance levels outside \( I \), and assumes that the parameters of the remaining resonances are statistically random, one can neglect the average of the cross terms and obtain

\[
< \sigma_{\beta\alpha} > \approx \frac{\pi}{k_\alpha^2} \left| \frac{g_{\beta \alpha}}{E - E_\alpha + i\Gamma_\alpha} \right|^2 \approx \frac{\pi}{k_\alpha^2} \sum_{\lambda} \frac{2\pi \Gamma_{\lambda \beta} \Gamma_{\lambda \alpha}}{\Gamma_\lambda} \approx \frac{\pi}{k_\alpha^2} \frac{2\pi}{< D >} < \Gamma_{\lambda \beta} \Gamma_{\lambda \alpha} / \Gamma_\lambda >
\]

where \(< D >\) is the average level distance and \(< \Gamma_{\lambda \beta} \Gamma_{\lambda \alpha} / \Gamma_\lambda >\) is the average value of the ratio in \( I \). If the \( \Gamma' \) are uncorrelated to each other, \(< \Gamma_{\lambda \beta} \Gamma_{\lambda \alpha} / \Gamma_\lambda > = < \Gamma_{\lambda \beta} > < \Gamma_{\lambda \alpha} > / < \Gamma_\lambda >\), hence

\[
< \sigma_{\beta\alpha} > \approx \frac{\pi}{k_\alpha^2} \frac{T_\beta T_\alpha}{\sum_{\gamma} T_\gamma} \quad \text{(III.2)}
\]

where

\[
T_\gamma = 2\pi < \Gamma_{\lambda \gamma} / D > , \quad \text{(III.3)}
\]

under the assumption that \( D \) is uncorrelated to the widths. Eq. (III.2) is the well-known Hauser-Feshbach formula\(^{13}\) for a transition to a discrete final state of the residual nucleus. \( T_\gamma \) is called a transmission coefficient. The sum of \(< \sigma_{\beta\alpha} >\) over all the open channels, \( \beta \), is the compound nucleus formation cross section through channel \( \alpha \). It is given by

\[
\sigma_{\alpha}^{CN} = \sum_{\beta} < \sigma_{\beta\alpha} > = \frac{\pi}{k_\alpha^2} \sum_{\gamma} T_\gamma \quad \text{(III.4)}
\]

since the denominator is canceled by the summed numerator. One can rewrite (III.4) as

\[
< \sigma_{\beta\alpha} > = \sigma_{\alpha}^{CN} R_\beta \quad \text{(III.5)}
\]

where \( R_\beta = T_\beta / \sum_{\gamma} T_\gamma \) is the branching ration of the decay from channel \( \beta \). Eq. (III.4') shows the statistical independence of the formation and decay of the compound nucleus since \(< \sigma_{\beta\alpha} >\) is given by a product of the probabilities of those processes, The memory of how the compound nucleus is formed is lost by the time of it decays.

If the residual nucleus \( B \) is in the continuum, the cross section per unit energy interval of the outgoing particle is given by

\[
\frac{d\sigma_{\beta\alpha}}{dE_\beta} = \sigma_{\alpha}^{CN} R_\beta (E_\beta) \omega_\beta (U_\beta) \quad \text{(III.6)}
\]
where \( \omega_b(U_B) \) is the level density of B at excitation energy \( U_B = U_{B_{\text{max}}} - E_p \), where \( U_{B_{\text{max}}} \) is the maximum value of \( U_B \). The energy spectrum is determined by \( R_B(E_p)\omega_B(U_B) = T_B\omega_B(U_B) \) since the denominator, \( \sum \gamma T_\gamma \) of, \( R_B(E_p) \) does not depend on \( E_p \). Using \( T_B \propto E_B\sigma_{BN}^CN(E_B) \), one gets

\[
\frac{\partial \sigma_{BA}}{\partial E_p} \propto E_B\sigma_{BN}^CN(E_B)\omega_B(U_B).
\] (III.7)

Thus, the energy spectrum is determined by the cross section of the inverse process, i.e., formation of the compound state through channel \( \beta \), and the level density of the residual nucleus, not the compound nucleus. The level density, \( \omega_B(U_B) \), is an exponentially increasing function of \( U_B \). Expanding its logarithm around \( U_{B_{\text{max}}} \) and keeping only the first order term, one obtains

\[
\omega_B(U_B) = \omega_B(U_{B_{\text{max}}}) \exp\left(-\frac{E_B}{T_B}\right)
\] (III.8)

where

\[
T_B = \left( \frac{\partial \omega_B}{\partial U_B} \right)^{-1}
\] (III.9)

is called the temperature of B. Substituting (III.9) into (III.8), one obtains

\[
\frac{\partial \sigma_{BA}}{\partial E_p} \propto E_B\sigma_{BN}^CN(E_B)\exp\left(-\frac{E_B}{T_B}\right).
\] (III.10)

Since \( \sigma_{BN}^CN(E_B) \) varies slowly with \( E_B \), the energy spectrum is mainly determined by \( E_B\exp\left(-\frac{E_B}{T_B}\right) \), which is called Maxwellian, and is essentially the energy spectrum of molecules evaporated from a liquid. The model is therefore called evaporation model \(^{14}\) It should be noted that the evaporation model is derived straightforwardly from the dispersion formula of the S-matrix under the assumptions of uncorrelated randomness of resonance parameters. No additional assumptions such, as the principle of detailed balance, is made. The underlying statistical assumption on resonance parameters, however, is still questionable, though look plausible. That will be discussed in VIII.

IV. Direct reactions at high energies

In the late 1940s, a series of experiments with 90 MeV neutrons and 200 MeV deuterons was done at Berkeley. It was found that much more high energy particles are emitted at forward angles than the prediction of compound nucleus model. R. Serber explained it with a model \(^{15}\) that at such high energies the first step in the process of a reaction is collisions of the incident particle with individual target nucleons since the collision time is so short that the interaction with the third nucleon can be neglected. Since the nucleon-nucleon scattering cross section is small at high energies, the mean free path of the projectile in the nuclear medium is comparable with the nuclear radius. In many cases, therefore, the
projectile gets out of the nucleus without or after small number of collision(s) with target nucleons. It is emitted at high energies and in forward directions. The struck target nucleons follow similar fate if their energies are high. If the energies of the projectile or the struck nucleons get too low, they are trapped in the nucleus and eventually form a compound nucleus.

Serber's picture was immediately translated into various models of direct reactions, such as optical model\textsuperscript{16}, intra-nuclear-cascade model\textsuperscript{17} stripping and pick-up model of reactions involving deuterons\textsuperscript{18}, etc. The picture was given quantum mechanical foundations in impulse approximation\textsuperscript{19} that was then developed into the multiple scattering theory\textsuperscript{20} and the Brueckner theory of nuclear matter.

V. Optical model

Soon after direct reactions were found at high energies, it was also found at low energies. Besides stripping and pick up reactions of deuterons, which were first found, perhaps the most remarkable discovery was the gross structure in the neutron total cross section $\sigma$, at low energies, $E_n = 0 \sim 3\text{MeV}$. In his poor resolution measurements of $\sigma$, as a function of $E_n$, and the target nucleus mass number, $A$, H. H. Barshall observed\textsuperscript{21} peaks and valleys with spacing and widths of the order of MeV, in contradiction to a monotonous dependence on $E_n$ and $A$ predicted by the compound nucleus model.

Feshbach, Porter and Weisskopf\textsuperscript{22} (hereafter abbreviated as FPW) interpreted this as due to the interference between the incident and the elastically scattered waves. The interference is possible only if the scattered wave is emitted at the same time as the incident wave. Compound nucleus process is too slow for that. FPW assumed that the prompt scattering can be described as a scattering by a complex one-body potential,

$$U(r,\sigma) = V(r,\sigma) + iW(r,\sigma),$$

where $r$ is the relative coordinate between the incident neutron, $n$, and the target nucleus, $A$, and $\sigma$ is the spin of $n$. The imaginary part, $iW(r,\sigma)$, represents the absorption of the flux of the entrance channel into non-elastic channels. The absorption may be virtual, and the flux may come back to the elastic channel later, after the incident wave has gone. It is important to note that $W$ is not necessarily zero even if no non-elastic channel is open.

FPW succeeded in reproducing Barshall’s data with a square-well potential of radius $R = 1.45A^{1/3}\text{fm}$ and depths,

$$V = 42\text{MeV} \text{ and } W = 0.03V.$$ 

It is remarkable that $W$ is only 3% of $V$. That shows that in most cases the incident neutron is scattered off by the mean field of the target nucleus.

Compound nucleus is formed, however. Resonance peaks do show up in good resolution experiments at the energies of Barshall’s experiments. How could this be compatible with the small $W$ that FPW found? They emphasized the fact that optical model
describes a wave function, averaged over the energy interval, I, of the incident beam, and its scattering amplitude, \( f_{se} \), called \textit{shape elastic scattering} amplitude, is an energy-average of the elastic scattering amplitude, \( f_{el} \),

\[
f_{se} = \langle f_{el} \rangle. \tag{V.1}
\]

Friedman and Weisskopf\textsuperscript{23} argued that the energy-averaged wave function was a wave packet of length \( h/I \) in time that was of the order of \( 10^{-22} \) sec in Barshall's experiment, and optical model only described what happened in that short time.

How about the total cross section, \( \sigma_t \), which includes everything, then? It \textit{can} be calculated with optical model because of the optical theorem,

\[
\sigma_t = 4\pi \text{Im} f_{el}(0)/k_n \]

where \( f_{el}(0) \) is the forward elastic scattering amplitude. Thus,

\[
\langle \sigma_t \rangle = \frac{4\pi}{k_n} \text{Im} f_{el}(0) = \frac{4\pi}{k_n} \text{Im} f_{se}(0). \tag{V.2}
\]

The physical reason for this is simple. When a short wave packet hits the target nucleus, all reaction processes \textit{begin} at once, although slow ones finish much later. The total cross section, which is the sum of the Cross sections of them all, can therefore be calculated from the instantaneous change of the wave packet, i.e., from \( f_{se} \). Obviously, this argument does not apply to cross sections of individual processes. For example, elastic scattering through compound nucleus formation is not included in \( f_{se} \), but in

\[
f_{se} = f_{el} - f_{re} \tag{V.3}
\]

The corresponding cross section, \( \sigma_{ce} = \|f_{ce}\|^2 \), cannot be calculated with optical model. It also contributes to the energy averaged elastic scattering cross section, hence

\[
\langle \sigma_{ce} \rangle = \langle \sigma_{se} \rangle \tag{V.4}
\]

The energy-average of total reaction cross section, \( \sigma_r \), is

\[
\langle \sigma_r \rangle = \langle \sigma_t \rangle - \langle \sigma_{el} \rangle = \langle \sigma_t \rangle - \langle \sigma_{se} \rangle = \langle \sigma_{re} \rangle. \]

Thus, e.g., for s-wave neutrons,

\[
\sigma_r = \sigma_{re}^{CN} = \frac{\pi}{k_n^2} \langle T_{n0} \rangle = \frac{2\pi^2}{k_n^2} \langle \Gamma_{n0} \rangle. \tag{V.5}
\]

As we saw in III., \( \sigma_{re}^{CN} \) and transmission coefficients are essential ingredients of the statistical compound nucleus model. \( \langle \Gamma_{n0} \rangle/D \rangle \), is a directly measurable quantity. FPW calculated it with their optical potential for 0 energy neutrons as a function of \( A \) and reproduced experimental data very well, except for some cases that could be understood from the structure of the target nuclei.
In the continuum region, \( f_{\eta} \) is a slowly varying function of energy, therefore \( f_{\eta} = \langle f_{\eta} \rangle = f_{\eta0} \). One can therefore calculate \( f_{\eta} \) itself and consequently every measurable quantity in the elastic channel with optical model and compare with experimental data. Thus, measurements of not only cross sections, but also various polarization quantities have been made for various combinations of projectiles and target nuclei in wide range of incident energies. Optical potentials have been searched for to reproduce individual experimental data. Global optical potentials have also been found for nucleons, deuteron, and alpha particle in certain regions of incident energies and target nuclei\(^{24}\). Optical model is one of the most basic concepts of nuclear reaction studies today.

VI. Direct reactions at low energies

Success of the optical model shows that direct reactions occur not only at high energies, as Serber's model implied, but also at low energies. In fact, stripping of deuteron\(^{25}\) was discovered even before the optical mode. Excitation of collective nuclear states was then found in (p,p') and (α,α').\(^{26}\) The characteristic diffraction-pattern like angular distribution in those reactions was well reproduced by the differential cross section formula

\[
\frac{d\sigma}{d\Omega} \propto (qR)^2
\]  

(VI.1)

which can be obtained with simple plane wave Born approximation (PWBA) under the assumption of surface reaction. In (VI.1), \( q \) and \( l \) are the linear and angular momentum transfer and \( R \) is the nuclear radius. \( j_i \) is the spherical Bessel function of order \( l \). (VI.1) enable one conversely to determine \( l \) from the observed angular distribution. Since \( l \) and \((-1)^l\) are the difference between the spins and parities, respectively, of the initial and the final nuclear states, this provides one with a means of determining the spin and parity of one of the states from the other. That was the advent of the nuclear spectroscopy with direct reactions.

PWBA reproduced angular distributions well enough to determine \( l \), and often relative magnitude of cross sections, but not their absolute magnitude: Calculated values were off experimental values sometimes by a factor of 1000. DWBA\(^{24,27}\) was then introduced as a more reliable method of analysis. For a transition \( \alpha \rightarrow \beta \), the wave functions of relative motion, \( \chi_\alpha \) and \( \chi_\beta \), are, in stead of plane waves, now solutions of

\[
(K_\gamma + U_\gamma)\chi_\gamma = E_\gamma \chi_\gamma \quad (\gamma = \alpha, \beta)
\]  

with the distorting potential, \( U_\gamma \). The S-matrix element is given by

\[
S_{\beta\alpha}^{\text{DWBA}} = C < \chi^-_{\beta}\phi_\beta |V_\gamma - U_\gamma|\chi^+_{\alpha}\phi_\alpha > = C < \chi^+_{\beta}\phi_\beta |V_\alpha - U_\alpha|\chi^-_{\alpha}\phi_\alpha >
\]  

(VI.2)

where \( C \) is a constant, \( V_\gamma \), \( \gamma = \alpha \) or \( \beta \), is the potential that causes the transition. The superscript \( (+) \) \((-)\) on the \( \chi \) stands for an outgoing (incoming) scattered wave. The first form of \( S_{\beta\alpha}^{\text{DWBA}} \) in (VI.2) is called post form and the second one prior form. The post-prior identity may not hold if the expressions are calculated approximately as is done in most calculations.
The $U_\gamma$ are normally taken to be the optical potentials that describe elastic scattering in the respective channels. Tacit assumptions are that the projectile (ejectile) feels the same mean field as in elastic scattering before (after) the transition takes place. This procedure, however, is theoretically questionable, as Satchler pointed out, because it double-counts $V_\gamma$ once as the transition potential, and once in the optical potential. Satchler proposed asymmetric DWBA, in which the distorting potential is the optical potential minus a term linear in $V_\gamma$. This model is a step closer to the method of coupled channels discussed shortly than DWBA, although it is not always easy to implement in practice. Since the modified potential cannot be determined from elastic scattering data.

In some cases, direct transition $\alpha \rightarrow \beta$ is hindered or forbidden but two step processes $\alpha \rightarrow \gamma \rightarrow \beta (\alpha=1,2,\ldots)$ are allowed, but not so strong. Examples are $({}^3\text{He} \rightarrow \alpha \rightarrow t)$, $({}^3\text{He} \rightarrow d \rightarrow t)$, $(p \rightarrow d \rightarrow t)$ etc. In such cases second order DWBA is useful.

DWBA should not work for strong transitions. In such cases, multi-step processes are important. The method of coupled channels (CC) is a general theoretical framework to deal with such reactions. It was first used by S. Yoshida and was extensively developed by T. Tamura for analyses of inelastic collective excitations. In CC, the total wave function, \( \Psi^{(\alpha)} \), is expanded as

\[
\Psi^{(\alpha)} = \sum_s \chi_s (r_s) \phi_s (\xi_s)
\]  

in the wave functions of those channels, \( s \), which are strongly coupled, either directly or indirectly, to the entrance channel \( \alpha \) and/or the exit channel \( \beta \). Assuming that the internal wave functions, \( \phi_s (\xi_s) \), are known, one solves the Schrödinger equation,

\[
(E_s - K_s - U_s) \chi_s = \sum_j \left< \phi_j | H - E | \phi_s \right> \chi_s .
\]  

for the unknown wave functions of relative motion, \( \chi_s (r_s) \). \( E_s (K_s) \) is the total (kinetic) energy of relative motion, and \( U_s \) is the diagonal matrix element of the interaction potential. A basic assumption of CC is that one can replace the \( U_s \) for all \( \gamma \), by phenomenological complex potentials and include in them all the effects of all the eliminated channels. The boundary condition for the \( \chi_s (r_s) \) is that the incident wave is only in \( \chi_\alpha \) and outgoing waves are in all the \( \chi_\gamma \).

The coupling potentials, \( \left< \phi_j | H - E | \phi_s \right> \), on the RHS of (VI.4) are local operators in the coordinate space if only inelastic channels are coupled. They are non-local operators if rearrangement channels are coupled, and the method is called the method of coupled reaction (rearranged) channels. CC wave functions are sometimes used as distorted waves in DWBA calculations to take account of strong coupling of inelastic channel(s) in the entrance and/or exit channels. The approximation is called CCBA.
If the projectile or ejectile is weakly bound, they can easily break up, really or virtually, in the course of the reaction. The break-up channels couple strongly with the elastic channel and with each other. Since the break-up states are in the continuum, there are continuously infinite numbers of coupled channels. The method of Continuum Discretized Coupled Channel (CDCC) is designed to deal with this problem, with truncation and discretization of the linear and angular momenta of the weakly bound sub-system. The model space then contains only a finite number of channels. CDCC has been highly successful as a method of analysis of reactions involving weakly bound systems, in particular deuteron, $^6\text{Li}$, and $^{12}\text{C}$.

The method of coupled channels has been highly successful in reproducing variety of experimental data on multi-step direct reactions in light and light heavy ions leading to discrete states of the residual nucleus. It is a standard theoretical frame for theories of direct nuclear reaction with discrete final nuclear states.

VII Direct and compound nucleus processes

Coexistence of direct and compound nucleus processes poses a serious problem to the basic assumption of the statistical compound nucleus model described in III that the resonance parameters are statistically uncorrelated. Separation of the shape elastic scattering, a direct process, from the compound nucleus process discussed in V can naturally be extended to a general reaction, $\alpha \to \beta , (\alpha \neq \beta$): A direct process is described by an energy averaged S-matrix element,

$$S^{\text{dir}}_{\alpha \beta} = \langle S_{\alpha \beta} \rangle,$$

and the slow processes by fluctuation part of the S-matrix element,

$$S^{\text{fl}}_{\alpha \beta} = S_{\alpha \beta} - S^{\text{dir}}_{\alpha \beta}.$$

The cross section is given by

$$\sigma_{\alpha \beta} = \sigma^{\text{dir}}_{\alpha \beta} + \sigma^{\text{fl}}_{\alpha \beta},$$

where

$$\sigma^{\text{dir}}_{\alpha \beta} = \frac{\pi}{k_\alpha} |S^{\text{dir}}_{\alpha \beta}|^2 \quad \text{and} \quad \sigma^{\text{fl}}_{\alpha \beta} = \frac{\pi}{k_\alpha} |S^{\text{fl}}_{\alpha \beta}|^2.$$

Question now is how this separation be related to the dispersion formula of the S-matrix? Suppose one takes energy average of (II.5),

$$S^{\text{dir}}_{\alpha \beta} = \langle S_{\alpha \beta} \rangle = -i \sum \frac{g_{\lambda \gamma} \delta_{\lambda \alpha}}{E - W_\lambda}.$$

If the $g_{\lambda \gamma}$ are statistically uncorrelated as assumed in III, the RHS of (VII.4) would be 0. Since $S^{\text{dir}}_{\alpha \beta}$ is not 0 in reality, $g_{\lambda \gamma}$ must be correlated, no matter how complicated and random the compound nucleus levels may be. The resonance terms implicitly contain non-resonant $S^{\text{dir}}_{\alpha \beta}$. How can then the statistical compound nucleus model be derived from the dispersion formula?
It has been shown\(^3\) that the dispersion formula can be rewritten in the form,

\[ \frac{S_{\text{dir}}^{\text{dir}}}{I} - i \sum_{k} \frac{\tilde{g}_{k \alpha} \tilde{g}_{k \alpha}^{\ast}}{E - \tilde{W}_{k}}, \]  

(VII.5)

so that

\[ \left< \sum_{k} \frac{\tilde{g}_{k \alpha} \tilde{g}_{k \alpha}^{\ast}}{E - \tilde{W}_{k}} \right> = 0, \]  

(VII.6)

i.e., the modified resonance terms no longer contain non-resonant \(S_{\text{dir}}^{\text{dir}}\). The modified resonance parameters in (VII.6), denoted with tilde, depend on the interval of energy averaging, \(I\), as does \(S_{\text{dir}}^{\text{dir}}\). Thus, what are included in the direct and resonance components of the S-matrix depend on \(I\). The gross structure of neutron total cross sections at very low energies can be seen only when \(I\) is several hundreds keV, containing many compound nucleus resonances. The fine structure can be seen only if \(I\) is less than the distances of compound nucleus levels. If \(I\) is between those two extremes, \(S_{\text{dir}}^{\text{dir}}\) contains intermediate structures such as door way state resonances. A door way state is a bound state imbedded in the continuum, like a compound nucleus state, but with excitation of only one degree of freedom\(^9\). The smaller \(I\) is, the finer is the structure included in \(S_{\text{dir}}^{\text{dir}}\), although it may not be experimentally observable.

The fluctuation cross section is defined by

\[ \sigma_{f}^{\text{fl}} = \frac{\pi}{k_{\text{eff}}^{2}} \left< \left| \sum_{k} \frac{\tilde{g}_{k \alpha} \tilde{g}_{k \alpha}^{\ast}}{E - \tilde{W}_{k}} \right|^{2} \right>, \]  

(VII.7)

just as the compound elastic scattering cross section. It again depends on \(I\). One can only calculate it under some statistical assumptions\(^3\). It should be noted that the \(\tilde{g}_{k \alpha}\) are in general still correlated to each other which may result in modifications of the Hauser-Feshbach formula, even its product form of probabilities of the compound nucleus formation and decay, depending on the assumptions on the statistical distribution of the \(\tilde{g}_{k \alpha}\).

VIII Summary and Discussions

Developments of some basic concepts of nuclear reaction theories is surveyed in a historical perspective. Many of the more recent important developments which are still in progress are left out. In particular, pre-equilibrium processes in light and heavy ion reactions have been the subjects of great interest in recent years,\(^3\) and much relevant to the present symposium. Intensive studies, both experimental and theoretical are being made\(^3\). Hopefully in the near future, they will be understood as well as direct and compound nucleus processes, and a unified picture of nuclear reactions will emerge that quantitatively describe all stages of nuclear reactions from shape elastic scattering to compound nucleus decay.
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