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GENERALIZED ZEROPOINT ANOMALY

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Abstract

We define Zeropoint Anomaly (ZPA) as the difference between the Effective Potential (EP) and the Zeropoint Energy (ZPE). We show, for a massive and interacting scalar field that, in very general conditions, the renormalized ZPA vanishes and then the renormalized EP and ZPE coincide.

1. Zeropoint Anomaly

Some years ago Eric Myers[1] published a very interesting work on a possible ambiguity of the definition of vacuum energy. He pointed out that we can define vacuum energy as due to Zero-Point fields fluctuations and on the other hand as the minimum of the Effective Potential. Although both definitions coincide for Casimir Effect in electrodynamics, they will give different answers for interacting and/or massive fields. The difference between these two vacuum energies we name Zero-Point Anomaly (ZPA).

We show below, for a massive and interacting scalar field that, in very general conditions the renormalized ZPA vanishes and then the renormalized Effective Potential and Zeropoint Energy coincide.

Let $\phi(x)$ be a single real scalar field in a N-dimensional Minkowsky space-time M, where $N = m + 1$ and m is the number of spatial dimensions.

The Effective Potential, to the first order in the loop expansion (or equivalently in powers of $\hbar$), is given by [1]

$$V_{\rm eff}(\overline{\phi}) = V_{cl}(\overline{\phi}) + \frac{1}{2}\frac{\hbar}{\Omega_M} \ln\ \mathrm{Det}\left[\frac{\delta^2 S[\overline{\phi}]}{\delta\phi(x)\delta\phi(y)}\right], \qquad (1)$$

where $\overline{\phi} = \langle\phi\rangle$ is the classical field, $S[\phi]$ is the Classical Action, $\Omega_M = L^m T$ is the volume of the background space-time manifold and in the Classical Potential $V_{cl}(\phi)$ is included mass and interactions terms. Making the usual analytic continuation to the Euclidian space-time, the classical Action can be written as

$$S[\phi] = \int d^N x [\frac{1}{2}\partial_\mu\phi\partial_\mu\phi + V_{cl}(\phi)]. \tag{2}$$

From (2) we get the matrix $\mathcal{M}(x,y)$ of the quadratic variation of the action $S[\phi]$

$$\mathcal{M}(x,y) \equiv \frac{\delta^2 S[\overline{\phi}]}{\delta\phi(x)\delta\phi(y)} = \delta^N(x-y)[-\delta^{\mu\nu}\partial_\mu\partial_\nu + V''_{cl}(\overline{\phi})]. \tag{3}$$

Let $\{\lambda_i\}$ be the eigenvalues of the operator $\mathcal{M}(x,y)$. The Generalized Zeta Function associated to $\mathcal{M}$ is defined by

$$\zeta_{\mathcal{M}}(s) = \sum_i (\lambda_i)^{-s}. \tag{4}$$

The relation below is well known[2]

$$\ln \mathrm{Det}\, \mathcal{M} = -\frac{d\zeta_{\mathcal{M}}(s)}{ds}\bigg|_{s=0} - \ln(2\pi\mu^2)\zeta_{\mathcal{M}}(0), \tag{5}$$

where μ is a scale parameter (with units of mass).

In general, the eigenvalues λ_i can be written as

$$\lambda_i = \omega^2 + h_i \tag{6}$$

where h_i are the eigenvalues of the Hamiltonian operator H, and w is a continuous variable.

Substituting (6) into (4) we get

$$\zeta_{\mathcal{M}}(s) = \frac{1}{2} \frac{\Gamma(s-\frac{1}{2})}{\Gamma(s)} \zeta_H(s-\frac{1}{2})T, \tag{7}$$

where $\zeta_H\left(s - \frac{1}{2}\right)$ is the Generalized Zeta Function associated to the Hamiltonian operator H. It is well known that the Zeropoint Energy density is given by

$$\varepsilon = \frac{1}{2}\frac{\hbar}{L^m} \lim_{s\to 0} \zeta_H\left(s - \frac{1}{2}\right). \tag{8}$$

From Eq(s) (1), (3),(5),(7) and (8) we find

$$V_{\text{eff}}(\overline{\phi}) = \epsilon - \frac{\hbar}{2L^m} \lim_{s\to 0}\left\{ \frac{\psi(s-\frac{1}{2})\Gamma(s-\frac{1}{2})\zeta_H(s-\frac{1}{2})}{2\,\pi\Gamma(s)} + \frac{\Gamma(s-\frac{1}{2})\zeta'_H(s-\frac{1}{2})}{2\,\pi\Gamma(s)} \right\} \tag{9}$$

The Eq. (9) shows that the Effective Potential can be different from the Zeropoint Energy. We call the additional term

$$\mathcal{A} = \frac{-\hbar}{2L^m} \lim_{s\to 0}\left\{ \frac{\psi(s-\frac{1}{2})\Gamma(s-\frac{1}{2})\zeta_H(s-\frac{1}{2})}{2\,\pi\,\Gamma(s)} + \frac{\Gamma(s-\frac{1}{2})\zeta'_H(s-\frac{1}{2})}{2\,\pi\,\Gamma(s)} \right\} \tag{10}$$

Zeropoint Anomaly (ZPA).

2. The ZPA and the divergencies in the ZPE

It is well known that $\zeta_{\mathcal{M}}(s)$ is analytic[2] at $s = 0$. Then $\zeta_{\mathcal{M}}(0)$ is finite. Now, in Eq. (7) we can distinguish two cases. In the first one $\zeta_H\left(s-\frac{1}{2}\right)$ is analytic at $s = 0$ and then $\xi_H(-\frac{1}{2})$ and $\zeta'_H(-1/2)$ are finite and it is clear from Eq. (9) that the ZPA vanishes.

In the second case $\zeta_H\left(s-\frac{1}{2}\right)$ is not analytic at $s = 0$. Now, since $\zeta_{\mathcal{M}}(s)$ is analytical at $s = 0$, we conclude from Eq. (7) that $\zeta_H\left(s-\frac{1}{2}\right)$ has the same simple poles as the Gamma Function $\Gamma(s)$. Thus after regularization we can write

$$\zeta_H\left(s-\frac{1}{2}\right) = C(s) + \Gamma(s)\, B(s)\,, \tag{11}$$

where $C(s)$ and $B(s)$ are analytic functions at $s = 0$.

Substituting Eq. (11) into (10) we obtain for the ZPA

$$\mathcal{A} = \frac{\hbar}{2L^m}\,[\psi(-\frac{1}{2})\, B(0) + B'(0)] - \frac{\hbar}{2L^m} \lim_{s\to 0} [\Gamma(s)\, B(s)]. \tag{12}$$

This equation shows that the ZPA depends only on the divergent part of the ZPE. We can hope, on physical grounds, that it is not observable. We show bellow, that this is just the case.

3. The renormalized ZPA

It is well known that the ZPE density is divergent, but as in the case of the Casimir Effect, only energy differences are observable. So, we define the renormalized vacuum energy densities as the difference between two different configurations. As it is well known, this subtraction procedure cancels the poles of the ZPE and then the renormalized vacuum energy density is finite[3].

Let ε^0 and ε the ZPE density in two different configurations. Then the renormalized density is

$$\varepsilon^R = \varepsilon - \varepsilon^0, \tag{13}$$

where, in analogy with Eq. (8), ε^0 can be written as

$$\varepsilon^0 = \frac{\hbar}{2L^m} \lim_{s\to 0} \zeta_H^0\left(s - \frac{1}{2}\right) \tag{14}$$

As observed above, ε^R is finite. This implies, using Eq(s) (8), (13) and (14,) that the function

$$F(s) = \zeta_H\left(s - \frac{1}{2}\right) - \zeta_H^0\left(s - \frac{1}{2}\right) \tag{15}$$

is analytic at $s = 0$.

We can write the renormalized Zeropoint Anomaly as

$$\mathcal{A}^R = V_{\text{eff}}^R - \varepsilon^R = \mathcal{A} - \mathcal{A}^0 \tag{16}$$

where $\mathcal{A} = V_{\text{eff}} - \varepsilon$ and $\mathcal{A}^0 = V_{\text{eff}}^0 - \varepsilon^0$.

From Eq(s) (16), (15) and (10) we find that

$$\mathcal{A}^R = \frac{\hbar}{2L^m} \lim_{s\to 0}\left\{\frac{\psi(s-\frac{1}{2})\Gamma(s-\frac{1}{2})}{2\ \pi}\ \frac{F(s)}{\Gamma(s)} + \frac{\Gamma(s-\frac{1}{2})}{2}\ \frac{F'(s)}{\Gamma(s)}\right\}. \tag{17}$$

The fact that $F(s)$ is analytic at $s = 0$ implies that $F(0)$ and $F'(0)$ are finite. Then, from (17) we find that

$$\mathcal{A}^R = 0. \tag{18}$$

Therefore, we conclude, finally, that notwithstanding the renormalization procedure, the renormalized Effective Potential coincides with the Zeropoint Energy density.

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