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A. Tidjane Diallo

INTERNATIONAL ATOMIC ENERGY AGENCY

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A. Tidjane Diallo[1]
International Centre for Theoretical Physics, Trieste, Italy.

ABSTRACT

We show that if $f \in C[0,1]$ and $B_n^{(2r-1)}f$ (r integer ≥ 1) is the Bernstein Quasi-Interpolant defined by Sablonnière, then $\|B_n^{(2r-1)}f - f\|_{C[0,1]} \leq \omega_\varphi^{2r}\left(f, \frac{1}{\sqrt{n}}\right)$ where ω_φ^{2r} is the Ditzian-Totik modulus of smoothness with $\varphi(x) = \sqrt{x(1-x)}$, $x \in [0,1]$.

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[1]Permanent address: Institut Superiéur des Sciences de l'Education de Manéah, BP 795, Conakry, Republic of Guinea.

1. Introduction and Definitions

The aim of this work is to estimate the rate of convergence of Bernstein Quasi-Interpolants on $C[0,1]$.

This paper is divided into three sections.

We first recall the construction of these approximation processes, then we summarize some results needed in the sequel and finally we state our theorem.

Let $C[0,1]$ be the space of real continuous functions on $[0,1]$ endowed with the supremum norm. For a function $f \in C[0,1]$, the Bernstein operator B_n is the linear positive operator given by:

$$B_n(f,x) = \sum_{k=0}^{n} f\left(\frac{k}{n}\right) p_{nk}(x) \quad \text{where} \quad p_{nk}(x) = \binom{n}{k} x^k(1-x)^{n-k}, \quad 0 \leq k \leq n$$

The approximation order of B_n on $C[0,1]$ is $O\left(\frac{1}{n}\right)$. To obtain faster convergence speed, various Bernstein-type Quasi-Interpolants have been constructed.

Since the publication of Butzer's work [1], a great amount of papers have been devoted to combinations of Bernstein-type operators.

In 1989, Sablonnière [4] introduced a new type of quasi-interpolants. To construct these operators, we recall some notations.

Denote by $\mathcal{P}_n$ the set of all polynomials of degree at most n. On the space $\mathcal{P}_n$ the operators B_n can be considered as linear differential operators

$$B_n = \sum_{i=0}^{n} \beta_i^n D^i$$

where $\beta_i^n \in \mathcal{P}_n$ are defined by the recursion relation

$$n(i+1)\,\beta_{i+1}^n(x) = X\left(D\beta_i^n(x) + \beta_{i-1}^n(x)\right)$$

$$\beta_0^n = 1, \quad \beta_1^n = 0 \quad \text{where} \quad X = x(1-x), \quad x \in [0,1]$$

Since B_n is one-to-one mapping on $\mathcal{P}_n$, there exists the inverse operator B_n^{-1} defined on $\mathcal{P}_n$ which can also be considered as linear differential operator

$$B_n^{-1} = \sum_{j=0}^{n} \alpha_j^n D^j$$

For $0 \le \nu \le n$, Sablonnière [4] introduced the truncated inverse of B_n

$$A_n^\nu = \sum_{j=0}^{\nu} \alpha_j^n \ D^j$$

α_j^n are defined by the recursion relation

$$\alpha_0^n(x) = 1, \quad \alpha_1^n(x) = 0 \quad \text{and for} \quad s \ge 2$$

$$\sum_{\ell=0}^{s} \delta_{s\ell}^n \ \alpha_\ell^n = 0$$

$$\delta_{s\ell}^n = \sum_{i=0}^{\ell} \binom{\ell}{i} \ D^i \ \beta_{i+s-\ell}^n$$

For ν fixed, he defined the so-called left Bernstein quasi-interpolant (of order ν)

$$B_n^{(\nu)} = A_n^{(\nu)} \ B_n$$

That is

$$B_n^{(\nu)} = A_n^{(\nu)} \ B_n = \sum_{j=0}^{\nu} \alpha_j^n \ D^j B_n$$

2. Auxiliary Results

In this section we summarize properties which will be needed later.

a) The main tool for proving our result is the K-functional characterization of the Ditzian–Totik Modulus of smoothness [3]

$\omega_\varphi^\nu(f,t)$:

$$\omega_\varphi^\nu(f,t) = \sup_{0 \le h \le t} \|\Delta_{h\varphi}^\nu f\|_\infty, \quad \varphi(x) = \sqrt{x(1-x)}, \quad x \in [0,1]$$

where

$$\Delta_{h\varphi}^\nu f(x) = \sum_{k=0}^{\nu} \binom{\nu}{k} (-1)^k f\left(x + \left(\frac{\nu}{2} - k\right) h\varphi\right), \quad \text{if} \quad \left[x - \frac{\nu}{2} \ h\varphi, x + \frac{\nu}{2} \ h\varphi\right] \subset [0,1]$$

$\Delta_h^\nu f(x) = 0$ otherwise

The K-functional in question is given by

$$K_{\nu,\varphi}(f,t^\nu) = \inf_g \left\{\|f-g\| + t^\nu \|\varphi^\nu g^{(\nu)}\|\right\}$$

where the infimum is taken on all g such that $g^{(\nu-1)} \in A.C._{loc}$ (that is absolutely continuous in $[a,b]$ for every $a, b \quad 0 < a < b < 1$) Ditzian and Totik ([3], Theorem 2.1.1, p.11) proved the equivalence between $\omega_\varphi^\nu(f,t)$ and $K_{\nu,\varphi}(f,t^\nu)$. This means the existence of $M > 0$ such that:

$$M^{-1} \ \omega_\varphi^\nu(f,t) \le K_{\nu,\varphi}(f,t^\nu) \le M \ \omega_\varphi^\nu(f,t)$$

b) Properties of $B_n^{(\nu)}, \alpha_j^n(x), \quad j = 2, \ldots, \nu$ and $T_{n,2s}(x) = \sum_{k=0}^{n} (k-nx)^{2s} p_{nk}(x)$ where s is an integer ≥ 1.

Lemma 1 *For a fixed integer $\nu \ge 1$, we have:*

A) Sablonnière [4]: *$B_n^{(\nu)}(1,x) \equiv 1$ and $B_n^{(\nu)}$ preserves polynomials of degree ν.*

B) Wu [6]: *There exists a constant $C^{(\nu)}$ independent of $n \ge \nu$ such that*

$$\|B_n^{(\nu)}\| \le C^{(\nu)}$$

Therefore $B_n^{(\nu)} f$ converges uniformly to f when n tends to infinity.

C) Wu [6]: *We have $\alpha_0^n(x) = 1, \alpha_1^n(x) = 0$ and*

$$\alpha_s^n(x) = H_{s-1}^{ns}(x) \ \frac{X}{n^{s-1}} + H_{s-2}^{ns} \ \frac{X^2}{n^{s-2}} + \ldots + H_{s'}^{ns}(x) \ \frac{X^{s-s'}}{n^{s'}}, \quad s \ge 2$$

where $s' = \left[\frac{s+1}{2}\right]$ and $H_j^{ns}(x)$ are functions uniformly bounded in n and $x \in [0,1]$.

D) Ditzian–Totik ([3], Lemma 9.4.4, p 128): *Set $E_n = \left[\frac{1}{n}, 1 - \frac{1}{n}\right]$. For $T_{n,2s}(x) = \sum_{k=0}^{n} (k-nx)^{2s} p_{nk}(x)$ the following estimate holds*

$$T_{n,2s}(x) \le C \ n^s \ X^s \quad \text{for} \quad x \in E_n \tag{1}$$

Observe that from Lemma 1 point C), it results

$$\left|\alpha_{2\ell}^n(x)\right| \le C \ \frac{X^\ell}{n^\ell} \quad \text{for} \quad x \in E_n \tag{2}$$

$$\left|\alpha_{2\ell+1}^n(x)\right| \le C \ \frac{X^\ell}{n^{\ell+1}} \quad \text{for} \quad x \in E_n \tag{3}$$

Notice also that

$$B_n^{(\nu)}(f,0) = B_n(f,0) = f(0) \quad \text{and} \quad B_n^{(\nu)}(f,1) = B_n(f,1) = f(1)$$

3. Main Result

In this section we establish the rate of convergence of $B_n^{(2r-1)}$ on $C[0,1]$.

Theorem 1: *Assume that r is a fixed integer ≥ 1 and $\varphi(x) = \sqrt{x(1-x)}$,* $\quad x \in [0,1]$.

If $f \in C[0,1], \omega_\varphi^{2r}\left(f, \frac{1}{\sqrt{n}}\right)$ is the Ditzian-Totik modulus of smoothness and $B_n^{(2r-1)}f$ is the Bernstein quasi-interpolant of order $2r-1$, then there exists a constant M independent of n and f such that the estimate

$$\|B_n^{(2r-1)}f - f\|_{C[0,1]} \leq M\ \omega_\varphi^{2r}\left(f, \frac{1}{\sqrt{n}}\right)$$

holds for all $n > 2r-1$.

Proof In general we denote by C the constants and point out that they are not the same at each occurrence.

The proof utilizes some ideas found in [3] for combinations of Bernstein polynomials.

Observe that Lemma 1 point A) implies

$$\|B_n^{(2r-1)}f - f\| \leq (C+1)\|f\|$$

Set $f = f - g + g$ where $g^{(2r-1)} \in A.C._{loc}$ and $\varphi^{2r}\ g^{(2r)}$ is bounded on $[0,1]$. We have

$$\|B_n^{(2r-1)}f - f\| \leq (C+1)\|f-g\| + \|B_n g - g\|$$

Let P_k be a polynomial of degree $k < n$. Since for $k < n, B_n(P_k, x)$ are polynomials of degree at most k, it follows that $D^j B_n(P_k, x)$ are polynomials of degree at most $k-j$. As $\alpha_n^j(x)$ are polynomials of degree at most j ([5], Theorem 2, iii) it results that

$$B_n^{(2r-1)}(P_k, x) = \sum_{j=0}^{2r-1} \alpha_j^n(x)\ D^j B_n(P_k, x)$$

are polynomials of degree at most k.

On the other hand, let $P_{[\sqrt{n}]}$ be the best $[\sqrt{n}\,]$-th polynomial approximation of $f \in C[0,1]$. By ([3], Theorem 7.2.1, p.79) it follows that $\|f - P_{[\sqrt{n}\,]}\| \leq MK_{2r,\varphi}(f, n^{-r})$, $[\sqrt{n}\,] > 2r$ where $M = M(r)$ is independent of $\sqrt{n} > 2r$ and f.

Reasoning as Ditzian and Totik ([3], p.119) we can state the existence of a constant C which does not depend on n and $P_{[\sqrt{n}\,]} \in \mathcal{P}_{[\sqrt{n}\,]}$ such that

$$\|B_n^{(2r-1)}(P_{[\sqrt{n}\,]}) - P_{[\sqrt{n}\,]}\|_{C[0,1]} \leq C\|B_n^{(2r-1)}P_{[\sqrt{n}\,]} - P_{[\sqrt{n}\,]}\|_{C(E_n)}$$

Recall that $E_n = \left[\frac{1}{n}, 1-\frac{1}{n}\right]$.

Since $P_{[\sqrt{n}\,]} \in C[0,1]$ and $P_{[\sqrt{n}\,]}^{(2r-1)}$ is $A.C._{loc}$, to prove our theorem, it is enough to establish the following estimate

$$\|B_n^{(2r-1)}\ g - g\|_{C(E_n)} \leq C\ \frac{\|\varphi^{2r}g^{(2r)}\|}{n^r} \tag{4}$$

where $g^{(2r-1)}$ is $A.C._{loc}$ and $\varphi^{2r}g^{(2r)}$ is bounded on $[0,1]$. To this end, expand g by Taylor formula

$$g(t) = g(x) + (t-x)g'(x) + \ldots + \frac{(t-x)^{2r-1}}{(2r-1)!}g^{(2r-1)}(x) + \frac{1}{(2r-1)!}\int_x^t (t-u)^{2r-1}\ g^{(2r)}(u)\ du \tag{5}$$

Set

$$R_{2r}(g,t,x) = \frac{1}{(2r-1)!}\int_x^t (t-u)^{2r-1}\ g^{(2r)}(u)\ du \tag{6}$$

the remainder in the Taylor expansion (5).

Apply $B_n^{(2r-1)}$ to both sides of (5). Since $B_n^{(2r-1)}$ preserves polynomials of degree $2r-1$, we have

$$B_n^{(2r-1)}(g,x) - g(x) = B_n^{(2r-1)}(R_{2r}(g,t,x))$$

Our theorem is proved if we show that

$$\|B_n^{(2r-1)}(R_{2r}(g,t,x))\|_{C(E_n)} \leq C\ \frac{\|\varphi^{2r}g^{(2r)}\|}{n^r} \tag{7}$$

Let us consider the expression

$$B_n^{(2r-1)}(R_{2r}(g,t,x)) = \sum_{j=0}^{2r-1} \alpha_j^n\ D^j B_n(R_{2r}(g,t,x)) =$$

$$= B_n(R_{2r}(g,t,x)) + \sum_{j=2}^{2r-1} \alpha_j^n\ D^j B_n(R_{2r}(g,t,x))$$

The estimate (7) is established if we prove that

$$\|B_n(R_{2r}(g,t,x))\|_{C(E_n)} \leq \frac{C\|\varphi^{2r}g^{(2r)}\|}{n^r} \tag{8}$$

and

$$\|\alpha_j^n D^j B_n(R_{2r}(g,t,x))\|_{C(E_n)} \leq \frac{C\|\varphi^{2r}g^{(2r)}\|}{n^r} \tag{9}$$

$$j = 2, \dots, 2r-1$$

To prove these inequalities, we need the following:

Denote by $\mathcal{M}(G,x) = \sup_t \left| \frac{\int_x^t |\varphi^{2r}(u)\ g^{(2r)}(u)| du}{t-x} \right|$ the maximal function of $G(u) = \varphi^{2r}(u)\ g^{(2r)}(u), u \in [0,1]$. From [3] (Lemma 9.6.1, p.141) we get the estimate.

For $x, t \in (0,1)$

$$|R_{2r}(g,t,x)| \le \frac{C\mathcal{M}(G,x)(t-x)^{2r}}{\varphi^{2r}(x)} \tag{10}$$

By (10) and (1), we get from the expression $|B_n(R_{2r}(g,t,x))| \le \sum_{k=0}^{n} |R_{2r}\left(g, \frac{k}{n}, x\right)| p_{nk}$:

$$\begin{aligned}
|B_n(R_{2r}(g,t,x))| &\le \frac{C\mathcal{M}(G,x)}{\varphi^{2r}(x)} \sum_{k=0}^{n} \left(\frac{k}{n} - x\right)^{2r} p_{nk}(x) = \\
&= \frac{C\mathcal{M}(G,x)}{\varphi^{2r}(x)\ n^{2r}} \sum_{k=0}^{n} (k-nx)^{2r} p_{nk}(x) = \frac{C\mathcal{M}(G,x)}{X^r\ n^{2r}} T_{n,2r}(x) \le \\
&\le \frac{C\mathcal{M}(G,x)\ n^r X^r}{X^r\ n^{2r}} = \frac{C\mathcal{M}(G,x)}{n^r} \le \frac{C\|\varphi^{2r} g^{(2r)}\|}{n^r} \quad \text{since}
\end{aligned}$$

$$\mathcal{M}(G,x) \le \|G\|$$

Thus the estimate (8) is proved.

To prove (9), compute the derivative $D^m B_n(R_{2r}(g,t,x))$ (see Devore and Lorentz [2], p.112). For $x \in E_n$ we have

$$D^m B_n(R_{2r}(g,t,x)) = X^{-m} \sum_{k=0}^{n} R_{2r}\left(g, \frac{k}{n}, x\right) p_{nk}(x) \sum_{2i+j \le m} [nX]^i (k-nx)^j p_{i,j,m}(x)$$

where $p_{i,j,m}(x)$ are polynomials independent of n.

By (10), we have

$$|D^m B_n(R_{2r}(g,t,x))| \le \frac{C\mathcal{M}(G,x)}{X^m \varphi^{2r}(x) n^{2r}} \sum_{k=0}^{n} (k-nx)^{2r} p_{nk}(x) \sum_{2i+j \le m} (nX)^i |k-nx|^j |p_{i,j,m}(x)| \tag{11}$$

From (11), it follows

$$|D^m B_n(R_{2r}(g,t,x))| \le \frac{C\mathcal{M}(G,x)}{X^m \varphi^{2r}(x) n^{2r}} \sum_{k=0}^{n} (k-nx)^{2r} p_{nk}(x) \left\{ \sum_{2i+2\ell \le m} (nX)^i (k-nx)^{2\ell} + \right.$$

$$\left. + \sum_{2i+2\ell+1 \le m} (nX)^i (k-nx)^{2\ell} |k-nx| \right\}$$ since the polynomials $p_{i,j,m}(x)$ are bounded on $[0,1]$

Thus

$$|D^m B_n(R_{2r}(g,t,x))| \le \frac{C\mathcal{M}(G,x)}{X^m \varphi^{2r}(x) n^{2r}} \left[\sum_{2i+2\ell \le m} (nX)^i \sum_{k=0}^{n} (k-nx)^{2r+2\ell} p_{nk}(x) + \right.$$

$$\left. + \sum_{2i+2\ell+1 \le m} (nX)^i \sum_{k=0}^{n} (k-nx)^{2r+2\ell} |k-nx| \right] \tag{12}$$

By Cauchy-Schwartz inequality, it follows

$$\begin{aligned}
\sum_{k=0}^{n} (k-nx)^{2r+2\ell} |k-nx| p_{nk}(x) &\le \left(\sum_{k=0}^{n} p_{nk}(x)(k-nx)^{4(r+\ell)} \right)^{1/2} \left(\sum_{k=0}^{n} (k-nx)^2 p_{nk}(x) \right)^{1/2} = \\
&= \left(T_{n,4(r+\ell)}(x) \right)^{1/2} (T_{n,2}(x))^{1/2}
\end{aligned}$$

From (12) we obtain

$$|D^m B_n(R_{2r}(g,t,x))| \le \frac{C\mathcal{M}(G,x)}{X^m \varphi^{2r}(x) n^{2r}} \left\{ \sum_{2i+2\ell \le m} (nX)^i T_{n,2(r+\ell)} + \right.$$

$$\left. + \sum_{2i+2\ell+1 \le m} (nX)^i \left(T_{n,4(r+\ell)} \right)^{1/2} (T_{n,2}(x))^{1/2} \right\}$$

By (1) we get

$$\begin{aligned}
|D^m B_n(R_{2r}(g,t,x))| &\le \frac{C\mathcal{M}(G,x)}{X^m \varphi^{2r}(x) n^{2r}} \left[\sum_{2i+2\ell \le m} (nX)^{i+r+\ell} + \right. \\
&\qquad \left. + \sum_{2i+2\ell+1 \le m} (nX)^{i+r+\ell+\frac{1}{2}} \right] \\
&= \frac{C\mathcal{M}(G,x) n^r X^r}{X^m X^r n^{2r}} \left[\sum_{2i+2\ell \le m} (nX)^{i+\ell} + \sum_{2i+2\ell+1 \le m} (nX)^{i+\ell+\frac{1}{2}} \right]
\end{aligned}$$

Thus for $x \in E_n$ we have

$$|D^m B_n(R_{2r}(g,t,x))| \le \frac{C\mathcal{M}(G,x)}{X^m\ n^r} \left[\sum_{2i+2\ell \le m} (nX)^{i+\ell} + \sum_{2i+2\ell+1 \le m} (nX)^{i+\ell+\frac{1}{2}} \right] \tag{13}$$

From now, we shall distinguish two cases: $m = 2s$ and $m = 2s+1$

$$s = 1, 2, \dots, r-1$$

<u>Case 1</u>: $m = 2s$

The estimate (13) yields

$$|D^{2s} B_n(R_{2r}(g,t,x))| \le \frac{C\mathcal{M}(G,x)}{X^{2s}\ n^r} \left[\sum_{i+\ell \le s} (nX)^{i+\ell} + \sum_{i+\ell \le s-1} (nX)^{i+\ell+\frac{1}{2}} \right] \tag{14}$$

Observe that the dominant term in the sums of the above expression is $n^s X^s$. Hence, from (14) we get:

$$|D^{2s}B_n(R_{2r}(g,t,x))| \le \frac{C\mathcal{M}(G,x)}{X^{2s}\,n^r} n^s X^s \left[\sum_{i+\ell\le s} \frac{1}{(nX)^{s-\ell-i}} + \sum_{i+\ell\le s-1} \frac{1}{(nX)^{s-i-\ell-\frac{1}{2}}}\right] \quad (15)$$

Notice that in the expression (15) $s-\ell-i\ge 0$ in the first sum and $s-\ell-i-\frac{1}{2}>0$ in the second.

As for $x\in E_n$ we have $nX\ge C$, from (15) we can deduce

$$|D^{2s}B_n(R_{2r}(g,t,x))| \le \frac{C\mathcal{M}(G,x)n^sX^s}{X^{2s}n^r} = \frac{C\mathcal{M}(G,x)n^s}{X^s\,n^r}$$

In view of (2), we obtain:

For $x\in E_n$

$$|\alpha^n_{2s}(x)D^{2s}B_n(R_{2r}(g,t,x))| \le \frac{C\mathcal{M}(G,x)n^s}{X^s\,n^r}|\alpha^n_{2s}(x)| \le \frac{C\mathcal{M}(G,x)n^sX^s}{X^s\,n^r\,n^s} \le \frac{C\|\varphi^{2r}g^{(2r)}\|}{n^r}$$

That is

$$|\alpha^n_{2s}(x)D^{2s}B_n(R_{2r}(g,t,x))| \le \frac{C\|\varphi^{2r}g^{(2r)}\|}{n^r} \quad (16)$$

$$s=1,2,\dots,r-1$$

Case 2: $m=2s+1$

By (13) we have

$$|D^{2s+1}B_n(R_{2r}(g,t,x))| \le \frac{C\mathcal{M}(G,x)}{X^{2s+1}n^r}\left[\sum_{2i+2\ell\le 2s+1}(nX)^{i+\ell} + + \sum_{2i+2\ell+1\le 2s+1}(nX)^{i+\ell+\frac{1}{2}}\right]$$

$$= \frac{C\mathcal{M}(G,x)}{X^{2s+1}\,n^r}\left[\sum_{i+\ell\le s}(nX)^{i+\ell} + \sum_{i+\ell\le s}(nX)^{i+\ell+\frac{1}{2}}\right] \quad (17)$$

The dominant term in the sums of the expression (17) is $(nX)^{s+\frac{1}{2}}$.

From (17) it follows

$$|D^{2s+1}B_n(R_{2r}(g,t,x))| \le \frac{C\mathcal{M}(G,x)}{X^{2s+1}\,n^r}(nX)^{s+\frac{1}{2}}\left[\sum_{i+\ell\le s}\frac{1}{(nX)^{s+\frac{1}{2}-i-\ell}} + \sum_{i+\ell\le s}\frac{1}{(nX)^{s-i-\ell}}\right] \quad (18)$$

Notice that $s+\frac{1}{2}-i-\ell>0$ in the first sum and $s-i-\ell\ge 0$ in the second. As for $x\in E_n$, we have $nX\ge C$, from (18) we deduce

$$|D^{2s+1}B_n(R_{2r}(g,t,x))| \le \frac{C\mathcal{M}(G,x)n^{s+\frac{1}{2}}}{X^{s+\frac{1}{2}}\,n^r}$$

By the estimate (3), we get:

$$|\alpha^n_{2s+1}(x)D^{2s+1}B_n(R_{2r}(g,t,x))| \le \frac{C\mathcal{M}(G,x)n^{s+\frac{1}{2}}X^s}{X^{s+\frac{1}{2}}\,n^r\,n^{s+1}} =$$

$$= \frac{C\mathcal{M}(G,x)}{n^r(nX)^{1/2}} \le \frac{C\|\varphi^{2r}g^{(2r)}\|}{n^r} \quad \text{for } x\in E_n$$

That is

$$|\alpha^n_{2s+1}(x)D^{2s+1}B_n(R_{2r}(g,t,x))| \le \frac{C\|\varphi^{2r}g^{(2r)}\|}{n^r} \quad \text{for } x\in E_n \quad (19)$$

$$s=1,2,\dots,r-1$$

The inequalities (16) and (19) yield the estimate (9).

The proof of our theorem is complete.

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