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A MIN–MAX VARIATIONAL PRINCIPLE

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A MIN–MAX VARIATIONAL PRINCIPLE

Pando Gr. Georgiev[1]
International Centre for Theoretical Physics, Trieste, Italy.

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[1]Permanent address: Department of Mathematics and Informatics, University of Sofia, 5 James Bourchier Blvd., 1126 Sofia, Bulgaria.

In this paper we prove a variational principle for min-max problems, which is of the same spirit as Deville-Godefroy-Zizler's variational principle [D-G-Z] for minimization problems. We present a localization theorem in which we localize min-max points for the perturbed function with respect to a given ε-min-max point. This theorem allows to prove in a new way (not by mountain-pass techniques) existence of Palais-Smale sequences, which are near to a given optimizing sequence with respect to some subspaces.

About abstract variational principles we refer to [Ph].

Notations and definitions.

Let $(E, \|.\|)$ be a Banach space and $g : E \to \mathbf{R}$ be a function. We denote by $B(x;r)$ (resp. $B[x;r]$) - the open (resp. closed) ball with center x and radius r.

The point $x_0 \in X \subset E$ is said to be a strong minimum point (resp. strong maximum point) of the function g over the subset X of E, if $x_n \to x_0$ whenever $g(x_n) \to \inf g(X)$ (resp. $g(x_n) \to \sup g(X)$).

If T_1, T_2 are topological spaces and $f : T_1 \times T_2 \to \mathbf{R}$ is a function, then the (inf – sup) problem (f, T_1, T_2): find $x_0 \in T_1$ and $y_0 \in T_2$ such that

$$f(x_0, y_0) = \max_{y \in Y} f(x_0, y) = \min_{x \in X} \sup_{y \in Y} f(x, y) =: c$$

is called well posed in sense of Kenderov-Lucchetti (see [K-L]), if every optimizing sequence (x_n, y_n) (i.e. $f(x_n, y_n) \to c$ and $\sup_{y \in Y} f(x_n, y) \to c$) converges to (x_0, y_0).

A function $b : E \to \mathbf{R}$ is said to be a bump function, if the set *suppb* $:= \{x \in E : b(x) \neq 0\}$ is bounded.

Let E_1 and E_2 be Banach spaces, G and H be Banach spaces of bounded continuous functions on E_1 and E_2 respectively, satisfying the following conditions (H1),...,(H4) with respect to E_1 and E_2 respectively i.e. (we present the conditions with respect to E_1):

(H1) $\|g\|_\infty \leq \|g\|_G \quad \forall g \in G$;

(H2) for every $g \in G$ and $x \in E_1$ the function $\varphi_x g : y \mapsto g(x + y)$ belongs to G and $\|\varphi_x g\|_G = \|g\|_G$;

(H3) for every $g \in G$ and $\alpha \in \mathbf{R}$ the function $\psi_\alpha g : y \mapsto g(\alpha y)$ belongs to G and $\|\psi_\alpha g\|_G = |\alpha| \|g\|_G$;

(H4) there exists a bump function in G.

Let, in addition, the functions from G be Lipschitz and from class $C^{1,1}$, i.e. differentiable with locally Lipschitz derivatives.

Let the Banach space $W := G \times H$ be equipped with the "max" norm: $\|(g, h)\| := \max\{\|g\|_G, \|h\|_H\}$.

Theorem 1 *Let X and Y be closed non-empty subsets of E_1 and E_2 respectively, $f : X \times Y \to \mathbf{R}$ be a function, differentiable with respect to the first variable for every $y \in Y$ such that*

(a) the functions $\{f'(., y), y \in Y\}$ are equi-locally Lipschitz;

(b) the function $f(x, .)$ is upper semi-continuous for every $x \in X$;

(c) $\sup_{y \in Y} f(x, y) < +\infty \quad \forall x \in X$,

(d) $\inf_{x \in X} \sup_{y \in Y} f(x, y) > -\infty$.

Then there exists a dense G_δ subset $W_0 \subset W$ such that for every $(g, h) \in W_0$ there exists $(x_0, y_0) \in X \times Y$ for which

(e) $f_2(x_0, y_0) = \max_{y \in Y} f_2(x_0, y) = \min_{x \in X} \sup_{y \in Y} f_2(x, y)$,

where $f_2(x,y) = f(x,y) + g(x) + h(y)$,

(f) x_0 *and* y_0 *in (e) are strong minimum and strong maximum points respectively for the functions* $\sup_{y\in Y} f_2(.,y)$ *and* $f_2(x_0,.)$.

(g) the inf-sup problem for the function f_2 *is well posed in the sense of Kenderov-Lucchetti [K-L].*

Proof. For a function $\psi : E \to \mathrm{R}$, where E is a Banach space, $X \subset E$, define

$$S(X,\psi,\alpha) = \{x \in X : \psi(x) < \inf \psi(X) + \alpha\}$$

and

$$\overline{S}(X,\psi,\alpha) = \{x \in X : \psi(x) \le \inf \psi(X) + \alpha\}.$$

Define:

$$\begin{aligned} W_n = \ & \Big\{(g,h) \in G \times H : \exists m > n : \\ & diam S(X, \sup_{y\in Y}(f+g+h)(.,y); \alpha_1) < 1/m, \\ & \text{for some } \ \alpha_1 > 0 \ \text{ and} \\ & diam S(Y, -(f+g+h)(x',.); \alpha_2) < 1/m \\ & \text{for some } \ x' \in S(X, \sup_{y\in Y}(f+g+h)(.,y); \alpha_1) \\ & \text{and some } \ \alpha_2 > \tfrac{n}{m^2}\Big\} \end{aligned}$$

We shall show that W_n is dense and open in $G \times H$ and then their intersection would be the desired set W_0.

Claim 1. W_n is open.

Let $(g_0,h_0) \in W_n$ with corresponding $\alpha_1 > 0, \alpha_2 > 0, x'$ and m. Take $\alpha_1' < \alpha_1' < \alpha_1$ such that $\alpha_1 - \alpha_1' = \frac{\alpha_1 - \alpha_1'}{2}$ and

$$\sup_{y\in Y}(f+g_0+h_0)(x',y) < \inf_{x\in X}\sup_{y\in Y}(f+g_0+h_0)(x,y) + \alpha_1'.$$

Take

$$0 < \delta < \min\{(\frac{\alpha_1 - \alpha_1)}{4}, \frac{1}{4}(\alpha_2 - \frac{n}{m^2})\}$$

and let $(g,h) \in W$ be such that $\|(g,h) - (g_0,h_0)\| < \delta$.

By (H1) we have $\|g - g_0\|_\infty < \delta$, and $\|h - h_0\|_\infty < \delta$. If we denote $\varphi_0 := \sup_{y\in Y}(f + g_0 + h_0)(.,y)$, and $\varphi := \sup_{y\in Y}(f+g+h)(.,y)$, then we have: $\|\varphi - \varphi_0\|_\infty < 2\delta$.

It is easy to see that

$$S(X,\varphi;\alpha_1) \subset S(X,\varphi;\alpha_1 - 4\delta) \subset S(X,\varphi_0;\alpha_1). \qquad (1)$$

Therefore $diam S(X,\varphi;\alpha_1) < 1/m$.

Let now $\alpha_2 = \alpha_2 - 4\delta$. Then, we see as (1), that

$$S(Y, -(f+g+h)(x',.);\alpha_2) \subset S(Y, -(f+g_0+h_0)(x',.);\alpha_2)$$

therefore

$$diam S(Y, -(f+g+h)(x',.);\alpha_2) < 1/m.$$

By the choice of δ and α_1' we have

$$\begin{aligned} \sup_{y\in Y}(f+g+h)(x',y) \ & < \sup_{y\in Y}(f+g_0+h_0)(x',y) + 2\delta < \\ & < \inf_{x\in X}\sup_{y\in Y}(f+g_0+h_0)(x,y) + \alpha_1' + 2\delta \\ & \le \inf_{x\in X}\sup_{y\in Y}(f+g+h)(x',.) + \alpha_1 \end{aligned}$$

which means $x' \in S(X, -\sup_{y\in Y}(f+g+h)(.,y);\alpha_1)$. Claim 1 is proved.

Claim 2. The set W_n is dense in W.

Let $(g,h) \in G \times H$ and $\varepsilon > 0$. By the hypothesis (H4), G and H contain bump functions b_1 and b_2 respectively. Without loss of generality we may assume that the values of b_1 and b_2 are non-negative (by replacing, for instance, $b_1(x)$ by $\varphi(b_1(x))$, where $\varphi : \mathrm{R} \to [0,1]$ is continuous differentiable). By hypotheses (H2) and (H3), we can assume that $b_1(0) > \frac{3}{4}, supp b_1 \subset B_1(0;1), \|b_1\|_G < 1, \|b_2\|_H < 1, supp b_2 \subset B_2(0;1), b_2(0) > 0$, where $B_1(0;1)$ and $B_2(0;1)$ are the open unit balls in E_1 and E_2 respectively. Take $\varepsilon_1 = \frac{\varepsilon}{2m}, \varepsilon_2 = \frac{\varepsilon_1}{8}, m = \frac{2^6 n}{\varepsilon b_2(0)}, \alpha_1 = \frac{\varepsilon_1}{2}, \alpha_1' = \frac{\varepsilon_1}{8}, \alpha_2 = \frac{\varepsilon_2 b_2(0)}{2}$.

Choose $x' \in X$ and $y' \in Y$ such that

$$\sup_{y\in Y}(f+g+h)(x',y) < \inf_{x\in X}\sup_{y\in Y}(f+g+h)(x,y) + \alpha_1'$$

and

$$(f+g+h)(x',y') > \sup_{y\in Y}(f+g+h)(x',y) - \alpha_2.$$

Define $h_1(y) = \varepsilon_2 b_2(2m(y-y'))$ and $g_1(x) = -\varepsilon_1 b_1(2m(x-x'))$.

By (H2) and (H3), $g_1 \in G, h_1 \in H$ and $\|g_1\|_G < \varepsilon$, $\|h_1\|_H < \varepsilon$. We shall show that

$$x \in S(X, \sup_{y\in Y}(f+g+g_1+h+h_1)(.,y);\alpha_1) \Rightarrow \|x - x'\| < \frac{1}{2m}. \qquad (2)$$

Let $x_1 \in S(X, \sup_{y\in Y}(f+g+g_1+h+h_1)(.,y)\alpha_1)$ and assume that $\|x_1 - x'\| \ge \frac{1}{2m}$. Then $g_1(x_1) = 0$ and

$$\begin{aligned} & \sup_{y\in Y}(f+g+g_1+h+h_1)(x_1,y) \\ & = \sup_{y\in Y}(f+g+h+h_1)(x_1,y) \\ & \ge \inf_{x\in X}\sup_{y\in Y}(f+g+h)(x,y) \\ & \ge \sup_{y\in Y}(f+g+h)(x',y) - \alpha_1' \\ & = \sup_{y\in Y}(f+g+g_1+h)(x',y) - \alpha_1' + \varepsilon_1 b_1(0) \\ & \ge \sup_{y\in Y}(f+g+g_1+h+h_1)(x',y) - \alpha_1' + \varepsilon_1 b_1(0) - \varepsilon_2 \\ & > \inf_{x\in X}\sup_{y\in Y}(f+g+g_1+h+h_1)(x,y) + \alpha_1 \end{aligned}$$

which is a contradiction, and (2) is proved. Therefore

$$diam S(X, \sup_{y\in Y}(f+g+g_1+h+h_1)(.,y);\alpha_1) < \frac{1}{m}.$$

We shall prove:

$$y \in S(Y, -(f+g+h+g_1+h_1)(x',.);\alpha_2) \Rightarrow \|y-y'\| < \frac{1}{2m} \tag{3}$$

Let $y_1 \in S(Y, -(f+g+h+g_1+h_1)(x',.);\alpha_2)$ and assume that $\|y_1 - y'\| \geq \frac{1}{2m}$. Then $h_1(y_1) = 0$ and we have

$$\begin{aligned}&(f+g+g_1+h+h_1)(x',y_1)\\ &= (f+g+g_1+h)(x',y_1)\\ &\leq \sup_{y\in Y}(f+g+g_1+h)(x',y)\\ &< (f+g+g_1+h)(x',y')+\alpha_2\\ &= (f+g+g_1+h+h_1)(x',y') - \varepsilon_2 b_2(0) + \alpha_2\\ &\leq \sup_{y\in Y}(f+g+g_1+h+h_1)(x',y) - \alpha_2,\end{aligned}$$

which is a contradiction, and (3) is proved. Therefore

$$diam S(Y, -(f+g+h+g_1+h_1)(x',.);\alpha_2) < \frac{1}{m}.$$

We shall prove that

$$x' \in S(X, \sup_{y\in Y}(f+g+g_1+h+h_1)(.,y);\alpha_1). \tag{4}$$

Indeed,

$$\begin{aligned}&\sup_{y\in Y}(f+g+g_1+h+h_1)(x',y)\\ &\sup_{y\in Y}(f+g+h)(x',y)+\varepsilon_2-\varepsilon_1 b_1(0)\\ &< \inf_{x\in X}\sup_{y\in Y}(f+g+h)(x,y)+\alpha_1'+\varepsilon_2-\varepsilon_1 b_1(0)\\ &\leq \inf_{x\in X}\sup_{y\in Y}(f+g+g_1+h)(x,y)+\alpha_1'+\varepsilon_1+\varepsilon_2-\varepsilon_1 b_1(0)\\ &\leq \inf_{x\in X}\sup_{y\in Y}(f+g+g_1+h+h_1)(x,y)+\alpha_1\end{aligned}$$

and (4) and Claim 2 are proved.

By the Baire category theorem, the set $W_0 := \cap_{n=1}^{\infty} W_n$ is dense and G_δ in W. Let $(g,h) \in W_0$. By definition of W_n, for every $n \in N$ there exist $m > n, \alpha_{1,n} > 0$ such that

$$diam S(X, \sup_{y\in Y}(f+g+h)(.,y);\alpha_{1,n}) < 1/m_n. \tag{5}$$

Therefore

$$\cap_{n=1}^{\infty}\overline{S}(X, \sup_{y\in Y}(f+g+h)(.,y);\alpha_{1,n}) = \{x_0\}.$$

Here we used the fact that the function $\sup_{y\in Y}(f+g+h)(.,y)$ is lower semicontinuous, i.e. the set $\overline{S}(X, \sup_{y\in Y} f+g+h)(.,y);\alpha_{1,n})$ is closed.

Let $\{x_k\}$ be a minimizing sequence for the function $\sup_{y\in Y}(f+g+h)(.,y)$, i.e.

$$\sup_{y\in Y}(f+g+h)(x_k,y) \to \inf_{x\in X}\sup_{y\in Y}(f+g+h)(x,y).$$

Then, for every n there exist μ such that for every $k > \mu$ we have $x_k \in \overline{S}(X, \sup_{y\in Y}(f+g+h)(.,y);\alpha_{1,n})$, therefore $x_k \to x_0$.

Again by definition of W_n, there exist $x_n' \in S(X, \sup_{y\in Y}(f+g+h)(.,y);\alpha_{1,n})$ and $\alpha_{2,n} > \frac{n}{m_n^2}$ such that

$$diam S(Y, -(f+g+h)(x_n',.);\alpha_{2,n}) < \frac{1}{m_n}.$$

We shall prove that

$$S(Y, -(f+g+h)(x_0,.);\alpha_{2,n}/2)) \subset S(Y, -(f+g+h)(x_n',.);\alpha_{2,n}). \tag{6}$$

Let $y' \in S(Y, -(f+g+h)(x_0,.);\alpha_{2,n}/2), \varphi = f+g+h$. By (5), $\|x_n' - x_0\| < 1/m_n$ and by the mean-value theorem for large n we can write:

$$\begin{aligned}&\varphi(x_n',y') = \varphi(x_0,y') + \langle\varphi'(z_n,y'), x_n'-x_0\rangle\\ &\geq \sup_{y\in Y}\varphi(x_0,y) - \alpha_{2,n}/2 - L\|x_n'-x_0\|\|z_n-x_0\|\\ &\geq \sup_{y\in Y}\varphi(x_n',y) - \sup_{y\in Y}[\varphi(x_n',y)-\varphi(x_0,y)] - \frac{L}{m_n^2} - \alpha_{2,n}/2\\ &= \sup_{y\in Y}\varphi(x_n',y) - \sup_{y}\langle\varphi'(z_{n,y},y), x_n'-x_0\rangle - \frac{L}{m_n^2} - \alpha_{2,n}/2\\ &= \sup_{y\in Y}\varphi(x_n',y) - L\|x_n'-x_0\|\|z_{n,y}-x_0\| - \frac{L}{m_n^2} - \alpha_{2,n}/2\\ &\geq \sup_{y\in Y}\varphi(x_n',y) - \frac{2L}{m_n^2} - \alpha_{2,n}/2\\ &\geq \sup_{y\in Y}\varphi(x_n',y) - \alpha_{2,n},\end{aligned}$$

where $z_n \in [x_n', x_0], z_{n,y} \in [x_n', x_0]$ and L is a Lipschitz constant of $\varphi'(.,y)$ on a neighborhood of x_0.

This means $y' \in S(Y, -(f+g+h)(x_n',.);\alpha_{2,n})$, and (6) is proved. Therefore

$$diam S(Y, -(f+g+h)(x_0,.);\alpha_{2,n}/2) < \frac{1}{m_n}.$$

and

$$\cap_{n=1}^{\infty}\overline{S}(Y, -(f+g+h)(x_0,.);\alpha_{2,n}) = \{y_0\}.$$

Here we used the fact that $f(x_0,.)$ is upper semicontinuous, i.e. $\overline{S}(Y, -(f+g+h)(x_0,.);\alpha_{2,n})$ is a closed set.

Let $\{y_k\}$ be a maximizing sequence for the function $(f+g+h)(x_0,.)$, i.e. $(f+g+h)(x_0,y_k) \to \sup_{y\in Y}(f+g+h)(x_0,y)$. Then for every n there exists ν such that for every $k > \nu$ we have $y_k \in S(Y, -(f+g+h)(x_0,.);\alpha_{2,n})$, which implies that $y_k \to y_0$.

Let $\{(x_n,y_n)\}$ be an optimizing sequence, i.e. $f(x_n,y_n) \to c := \inf_{x\in X}\sup_{y\in Y} f(x,y)$ and $\sup_{y\in Y} f(x_n,y) \to c$. Let $\varepsilon > 0$ be given. Then for large n we have

$$|f(x_0,y_n) - c| \leq |f(x_n,y_n) - c| + \varepsilon < 2\varepsilon$$

therefore $f(x_0,y_n) \to c$. By (f) we conclude that $x_n \to x_0$ and $y_n \to y_0$. The condition (g) can be verified directly. ■

Theorem 2 (Localization) *Let the assumptions of Theorem 1 be satisfied and let, in addition, the bump functions $b_1 \in G$ and $b_2 \in H$ be such that $b_1(0) = b_2(0) = 1, supp b_1 \subset B_1[0;1], supp b_2 \subset B_2[0;1], b_1(x) \in [0,1] \quad \forall x \in E_1, b_2(y) \in [0,1] \quad \forall y \in E_2$. Let $\varepsilon_1, \varepsilon_2, \lambda_1, \lambda_2 > 0, \varepsilon_2' > \varepsilon_2$ be given and $x' \in X$ and $y' \in Y$ be such that:*

$$\sup_{y \in Y} f(x', y) < \inf_{x \in X} \sup_{y \in Y} f(x, y) + \varepsilon_1$$

$$f(x', y') > \sup_{y \in Y} f(x', y) - \varepsilon_2.$$

Let the functions $\{f(., y) : y \in Y\}$ be equi-Lipschitz in $B(x'; \lambda_1)$ with a Lipschitz constant L and

$$\varepsilon_2' > \varepsilon_2 + 2\lambda_1 L. \tag{7}$$

Then there exist $g \in G, h \in H, x_0 \in X$ and $y_0 \in Y$ such that e), f) and g) of Theorem 1 are satisfied and

$$\|x_0 - x'\| < \lambda_1, \tag{8}$$

$$\|g\|_G < \frac{\varepsilon_1 + \varepsilon_2'}{\lambda_1} \|b_1\|_G, \tag{9}$$

$$\|h\|_H < \frac{\varepsilon_2'}{\lambda_2} \|b_2\|_H, \tag{10}$$

$$\|y_0 - y'\| < \lambda_2. \tag{11}$$

Proof . Take $\varepsilon_1' < \varepsilon_1$ such that

$$\sup_{y \in Y} f(x', y) < \inf_{x \in X} \sup_{y \in Y} f(x, y) + \varepsilon_1'.$$

and $\varepsilon_2'' < \varepsilon_2'$ such that $\varepsilon_2'' > \varepsilon_2 + \lambda_1 L$. Put $\varepsilon_1'' = \varepsilon_1' + \varepsilon_2''$ and define

$$f_1(x, y) = f(x, y) - \varepsilon_1'' b_1(\frac{x - x'}{\lambda_1}) + \varepsilon_2'' b_2(\frac{y - y'}{\lambda_2}).$$

Then $f_1(x', y) = f(x', y) - \varepsilon_1'' + \varepsilon_2'' b_2(\frac{y-y'}{\lambda_2})$ and therefore

$$\sup_{y \in Y} f_1(x', y) = \sup_{y \in Y} \{f(x', y) + \varepsilon_2'' b_2(\frac{y - y'}{\lambda_2})\} - \varepsilon_1'' \leq$$

$$\leq \sup_{y \in Y} f(x', y) - \varepsilon_1' < \inf_{x \in X} \sup_{y \in Y} f(x, y)$$

Putting

$$\delta_1 := \inf_{x \in X} \sup_{y \in Y} f(x, y) - \sup_{y \in Y} f_1(x', y),$$

we see that $\delta_1 > 0$. By Theorem 1, there exists a function $k(x, y) = k_1(x) + k_2(y)$ such that $k_1 \in G, k_2 \in H$,

i) $\|k_i\| < min\{\delta_1/4, \varepsilon_1 - \varepsilon_1', \varepsilon_2' - \varepsilon_2'', \frac{\varepsilon_1 - \varepsilon_1'}{\lambda_1}\|b_1\|_G, \frac{\varepsilon_2' - \varepsilon_2''}{\lambda_2}\|b_2\|_H\}, i = 1, 2$
and $\|k_2\|_H < (\varepsilon_2'' - \varepsilon_2)/2 - \lambda_1 L$,

ii) $\sup_{y \in Y}(f_1 + k)(., y)$ attains its strong minimum at x_0;
iii) $(f_1 + k)(x_0, .)$ attains its strong maximum at y_0.
Assume that

$$\frac{x_0 - x'}{\lambda_1} \notin supp b_1 . \tag{12}$$

Then

$$f_1(x_0, y) = f(x_0, y) + \varepsilon_2'' b_2(\frac{y - y'}{\lambda_2})$$

and by i) and ii),

$$\begin{aligned}
&-\delta_1/2 < -2\|k_1\|_G \leq k_1(x_0) - k_1(x') \\
&\leq \sup_{y \in Y}(f_1 + k_2)(x', y) - \sup_{y \in Y}(f_1 + k_2)(x_0, y) \\
&\leq \sup_{y \in Y} f_1(x', y) + \|k_2\|_H - \sup_{y \in Y} f_1(x_0, y) + \sup_{y \in Y}(-k_2)(y) \\
&\leq \inf_{x \in X} \sup_{y \in Y} f(x, y) - \delta_1 - \sup_{y \in Y}\{f(x_0, y) + \varepsilon_2'' b_2(\frac{y - y'}{\lambda_2})\} + 2\|k_2\|_H \\
&\leq \inf_{x \in X} \sup_{y \in Y} f(x, y) - \delta_1 - \sup_{y \in Y} f(x_0, y) + 2\|k_2\|_H \\
&\leq 2\|k_2\|_H - \delta_1.
\end{aligned}$$

Hence

$$2\|k_2\|_H \geq \delta_1/2$$

a contradiction and (12) is proved, which implies (8).
Define $g(x) = -\varepsilon_1'' b_1(\frac{x-x'}{\lambda_1}) + k_1(x)$ and $h(y) = \varepsilon_2'' b_2(\frac{y-y'}{\lambda_2}) + k_2(y)$.
Then

$$\|g\|_G \leq \frac{\varepsilon_1''}{\lambda_1}\|b_1\|_G + \|k_1\|_G < \varepsilon_1 + \varepsilon_2''$$

which is (9).
Assume that $\frac{y_0 - y'}{\lambda_2} \notin supp b_2$. Then by iii) we have

$$\begin{aligned}
&2\|k_2\|_H \geq k_2(y_0) - k_2(y') \\
&\geq f_1(x_0, y') - f_1(x_0, y_0) = f(x_0, y') + \varepsilon_2'' - f(x_0, y_0) \\
&\geq f(x', y') + \varepsilon_2'' - 2L\|x_0 - x'\| - f(x', y_0) \\
&> \varepsilon_2'' - \varepsilon_2 - 2\lambda_1 L > 2\|k_2\|_H,
\end{aligned}$$

which is a contradiction.
We have

$$\|h\|_H \leq \frac{\varepsilon_2''}{\lambda_2}\|b_2\|_H + \|k_2\|_H < \frac{\varepsilon_2'}{\lambda_2}\|b_2\|_H.$$

The theorem is proved, since $f + g + h = f_1 + k$. ∎

Let in the following theorem G, (resp. H) be the Banach space of all differentiable Lipschitz functions on E_1 (resp. on E_2) with locally Lipschitz derivatives, with the norm

$$\|g\|_G = \max_{x \in E_1}\|g(x)\| + \max_{x \in E_1}\|g'(x)\|$$

(the norm in H is defined in analogous way).

Theorem 3 *Let the assumptions of Theorem 1 be satisfied and $\{x'_n\}$ be a minimizing sequence for the function $\sup_{y\in Y} f(.,y)$ Then there exists a sequence $\{y'_n\} \subset Y$ such that the sequence $\{(x'_n, y'_n)\} \in X \times Y$ is an optimizing sequence, (i.e. $f(x'_n, y'_n) \to c := \inf_{x\in X}\sup_{y\in Y} f(x,y)$ and $\sup_{y\in Y} f(x'_n, y) \to c$) and there exists an optimizing sequence $\{(x_n, y_n)\}$ such that*

$$\|x_n - x'_n\| \to 0 \qquad \|y_n - y'_n\| \to 0$$

and

$$f'_x(x_n, y_n) \to 0, \qquad f'_y(x_n, y_n) \to 0.$$

Proof. Let , $\varepsilon_n = \frac{1}{2}(\sup_{y\in Y} f(x'_n, y) - c)$, and put $\varepsilon_{n,1} = \varepsilon_{n,2} = \varepsilon_n, \varepsilon'_{n,2} = 2\varepsilon_n$. Then $\varepsilon_n > 0, \varepsilon_n \to 0$. By Theorem 2 for $\lambda_{n,1} = \lambda_{n,2} = \varepsilon_n^{\frac{3}{4}}$ we obtain that there exist $g_n \in G, \|g_n\|_G < 3\varepsilon_n^{\frac{1}{4}}$, and $h_n \in H, \|h_n\|_H < 2\varepsilon_n^{\frac{1}{4}}$, there exist $x_n \in B(x'_n; \lambda_{n,1})$ and $y_n \in Y$ such that

$$(f + g + h)(x_n, y_n) = \max_{y\in Y}(f + g + h)(x_n, y) = \min_{x\in X}\sup_{y\in Y}(f + g + h)(x, y) \qquad (13)$$

By (13) we obtain

$$f'_y(x_n, y_n) = -h'(y_n), \qquad \|f'_y(x_n, y_y)\| < \varepsilon_n^{\frac{3}{4}}$$

Denote

$$\varphi(x) = \sup_{y\in Y}(f + g + h)(x, y).$$

It is easy to prove that

$$\varphi'(x_n) = f'_x(x_n, y_n) + g'(x_n) \;, \qquad (14)$$

due to the fact that y_n is a strong minimum in (13). By (13), $\varphi'(x_n) = 0$, therefore

$$f'_x(x_n, y_n) = -g'(x_n), \qquad \|f'_x(x_n, y_n)\| = \|g'(x_n)\|. \qquad (15)$$

We shall check condition (7) from Theorem 2. By the mean-value theorem and by the uniform Lipschitz property of the derivatives, for any $y \in Y$ we have

$$\begin{aligned} |f(x,y) - f(x_n, y)| &= \langle f'(z_n, y), x'_n - x_n\rangle \\ &\le L\|x_n - x'_n\|^2 \le L\lambda_{n,1}\|x - x_n\|, \end{aligned}$$

where L is a Lipschitz constant of $f'_x(.,y)$ for every $y \in Y$ in a neighborhood of x_n, and $z_n \in [x, x_n]$.

Therefore $L\lambda_{n,1}$ is a uniform Lipschitz constant of $f(.,y), y \in Y$ on $B(x_n; \lambda_{n,1})$, whence for large n we have

$$\varepsilon_{n,2} + 2L\lambda_{n,1}^2 = \varepsilon_n + \varepsilon_n^{\frac{3}{2}} < 2\varepsilon_n = \varepsilon'_{n,2},$$

which is condition (7) of Theorem 2. Hence, by (11),

$$\|y_n - y'_n\| \le \lambda_{n,2}$$

and the theorem is proved. ∎

This theorem can be useful for finding critical points, when Palais-Smale condition holds, and when we can find optimizing sequence with respect to some subspaces.

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REFERENCES

[D-G-Z] R.Deville, G.Godefroy, V.Zizler, A smooth variational principle with application to Hamilton-Jacoby equations in infinite dimensions, J. Funct. Anal., 111, 197-212, 1993.

[K-L] Well posed sup-inf problems, preprint.

[Ph] R.R.Phelps, Convex functions, monotone operators and differentiability. Lecture Notes in Mathematics.Springer Verlag, Vol. 1364.