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ABSTRACT

We give a q-deformation $\hat{\mathcal{S}}_q$ of the centrally extended Schrödinger algebra. We construct the lowest weight representations of $\hat{\mathcal{S}}_q$, starting from the Verma modules over $\hat{\mathcal{S}}_q$, finding their singular vectors and factoring the Verma submodules built on the singular vectors. We also give a vector-field realization of $\hat{\mathcal{S}}_q$ which provides polynomial realization of the lowest weight representations and an infinite hierarchy of q-difference equations which may be called generalized q-deformed heat equations. We also apply our methods to the on-shell q-Schrödinger algebra proposed by Floreanini and Vinet.

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1. Introduction

Quantum groups attracted a lot of attention about ten years ago after the seminal papers of Drinfeld [1], Jimbo, [2], L.D. Faddeev, N.Yu. Reshetikhin and L.A. Takhtajan [3]. Yet most of the research is related to the quantum group deformations of simple Lie algebras and groups, while there are not so many examples of q-deformations of nonsemisimple Lie algebras.

We are addressing the latter question in the present paper. We are motivated by the essential role played in physics by nonsemisimple Lie algebras, recall, e.g., that the quantum mechanics of a free particle in $\mathbb{R}^n$ is governed by the Schrödinger algebra $\mathcal{S}(n)$, (for other examples, cf., e.g., [4]). This is furthermore interesting from a mathematical point of view because a general deformation theory for nonsemisimple Lie algebras is unknown in general even in the case when one looks for a q-deformation with q-difference operators for which a Hopf structure may not exist. Until now q-deformations of nonsemisimple Lie algebras were obtained by *contractions of q-deformations of semisimple Lie algebras*, (cf. the first examples in [5], [6], and for more references the recent paper [7]).

In the present paper we give the first example of a q-deformation which is not obtained by the standard method of contraction of commutator relations. Namely, we give a q-deformation of the centrally extened Schrödinger algebra in $1+1$ dimensional space-time and construct and study some of its representations and realizations. (The Schrödinger algebra was introduced for $3+1$ dimensional space-time in [8], [9].) We note that the somewhat indirect approach in [10] starts with a special q-deformed heat equation and then looks for a q-symmetry algebra on the solution variety. The resulting q-deformation of the Schrödinger algebra in [10] is different from ours and is (expectedly) valid only on the solutions of the q-deformed heat equation under consideration.

The paper is organized as follows. In Section 2 we give and explain our q-deformation $\hat{\mathcal{S}}_q$ of the centrally extended Schrödinger algebra and discuss some of its properties: subalgebras, grading, conjugation. In Section 3 we construct the lowest weight representations of $\hat{\mathcal{S}}_q$. We first construct the Verma modules over $\hat{\mathcal{S}}_q$, then we find their singular vectors and finally factor the Verma submodules built on the singular vectors. In Section 4 we give the vector-field realization of $\hat{\mathcal{S}}_q$ which provides polynomial realization of the lowest weight representations constructed in Section 3 and an infinite hierarchy of q-difference equations which may be called generalized q-deformed heat equations. In Section 5 we apply our methods to the on-shell q-deformation of [10].

2. q–deformed Schrödinger algebra $\hat{S}_q(1)$

Here we give explicitly a q-deformation $\hat{S}_q(1)$ of the centrally extended Schrödinger algebra $\hat{S}(1)$. We first recall the classical commutation relations of $\hat{S}(1)$ [4] :

$$[P_t, G] = P_x \tag{2.1a}$$
$$[K, P_x] = -G \tag{2.1b}$$
$$[D, G] = G \tag{2.1c}$$
$$[D, P_x] = -P_x \tag{2.1d}$$
$$[D, P_t] = -2P_t \tag{2.1e}$$
$$[D, K] = 2K \tag{2.1f}$$
$$[P_t, K] = D \tag{2.1g}$$
$$[P_x, G] = m \tag{2.1h}$$

We need some notation for the q-deformed case. The basic q-number notations we shall use are:

$$[a]_q \doteq \frac{q^a - q^{-a}}{q - q^{-1}}, \qquad [a]'_q \doteq [a]_{q^{1/2}} = \frac{q^{a/2} - q^{-a/2}}{q^{1/2} - q^{-1/2}} = \frac{[a/2]_q}{[1/2]_q} \tag{2.2}$$

and they will be used also for diagonal operators H replacing a in (2.2).

Now we can give our q-deformation of the Schrödinger algebra. We have the following nontrivial relations instead of (2.1) :

$$P_t\, G - q\, G\, P_t = P_x \tag{2.3a}$$
$$[P_x, K] = G\, q^{-D} \tag{2.3b}$$
$$[D, G] = G \tag{2.3c}$$
$$[D, P_x] = -P_x \tag{2.3d}$$
$$[D, P_t] = -2P_t \tag{2.3e}$$
$$[D, K] = 2K \tag{2.3f}$$
$$[P_t, K] = [D]_q \tag{2.3g}$$
$$P_x\, G - q^{-1}\, G\, P_x = m \tag{2.3h}$$
$$P_t\, P_x - q^{-1}\, P_x\, P_t = 0 \tag{2.3i}$$

Note that this deformation preserves the subalgebra structure. Namely, it has a q-deformed centrally extended Galilei subalgebra and a q-deformed $sl(2,\mathbb{C})$ ($sl(2,\mathbb{R})$) subalgebra, generated, resp., by P_t, P_x, G, and D, K, P_t. Note that the Galilei algebra deformation is a mild one, namely, commutators being turned into q-commutators, cf. (2.3a, h, i). It differs from the Galilei algebra q-deformation given in [6], which is not a surprise taking into account that the latter is not a subalgebra of a (q-deformed) Schrödinger algebra. On the other hand the commutation relations (2.3e, f, g) are the standard commutation relations of the Drinfeld-Jimbo deformation $U_q(sl(2,\mathbb{C}))$.

The commutation relations (2.3) are graded as the undeformed ones, if we define:

$$\deg D = 0 \tag{2.4a}$$
$$\deg G = 1 \tag{2.4b}$$
$$\deg K = 2 \tag{2.4c}$$
$$\deg P_x = -1 \tag{2.4d}$$
$$\deg P_t = -2 \tag{2.4e}$$
$$\deg m = 0 \tag{2.4f}$$

For future reference we record also the following involutive antiautomorphism of the q-Schrödinger algebra valid for *real* q:

$$\begin{aligned} &\omega(P_t) = K, \quad \omega(P_x) = G, \quad \omega(D) = D, \\ &\omega(m) = m, \quad \omega(q) = q \end{aligned} \tag{2.5}$$

WIth this conjugation the subalgebra generated by P_t, K, D is $U_q(sl(2,\mathbb{R}))$.

3. Lowest weight modules of $\hat{S}_q(1)$

Let us denote by $\mathcal{S}^+ = \mathcal{S}(1)^+$ the subalgebra generated by the positively graded generators G, K, by $\mathcal{S}^- = \mathcal{S}(1)^-$ the subalgebra generated by the negatively graded generators P_x, P_t.

Now we consider lowest weight modules (LWM) of $\hat{S}(1)$, in particular, Verma modules, which are standard for semisimple Lie algebras (SSLA) and their q-deformations. A lowest weight module is characterized by its lowest weight vector v_0 and its lowest weight. The lowest weight vector is characterized by the property of being annihilated by $\mathcal{S}^-$ and by being an eigenvector of the Cartan generators. The lowest weight is given by the eigenvalues of the Cartan generators on v_0. In our case the Cartan generator is D and we write the properties we mention as:

$$D\, v_0 \;=\; -d\, v_0\,, \quad P_x\, v_0 \;=\; 0, \quad P_t\, v_0 \;=\; 0 \tag{3.1}$$

where $d \in I\!R$ will be called the (conformal) weight, and the minus sign is for later convenience.

We denote by $\mathcal{B}$ the nonpositively graded subalgebra generated by D, P_x, P_t. (This is an analogue of a Borel subalgebra.) A Verma module V^d is defined as the LWM with lowest weight $-d$, induced from a one-dimensional representation of $\mathcal{B}$ spanned by v_0, on which the generators of $\mathcal{B}$ act as in (3.1). The Verma module is given explicitly by $V^d = U_q(\mathcal{S}^+) \otimes v_0$, where $U_q(\mathcal{S}^+)$ is the q-deformed universal enveloping algebra of $\mathcal{S}^+$. Clearly, $U_q(\mathcal{S}^+)$ has the basis elements $p_{k,\ell} = G^k K^\ell$. The basis vectors of the Verma module are $v_{k,\ell} = p_{k,\ell} \otimes v_0$, (with $v_{0,0} = v_0$). The action of the q-Schrödinger algebra on this basis is derived easily from (2.3):

$$\begin{aligned} D\, v_{k,\ell} &= (k+2\ell-d)\, v_{k,\ell} && (3.2a)\\ G\, v_{k,\ell} &= v_{k+1,\ell} && (3.2b)\\ K\, v_{k,\ell} &= v_{k,\ell+1} && (3.2c) \end{aligned}$$

$$P_x v_{k\ell} = q^{\frac{1-k}{2}}\, m\, [k]'_q\, v_{k-1,\ell} + q^{d+1-\ell-k}\, [\ell]_q\, v_{k+1,\ell-1} \tag{3.2d}$$

$$P_t\, v_{k\ell} = [\ell]_q\, [k+\ell-1-d]_q\, v_{k,\ell-1} + m\, \frac{[k]'_q\, [k-1]'_q}{[2]'_q}\, v_{k-2,\ell} \tag{3.2e}$$

For the derivation of (3.2) one uses the relations:

$$\begin{aligned} P_x G^k - q^{-k} G^k P_x &= m\, q^{(1-k)/2} [k]'_q\, G^{k-1} && (3.3a)\\ P_x K^\ell - K^\ell P_x &= q^{1-\ell} [\ell]_q\, G\, K^{\ell-1}\, q^{-D} && (3.3b) \end{aligned}$$

$$P_t G^k - q^k G^k P_t = [k]_q\, G^{k-1}\, P_x + \frac{[k]'_q\, [k-1]'_q}{[2]'_q}\, G^{k-2} \tag{3.3c}$$

$$P_t K^\ell - K^\ell P_t = [\ell]_q\, K^{\ell-1}\, [D+\ell-1]_q \tag{3.3d}$$

which follow from (2.3).

Because of (3.2a) we notice that the Verma module V^d can be decomposed in homogeneous (w.r.t. D) subspaces as follows:

$$\begin{aligned} V^d &= \oplus_{n=0}^{\infty} V^d_n && (3.4a)\\ V^d_n &= \text{lin.span.}\ \{v_{k,\ell} | k+2\ell = n\} && (3.4b)\\ \dim V^d_n &= 1 + \left[\frac{n}{2}\right]_{\rm int} && (3.4c) \end{aligned}$$

where $[s]_{\rm int}$ (not to be confused with $[s]_q$) is the biggest integer smaller or equal to s.

Next we analyze the reducibility of V^d through the so-called singular vectors. In analogy to the SSLA situation a singular vector v_s in our case is a homogeneous element of V^d, such that $v_s \notin \mathbb{C} v_0$, and

$$P_x\, v_s \;=\; 0\,, \quad P_t\, v_s \;=\; 0 \tag{3.5}$$

Now we give the possible singular vectors explicitly. Fix the grade $p > 0$ and denote the singular vector as v^p_s. Consider first the case of *even* grade, $p \in 2I\!N$. Since $v^p_s \in V^d_p$ the general expression is:

$$v^p_s = \sum_{\ell=0}^{p/2} a_\ell\, v_{p-2\ell,\ell} = \mathcal{Q}^p(G,K) \otimes v_0\,, \quad p \text{ even} \tag{3.6}$$

Applyng (3.5) we obtain that a singular vector exists only for $d = \frac{p-3}{2}$ (as for $q = 1$ [11]) and is given explicitly for arbitrary q by the formula:

$$\begin{aligned} v^p_s &= a_0 \sum_{\ell=0}^{\frac{p}{2}} \left(-m[2]'_q\right)^\ell \binom{\frac{p}{2}}{\ell}_q v_{p-2\ell,\ell} = a_0\, \left(G^2 - m\, [2]'_q\, K\right)^{\frac{p}{2}}_q \otimes v_0\\ \mathcal{Q}^p(G,K) &= a_0\, \left(G^2 - m\, [2]'_q\, K\right)^{\frac{p}{2}}_q \end{aligned} \tag{3.7}$$

where

$$\binom{p}{s}_q \doteq \frac{[p]_q!}{[s]_q![p-s]_q!}\,, \qquad [n]_q! \doteq [n]_q [n-1]_q \ldots [1]_q \tag{3.8}$$

As for $q = 1$ [11] there are no singular vectors for *odd* grade.

Further we consider the consequences of the reducibility of our Verma modules. We consider the subspace of $V^{(p-3)/2}$:

$$I^{(p-3)/2} = U(\mathcal{S}^+)\, v^p_s \tag{3.9}$$

It is invariant under the action of the Schrödinger algebra, and is isomorphic to a Verma module $V^{d'}$ with shifted weight $d' = d - p = -(p+3)/2$. The latter Verma module has no singular vectors, since its weight is restricted from above : $d' \leq -5/2$, while it is clear that the necessary weight is $\geq -1/2$.

Let us denote the factor-module $V^{(p-3)/2}/I^{(p-3)/2}$ by $\mathcal{L}^{(p-3)/2}$. Let us denote by $|p\rangle$ the lowest weight vector of $\mathcal{L}^{(p-3)/2}$. It satisfies the following null conditions as a consequence of (3.5) and (3.7) :

$$P_x\, |p\rangle = 0 \tag{3.10a}$$

$$P_t\, |p\rangle = 0 \tag{3.10b}$$

$$\sum_{\ell=0}^{\frac{p}{2}} (\,-\,m\,[2]'_q\,)^\ell \binom{\frac{p}{2}}{\ell}_q G^{p-2\ell}\, K^\ell\, |p\rangle = 0 \tag{3.10c}$$

Now from (3.10c) we see that:

$$K^{p/2}\, |p\rangle = -\sum_{\ell=0}^{p/2-1} \frac{1}{(-m[2]'_q)^{p/2-\ell}} \binom{p/2}{\ell}_q G^{p-2\ell}\, K^\ell\, |p\rangle \tag{3.11}$$

Applying repeatedly this relation to the basis one can get rid of all powers of K which are $\geq p/2$. Thus the basis of $\mathcal{L}^{(p-3)/2}$ will be *singleton* for $p=2$, and *quasi-singleton* for $p \geq 4$:

$$\dim V_n^{(p-3)/2} = 1\,, \quad \text{for } n = 0, 1 \text{ or } n \geq p \tag{3.12}$$

and it is given by:

$$v^p_{k\ell} \equiv G^k\, K^\ell\, |p\rangle\,, \qquad p \in 2\mathbb{N},\ k,\ell \in \mathbb{Z}_+\,,\ \ell \leq p/2-1,\ d = \frac{p-3}{2} \tag{3.13}$$

The transformation rules of this basis are given by the formulae in (3.2) except (3.2c) for $\ell = p/2 - 1$, when the transformation is:

$$K\, v^p_{k,p/2-1} = -\sum_{s=0}^{p/2-1} \frac{1}{(-m[2]'_q)^{p/2-s}} \binom{p/2}{s}_q v^p_{k+p-2s,s} \tag{3.14c'}$$

From the transformation rules we see that $\mathcal{L}^{(p-3)/2}$ is irreducible. It is also clear that in the simplest case $p=2$ the irrep $\mathcal{L}^{-1/2}$ is also an irrep of the q- deformed centrally extended Galilean subalgebra $G_q(1)$ generated by P_x, P_t, G.

Thus, the list of the irreducible lowest weight modules over the q-deformed centrally extended Schrödinger algebra is given by:

- V^d , when $d \neq (p-3)/2,\ p \in 2\mathbb{N}$;
- $\mathcal{L}^{(p-3)/2}$, when $d = (p-3)/2,\ p \in 2\mathbb{N}$.

These irreps are infinite-dimensional.

4. Vector–field realization of $\hat{S}_q(1)$ and generalized q-deformed heat equations

Let us introduce the number operator N_y for the coordinate $y = x, t$, i.e., $N_y\, y^k = k y^k$, and the q - difference operators $\mathcal{D}_y$, $\mathcal{D}'_y$, which admit a general definition on a larger domain than polynomials, but on polynomials are well defined as follows:

$$\mathcal{D}_y \doteq \frac{1}{y}[N_y]_q \tag{4.1a}$$

$$\mathcal{D}'_y \doteq \frac{1}{y[\frac{1}{2}]_q}\left[\frac{N_y}{2}\right]_q = \frac{1}{y}[N_y]'_q \tag{4.1b}$$

Clearly, for $q \to 1$ one has: $N_y \to y\partial_y$, $\mathcal{D}_y, \mathcal{D}'_y \to \partial_y$.

With this notation we write down a five-parameter vector-field realization of our generators:

$$P_t = q^{c_1}\, \mathcal{D}_t\, q^{(1-c_5)N_t + (1-c_4)N_x} \tag{4.2a}$$

$$P_x = q^{c_2}\, \mathcal{D}'_x\, q^{-c_4 N_t + (c_3 + \frac{1}{2})N_x} \tag{4.2b}$$

$$D = 2N_t + N_x - d \tag{4.2c}$$

$$G = q^{c_2 - c_1 - c_4 + c_5}\, t\, \mathcal{D}'_x\, q^{(c_5 - c_4)N_t + (c_3 + c_4 - \frac{1}{2})N_x} + q^{-c_2 - c_3 - \frac{1}{2}}\, m x\, q^{c_4 N_t - (c_3+1)N_x} \tag{4.2d}$$

$$
\begin{aligned}
K \;=\;& q^{-c_1+c_5-1+d}\, t^2\, \mathcal{D}_t\, q^{(c_5-1)N_t+c_4N_x} \\
&+\; q^{-c_1+c_5-1+d}\, tx\, \mathcal{D}_x\, q^{(c_5-2)N_t+(c_4-1)N_x} \\
&-\; q^{-c_1+c_5-1}\, [d]_q\, t\, q^{c_5N_t+c_4N_x} \\
&+\; q^{-2c_2-3c_3-\frac{3}{2}+d}\, [\tfrac{1}{2}]_q\, mx^2\, q^{2(c_4-1)N_t-2(c_3+1)N_x}
\end{aligned}
\qquad (4.2e)
$$

where c_1, c_2, c_3, c_4, c_5 are arbitrary parameters.

For $q = 1$ we recover the standard vector-field realization of $\hat{\mathcal{S}}(1)$, namely,

$$P_t = \partial_t \qquad (4.3a)$$

$$P_x = \partial_x \qquad (4.3b)$$

$$D = 2t\partial_t + x\partial_x - d \qquad (4.3c)$$

$$G = t\partial_x + mx \qquad (4.3d)$$

$$K = t^2\partial_t + tx\partial_x - td + \tfrac{m}{2}x^2 \qquad (4.3f)$$

This realization may be used to construct a polynomial realization of the irreducible lowest weight modules considered in Section 3. For this realization we represent the lowest weight vector by the function 1. Indeed, the constants in (4.2) are chosen so that (3.1) is satisfied:

$$D\,1 \;=\; -d\,, \quad P_x\,1 \;=\; 0, \quad P_t\,1 \;=\; 0 \qquad (4.4)$$

Applying the basis elements $p_{k,\ell} = G^k K^\ell$ of the universal enveloping algebra $U_q(\mathcal{S}^+)$ to 1 we get polynomials in x, t. These polynomials we shall denote by $f_{k,\ell} \equiv p_{k,\ell}\,1$. We have in special cases:

$$
\begin{aligned}
f_{0,\ell} \;=\;& q^{\ell^2\frac{c_5-1}{2}+\ell\left(-c_1+\frac{c_5-1}{2}\right)}\, (-d)^q_\ell\, t^\ell \;\times \\
&\times \sum_{s=0}^{\ell} \frac{q^{-s^2\left(2c_3+2c_4-\frac{c_5}{2}+\frac{1}{2}\right)}}{(-d)^q_\ell} \binom{\ell}{s}_q \times \\
&\times \left(\frac{q^{\ell(2c_4-c_5-1)+c_1-2c_2-c_3-\frac{c_5}{2}+1+d}\, mx^2}{[2]'_q t}\right)^s
\end{aligned}
\qquad (4.5a)
$$

$$
\begin{aligned}
f_{2k,0} \;=\;& q^{k^2\frac{c_5-1}{2}+k\left(-c_1+\frac{c_5-1}{2}\right)}\, (\tfrac{1}{2})^q_k\, \left([2]'_q mt\right)^k \;\times \\
&\times \sum_{s=0}^{k} \frac{q^{-s^2\left(2c_3+2c_4-\frac{c_5}{2}+\frac{1}{2}\right)}}{(\frac{1}{2})^q_s} \binom{k}{s}_q \times \\
&\times \left(\frac{q^{k(2c_4-c_5-1)+c_1-2c_2-c_3-\frac{c_5}{2}+\frac{1}{2}}\, mx^2}{[2]'_q t}\right)^s
\end{aligned}
\qquad (4.5b)
$$

$$
\begin{aligned}
f_{2k+1,0} \;=\;& mx\, q^{k^2\frac{c_5-1}{2}+k\left(-c_1+c_4+\frac{c_5-3}{2}\right)-c_2-c_3-\frac{1}{2}}\, (\tfrac{3}{2})^q_k\, \left([2]'_q mt\right)^k \;\times \\
&\times \sum_{s=0}^{k} \frac{q^{-s^2\left(2c_3+2c_4-\frac{c_5}{2}+\frac{1}{2}\right)}}{(\frac{3}{2})^q_s} \binom{k}{s}_q \times \\
&\times \left(\frac{q^{k(2c_4-c_5-1)+c_1-2c_2-3c_3-c_4-\frac{c_5}{2}-\frac{1}{2}}\, mx^2}{[2]'_q t}\right)^s
\end{aligned}
\qquad (4.5c)
$$

where $(a)^q_p$ is the q-Pochhammer symbol

$$(a)^q_p \;=\; [a+p-1]_q\, [a+p-2]_q \ldots [a]_q \qquad (4.6)$$

If we choose the constants so that $2c_3 + 2c_4 - c_5/2 \;=\; 0$ then the above sums are standard degenerate q-hypergeometric polynomials:

$${}_1F^q_1(-a, b; y) \;\equiv\; \sum_{s=0}^{a} \binom{a}{s}_q \frac{q^{-\frac{s^2}{2}}}{(b)^q_s}\, (-y)^s \qquad (4.7)$$

One can show that the basis $f_{k,\ell}$ is a realization of the irreducible lowest weight representations of $\hat{\mathcal{S}}(1)$ listed at the end of the previous section. Indeed, there is 1-to-1 correspondence between the states $v_{k,\ell}$ of the Verma modules over $\hat{\mathcal{S}}_q(1)$ and the polynomials $f_{k,\ell}$. The irreducible lowest weight representations of $\hat{\mathcal{S}}_q(1)$ are factor-modules of Verma modules, with factorization over the invariant subspaces generated by singular vectors. Thus, the statement is trivial when there is no singular vector. When there is a singular vector, i.e., for the irreps $\mathcal{L}^{(p-3)/2}$, we first obtain a q-difference operator by substituting in $\mathcal{Q}^p(G, K)$ (cf. (3.6), (3.7)) each generator with its vector-field realization.

Now for the irreducibility of $\mathcal{L}^{(p-3)/2}$ it is enough to show that this q-difference operator vanishes identically when applied to 1. There is more information in this, namely, this operator gives also an invariant q-difference equation, (thus extending the $q=1$ semisimple procedure of [12] to this q-deformed non-semisimple situation). Because of invariance the solutions of this equation are elements of $\mathcal{L}^{(p-3)/2}$. Thus we are provided with an infinite hierarchy of q-difference equations which may be called generalized q-deformed heat equations. The case $p=2$ is a q-difference analog of the ordinary heat equation.

Before making the last example explicit we make a choice of constants in (4.2), namely we set $c_1 = c_2 = c_3 = c_4 = c_5 = 0$ so that to work with simpler expressions for the generators:

$$P_t \;=\; \mathcal{D}_t\, q^{N_t+N_x} \tag{4.8a}$$

$$P_x \;=\; \mathcal{D}'_x\, q^{\frac{1}{2}N_x} \tag{4.8b}$$

$$D \;=\; 2N_t + N_x - d \tag{4.8c}$$

$$G \;=\; t\,\mathcal{D}'_x\, q^{-\frac{1}{2}N_x} + q^{-\frac{1}{2}}\, mx\, q^{-N_x} \tag{4.8d}$$

$$\begin{aligned} K \;=\; & q^{d-1}\, t^2\, \mathcal{D}_t\, q^{-N_t} + q^{d-1}\, tx\, \mathcal{D}_x\, q^{-2N_t-N_x} \\ & - q^{-1}\, [d]_q\, t + q^{d-\frac{3}{2}}\, [\tfrac{1}{2}]_q\, mx^2\, q^{-2N_t-2N_x} \end{aligned} \tag{4.8e}$$

The polynomials from (4.5) simplify to:

$$f_{0,\ell} \;=\; q^{-\frac{\ell(\ell+1)}{2}}\, (-d)^q_\ell\, t^\ell\, {}_1F^q_1\,(-\ell,-d;-\frac{q^{1+d-\ell}mx^2}{[2]'_q t}) \tag{4.9a}$$

$$f_{2k,0} \;=\; q^{-\frac{k(k+1)}{2}}\, (\tfrac{1}{2})^q_k\, \left([2]'_q mt\right)^k\; {}_1F^q_1\,(-k,\tfrac{1}{2};-\frac{q^{\frac{1}{2}-k}mx^2}{[2]'_q t}) \tag{4.9b}$$

$$f_{2k+1,0} \;=\; q^{-\frac{k(k+3)}{2}-\frac{1}{2}}\, (\tfrac{3}{2})^q_k\, \left([2]'_q mt\right)^k\, mx\; {}_1F^q_1\,(-k,\tfrac{3}{2};-\frac{q^{-\frac{1}{2}-k}mx^2}{[2]'_q t}) \tag{4.9c}$$

For the operator $S_q = \mathcal{Q} = G^2 - [2]'_q m\, K$ determining the singular vectors we get:

$$\begin{aligned} S_q \;=\; & t^2\, q^{\frac{1}{2}} \left(\mathcal{D}'^2_x\, q^{-N_x} - q^{d-\frac{3}{2}}\, [2]'_q\, m\, \mathcal{D}_t\, q^{-N_t} \right) + \\ & + mtx\, \mathcal{D}'_x\, \left([2]'_q - (1+q^{N_x})\, q^{d-1-2N_t} \right)\, q^{-\frac{3}{2}N_x} + \\ & + q^{-1}\, mt\, \left(q^{-2N_x} + [2d]'_q \right) + \\ & + q^{-2}\, m^2x^2 \left(1 - q^{d+\frac{1}{2}-2N_t} \right) q^{-2N_x} \end{aligned} \tag{4.10}$$

which for $q=1$ gives:

$$S \;=\; t^2\, (\, \partial_x^2 - 2m\, \partial_t\,) + m\, t\, (2d+1) \tag{4.11}$$

So for $d=-\frac{1}{2}$ (which corresponds to the lowest singular vector ($p=2$)) we can interpret the equation $S_q f = 0$ as a q-deformed heat equation. The explicit form of this equation is:

$$S_q\, f \;=\; 0$$

$$\begin{aligned} S_q \;=\; & t^2\, q^{\frac{1}{2}}\, \left(\mathcal{D}'^2_x\, q^{-N_x} - q^{-2}\, [2]'_q\, m\, \mathcal{D}_t\, q^{-N_t} \right) + \\ & + mtx\, \mathcal{D}'_x \left([2]'_q - (1+q^{N_x})\, q^{-\frac{3}{2}-2N_t} \right) q^{-\frac{3}{2}N_x} - \\ & - \lambda q^{-1}\, mtx\, \mathcal{D}_x\, q^{-N_x} + \\ & + \lambda q^{-2}\, m^2x^2t\, \mathcal{D}_t\, q^{-N_t-2N_x} \end{aligned} \tag{4.12}$$

where $\lambda \doteq q - q^{-1}$. This is our proposal for a *q-deformed heat equation.* Naturally, for $q \mapsto 1$ $(\lambda \mapsto 0)$ our equation goes into the ordinary heat equation.

5. A q-deformed Schrödinger algebra on shell

Here we shall use the q-deformation of the vector-field realization of $\hat{S}(1)$ given by Floreanini and Vinet [10]. The expressions for the generators are [10] :

$$P_t \;=\; \mathcal{D}_t\, q^{-1-N_t} \tag{5.1a}$$

$$P_x \;=\; \mathcal{D}'_x\, q^{-\frac{1}{2}N_x-\frac{1}{2}} \tag{5.1b}$$

$$D \;=\; 2N_t \;+\; N_x + \tfrac{1}{2} \tag{5.1c}$$

$$G \;=\; t\,\mathcal{D}'_x\, q^{-\frac{1}{2}N_x-\frac{3}{2}} \;+\; x\,[\tfrac{1}{2}]_q\, q^{-N_x-\frac{3}{2}} \tag{5.1d}$$

$$\begin{aligned} K \;=\; & t^2\,\mathcal{D}_t\, q^{-N_t-2N_x-4} \;+\; tx\,\mathcal{D}'_x\, q^{-\frac{3}{2}N_x-\frac{5}{2}} \;+ \\ & +\; x^2\,[\tfrac{1}{2}]_q^2\, q^{-2N_x-4} \;+\; t\,[\tfrac{1}{2}]_q\, q^{-2N_x-\frac{7}{2}} \end{aligned} \tag{5.1d}$$

Note that the explicit form of the above expressions differs from the one in [10], formulae (9). From the latter our formulae are obtained by the change $t \to (1-q^{-2})t$, $x \to (1-q)x$ (employed in [10] only for the limit $q \to 1$); also our definition for the q - difference operators (4.1) is slightly different; we use N_y instead of $T_y = q^{N_y}$ used in [10]; finally, our generator D is essentially the log of their D.

The advantage of the above form of the q-deformed generators is that the $q \to 1$ limit is transparent. In this limit we recover (4.3) with $m = 1/2$, $d = -1/2$. The value of d is not accidental since this realization was achieved in [10] as the symmetry algebra of the *solutions* of a q - deformation of the heat equation. Indeed, these generators do not form a closed algebra. We have instead of (2.1) :

$$[P_t, G] \;=\; P_x\, q^{-D-\frac{1}{2}} \tag{5.2a}$$
$$q^2 P_x K - K P_x \;=\; G \tag{5.2b}$$
$$[D, G] \;=\; G \tag{5.2c}$$
$$[D, P_x] \;=\; -P_x \tag{5.2d}$$
$$[D, P_t] \;=\; -2P_t \tag{5.2e}$$
$$[D, K] \;=\; 2K \tag{5.2f}$$
$$[P_t, K] \;=\; \frac{1}{[\frac{1}{2}]_q} \left[\frac{D}{2}\right]_q q^{-\frac{3}{2}D-2} \;-\; \lambda[\tfrac{1}{4}]_q[\tfrac{3}{4}]_q\, q^{-2D-2} \tag{5.2g}$$
$$qP_xG - GP_x \;=\; [\tfrac{1}{2}]_q q^{-1/2} \tag{5.2h}$$

However, instead of $[G, K] = 0$ (or some more complicated relation, but closed in the algebra) one has:

$$[G, K] \;=\; L \;=\; -\lambda\, \frac{t^2}{x[\frac{1}{2}]_q} \left[\frac{N_x}{2}\right]_q^2 q^{-2N_x-\frac{7}{2}} \;-\; \lambda t x [\tfrac{1}{2}]_q\, [N_t]_q\, q^{-N_t-3N_x-\frac{13}{2}} \tag{5.3}$$

where L is a new generator. For our purposes it is enough that L annihilates all functions $f_{k,\ell} \;=\; G^k K^\ell\, 1$. Thus we may again use the basis $f_{k,\ell}$. We have:

$$f_{0,\ell} \;=\; f_{2\ell,0} \tag{5.4a}$$
$$f_{2k,0} \;=\; (\tfrac{1}{2})_k^q\, t^k\, q^{-\frac{1}{2}k^2-3k}\; {}_1F_1^q\left(-k;\, \tfrac{1}{2};\, -\frac{q^{1-k}x^2[\frac{1}{2}]_q^2}{t}\right) \tag{5.4b}$$
$$f_{2k+1,0} \;=\; (\tfrac{3}{2})_k^q\, [\tfrac{1}{2}]_q\, xt^k\, q^{-\frac{1}{2}(k^2+3)-4k}\; {}_1F_1^q\left(-k;\, \tfrac{3}{2};\, -\frac{q^{-k}x^2[\frac{1}{2}]_q^2}{t}\right) \tag{5.4c}$$

(cf. (4.7)). For $q = 1$ these expressions were obtained in [11].

Formula (5.4a) is equivalent to $(G^2 - K)\,1 \;=\; 0$, i.e., we have the q-deformed version of the irrep $\mathcal{L}^{-1/2}$, $(p = 2)$, and the basis consists only of $f_k \equiv f_{k,0} = G^k\, 1$. The generators act on this basis as follows:

$$D\, f_k \;=\; (k + \tfrac{1}{2})\, f_k \tag{5.5a}$$
$$G\, f_k \;=\; f_{k+1} \tag{5.5b}$$
$$K\, f_k \;=\; f_{k+2} \tag{5.5c}$$
$$P_x\, f_k \;=\; k[\tfrac{1}{2}]_q q^{-\frac{3}{2}}\, f_{k-1} \tag{5.5d}$$
$$P_t\, f_k \;=\; [\tfrac{1}{2}]_q q^{-\frac{5}{2}} b_k\, f_{k-2}\,, \quad b_k \doteq \sum_{s=0}^{k-1} s q^{-s} \tag{5.5e}$$

where, by summation convention, $b_0 = b_1 = 0$.

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