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OPERATIONS ON N-VARIETIES OF REGULAR ω-LANGUAGES AND PRODUCTS ON REGULAR ω-LANGUAGES

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ABSTRACT

In an earlier paper (1986) we have shown the existence of an Eilenberg correspondence between varieties of regular ω-languages (*N-varieties*, for short) and varieties of finite monoids (*M-varieties*) not being a variety of groups. The aim of this paper is to establish the correspondence in the connection with some operations on languages and on M-varieties. For this, some operations on N-varieties concerning with the shuffle product on ω-languages are studied, explicit forms of some N-varieties closed under shuffle product are given. In particular, a new product on ω-language and a new operation on N-varieties are introduced and considered.

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0. Introduction

In [11] D. Perrin has established a correspondence between varieties of finite semigroups (resp. of finite monoids) and varieties of regular ω-languages. For the case of semigroups, this correspondence is not one-to-one as shown by J.P. Pecuchet in [10]. But, for the case of monoids, as shown in [7,9], this is an one-to-one correspondence between N- varieties and varieties of finite monoids not being a variety of groups (U_1-variety, for short). Our aim in this paper is to study the mentioned above correspondence in the connection with some operations on languages and on varieties.

In Section 1, some new operations on ω-languages, namely a product of regular ω-languages, denoted by $\circ$, and a quotient of an ω-languages by an infinite word are introduced. In Section 2, we prove that an N- variety is closed under the product $\circ$ if and only if the corresponding U_1-variety is closed under the Schützenberger product (Theorem 2.5). Also a language theoretical characterization of the N- varieties closed under the product $\circ$ is established (Theorem 2.8). In section 3, some results of Ch. Reutenauer [15], H. Straubing [16,17], J.E. Perrot [12] concerning the shuffle product, alphabetical morphisms and inverse substitutions are nontrivially extended to the case of ω-languages (Theorem 3.4, Corollaries 3.5, 3.6 and Theorem 3.7, Corollary 3.8). Finally, in section 4, explicit forms of the smallest N- variety closed under the shuffle product and of the N- variety of commutative regular ω-languages are given (Proposition 4.2, Corollary 4.3, Proposition 4.4).

Throughout this paper, the alphabets Σ, Γ..., the monoids M, N... are assumed finite, the morphisms are morphisms of monoids. For background we refer to [2,4,5,6,14]. Denote by: Σ^*- the free monoid on Σ with the unit ϵ; $\Sigma^+ = \Sigma^* - \{\epsilon\}$; Σ^ω- the set of infinite words on Σ; $\Sigma^\infty = \Sigma^\omega \cup \Sigma^*$; and by ΣRec^ω (resp., by ΣRec^*) the class of all regular ω-languages (resp., of all regular languages) on Σ. Let M, N be monoids and $X, Y \subseteq \Sigma^*$, $W \subseteq \Sigma^\omega$. We define the sets

$$X^\omega = \{x_1x_2... \in \Sigma^\omega | x_1, x_2, ... \in X - \{\epsilon\}\}$$

(with the convention $\forall x \in \Sigma^*, x\epsilon^\omega = \emptyset \notin \Sigma^\omega$);

$$\vec{X} = \{x_1x_2... \in \Sigma^\omega | x_1x_2...x_k \in X - \{\epsilon\}, \forall k \geq 1\};$$

$$YW = \{yw \in \Sigma^\omega | y \in Y, w \in W\};$$

and denote $M \prec N$ if M is a homomorphic image of some submonoid of N. A class $\underline{V}$ of finite monoids is called an *M- variety* if it holds for every monoid P: $\forall M, N \in \underline{V}$, $P \prec M \times N$ implies $P \in \underline{V}$, moreover if the monoid $U_1 = \{0,1\}$ in $\underline{V}$ then $\underline{V}$ is called a *U_1-vatiety*. Let $\varphi : \Sigma^* \to M$ be a morphism, we say that a subset L of Σ^* (resp., W of Σ^ω) is *saturated by* φ if for every $e \in M, \varphi^{-1}(e) \cap L \neq \emptyset$ implies $\varphi^{-1}(e) \subseteq L$ (resp., for every $e, f \in M, \varphi^{-1}(e)[\varphi^{-1}(f)]^\omega \cap W \neq \emptyset$ implies $\varphi^{-1}(e)[\varphi^{-1}(f)]^\omega \subseteq W$). Then we also say that L (resp., W) is *saturated by the kernel congruence* $\underline{\varphi}$ of φ and with this context we also say that L (resp., W) is *saturated by monoid M*. The largest congruence on Σ^* saturating L (resp., W) is called the *syntactic congruence* of L (resp., of W) and denoted by $\sim_L$ (resp., by $\approx_W$), the quotient monoid $M_L = \Sigma^*/\sim_L$ (resp., $I_W = \Sigma^*/\approx_W$) is called the *syntactic monoid* of L (resp., of W). In details the reader can see [1,4,8].

Let $\underline{V}$ be an M- variety. For each alphabet Σ we define the families

$$\Sigma V^* = \{L \in \Sigma Rec^* | M_L \in \underline{V}\}; \ \Sigma V^N = \{W \in \Sigma Rec^\omega | I_W \in \underline{V}\};$$

$$\Sigma V^\omega = \{L \in \Sigma Rec^\omega \mid \exists n \in \mathbb{N}, \exists X_i, Y_i^*, Y_i^* \in \Sigma V^*(i = 1, ..., n) : L = \bigcup_{i=1}^{n} X_i Y_i^{*\omega}\};$$

and $\Sigma\overrightarrow{V} = \{W \in \Sigma Rec^\omega \mid \exists X \in \Sigma V^* : \overrightarrow{X} = W\}^B$ the boolean closure of the class of languages $\overrightarrow{X}$ with every X belongs to ΣV^*. Then the family V^* of classes ΣV^* is called a *$*$-variety* (see [4]). Due to [4], a family $V^* = \{\Sigma V^*\}$ is a $*$-variety if the following conditions hold:

α_1) ΣV^* is closed under the boolean products;

α_2) for any $x, y \in \Sigma^*$; $L \in \Sigma V^*$ implies $x^{-1}Ly^{-1} \in \Sigma V^*$;

α_3) for any morphism $h : \Sigma^* \to \Gamma^*$, $Y \in \Gamma V^*$ implies $h^{-1}(Y) \in \Sigma V^*$.

For each U_1-variety $\underline{V}$, the family V^N of the sets ΣV^N is called an *N- variety*. As the same as the *Eilenberg correspondence* $\underline{V} \Longleftrightarrow V^*$, by our results in [7], the N- variety V^N is also uniquely defined by $\underline{V}$, with a language theoretic definition established in [9] and conversely, for each N- variety V^N we have the unique corresponding U_1-variety $\underline{V}$ generated by syntactic monoids I_X with X in ΣV^N for all Σ. Hence, we can establish the *one-to-one corresspondences* $V^N \Longleftrightarrow \underline{V} \Longleftrightarrow V^*$.

1. New operations on regular ω-languages

In the first of this section, we introduce a new product on regular ω-languages. As they thougth in mind, a product between two ω-languages, say W_1 and W_2, should be given so as to represent some "information" of W_1, W_2 in a similar manner of the concatenation of two languages and the result is also an ω-language. Such as for two singleton ω-languages $W_1 = xy^\omega, W_2 = zt^\omega$: by replacing W_1 with $W_1' = xy^*$, then the concatenation xy^*zt^ω seems to satisfy our order. But the problem is that the result must not depend on any decomposition of W_1 and W_2. For the case of regular ω-languges, this problem is settled completely.

Firstly, we need some lemmas, notions and notations. For each monoid M we define a *link relation* $\sim$ on $M \times M$ as follows: $(u, v) \sim (u', v')$ if and only if there exist an infinite chain $s_1, s_2, ...$ of elements of M, two infinite chains of integers $n_0 \leq n_1 \leq ...$; $m_1 \leq m_2 \leq ...$ such that the following conditions hold:

$$(*) \qquad \begin{cases} u &= s_0 s_1 ... s_{n_0}, \quad v = s_{n_i+1} s_{n_i+2} ... s_{n_{i+1}}; \\ u' &= s_0 s_1 ... s_{m_0}, \quad v' = s_{m_i+1} s_{m_i+2} ... s_{m_{i+1}}; i = 0, 1, ... \end{cases}$$

Then the equivalent relation $\dot{\sim}$ generated by $\sim$ is the transitive closure of $\sim$. Denote by $\mathcal{S}(M)$ the class of all subsets of $M \times M$ which are closed under $\dot{\sim}$. It is easily seen that $J \in \mathcal{S}(M)$ if and only if

$$(1.1) \qquad \forall (u, v) \in J, \forall (u', v') \in M \times M : (u, v) \sim (u', v') \Rightarrow (u', v') \in J.$$

Let $\varphi : \Sigma^* \to M$ be a morphism. For each $X \subseteq \Sigma^\omega, J \subseteq M \times M$ we define

$$(**) \qquad \begin{cases} J_X &= \{(e, f) \in M^{(2)} \mid \emptyset \neq \varphi^{-1}(e)[\varphi^{-1}(f)]^\omega \subseteq X\}; \\ C_J &= \bigcup_{(e,f) \in J} \varphi^{-1}(e)[\varphi^{-1}(f)]^\omega. \end{cases}$$

Due to [8] we have

Lemma 1.1. *Let $\varphi : \Sigma^* \to M$ be a sujective morphism, $X \in \Sigma Rec^\omega$ and $J \in \mathcal{S}(M)$. Then*

(1) *If φ saturated X, then $I_X \prec M$ and $J_X \in (M)$ (then we say that J_X is defined by φ and (**));*

(2) *φ saturates C_J (then we say that C_J is defined by φ and (**)).*

For each $W \in \Sigma Rec^\omega$, we set $up(W) = \{xy^\omega \in W \mid X \in \Sigma^*, y \in \Sigma^+\}$. The following important characterization of regular ω-languages is given in [3]

Lemma 1.2. *[3] Let L, $L' \in \Sigma Rec^\omega$. $L = L'$ if and only if $up(L) = up(L')$.*

The definition of the new product on regular ω-languages is based on

Proposition 1.3. *Let $X, Y \in \Sigma Rec^\omega$. there exists uniquely a regular ω- language $Z \in \Sigma Rec^\omega$ such that*

$$up(Z) = \bigcup_{xy^\omega \in X;\, zt^\omega \in Y} xy^*.zt^\omega.$$

Using the notations as above, now we state

Definition 1.1. *Z is called the product of X, Y and denoted by $Z = X \circ Y$.*

Proof of Proposition 1.3. Let $\varphi : \Sigma^* \to M$ be a surjective morphism saturating X (for the existence of such a φ the reader can see [1,11]). We define

$$L = \bigcup_{\exists f:\, (e,f) \in J} \varphi^{-1}(e); \; Z = LY$$

It follows from [4,11] that Z is in ΣRec^ω and L is saturated by φ. With a simple verification we have

$$L = \{x \in \Sigma^* \mid \exists y \in \Sigma^+ : xy^\omega \in X\} = \bigcup_{xy^\omega \in X} xy^*;$$

$$up(Z) = L.up(Y) = \bigcup_{xy^\omega \in X,\, zt^\omega \in Y} xy^*.zt^\omega.$$

Uniqueness of Z is directly implied from Lemma 1.2.□

Combining with each $X \in \Sigma Rec^\omega$ we define a set

$$Tr(X) = \{X(u, -); X(-, v) \mid u, v \in \Sigma^*\}$$

which is called the *trace* of X where

$X(u, -) = \{v \in \Sigma^+ : uv^\omega \in X\}$;

$X(-, v) = \{u \in \Sigma^* : uv^\omega \in X\}, v \in \Sigma^+$; $X(-, \epsilon) = \emptyset$.

In fact, this set is finite and effective defineed as shown by Lemma 1.4 below. Essentially, the trace of a regular ω-language (in view of ω-behaviours) consists of typical finite behaviours (closely related with these ω-behaviours) which one can hold and compute, and it is very usefull tool in the paper. For a related aspect the reader can see in [3]. Denote by $A^{(m)}$ the n- fold cartesian product of the set A. Due to [7,9] we have

Lemma 1.4. *For every $L \in Tr(X)$, then*

(1) $L \in \Sigma Rec^*$;
(2) $M_L \prec I_X \times U_1$;
(3) $I_X \prec [\prod_{L \in Tr(X)} M_L \times U_1]^{(m)}$ *with* $m = card\, I_X$.

Let $\alpha, \beta \in up(\Sigma^\omega)$. For simplicity we denote the product $\{\alpha\} \circ \{\beta\}$ by $\alpha \circ \beta$. Given $X \subseteq \Sigma^\omega$, we form

$$(\alpha \circ \beta) \widehat{\cap} X = \bigcup_{xy^\omega = \alpha,\, xy^*\beta \subseteq X} xy^*\beta.$$

The definition of the *left,right quotient* of a regular ω-languages by an infinite word is now based on

Proposition 1.5. *Let $X \in \Sigma Rec^\omega, w \in up(\Sigma^\omega)$. There exist uniquely $Y, Z \in \Sigma Rec^\omega$ such that:*

(1) $up(Y) = \{\alpha \in up(\Sigma^\omega) | (\alpha \circ w) \widehat{\cap} X \neq \emptyset\}$;
(2) $up(Z) = \{\alpha \in up(\Sigma^\omega) | (w \circ \alpha) \widehat{\cap} X \neq \emptyset\}$.

With notations as above, we have

Definition 1.2. *The set Y is called the right quotient of X by w, denoted by $Y = X \circ w^{-1}$. The set Z is called the left quotient of X by w, denoted by $Z = w^{-1} \circ X$.*

Proof of Proposition 1.5.

(1) Given $w \in \Sigma^\omega$, with any decomposition $w = uv^\omega$ we have $X(-,v).u^{-1} = \{x \in \Sigma^* | xu \in X(-,v)\}$. Put $L = X(-,v).u^{-1}$, $Y = \overrightarrow{L}$. It follows from [4] and lemma 1.4 that $L \in \Sigma Rec^*$ and then $Y \in \Sigma Rec^\omega$. One can deduces:

$$\begin{aligned} up(Y) &= \{\alpha \in up(\Sigma^\omega) | \exists x, y \in \Sigma^*, xy^\omega = \alpha,\ xy^* \subseteq L\} \\ &= \{\alpha \in up(\Sigma^\omega) | \exists x, y \in \Sigma^* : xy^\omega = \alpha,\ xy^*uv^\omega \subseteq X\} \\ &= \{\alpha \in up(\Sigma^\omega) | (\alpha \circ w) \widehat{\cap} X \neq \emptyset\}. \end{aligned}$$

Uniqueness of Y follows from Lemma 1.2.

(2) Let $\varphi : \Sigma^* \to M$ be a surjective morphism saturating X. For an arbitrary $u \in \Sigma^*$, from the definition of saturation it implies that the set $u^{-1}X = \{\alpha \in \Sigma^\omega | u\alpha \in X\}$ is saturated by φ. Set $Z = \cup_{u \in S} u^{-1}X$, where

$$S = \{u \in \Sigma^* | \exists v \in \Sigma^+ : uv^\omega = w, \varphi(uv) = \varphi(u), \varphi(v^2) = \varphi(v)\}.$$

Since the class of languages saturated by φ is finite and closed under the boolean products, it follows that Z is saturated by φ and $Z \in \Sigma Rec^\omega$. We have now

$$\begin{aligned} up(Z) &= \{\alpha \in up(\Sigma^\omega) | \exists u \in \Sigma^*, \exists v \in \Sigma^+ : uv^\omega = w, \varphi(uv) = \varphi(u), \varphi(v^2) = \varphi(v), u\alpha \in X\} \\ &= \{u \in up(\Sigma^\omega) | \exists u \in \Sigma^*, \exists v \in \Sigma^+ : uv^\omega = w, uv^*\alpha \subseteq X\} \\ &= \{\alpha \in up(\Sigma^\omega) | (w \circ \alpha) \widehat{\cap} X \neq \emptyset\}. \end{aligned}$$

Uniqueness of Z follows from Lemma 1.2.□

Remark from the proof, we deduce that the notion "left, right quotient" is correctly defined, not depend on any decomposition of the form $w = uv^\omega$.

2. <u>N- varieties closed under the product $\circ$</u>

An N- variety V^N is said to be *closed under the product* $\circ$ if for every alphabet Σ, the following condition holds: $W_1, W_2 \in \Sigma V^N \Rightarrow W_1 \circ W_2 \in \Sigma V^N$.

Due to [8] we have the following result

Lemma 2.1. *[8] An N- class is an N- variety if and only if there exists a $*$- variety V^* containing $\{\epsilon\}$ (i.e. $\forall \Sigma : \epsilon \in \Sigma V^*$) such that for every alphabet Σ.*

$$\Sigma V^N = \{W \in \Sigma Rec^\omega | Tr(W) \subseteq \Sigma V^*\}.$$

Let M be a monoid. Denote by M^1 the monoid obtained from M by adjoining a new unit $1 \notin M$. We say that an M- variety $\underline{V}$ is a *G-variety* if every monoid in $\underline{V}$ is a group. From [4] one can easily deduce

Lemma 2.2. *An M-variety $\underline{V}$ is a U_1-variety if and only if one of the following conditions hold*

(1) *$\underline{V}$ is not a G-variety;*
(2) *$M \in \underline{V} \Rightarrow M^1 \in \underline{V}$;*
(3) *if V^* is the $*$-variety corresponding to $\underline{V}$, then V^* contains $\{\epsilon\}$;*
(4) *if V^* is the $*$-variety corresponding to $\underline{V}$, then for any Σ, Γ, $\Sigma V^* \subseteq (\Sigma \cup \Gamma)V^*$;*
(5) *$M \in \underline{V} \Rightarrow M^0 \in \underline{V}$;*

Using notations in (**) due to [9] we have

Lemma 2.3. *[9] Let M be a monoid with an unit 1 having no proper divisors (i.e. $\forall a, b \in M : ab = 1 \Rightarrow a = b = 1$). Then for all $B \subseteq M$, $J_B = B \times \{1\} \in \mathcal{S}(M)$.*

Given $X, Y \in \Sigma^\infty$. We define the *shuffle product* of X, Y the following set

$$X \sqcup\!\sqcup Y = \{x_1y_1x_2y_2... \in \Sigma^\infty | x_1x_2... \in X, y_1y_2... \in Y, \forall x_i, y_i \in \Sigma^*\}.$$

From now on in expressions, for simplicity, we use the convention that *the operations* $^*, ^+, ^\omega, ^\infty, ^\rightarrow$ *are stronger than the concatenation, the concatenation is stronger than the shuffle product, and the shuffle product is stronger than the boolean products.*

Lemma 2.4. *Let $\underline{V}$ be a U_1-variety, V^N, V^* be the corresponding N- variety and $*$-variety. Let Σ, Γ be alphabets such that $\Sigma \cap \Gamma = \emptyset$. Then for every language L,*

(1) $L \in \Sigma V^* \Rightarrow L \sqcup\!\sqcup \Gamma^* \in (\Sigma \cup \Gamma)V^*$ & $L \sqcup\!\sqcup \Gamma^\omega \in (\Sigma \cup \Gamma)V^N$.
(2) $L \in \Sigma V^N \Rightarrow L \sqcup\!\sqcup \Gamma^\omega \in (\Sigma \cup \Gamma)V^N$.

Proof. (1) Assume that $L \in \Sigma V^*$. There exist $M \in \underline{V}$ and a surjective morphism $\varphi : \Sigma^* \to M$ saturating L. Then Lemma 2.2 with $U_1 \in \underline{V}$ implies that $M^1 \in \underline{V}$. Consider the morphism $\overline{\varphi} : (\Sigma \cup \Gamma)^* \to M^1$ defined by: $\overline{\varphi}|\Sigma = \varphi$, and $\overline{\varphi}(\Gamma^*) = 1$

(the unit of M^1). Since φ saturates L, it follows that $\exists B \subseteq M : \varphi^{-1}(B) = L$. The equality $\overline{\varphi}^{-1}(B) = L\sqcup\!\sqcup\Gamma^*$ implies that $L\sqcup\!\sqcup\Gamma^*$ is saturated by φ.By $M^1 \in \underline{V}$, we have $L\sqcup\!\sqcup\Gamma^* \in (\Sigma\cup\Gamma)V^*$. Using the equality

$$L\sqcup\!\sqcup\Gamma^\omega = \bigcup_{(e,1)\in J} \overline{\varphi}^{-1}(e)[\overline{\varphi}^{-1}(f)]^\omega, \text{ where } J = B\times\{1\}\in\mathcal{S}(M^1),$$

with Lemma 2.3 we obtain $L\sqcup\!\sqcup\Gamma^\omega = C_J$. Applying Lemma 1.1, we deduce that $L\sqcup\!\sqcup\Gamma^\omega$ is saturated by $\overline{\varphi}$. Again the fact $M^1 \in \underline{V}$ implies $L\sqcup\!\sqcup\Gamma^\omega \in (\Sigma\cup\Gamma)V^N$ as required.

(2) Suppose now $L \in \Sigma V^N$. If L is saturatedby $\varphi : \Sigma^* \to M$, then L is also saturated by $\varphi' : (\Sigma\cup\Gamma)^* \to M$ with $\varphi'|\Sigma = \varphi, \varphi'(\Gamma) = 1$, following $L\sqcup\!\sqcup\Gamma^\omega \in (\Sigma\cup\Gamma)V^N$.

Similar to a result of Reutenauer [15], for the case of infinite words we have

Theorem 2.5. *Let V^N be an N- variety and $\underline{V}$ be the corresponding M- variety. V^N is closed under the product $\circ$ if and only if $\underline{V}$ is closed under the Schützenberger product $\diamond$.*

Proof. Let V^* be the $*$-variety corresponding to $\underline{V}$, we have the correspondences

$$V^N \Longleftrightarrow \underline{V} \Longleftrightarrow V^*.$$

(1) "only if condition": Given any V^N closed under the product $\circ$, we will verify that V^* is closed under concatenation. Set $\Gamma = \Sigma\cup\{a\}$ where a is an arbitrary letter not in Σ. For any $L_1, L_2 \in \Sigma V^*$, by Lemma 2.4 we have $L_1\sqcup\!\sqcup a^\omega$, $L_2\sqcup\!\sqcup a^\omega \in \Gamma V^N$. Put $W = (L_1\sqcup\!\sqcup a^\omega)\circ(L_2\sqcup\!\sqcup a^\omega)$. Then by assumption $W \in \Sigma V^N$. This together with $L_1L_2\sqcup\!\sqcup a^* = W(-,a)$ and Lemma 1.4 imply $L_1L_2\sqcup\!\sqcup a^* \in \Gamma V^N$. Using the embbeding $j : \Sigma^* \hookrightarrow \Gamma^*$ we get $J^{-1}(L_1L_2\sqcup\!\sqcup a^*) = L_1L_2$. By definition, the condition (α_3) implies that $L_1L_2 \in \Sigma V^*$, therefore V^* is closed under concatenation. It follows from [15] that $\underline{V}$ is closed under the product $\diamond$.

(2) "if condition": Assume that $\underline{V}$ is closed under the Schützenberger product $\diamond$. It follows from [15] that V^* is closed under concatenation. By the definition of V^N it is easily verified $U_1 \in \underline{V}$. Given any $W_1, W_2 \in \Sigma V^N$, take $W = W_1\circ W_2$. Then, there exist monoids $M, N \in \underline{V}$ and surjective morphisms $\varphi : \Sigma^* \to M$, $\psi : \Sigma^* \to N$ such that φ saturates W_1, ψ satutates W_2. *Denote by $J = J_{W_1}, J' = J_{W_2}$ the sets defined by φ, ψ* and (**) respectively. Set $B = \{e \in M \mid \exists f \in M : (e,f) \in J\}$, $L = \varphi^{-1}(B)$. It follows from the proof of Proposition 1.3 and Lemma 1.1 that

$$W = W_1\circ W_2 = L.W_2 = \bigcup_{(e,f)\in J'} \psi^{-1}(e)[\psi^{-1}(f)]^\omega.$$

Since N is finite, there exists an integer k such that for every element $e \in N$: $e^{2k} = e^k$. By the definition of the relation $\sim$ it implies that $\forall (e,f) \in J'$: $\psi^{-1}(e)[\psi^{-1}(f)]^\omega \subseteq \psi^{-1}(ef^k)[\psi^{-1}(f^k)]^\omega$, $(ef^k, f^k) \in J'$ and

$$W_2 = \bigcup_{(e,f)\in J'} \psi^{-1}(e)[\psi^{-1}(f)]^\omega = \bigcup_{(e,f)\in J' \mid ef=e, f^2=f} \psi^{-1}(e)[\psi^{-1}(f)]^\omega.$$

Then

$$W = L.W_2 = \bigcup_{(e,f)\in J' \mid ef=e, f^2=f} [L.\psi^{-1}(e)][\psi^{-1}(f)]^\omega.$$

As a consequence from [11], we get the equation $\Sigma V^\omega = \Sigma V^N$ and then can verify that for every $(e,f) \in J'$ with $ef = e, f^2 = f$, $[L.\psi^{-1}(e)][\psi^{-1}(f)]^\omega \in \Sigma V^N$, following $W \in \Sigma V^N$ since the preceding unions are finite. Thus V^N is closed under the product $\circ$. $\square$.

To establish Theorem 2.8 below we need two following Lemmas in [7,9]

Lemma 2.6. *Let V^N be an N- variety, $h : \Sigma^* \to \Gamma^*$ be an arbitrary morphism. for every Y in ΓV^N, the set*

$$h^{-1}(Y) = \{x_1x_2\ldots \in \Sigma^\omega \mid h(x_1)h(x_2)\ldots \in Y, \exists_n^\infty : h(x_n) \neq \epsilon\}$$

belongs to ΣV^N.

Remark If $w = x_1x_2,\ldots \in \Sigma^\omega, \exists_n^\infty$ (there are many infinitely n) such that $h(x_n) \neq \epsilon$, then for every decomposition $w = y_1y_2\ldots, \exists_n^\infty : h(y_n) \neq \epsilon$. Therefore $h^{-1}(Y)$ is well defined.

Lemma 2.7. *Let V^N be an N- variety. Then for every alphabet Σ, ΣV^N is closed under the boolean products.*

Now we introduce a characterization of N- varieties closed under the product $\circ$ which is similar to that of $*$-varieties in [4]. We call a family $V^N = \{\Sigma V^N \subseteq \Sigma Rec^\omega\}$ an N- *class* of regular ω-languages.

Theorem 2.8. *Let V^N be an N- class closed under the product $\circ$. V^N is an N- variety if and only if the following conditions are satisfied*

- (β_1) *For any alphabet Σ, ΣV^N is closed under the boolean products;*
- (β_2) *For any Σ, Γ such that $\Sigma\cap\Gamma = \emptyset$, then* $\forall W \in \Sigma V^N, \forall L \in Tr(W), \forall u, v \in \Sigma^* : (u^{-1}Lv^{-1})\Gamma^\omega \in (\Sigma\cup\Gamma)V^N$;
- (β_3) $\forall W \in \Sigma V^N, \forall a, b \in \Sigma : W\circ(ab^\omega)^{-1} \in \Sigma V^N$;
- (β_4) *For any morphism $h : \Sigma^* \to \Gamma^*$ and $Z \in \Gamma V^N$, $h^{-1}(Z)$ is in ΣV^N.*

Proof. "only if": Let V^N be an N- variety. By Theorem 2.5, the M- variety $\underline{V}$ corresponding to $\underline{V}$ is closed under the product $\diamond$. Then $U_1 \in \underline{V}$. It follows from Lemma 2.6 that V^N satisfies (β_4). Since V^N is an N- variety, using Lemma 2.7 we deduce that (β_1) is true. Let V^* be the $*$-variety corresponding to $\underline{V}$. As a result in [15], V^* is closed under concatenation. Combining this with the fact that $\underline{V}$ is closed under the Schützenberger product $\diamond$ and a result from [11] we get $\overrightarrow{V} = V^\omega = V^N$. A consequence of Lemmas 2.1, 2.2 is that $\forall L \in Tr(W)$: $L \in (\Sigma\cup\Gamma)V^*$. Since V^* satisfies (α_2) and $U_1 \in \underline{V}$, it follows $\forall u, v \in \Sigma^*, u^{-1}Lv^{-1} \in (\Sigma\cup\Gamma)V^*$ and $\Gamma^+ = \Gamma^* - \{\epsilon\} \in (\Sigma\cup\Gamma)V^*$. By $\Sigma\cap\Gamma = \emptyset$, $L\Gamma^\omega = (\overrightarrow{L\Gamma^+})$, this with the arguments above implies $L\Gamma^\omega = (\overrightarrow{L\Gamma^+}) \in \overrightarrow{V} = V^N$. Therefore $L\Gamma^\omega \in (\Sigma\cup\Gamma)V^N$. Analogously, $(u^{-1}Lv^{-1})\Gamma^\omega \in (\Sigma\cup\Gamma)V^N$. Thus V^N satisfies (β_2). From the proof of Proposition 1.5, it follows $W\circ(ab^\omega)^{-1} = (\overrightarrow{Xa^{-1}})$ with

$X = W(-,b)$. By the condition (α_3), $Xa^{-1} \in \Sigma V^*$, this with Lemma 2.1 implies $W \circ (ab^\omega)^{-1} \in \Sigma\overrightarrow{V} = \Sigma V^N$. Hence V^N satisfies (β_3).

"if": Suppose now that V^N is an N- class satisfying $(\beta_1) - (\beta_4)$. We verify that the M- variety $\underline{V}$ generated by the family $\{I_W | W \in V^N\}$ is closed under the product $\diamond$ and that V^N is the N- variety corresponding to it.

First, one can check $U_1 \in \underline{V}$ by the facts: for arbitrary alphabet $\Sigma, \Gamma \neq \emptyset, \Sigma\cap\Gamma = \emptyset$, the conditions $(\beta_1) - (\beta_2)$ imply $\emptyset, \Sigma^\omega \in \Sigma V^N$ and $\Sigma^*\Gamma^\omega \in (\Sigma \cup \Gamma)V^N$, $U_1 \prec I_{\Sigma^*\Gamma^\omega}$, $I_{\Sigma^*\Gamma^\omega} \in \underline{V}$. Moreover, it is easy to see that $U_1 \prec M_{\Sigma^*\Gamma^*}$.

Next, let V^* be the $*$-variety corresponding to $\underline{V}$. We show that V^* is closed under concatenation. Indeed, Lemma 1.4 together with the fact $U_1 \in \underline{V}$ imply $\forall W \in \Sigma V^N$: $Tr(W) \subseteq (\Sigma \cup \Gamma)V^*$. Given $L \in \Sigma V^*$ and an alphabet Γ such that $\Sigma \cap \Gamma = \emptyset$, we first prove $L\Gamma^\omega \in (\Sigma \cup \Gamma)V^N$. By Lemma 1.4, $\Sigma^*\Gamma^* = (\Sigma^*\Gamma^\omega)(-,b)$ for any b in Γ, this implies $M_{\Sigma^*\Gamma^*} \in \underline{V}$. Therefore $\Sigma^*\Gamma^* \in (\Sigma \cup \Gamma)V^*$. Lemm 1.4 with the fact $U_1 \prec M_{\Sigma^*\Gamma^*}$ follows that the set $\{M_X | X \in Tr(W), \forall W \in V^N\}$ generates $\underline{V}$. Hence, from the definitions of V^* and V^N, there exists a family

$$\{L_i \in \Sigma_i V^* | L_i \in Tr(W_i), W_i \in \Sigma_i V^N, i = 1, ..., n\}$$

such that $M_L \prec M_{L_1} \times M_{L_2} \times ... \times M_{L_n}$.

In virtue of the proof of Theorem 3.2 S [4], L is in the boolean closure of languages Y_i of the form $Y_i = h^{-1}(X_i)$ for some morphisms $h_i : \Sigma^* \to \Sigma_i^*$ with some X_i in the boolean closure of the sets $\{u^{-1}L_i v^{-1}\}, i = 1, 2, ..., n$. Choose an alphabet Γ such that $\Gamma \cap (\Sigma \cup \Sigma_1 \cup ... \cup \Sigma_n) = \emptyset$ and morphisms $\overline{h}_i : (\Sigma \cup \Gamma)^* \to (\Sigma_i \cup \Gamma)^*$ defined by $\overline{h}_i|\Sigma = h, \overline{h}_i(a) = a, \forall a \in \Gamma$. Using the facts $U_1 \in \underline{V}$ and $\forall X, Y \subseteq \Sigma_i^*$:

$$(X \cup Y)\Gamma^\omega = X\Gamma^\omega \cup Y\Gamma^\omega, (X \cap Y)\Gamma^\omega = X\Gamma^\omega \cup Y\Gamma^\omega,$$

$$(X - Y)\Gamma^\omega = X\Gamma^\omega - Y\Gamma^\omega, h^{-1}(X)\Gamma^\omega = \overline{h}_i^{-1}(X\Gamma^\omega),$$

from $(\beta_1) - (\beta_2)$ we have $X_i\Gamma^\omega \in (\Sigma_i \cup \Gamma)V^N$ and then, from (β_1) we deduce

$$Y_i\Gamma^\omega = \overline{h}_i^{-1}(X_i\Gamma^\omega) \in (\Sigma \cup \Gamma)V^N,$$

Hence, again by using (β_1) together with the facts above we obtain $L\Gamma^\omega \in (\Sigma \cup \Gamma)V^N$.

Now, we verify that $L_1L_2 \in \Sigma V^*$ for arbitrary $L_1, L_2 \in \Sigma V^*$. Using the same arguments as above, we have $L_1\Gamma^\omega, L_2\Gamma^\omega \in (\Sigma \cup \Gamma)V^N$. Since V^N is closed under the product $\circ$, it follows $L_1\Gamma^\omega \circ L_2\Gamma^\omega = (L_1\Gamma^*)(L_2\Gamma^\omega) \in (\Sigma \cup \Gamma)V^N$. Taking $W = L_1\Gamma^\omega \circ L_2\Gamma^\omega$, by $L \in (\Sigma\cup\Gamma)V^*$ for every $L \in Tr(W)$, from Lemma 1.4, for any b in Γ we have $W(-,b) = L_1\Gamma^*L_2\Gamma^* \in (\Sigma\cup\Gamma)V^*$. Considering the embbeding $j : \Sigma^* \hookrightarrow (\Sigma\cup\Gamma)^*$, we get $j^{-1}(L_1\Gamma^*L_2\Gamma^*) = L_1L_2$. From the condition (α_3) we deduce $L_1L_2 \in \Sigma V^*$. Therefore V^* is closed under concatenation. Thus, due to [15], $\underline{V}$ is closed under the product $\diamond$.

Finally, for any $L \in \Sigma V^*$, we assert that $\overrightarrow{L} \in \Sigma V^N$. Indeed, by the result above, $L\Gamma^\omega \in (\Sigma \cup \Gamma)V^N$. Choose a letter b in $\Gamma (b \notin \Sigma)$. Since V^N satisfies (β_3), it follows

that $(\overrightarrow{L\Gamma^*}) = L\Gamma^\omega \circ (b^\omega)^{-1} \in (\Sigma\cup\Gamma)V^N$. Then Lemma 2.2 with the fact $U_1 \in \underline{V}$ implies $\{\epsilon\} \in \Gamma V^*$. Applying $L_1 = \Sigma^\omega, L_2 = \{\epsilon\}$ we get $\Sigma^\omega = \{\epsilon\}\Sigma^\omega \in (\Sigma \cup \Gamma)V^N$. It follows from (β_1) that $\overrightarrow{L} = (\overrightarrow{L\Gamma^*})\cap\Sigma^\omega \in (\Sigma\cup\Gamma)V^N$. Consider the embbeding $j : \Sigma^* \hookrightarrow (\Sigma\cup\Gamma)^*$. Since $\underline{V}$ is closed under the product $\diamond$, from [7,9,11] one can see that $\overrightarrow{V}$ is the unique N- variety corresponding to $\underline{V}$. Hence, by virtue of the definition of the correspondence $\underline{V} \Rightarrow \overrightarrow{V}$ from M- variety $\underline{V}$ to N- variety $\overrightarrow{V}$, we obtain $V^N \subseteq \overrightarrow{V}$. Combining all arguments we have $V^N = \overrightarrow{V}$. Therefore V^N is the N- variety corresponding to $\underline{V}$. □

Remark replacing from the proof the condition "closed under the product $\circ$" by the condition $V^N = \overrightarrow{V}$ we can deduce that Theorem 2.7 remains valid.

3. N- varieties, alphabetical morphisms and inverse substitutions

This section is devided in two parts. In the first part, some results related to the shuffle product, alphabetical morphism, inverse substitution defined on ω-languages are presented. In the second one, some characterizations of N- varieties in connection with the shuffle product are studied. Recall that a morphism $\varphi : \Sigma^* \to \Gamma^*$ is called *alphabetical (weak alphabetical)* if for every a in Σ: $\varphi(a) \in \Gamma$ $(\varphi(a) \in \Gamma\cup\{\epsilon\})$ and written by a.m. (resp., w.a.m.) for short. A mapping $\psi : \Sigma^* \to \Gamma^*$ is called a *substitution (finite substitution)* if ψ can be extended to the the morphism $\overline{\psi} : \Sigma^* \to \mathcal{P}(\Gamma^*)$ from the free monoid Σ^* to the monoid of all subsets of Γ^* such that $\forall a \in \Sigma : \psi(a)$ is a subset (resp., a finite subset) of Γ^* and shortly written by s.m. (resp., f.s.m.). Let $\varphi : \Sigma^* \to \Gamma^*$ be an a.m., $\sigma : \Sigma^* \to \Gamma^*$ be a s.m. and $X \subseteq \Sigma^\omega, Y \subseteq \Gamma^\omega$. Define

$$\varphi(X) = \{\varphi(x_1)\varphi(x_2)... \in \Gamma^\omega | x_1x_2... \in X, x_i \in \Sigma^*, \forall i\}.$$

$$\sigma^{-1}(Y) = \{y_1y_2...\Sigma^\omega | \sigma(y_1)\sigma(y_2)... \cap Y \neq \emptyset, y_i \in \Sigma^*, \forall i\}.$$

Remark if $\varphi(x_1)\varphi(x_2)... \in \Gamma^\omega$, then $\exists_i^\infty : \varphi(x_i) \neq \epsilon$.

Definition 3.1. *Let $\underline{V}$ be a U_1-variety and V^N be the corresponding N- variety. Define*

(1) *$AM(V^N)$ (resp., $AM'(V^N)$) the smallest N- variety containing V^N such that for any a.m. (resp., for any w.a.m.) $h : \Sigma^* \to \Gamma^*$, for any $X \in \Sigma V^N$, one has $h(X) \in \Gamma AM(V^N)$ (resp., $h(X) \in \Gamma AM'(V^N)$). An N- variety V^N is called closed under alphabetical (resp., weak alphabetical) morphisms if for any a.m. (resp., w.a.m.) $\varphi : \Sigma^* \to \Gamma^*, \forall X \in \Sigma V^N$ implies $\varphi(X) \in \Gamma V^N$.*

(2) *$IS(V^N)$ (resp., $IS'(V^N)$) the smallest N- variety containing V^N such that for any f.s.m. (resp., for any s.m.) $\sigma : \Sigma^* \to \Gamma^*$, $\forall Y \in \Gamma V^N$ implies $\sigma^{-1}(Y) \in \Sigma IS(V^N)$ (resp., $\sigma^{-1}(Y) \in \Sigma IS'(V^N)$). An N- variety V^N is called closed under inverse finite substitution (resp., under inverse substitution) if for any f.s.m. (resp., for any s.m.) $\psi : \Sigma^* \to \Gamma^*$, $\forall Y \in \Gamma V^N$ implies $\psi^{-1}(Y) \in \Sigma V^N$.*

Lemma 3.1. *Let $\varphi : \Sigma^* \to \Gamma^*$ be an w.a.m. and $\sigma : \Sigma^* \to \Gamma^*$ a s.m. defined by the condition $\forall y \in \Gamma : \sigma(y) = \varphi^{-1}(y)$. Set $C = \varphi^{-1}(\epsilon)$, $D = \Sigma - C$. Then for every L in Σ^ω: $\varphi(L) = \sigma^{-1}(L - (D^* \sqcup\!\sqcup C^\omega))$.*

Proof. Note that $D^* \sqcup\!\sqcup C^\omega = \{w \in \Sigma^\omega | \varphi(w) = \emptyset\}$, hence $D^* \sqcup\!\sqcup C^\omega = \emptyset$ in the case $C = \{\epsilon\}$. Firstly, let us suppose that $y_1 y_2 ... \in \varphi(L), y_i \in \Gamma$. Since φ is an w.a.m., there exists $w \in L - (D^* \sqcup\!\sqcup C^\omega)$ such that w admits a decomposition

$$w = u_1 x_1 u_2 x_2 ..., u_i \in C^*, x_i \in \varphi^{-1}(y_i) \cap D, i \geq 1.$$

Then, $u_i x_i \in \sigma(y_i)$. Therefore $w = u_1 x_1 u_2 x_2 ... \in [\sigma(y_1)\sigma(y_2)...] \cap [L - (D^* \sqcup\!\sqcup C^\omega)]$. By definition, we have $y_1 y_2 ... \in \sigma^{-1}(L - (D^* \sqcup\!\sqcup C^\omega))$. Hence $\varphi(L) \subseteq \sigma^{-1}(L - (D^* \sqcup\!\sqcup C^\omega))$.

Conversely, consider an arbitrary $y_1 y_2 ...$ in $\sigma^{-1}(L - (D^* \sqcup\!\sqcup C^\omega))$. By definition, there exists $w \in [L - (D^* \sqcup\!\sqcup C^\omega)] \cap [\sigma(y_1)\sigma(y_2)...], w = x_1 x_2 ...$, for some $x_i \in \sigma(y_i)$, $i \geq 1$, this implies $y_1 y_2 ... = \varphi(x_1)\varphi(x_2) ... \in \varphi(L)$.

Lemma 3.2. *Let $\psi : \Sigma^* \to \Gamma^*$ be any s.m., $L \in \Sigma Rec^\omega$, $\theta_1 : \Gamma^* \to M$ be a morphism saturating L such that $\theta_1^{-1}(1) = \{\epsilon\}$ and $\overline{\psi} : \Sigma^* \to \mathcal{P}(\Gamma^*)$ be the morphism extended by ψ. Then the morphism $\theta_2 = \overline{\theta}_1 \circ \overline{\psi} : \Sigma^* \to \mathcal{P}(M)$ saturates $\psi^{-1}(L)$.*

Proof. Assume that $\psi^{-1}(L) \cap \theta_2^{-1}(A)[\theta_2^{-1}(B)]^\omega \neq \emptyset$ for some $A, B \in \mathcal{P}(M)$. By assumption, there exist $x_1, x_2, ... \in \Sigma^*$ such that $A = \theta_2(x_1), B = \theta_2(x_i), \forall i > 1$ and $x_1 x_2 x_3 ... \in \psi^{-1}(L) \cap \theta_2^{-1}(A)[\theta_2^{-1}(B)]^\omega$. It follows from the definition of ψ that $\exists y_i \in \psi(x_i), i \geq 1$ such that $y_1 y_2 ... \in [\psi(x_1)\psi(x_2)...] \cap L$. Denote $\alpha_i = \theta_1(y_i), i \geq 1$. then $\alpha_1 \in \theta_1 \psi(x_1) \subseteq \overline{\theta}_1 \overline{\psi}(x_1) = \theta_2(x_1) = A$, and similarly, $\alpha_i \in \theta_2(x_i) = B, \forall i > 1$. Thus $\theta_1^{-1}(\alpha_1)\theta_1^{-1}(\alpha_2) ... \subseteq L$ since $y_1 y_2 ... \in \theta_1^{-1}(\alpha_1)\theta_1^{-1}(\alpha_2) ... \cap L$ and θ_1 saturates L. Now consider any $w \in \theta_2^{-1}(A)[\theta_2^{-1}(B)]^\omega$. We have $w = z_1 z_2 ...$ for some $z_1 \in \theta_2^{-1}(A)$, $z_i \in \theta_2^{-1}(B), \forall i > 1$. By $\alpha_1 \in A, \alpha_i \in B, i > 1$, there exist $t_i \in \psi(z_i) \cap \theta_1^{-1}(\alpha_i), i \geq 1$. From $\theta_1(y_i) = \alpha_i, i \geq 1; \theta_1^{-1}(1) = \{\epsilon\}$, $y_1 y_2 ... \in L$, it implies $\exists^\infty_n : y_n \neq \epsilon$ and consequently, $\exists^\infty_n : t_n \neq \epsilon$. Hence, $t_1 t_2 ... \in L \cap [\psi(z_1)\psi(z_2)...]$. This means that $z_1 z_2 ... \in \psi^{-1}(L)$. Thus $\theta_2^{-1}(A)[\theta_2^{-1}(B)]^\omega \subseteq \psi^{-1}(L)$ as required.

Lemma 3.3. *Let $L \in \Sigma Rec^\omega$ and $\varphi : \Sigma^* \to M$ be a morphism saturating L. Then the morphism $\varphi_1 : \Sigma^* \to M^1$, defined by $\varphi_1(\epsilon) = 1; \varphi_1(a) = \varphi(a), \forall a \in \Sigma$, also satisfies L.*

Proof. Clearly, $\forall f \in M, \forall e \in M^1$: $e.f \in M$ and $\varphi_1^{-1}(f) \subseteq \varphi^{-1}(f)$. Assume that $\varphi_1^{-1}(e)[\varphi_1^{-1}(f)]^\omega \cap L \neq \emptyset$ for some $e, f \in M^1$, then we have $f \neq 1$. By $\varphi_1^{-1}(e)\varphi_1^{-1}(f) \subseteq \varphi_1^{-1}(e.f)$, it follows $\varphi_1^{-1}(e)[\varphi_1^{-1}(f)]^\omega \subseteq \varphi_1^{-1}(e.f)[\varphi_1^{-1}(f)]^\omega \subseteq \varphi^{-1}(e.f)[\varphi^{-1}(f)]^\omega$. Hence $\varphi^{-1}(e.f)[\varphi^{-1}(f)]^\omega \cap L \neq \emptyset$. It implies $\varphi^{-1}(e.f)[\varphi^{-1}(f)]^\omega \subseteq L$ since φ saturates L, this means that $\varphi^{-1}(e)[\varphi^{-1}(f)]^\omega \subseteq L$.

Denote by $\underline{PV}$ the M- variety generated by all monoids $\mathcal{P}(M)$ of subsets of M, $\forall M \in \underline{V}$. The following theorem is a similar to a generalized result from [12,13,15,17] for the case of ω-languages

Theorem 3.4. *Let $\underline{V}$ be an U_1-variety and V^N be the corresponding N- variety. Then $AM(V^N) = AM'(V^N) = IS(V^N) = IS'(V^N)$ and $\underline{PV}$ is in correspondence with these N- variety.*

Proof. Let V^* be the $*$-variety corresponding to $\underline{V}$. Replacing terminologies and denotations N- variety by $*$-variety, V^N by V^* in the definition 3.1 we immediately obtain the definition of $*$-varieties $AM(V^*)$, $AM'(V^*)$, $IS(V^*)$, $IS'(V^*)$ (see [13,15]). We now denote the correspondences between M- varieties, N- varieties and $*$-varieties as follows

$$\underline{V} \Leftrightarrow V^N \Leftrightarrow V^*; \underline{V}_1 \Leftrightarrow AM(V^N) \Leftrightarrow V_1^*; \underline{V}_2 \Leftrightarrow AM'(V^N) \Leftrightarrow V_2^*;$$

$$\underline{W}_1 \Leftrightarrow IS(V^N) \Leftrightarrow W_1^*; \underline{W}_2 \Leftrightarrow IS'(V^N) \Leftrightarrow W_2^*; \underline{PV} \Leftrightarrow W^N \Leftrightarrow W^*.$$

Remark that all these correspondences preserve inclusions. Throughout this proof, we use the equalities $W^* = AM(V^*) = AM'(V^*) = IS(V^*) = IS'(V^*)$ which are obtained from [13,15,17]. We first show that $W^N = AM(V^N) = AM'(V^N) = IS(V^N) = IS'(V^N)$. By definition , $AM(V^N) \subseteq AM'(V^N)$, $IS(V^N) \subseteq IS'(V^N)$, these imply $\underline{V}_1 \subseteq \underline{V}_2$, $\underline{W}_1 \subseteq \underline{W}_2$. By the fact that inclusions are preserved under the correspondences, to prove $\underline{V}_1 = \underline{V}_2 = \underline{W}_1 = \underline{W}_2$ it suffices to consider two following cases

i) $\underline{W}_2, \underline{V}_2 \subseteq \underline{PV}$: Instead of these, we prove that $IS'(V^N), AM'(V^N) \subseteq W^N$. Indeed, $V^* \subseteq IS'(V^*) = W^*$, this implies $V^N \subseteq W^N$. Let $\sigma : \Gamma^* \to \Sigma^*$ be an arbitrary s.m. and any $X \in \Sigma V^N$. Let $\varphi : \Sigma^* \to I_X$ be the syntactic morphism saturating X and $\theta : \Sigma^* \to I_X^1$ be the morphism defined as in Lemma 3.3, which also saturates X. Thus, $\theta^{-1}(X)$ is saturated by the monoid $\mathcal{P}(I_X^1)$. By $I_X^1 \in \underline{V}$ and $\mathcal{P}(I_X^1) \in \underline{PV}$, it implies $\sigma^{-1}(X) \in \Gamma W^N$. Since $IS(V^N)$ is the smallest N- variety satisfying definition 3.1 (2), we get $IS'(V^N) \subseteq W^N$.

Let now $\varphi : \Sigma^* \to \Gamma^*$ be an arbitrary w.a.m. and $X \in \Sigma V^N$. Define the s.m. $\sigma = \varphi^{-1} : \Gamma^* \to \Sigma^*$ by the condition $\sigma(y) = \varphi^{-1}(y), \forall y \in \Gamma^*$. By Lemma 2.1, $\varphi(X) = \sigma^{-1}(X - (D^* \sqcup\!\sqcup C^\omega))$ where $C = \varphi^{-1}(\epsilon) = \Sigma_0^*$ for some $\Sigma_0 \subseteq \Sigma$ and $D = \Sigma - C$. It follows from Lemma 2.4 that $D^* \sqcup\!\sqcup C^\omega = D^* \sqcup\!\sqcup \Sigma_0^\omega \in \Sigma V^N$. Therefore the set $Z = X - (D^* \sqcup\!\sqcup C^\omega)$ belongs to ΣV^N. This implies that $\varphi(X) = \sigma^{-1}(Z)$ is saturated by $\mathcal{P}(I_X^1)$. By $\mathcal{P}(I_X^1) \in \underline{PV}$ it follows $\varphi(X) \in \Sigma W^N$. Since $AM'(V^N)$ is the smallest satisfying definition 3.1 (1), we obtain $AM'(V^N) \subseteq W^N$.

ii) $\underline{PV} \subseteq \underline{V}_1, \underline{W}_1$: It suffices to show that $W^* \subseteq V_1^*$, $W^* \subseteq W_1^*$.Firstly, let $\varphi : \Sigma^* \to \Gamma^*$ be an arbitrary a.m. with any $X \in \Sigma V^*$. We will show that $\varphi(X) \in \Gamma V_1^*$ and then, from the fact that $W^* = AM(V^*)$ is the smallest satisfying definition 3.1(1) it follows $W^* \subseteq V_1^*$. Indeed, choose an arbitrary $b \notin \Sigma \cup \Gamma$. Take $\Sigma_1 = \Sigma \cup \{b\}, \Gamma_1 = \Gamma \cup \{b\}$ and $\overline{\varphi} : \Sigma_1^* \to \Gamma_1^*$ the extended morphism of φ ($\overline{\varphi}|\Sigma = \varphi$, $\overline{\varphi}(b) = b$). Then $\overline{\varphi}$ is also an a.m.. By Lemma 2.4 we have $X \sqcup\!\sqcup b^\omega \in \Sigma V^N$. Since $b \notin \Sigma \cup \Gamma$ and $\varphi, \overline{\varphi}$ are a.m., it implies that $\varphi(X) \sqcup\!\sqcup b^\omega = \overline{\varphi}(X \sqcup\!\sqcup b^\omega) \in \Gamma_1 AM(V^N)$. Using Lemma 2.1 we deduce $\varphi(X) \sqcup\!\sqcup b^* = (\varphi(X) \sqcup\!\sqcup b^\omega)(-, b) \in \Gamma_1 V_1^*$. Then $\Gamma^* \in \Gamma_1 V_1^*$ implies $\varphi(X) = (\varphi(X) \sqcup\!\sqcup b^*) \cap \Gamma^* \in \Gamma_1 V_1^*$, following $\varphi(X) \in \Gamma V_1^*$.

Secondly, to prove $W^* \subseteq W_1^*$, consider an arbitrary f.s.m. $\sigma : \Sigma^* \to \Gamma^*$ and $Y \in \Gamma V^*$. We will check that $\sigma^{-1}(Y) \in \Sigma W_1^*$. This with the fact that $W^* = IS'(V^*)$ is the smallest satisfying definition 3.1 (2) will imply $W^* \subseteq W_1^*$. Indeed, choose any letter $b \notin \Sigma \cup \Gamma$. Set $\Sigma_1 = \Sigma \cup \{b\}, \Gamma_1 = \Gamma \cup \{b\}$. Define the extended f.s.m. $\overline{\sigma}$ of σ, $\overline{\sigma} : \Sigma_1^* \to \Gamma_1^*$ by $\overline{\sigma}(b) = b, \overline{\sigma}|\Sigma = \sigma$. Since $Y \in \Gamma V^*$, this together with Lemma 2.4 implies $Y \sqcup\!\sqcup b^\omega \in \Gamma_1 V^N$. Then, by definition we get $\overline{\sigma}^{-1}(Y \sqcup\!\sqcup b^\omega) \in \Sigma_1 IS(V^N)$. Next, we check the equality $\overline{\sigma}^{-1}(Y \sqcup\!\sqcup b^\omega) = (\sigma^{-1}(Y) \sqcup\!\sqcup b^\omega) \cap (\Sigma^* \sqcup\!\sqcup b^\omega)$. Let $v \in \sigma^{-1}(Y) \sqcup\!\sqcup b^\omega$. v admits some decomposition $v = b^i c_1 b^j c_2 b^m c_3 ... c_n b^\omega$ with $u = c_1 c_2 ... c_n \in \sigma^{-1}(Y)$. The equality $\sigma(u) = \sigma(c_1)\sigma(c_2)...\sigma(c_n)$ implies $\sigma(u) \cap Y = \sigma(c_1)\sigma(c_2)...\sigma(c_n) \cap Y$. Since $c_1 c_2 ... c_n \in \sigma^{-1}(Y)$, it follows $\sigma(c_1)\sigma(c_2)...\sigma(c_n) \cap Y \neq \emptyset$. Therefore, there exist $a_i \in \sigma(c_i), i = 1, ..., n$ such that $a_1 a_2 ... a_n \in Y$. Put $\alpha = b^i a_1 b^j a_2 ... a_n b^\omega$. Then $\alpha \in \overline{\sigma}(v) \cap (Y \sqcup\!\sqcup b^\omega)$. Consequently, this implies $v \in \overline{\sigma}^{-1}(Y \sqcup\!\sqcup b^\omega)$. Clearly $v \in \Sigma^* \sqcup\!\sqcup b^\omega$.

Conversely, let $v \in \overline{\sigma}^{-1}(Y \sqcup\!\sqcup b^\omega) \cap (\Sigma^* \sqcup\!\sqcup b^\omega)$. By definition, v admits a decomposition $v = b^i c_1 b^j c_2 ... c_n b^\omega$ such that $\overline{\sigma}(v) \cap (Y \sqcup\!\sqcup b^\omega) \neq \emptyset$. We have

$$\overline{\sigma}(v) = b^i \sigma(c_1) b^j \sigma(c_2) ... \sigma(c_n) b^\omega.$$

Since $b \notin \Gamma$, it follows $\forall A, B \subseteq \Gamma^* : (A \sqcup\!\sqcup b^\omega) \cap (B \sqcup\!\sqcup b^\omega) \neq \emptyset \Leftrightarrow A \cap B \neq \emptyset$ Putting $A = \sigma(c_1)\sigma(c_2)...\sigma(c_n)$, $B = Y$, by $b^i \sigma(c_1) b^j \sigma(c_2) ... \sigma(c_n) b^\omega \cap (Y \sqcup\!\sqcup b^\omega) \neq \emptyset$, we get $\sigma(c_1)\sigma(c_2)...\sigma(c_n) \cap Y \neq \emptyset$. Therefore $c_1 c_2 ... c_n \in \sigma^{-1}(Y)$ and $v = b^i c_1 b^j c_2 ... c_n b^\omega \in \sigma^{-1}(Y) \cap b^\omega$. Thus, $\sigma^{-1}(Y \sqcup\!\sqcup b^\omega) \cap (\Sigma^* \sqcup\!\sqcup b^\omega) = \sigma^{-1}(Y) \sqcup\!\sqcup b^\omega$. Now, from Lemma 2.4 we can deduce $\Sigma^* \sqcup\!\sqcup b^\omega \in \Sigma_1 IS(V^N)$. Again by Lemma 2.4 we have $Y \sqcup\!\sqcup b^\omega \in \Gamma_1 V^N$. Hence $\overline{\sigma}^{-1}(Y \sqcup\!\sqcup b^\omega) \in \Sigma_1 IS(V^N)$. Since $\sigma^{-1}(Y) \sqcup\!\sqcup b^* = (\sigma^{-1}(Y) \sqcup\!\sqcup b^\omega)(-, b)$, using Lemma 2.1 we obtain $\sigma^{-1}(Y) = (\sigma^{-1}(Y) \sqcup\!\sqcup b^*) \cap \Sigma^* \in \Sigma_1 W_1^*$ and immediately, $\sigma^{-1}(Y) \in \Sigma W_1^*$. The proof is completed. □

As consequences of Theorem 3.4 and some results in [13,15,17] we have

Corollary 3.5. *Let $\underline{V}$ be an U_1-variety and V^N be the corresponding N- variety. The following conditions are equivalent*

(1) $\underline{V} = \underline{PV}$;
(2) *V^N is closed under a.m.;*
(3) *V^N is closed under w.a.m.;*
(4) *V^N is closed under inverse s.m.;*
(5) *V^N is closed under inverse f.s.m..*

Corollary 3.6. *The following conditions are equivalent*

(1) $\underline{V} = \underline{PV}$;
(2) *V^N is closed under alphabetical surjective morphisms;*
(3) *V^N is closed under weak alphabetical surjective morphisms.*

In the rest of this section we study the relationship between N- varieties in connection with the shuffle product $\sqcup\!\sqcup$. First we have

Definition 3.3. *Let V^N be an N- variety and V^* be a $*$-variety*

(1) *V^* is called closed under shuffle product if it holds*

$$\forall \Sigma, \forall X, Y \in \Sigma V^* \Rightarrow X \sqcup\!\sqcup Y \in \Sigma V^*.$$

Denote by $Sh(V^*)$ the smallest $*$-variety containing V^* such that

$$\forall \Sigma, \forall X, Y \in \Sigma V^* \Rightarrow X \sqcup\!\sqcup Y \in \Sigma Sh(V^*).$$

(2) *V^N is called closed under shuffle product if it holds*

$$\forall \Sigma, \forall X, Y \in \Sigma V^N \Rightarrow X \sqcup\!\sqcup Y \in \Sigma V^N.$$

Denote by $Sh(V^N)$ the smallest N-variety containing V^N such that

$$\forall \Sigma, \forall X, Y \in \Sigma V^N \Rightarrow X \sqcup\!\sqcup Y \in \Sigma Sh(V^N).$$

The main result of this part is

Theorem 3.7. *Let V^N be an N- variety and V^* be the corresponding $*$- variety. Then*

(1) *$Sh(V^N)$ is the N- variety corresponding to $Sh(V^*)$.*
(2) *V^N is closed under shuffle product if and only if V^* is closed under shuffle product.*

Proof. Denote the correspondences between these varieties as follows

$$\underline{V} \Leftrightarrow V^N \Leftrightarrow V^*, \underline{V}_1 \Leftrightarrow Sh(V^N) \Leftrightarrow V_1^*; \underline{V}_2 \Leftrightarrow V_2^N \Leftrightarrow Sh(V^*).$$

By definition, it suffices to prove that $Sh(V^N) \subseteq V_2^N$; $Sh(V^*) \subseteq V_1^*$.

i) Checking $Sh(V^*) \subseteq V_1^*$: Let $X, Y \in \Sigma V^*$. Choose any letter a not in Σ. Set $\Gamma = \Sigma \cup \{a\}$. By Lemma 2.1, we have $X \sqcup\!\sqcup a^\omega, Y \sqcup\!\sqcup a^\omega \subseteq \Gamma V^N$. It follows from definition that the set $Z = (X \sqcup\!\sqcup a^\omega) \sqcup\!\sqcup (Y \sqcup\!\sqcup a^\omega)$ belongs to $\Gamma Sh(V^N)$. Again, Lemma 2.1 yields $(X \sqcup\!\sqcup Y) \sqcup\!\sqcup a^* = Z(-, a) \in \Gamma V_1^*$, following $X \sqcup\!\sqcup Y = (X \sqcup\!\sqcup Y \sqcup\!\sqcup a^*) \cap \Sigma^* \in \Gamma V_1^*$. Since $Sh(V^*)$ is the smallest satisfying definition 3.3 (1), we obtain $Sh(V^*) \subseteq V_1^*$.

ii) Checking $Sh(V^N) \subseteq V_2^N$: Let $X, Y \in \Sigma V^N$. we prove $X \sqcup\!\sqcup Y \in \Sigma V_2^N$. Set $\backsim = \approx_X \cap \approx_Y$, $M = \Sigma^* / \backsim$. Then $M \prec I_X \times I_Y$, following $M \in \underline{V}$ since $I_X, I_Y \in \underline{V}$. Denote by h the canonical morphism defined by $\backsim$, $h : \Sigma^* \to M$. Put $T = X \sqcup\!\sqcup Y$, $T_{p,q} = h^{-1}(p) \sqcup\!\sqcup h^{-1}(q), \forall p, q \in M$. For short we use denotations

$$\sim_{p,q} = \sim_{T_{p,q}};\ M_{p,q} = M_{T_{p,q}};\ \sim_e = \sim_{h^{-1}(e)};\ M_e = M_{h^{-1}(e)}, \forall p, q, e \in M;$$

$$\rho = \bigcap_{p,q} \sim_{p,q} \bigcap \bigcap_{e \in M} \sim_e \bigcap \backsim;\ N = \Sigma^* / \rho.$$

Since $h^{-1}(p) \in \Sigma V^*$ for all $p \in M$, it implies $T_{p,q} \in \Sigma Sh(V^*)$. Moreover, by the facts

$$V^* \subseteq Sh(V^*), h^{-1}(e) \in \Sigma V^*, M \in \underline{V}, N \prec \prod_{p,q \in M} M_{p,q} \times \prod_{e \in M} M_e \times M,$$

it follows $N \in \underline{V}_2$. As a consequence of results in [1,4,9] we deduce that ρ saturates all the sets $T_{p,q}, h^{-1}(e), X, Y, \forall p, q, e \in M$. Define $\varphi : \Sigma^* \to N$ the canonical morphism defined by ρ. By definition φ also saturates all these sets. We prove that φ saturates $X \sqcup\!\sqcup Y$. Assume that there are some $e, f \in M$ such that $T \cap \varphi^{-1}(e)[\varphi^{-1}(f)]^\omega \neq \emptyset$. Then there exists w in this intersection and w admits two decompositions $w = x_1 y_1 x_2 y_2 ... = u_1 u_2 ...$ where $x_1 x_2 ... \in X$, $y_1 y_2 ... \in Y$, $u_1 \in \varphi^{-1}(e), u_i \in \varphi^{-1}(f), \forall i > 1$; $x_j, y_j, u_j \in$

$\Sigma^*, \forall j \geq 1$. Denote by x_{ij} the segment of the maximal length of x_j in common with u_i in the string w if it exists (namely x_{ij} is the longest subword of x_j and u_i satisfying the following condition: $\exists a, b, c, d \in \Sigma^*$ such that $a x_{ij} b = u_i, c x_{ij} d = x_j$ and $u_1 u_2 ... u_{i-1} a = x_1 y_1 x_2 y_2 ... x_{i-1} y_{i-1} c$), otherwise, put $x_{ij} = \epsilon$. In fact, there are only finitely many such $x_{ij} \neq \epsilon$. Set $\alpha_i = x_{i1} x_{i2} ...$. Similarly, denote by y_{ij} the segment of the maximal length of y_j in common with u_i and set $\beta_i = y_{i1} y_{i2} ...$. Clearly, $u_i \in \alpha_i \sqcup\!\sqcup \beta_i$. Let $p_i = h(\alpha_i), q_i = h(\beta_i), \forall i \geq 1$. Then $x_1 x_2 ... = \alpha_1 \alpha_2 ... \in h^{-1}(p_1) h^{-1}(p_2) ... \cap X$. This follows $h^{-1}(p_1) h^{-1}(p_2) ... \subseteq X$ since h saturates X. Similarly, we have $h^{-1}(q_1) h^{-1}(q_2) ... \subseteq Y$. Thus

$$[h^{-1}(p_1) \sqcup\!\sqcup h^{-1}(q_1)][h^{-1}(p_2) \sqcup\!\sqcup h^{-1}(q_2)] ... \subseteq X \sqcup\!\sqcup Y,$$

namely $Z_{p_1,q_1} Z_{p_2,q_2} Z_{p_3,q_3} ... \subseteq X \sqcup\!\sqcup Y$. Moreover, since $u_i \in \alpha_i \sqcup\!\sqcup \beta_i \subseteq h^{-1}(p_i) \sqcup\!\sqcup h^{-1}(q_i)$ and φ saturates Z_{p_i,q_i}, it follows $\varphi^{-1}\varphi(u_i) \subseteq Z_{p_i,q_i}, \forall i \geq 1$. Thus $\varphi^{-1}(e) = \varphi^{-1}\varphi(u_1) \subseteq Z_{p_1,q_1}, \varphi^{-1}(f) = \varphi^{-1}\varphi(u_i) \subseteq Z_{p_i,q_i}, \forall i > 1$ and $\varphi^{-1}(e)[\varphi^{-1}(f)]^\omega \subseteq Z_{p_1,q_1} Z_{p_2,q_2} ... \subseteq X \sqcup\!\sqcup Y$, following that φ saturates $X \sqcup\!\sqcup Y$. By $N \in \underline{V}_2$, we get $X \sqcup\!\sqcup Y \in \Sigma V_2^N$. Since $Sh(V^N)$ is the smallest satisfying definition 3.3 (2), we can verify $Sh(V^N) \subseteq V_2^N$ and then $Sh(V^N) = V_2^N$.

Next, assume that V^N is closed under shuffle product. Using the results above, we can check that $V^* = Sh(V^*)$ is the $*$-variety corresponding to V^N and the converse is also similarly proved. □

An M- variety $\underline{V}$ is called *commutative M- variety defined by groups* if there exists an M- variety $\underline{H}$ of groups such that $\underline{V} \cap \underline{G} = \underline{H}$ and $\underline{V} \subseteq \underline{Com}$ where $\underline{G}$ is the largest M- variety of groups and $\underline{Com}$ is the largest M- variety of commutative monoids. We say that an ω-language W is *commutative* if it satisfies

$$\forall w' \in \Sigma^\omega, w \in W, a \in \Sigma : |w|_a = |w'|_a \Rightarrow w' \in W$$

where $|w|_a$ is the number of the occurs of a letter a in the word w. Using Theorem 3.7, Lemma 2.1 and a result of Perrot [12], for the case of ω-language we can deduce

Corollary 3.8. *Let V^N be an N- variety of commutative regular ω- languages and $\underline{V}$ be the corresponding M- variety. The following conditions are equivalent*

(1) *V^N is closed under shuffle product;*
(2) *$\underline{V}$ is a commutative M- variety defined by groups;*
(3) *$\underline{V} = \underline{PV} \subseteq \underline{Com}$.*

4. Some examples for N- varieties closed under shuffle product

As examples, in Proposition 4.2, Corollary 4.3 we will give two explicit representations of the smallest N- variety A_C^N closed under shuffle product and in Proposition 4.4 a representation of the N- variety Com^N of all communtative regular ω-languages. By Proposition 4.1, the N- variety A_C^N is no longer the smallest N- variety closed under a.m. (also closed under inverse s.m.). Put $\underline{A}_C = \underline{A} \cap \underline{Com}$ where $\underline{A}$ is the M- variety of aperiodic monoids. So we can define the N- variety A_C^N and the $*$- variety A_C^* corresponding to $\underline{A}_C$. From [12] it follows that A_C^* is the nontrivial smallest $*$-variety closed under shuffle product. This combining with Theorem 3.4, Theorem 3.7 and some results in [12,13,17] imply

Proposition 4.1. *The N- variety A_C^N corresponding to M- variety $\underline{A}_C$ of communtative aperiodic monoids is*

(1) *the nontrivial smallest N- variety closed under shuffle product,*
(2) *the smallest N- variety closed under alphabetical morphisms,*
(3) *the smallest N- variety closed under weak alphabetical morphisms,*
(4) *the smallest N- variety closed under inverse s.m.,*
(5) *the smallest N- variety closed under inverse f.s.m..*

Proposition 4.2. *For every alphabet Σ, then*

$$\Sigma A_C^N = \{\Sigma_0^\omega \sqcup\!\sqcup \Sigma_1^{m_1} ... \Sigma_k^{m_k}; \Sigma_0^\omega \sqcup\!\sqcup \Sigma^\infty : \forall i : \Sigma_i \subseteq \Sigma, \Sigma_i \neq \Sigma_j, \forall j \neq i\}^B \tag{4.1}$$

is the boolean closure of ω-languages of the form $\Sigma_0^\omega \sqcup\!\sqcup \Sigma_1^{m_1} ... \Sigma_k^{m_k}$ or $\Sigma_0^\omega \sqcup\!\sqcup \Sigma^\infty$ with every family (Σ_i) of disjoint subsets of Σ.

Proof. Denote the right(left)-hand side of (4.1) by $\boldsymbol{R}$ (resp., by $\boldsymbol{L}$).

i) checking $\boldsymbol{R} \subseteq \boldsymbol{L}$: it is easily verified (see [13]) that $\underline{A}_C$ is generated by the monoids of the form

$$Z_{1,r} = \{1, a, a^2, ..., a^r \mid a^r = a^{r+1}, r \geq 1\}.$$

Since $\emptyset \sqcup\!\sqcup X = \emptyset, \forall X \subseteq \Sigma^\infty$, we could consider only the case $\Sigma_i \neq \emptyset, \forall i$. Firstly, by definition it implies that the morphism $h : \Sigma^* \to U_1 \cong Z_{1,1}$ defined by $h(\Sigma_0) = 0$ and $h(\Sigma - \Sigma_0) = 1$ saturates $\Sigma_0^\omega \sqcup\!\sqcup \Sigma^\infty = [h^{-1}(0)]^\omega$. Thus $\Sigma_0^\omega \sqcup\!\sqcup \Sigma^\infty \in \boldsymbol{L}$. Secondly, setting $\Gamma = \cup_{i=0}^k \Sigma_i$, it is can verified that the morphism

$$g : \Gamma^* \to \prod_{i=1}^k Z_{1,m_i+1} ; \forall x \in A_i, g(x) = (\underbrace{1, ..., 1, a_i^1}_{i}, 1, ..., 1), g(\Sigma_0) = (1, ..., 1)$$

satisfies the ω-language

$$S = \Sigma_0^\omega \sqcup\!\sqcup \Sigma_1^{m_1} \sqcup\!\sqcup ... \sqcup\!\sqcup \Sigma_k^{m_k} = g^{-1}((a_1^{m_1}, ..., a_k^{m_k}))[g^{-1}((1, ..., 1))]^\omega$$

Thus, $S \in \Gamma A_C^N$. By $\Gamma \subseteq \Sigma$, we deduce $S \in \boldsymbol{L}$. Since the sets S generate $\boldsymbol{R}$ and two sides of (4.1) are closed under the boolean products, one obtains $\boldsymbol{R} \subseteq \boldsymbol{L}$.

ii) checking $\boldsymbol{L} \subseteq \boldsymbol{R}$: Given any $X \in \Sigma A_C^N$, we prove that $X \in \boldsymbol{R}$. Recall that an ω-language $L \subseteq \Sigma^\omega$ is called *an h-simple part* with $h : \Sigma^* \to M$ is a morphism if there exist e, f in M, $ef = e, f^2 = f$ such that $L = h^{-1}(e)[h^{-1}(f)]^\omega$, and called *a simple part* if L is h-simple for some morphism h. For each M- variety $\underline{V}$ we define an ω-class V^ω the boolean closure of generated by all h-simple parts with every morphism $h : \Sigma^* \to M$, $M \in \underline{V}$. To complete the proof, first, one can verify without any difficulty the following facts

(i) Let $h : \Sigma^* \to M$, $g : M \to N$ be morphisms, if $g \circ h$ saturates $W \subseteq \Sigma^\omega$ then h also saturates W.

(ii) If $V^\omega \subseteq V^N$, then $V^\omega = V^N$. In particular, this holds if every h-simple part in V^ω is saturated by h.

(iii) (it can be checked by Lemmas 1.1, 2.3) For each morphism $h : \Sigma^* \to M$, $M \in \underline{A}_C$, then all h-simple parts are saturated by h.

Basing on these, secondly, one can check only the case that X is an h-simple part with any morphism of the form $h : \Sigma^* \to \prod_{i=1}^{k} Z_{1,i}$. Then one has a decomposition of h into the morphisms $h_i : \Sigma^* \to Z_{1,i}$, $h_i = p_i \circ h$ with the projections

$$p_i : \prod_{j=1}^{k} Z_{1,j} \to Z_{1,i}.$$

A simple verification shows that X is an intersection of some h_i-simple parts. Hence, it suffices now to check the only case that X is h-simple for some morphism h, $h : \Sigma^* \to Z_{1,n}$, $n \geq 1$. By Lemmas 2.3, 1.1, three following subcases maybe happen:

(+) $X = [h^{-1}(1)]^\omega$: setting $h^{-1}(1) \cap \Sigma = \Sigma_0$ one deduces that $X = \Sigma_0^\omega$ is in $\boldsymbol{R}$.

(+) $X = h^{-1}(a^k)[h^{-1}(1)]^\omega$, $0 \leq k < n$: putting $\Sigma_i = h^{-1}(a^i) \cap \Sigma$, $0 < i < n$, $\Sigma_0 = h^{-1}(1) \cap \Sigma$, one can check that

$$X = \bigcup_{(m_1, m_2, \ldots, m_{n-1}) \mid \sum_{i=1}^{n-1} i.m_i = k} [\Sigma_0^\omega \sqcup\!\sqcup \Sigma_1^{m_1} \sqcup\!\sqcup \ldots \sqcup\!\sqcup \Sigma_{n-1}^{m_{n-1}}].$$

This shows that X belongs to $\boldsymbol{R}$.

(+) $X = [h^{-1}(a_n)]^\omega$: It can be easily verified $X = \Sigma_n^\omega \sqcup\!\sqcup \Sigma^\infty$ where

$$\Sigma_n = h^{-1}(\{a, a^2, \ldots, a^n\}) \cap \Sigma.$$

Then $X \in \boldsymbol{R}$

Combining all arguments one obtains $\boldsymbol{L} = \boldsymbol{R}$ as required. □

Remark that by the facts (i)-(iii) one immediately deduces $\mathcal{A}_C^N = \mathcal{A}_C^\omega$. Denote by $\mathcal{P}$ the set of all prime numbers in $\mathbb{N}$ and set

$$L(a; k; p^m) = \{w \in \Sigma^\omega \mid |w|_a \equiv k (mod p^m)\}; \; L(a; n) = \{w \in \Sigma^\omega \mid |w|_a = n\}$$

with $a \in \Sigma, k, m \in \mathbb{N}, p \in \mathcal{P}, 0 \leq n \leq \infty$. Using (4.1) one can obtain another representation of $\mathcal{A}_C^N$ as follows

Corollary 4.3. *For each alphabet* Σ,

(4.2) $$\Sigma \mathcal{A}_C^N = \{L(a; n);\; L(a, \infty) \mid a \in \Sigma, n \in \mathbb{N}\}.$$

Finally, using Theorem 3.7, Lemma 2.1 and the presentation of $*$-variety *Com* of commutative languages in [4], one can obtain without any difficulty

Proposition 4.4. *The N- variety of all communtative regular ω-languages Com^N is the N- variety corresponding to* $\underline{Com}$. *For each alphabet* Σ,

(4.3) $$\Sigma Com^N = \{L(a; n);\; L(a; k; p^m) \mid 0 \leq n \leq \infty, m, k \in \mathbb{N}, p \in \mathcal{P}, a \in \Sigma\}^B.$$

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