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ABSTRACT

Feynman rules for the vacuum amplitude of fermions coupled to external gauge and Higgs fields in a domain wall lattice model are derived using time-dependent perturbation theory. They have a clear and simple structure corresponding to 1-loop vacuum graphs. Their continuum approximations are extracted by isolating the infrared singularities and it is shown that, in each order, they reduce to vacuum contributions for chiral fermions. In this sense the lattice model is seen to constitute a valid regularization of the continuum theory of chiral fermions coupled to weak and slowly varying gauge and Higgs fields. The overlap amplitude, while not gauge invariant, exhibits a well defined (modulo phase conventions) response to gauge transformations of the background fields. This response reduces in the continuum limit to the expected chiral anomaly, independently of the phase conventions.

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1 Introduction

Recent efforts to provide a lattice regularized model for chiral fermions have made some progress using Kaplan's idea of the domain wall [1]. This can be formulated as a 4-dimensional Euclidean lattice embedded in 4+1-dimensional spacetime. The continuous time coordinate is an artificial variable whose purpose is to accomodate a barrier, or domain wall, with which 4-dimensional chiral fermions can be associated. The barrier corresponds to a mass term discontinuity in the time-dependent Hamiltonian of the 4+1-dimensional system. Couplings to time-independent (external) gauge and Higgs fields are included in this Hamiltonian. The aim is to compute the vacuum transition amplitude for this 4+1-dimensional system and extract from it a quantity that can be interpreted as the Euclidean vacuum amplitude for chiral fermions on a 4-dimensional lattice.

An equivalent picture, developed by Narayanan and Neuberger [2,3], interprets the 4-dimensional Euclidean amplitude as the overlap of ground states belonging to the two distinct Hamiltonians that govern the 4+1-dimensional system on either side of the barrier.

The motivation behind these efforts is to obtain a model suitable for numerical, i.e., non-perturbative, studies of chiral theory. It is not yet clear that this aim will be achieved, but some encouraging results have been obtained by Narayanan and Neuberger [3,4] in the context of a 2-dimensional model whose continuum version is soluble.

Our purpose here is more limited. We want to show that the domain wall or overlap prescription is perturbatively correct. By this we mean that, for weak and slowly varying external gauge and Higgs fields, the prescription yields, in each order, a regulated version of the Feynman graph that one expects to find in a continuum theory of chiral fermions. Some work in this direction is already available in the literature. On the one hand, a set of rules for computing the perturbative contributions to the overlap amplitude was developed by Neuberger and Narayanan [3]. On the other hand, the continuum limit of low order contributions has been examined by several groups [2,5,6]. The latter work is concerned mainly with establishing that the expected chiral anomalies are indeed present.

Recently, however, we verified that the vacuum polarization tensor for chiral fermions of the $SU(2) \times U(1)$ standard model is recovered in the continuum limit [7]. Kaplan and Schmaltz [8] have shown in the continuum version, that the phase of the overlap coincides with the η-invariant of Alvarez-Gaumé *et al.* [9]. This is a non-perturbative result. Some non-perturbative analytic work in 2-dimensions is also available [10].

To set up the lattice model that corresponds to a collection of Weyl fermions coupled to background gauge and Higgs fields in a 4-dimensional Euclidean spacetime, one begins by doubling the number of fermion components. Corresponding to each 2-component Weyl fermion, $\psi_L(x)$ or $\psi_R(x)$, introduce a 4-component Dirac field, $\psi(n,t)$, defined on the sites of a 4-dimensional integer lattice, $n^\mu \in \mathbb{Z}^4$. The "time" coordinate, t, is continuous. Construct the Hamiltonian as a bilinear form,

$$H(t) = \sum_{n,m} \psi(n,t)^\dagger \; H(n,m,t) \; \psi(m,t) \tag{1.1}$$

where the matrices $H(n,m,t)$ are covariant functionals of the background fields. These background fields, $A_\mu(x)$ and $\phi(x)$, are assumed to be smooth and independent of t. The functional dependence of the matrices $H(n,m,t)$ is specified in detail in Appendix A. Here, we remark only that their time-dependence is confined to the mass-like term,

$$\varepsilon(t)\Lambda\gamma_5 T_c \delta_{nm} \tag{1.2}$$

where $\varepsilon(t) = sign(t)$, Λ is a positive parameter representing the height of the domain wall and T_c is a diagonal matrix with eigenvalues $+1(-1)$ corresponding to right(left) handed Weyl fermions. It commutes with the Dirac algebra. See Appendix A for details.

The structure of the Hamiltonian (1.1) is very simple. It has a discontinuity at $t = 0$ but is otherwise independent of time,

$$H(t) = \begin{cases} H_+(A), & t > 0 \\ H_-(A), & t < 0 \end{cases} \tag{1.3}$$

where the argument, A, stands collectively for the background gauge and Higgs fields. Although the Hamiltonian is discontinuous, the Heisenberg-picture field, $\psi(n,t)$, is continuous at $t = 0$ where it coincides with the Schroedinger-picture field, $\psi(n)$. When the background fields are weak, as we shall assume, there is a natural separation into free and interaction terms,

$$H(t) = H_0(t) + V \tag{1.4}$$

and one can set up the usual perturbation series. The only unusual feature here is the discontinuity in the free Hamiltonian.

With the two Hamiltonians, $H_+(A)$ and $H_-(A)$, one can construct two distinct normalized ground states, $|A+\rangle$ and $|A-\rangle$, respectively. Of particular interest is the functional $\Gamma(A)$ defined by the overlap,

$$\langle A + | A - \rangle = \langle + | - \rangle \; e^{-\Gamma(A)} \tag{1.5}$$

where $|\pm\rangle$ denotes the respective ground states of the free Hamiltonians, $H_{0\pm} = H_\pm(0)$. This is the functional whose perturbation development we shall consider and which, we shall show, reduces in the continuum limit to the connected vacuum amplitude for a set of Euclidean Weyl fermions. An efficient way to compute $\Gamma(A)$ is by means of time-dependent perturbation theory. The rules for expressing the contributions in terms of connected 1-loop vacuum graphs are obtained in Sec.2.

Since the free Hamiltonian depends on time, the free particle Green's function will not be invariant under time translations and, as a result, the detailed structure of the perturbative contributions to $\Gamma(A)$ is more complicated than in familiar theories. However, these complications tend to become unimportant in the continuum limit. When the 4-momenta carried by external fields are small compared to barrier height and inverse lattice spacing, amplitudes are dominated by infrared singularities, i.e. thresholds associated with the propagation of light fermions. The leading infrared singularities are insensitive to lattice structure and can be computed by an effective continuum field theory of chiral fermions in 4-dimensional Euclidean spacetime. It is precisely this continuum theory that the lattice model regulates. The emergence of a chiral continuum theory is discussed in Sec. 4. It depends crucially on the infrared behaviour of the free fermion Green's function whose detailed structure is considered in Appendix B.

The functional $\Gamma(A)$ is not gauge invariant. Under a gauge transformation of the background fields, $A \rightarrow A^\theta$, it responds according to

$$\Gamma(A^\theta) = \Gamma(A) - i\,\Phi_+(\theta, A) + i\,\Phi_-(\theta, A) \tag{1.6}$$

where $\Phi_\pm$ are real angles associated with transformations of the ground states, $|A\pm\rangle$. There is some arbitrariness in these angles that reflects the role of phase conventions in the construction of $|A\pm\rangle$. One possibility would be to impose the Brillouin–Wigner convention: that the overlaps $\langle\pm|A\pm\rangle$ shall be real and positive. This choice was adopted by Narayanan and Neuberger in their original formulation of the overlap prescription [2]. It is well adapted to time-independent perturbation theory and was used also in our low order computations [5,7,11]. Here we shall use another convention that is better adapted to time-dependent perturbation theory, to be explained in Sec.2. With either of these conventions the difference, $\Phi_+ - \Phi_-$, is non-vanishing in general. In both of them, however, it can be shown that $\Phi_+ - \Phi_-$ reduces to the standard chiral anomaly in the continuum limit. This means, in particular, that $\Gamma(A)$ becomes gauge invariant in the continuum limit if the Weyl fermions belong to an anomaly-free combination [5].

Gauge transformations are discussed in Sec.3 and the angles $\Phi_\pm$ are defined there. The relative phase between the ground states of this paper and the Brillouin–Wigner states used in earlier work is discussed in Appendix C where the time-independent formalism is briefly reviewed.

A subtle point concerning gauge transformations and the continuum limit is raised in Sec.4. This limit exhibits a lack of "uniformity". One finds that the gauge variation of the continuum limit of the effective action differs from the continuum limit of the gauge variation, $\Phi_+ - \Phi_-$. This is because the continuum limit of $\Gamma(A)$ is dominated by infrared singularities that are not present in $\Phi_+ - \Phi_-$. The latter quantity, it will be seen, is determined by massive fermions and reduces to a local form, the integral over 4-dimensional Euclidean spacetime of a pseudoscalar density in the slowly varying gauge fields and their derivatives. The coefficients in this density are finite lattice-dependent quantities. The chiral anomaly is a sub-dominant effect from the infrared point of view.

(This is only to be expected since, in continuum gauge theory the anomaly arises in a parity violating amplitude which is ultraviolet convergent and unambiguous, but whose gauge variation is, at least superficially, ultraviolet divergent.) In order to recover the standard consistent anomaly of continuum gauge theory from the local expression for $\Phi_+ - \Phi_-$ it is necessary to let the barrier height, Λ, become vanishingly small relative to the lattice cutoff, a^{-1}. The condition, $\Lambda a \ll 1$, was used in Ref.[5] where we obtained the chiral anomaly by computing $\Phi_+ - \Phi_-$. This condition is used implicitly in many studies of the overlap prescription – so called continuum models – where the fermions are represented by smooth fields in 5-dimensional spacetime, and ultraviolet questions are ignored [2,3,11,12,8]. However, it should be recognized as a non-essential technicality. In Sec.4 and Appendix B we show that the continuum theory emerges as the infrared dominant part of $\Gamma(A)$ provided only that the background fields are slowly varying on the scale of Λ^{-1}. It is not necessary to assume $\Lambda^{-1} \gg a$.[1]

In this paper we are exclusively concerned with the domain-wall-overlap formulation of chiral gauge theories on the lattice. For some other approaches see [13–17].

2 Perturbation theory

Since the domain wall problem is unusual in having a time dependent free Hamiltonian we begin with a brief description of perturbation theory in the interaction picture. We assume that the free Hamiltonian is invariant with respect to lattice translations so that Fourier transforms can be used. The interaction picture equations of motion take the form

$$\begin{aligned} i\,\partial_t\,\psi(p,t) &= [\psi(p,t), H_0(t)] \\ &= H(p,t)\psi(p,t) \end{aligned} \tag{2.1}$$

[1] We wish to acknowledge a useful correspondence with H. Neuberger who insisted on this point.

where $H(p,t)$ is an hermitian matrix, discontinuous at $t=0$ but otherwise independent of time (see Appendix A). The solution of (2.1), continuous at $t=0$, is given by

$$\psi(p,t) = \begin{cases} e^{-itH_+(p)}\ \psi(p), & t>0 \\ e^{-itH_-(p)}\ \psi(p), & t<0 \end{cases} \tag{2.2}$$

The time-dependent states of the interaction picture are governed by the usual unitary operator,

$$\Omega(t) = \begin{cases} T\left(e^{-i\int_0^t dt' V(t')}\right), & t>0 \\ \bar{T}\left(e^{-i\int_0^t dt' V(t')}\right), & t<0 \end{cases} \tag{2.3}$$

where $\bar{T}$ denotes antichronological ordering. In order that these integrals converge for $t \to \pm\infty$, the operator $V(t)$ should include the damping factor, $e^{-\varepsilon|t|}$, i.e.

$$i\,\partial_t\, V(t) = [V(t), H_0(t)] - i\varepsilon\ sgn(t)\ V(t) \tag{2.4}$$

There are two free fermion ground states, $|+\rangle$ and $|-\rangle$, defined by

$$H_{0+}|+\rangle = 0, \qquad H_{0-}|-\rangle = 0 \tag{2.5}$$

where the respective Dirac seas are filled. Expressions for the 1-body Hamiltonians $H_\pm(p)$ are given in Appendix A. The details are not important for now, except for the existence of a gap,

$$|H_\pm(p)| \geq \omega_{\min}(\Lambda) > 0$$

where Λ represents the height of the domain wall. Ground states of the interacting theory can be generated adiabatically from these states [18]. Thus, for small but finite ε define the asymptotic states

$$\begin{aligned} |in\ \pm\rangle &= \Omega(-\infty)^{-1}|\pm\rangle \\ |out\ \pm\rangle &= \Omega(\infty)^{-1}|\pm\rangle \end{aligned} \tag{2.6}$$

When ε tends to zero these states converge, apart from a singular phase, onto eigenstates of the Schroedinger-picture Hamiltonians,

$$H_\pm(A) = H_{0\pm} + V \tag{2.7}$$

where A denotes a collection of time-independent external fields. One can show [18]

$$\begin{aligned} \lim_{\varepsilon\to 0}\left(|in\ \pm\rangle\ e^{-\hat{E}_\pm/i\varepsilon}\right) &= |A\pm\rangle \\ \lim_{\varepsilon\to 0}\left(|out\ \pm\rangle\ e^{\hat{E}_\pm/i\varepsilon}\right) &= |A\pm\rangle \end{aligned} \tag{2.8}$$

where the functionals $\hat{E}_\pm(A)$ are related to the ground state energies defined by

$$H_\pm(A)|A\pm\rangle = |A\pm\rangle\ E_\pm(A) \tag{2.9}$$

The relation between $\hat{E}$ and E takes a simple form when both are expanded in powers of V, viz.

$$\begin{aligned} E &= E_1 + E_2 + E_3 + \ldots \\ \hat{E} &= E_1 + \frac{1}{2}E_2 + \frac{1}{3}E_3 + \ldots \end{aligned} \tag{2.10}$$

To compute these energies using time-dependent perturbation theory one writes, for example,

$$\begin{aligned} \langle out+|in+\rangle &= e^{2\hat{E}_+/i\varepsilon} \\ &= \langle +|\Omega(\infty)\ \Omega(-\infty)^{-1}|+\rangle \\ &= \langle +|T\left(e^{-i\int_{-\infty}^{\infty} dt\ V(t)}\right)|+\rangle \end{aligned}$$

which can be reduced to the computation of connected vacuum graphs,

$$\hat{E}_+ = \lim_{\varepsilon\to 0}\frac{i\varepsilon}{2}\ \langle +|T\left(e^{-i\int_{-\infty}^{\infty} dt\ V(t)}\right)|+\rangle_{con} \tag{2.11}$$

Since the interaction is bilinear,

$$V(t) = \int\left(\frac{dp}{2\pi}\right)^4\left(\frac{dq}{2\pi}\right)^4\ \psi(p,t)^\dagger\ V(p,q)\ \psi(q,t)\ e^{-\varepsilon|t|} \tag{2.12}$$

There is only one connected graph in each order. One finds,

$$\begin{aligned} \hat{E}_+ = &-\frac{i\varepsilon}{2}\sum_N\frac{(-i)^N}{N}\int_{-\infty}^{\infty} dt_1\ldots dt_N\ e^{-\varepsilon(|t_1|+\ldots+|t_N|)}\,. \\ &\cdot\int\left(\frac{dp_1}{2\pi}\right)^4\ldots\left(\frac{dp_N}{2\pi}\right)^4\ tr\Big[V(p_1,p_2)S_+(p_2,t_1-t_2)V(p_2,p_3)\ldots \\ &\ldots V(p_N,p_1)S_+(p_1,t_N-t_1)\Big] \end{aligned} \tag{2.13}$$

where the propagator S_+ is defined by

$$\langle +|T(\psi(q,t)\ \psi(p,t')^\dagger)|+\rangle = (2\pi)^4\delta_{2\pi}(q-p)\ S_+(q,t-t') \tag{2.14}$$

It is invariant with respect to time translations because the time dependence of $\psi(q,t)$ is determined here by the time-independent Hamiltonian, H_{0+}. Indeed, we can write

$$S_+(q,t-t') = \int_{-\infty}^{\infty}\frac{dE}{2\pi}\ \frac{i}{E-H_+(q)+i\eta\ sgn(H_+)}\ e^{-iE(t-t')} \tag{2.15}$$

with $\eta > 0$. The momentum integrations in (2.13) are compact. They range over the Brillouin zone, a cell of volume $(2\pi)^4$ in lattice units. The delta function in (2.14) is periodic with respect to reciprocal lattice translations.

An expression analogous to (2.13) can be written for $\hat{E}_-$ by replacing S_+ with S_-. More interesting is the transition amplitude that includes the barrier effect,

$$\langle out+|in-\rangle = \langle +|T\left(e^{-i\int_{-\infty}^{\infty}dt\ V(t)}\right)|-\rangle \tag{2.16}$$

where the time dependence of $\psi(q,t)$ is now determined by the time-dependent Hamiltonian, $H_0(t)$. Here one computes the connected vacuum graphs using the propagator S_F, defined by

$$\langle +|T(\psi(q,t)\ \psi(p,t')^\dagger)|-\rangle = \langle +|-\rangle\ (2\pi)^4\delta_{2\pi}(q-p)\ S_F(q,t,t') \tag{2.17}$$

This propagator is not invariant with respect to time translations. It is very much more complicated than (2.15) and it includes long range effects due to the propagation of chiral fermions. Its detailed structure is discussed in Appendix B.

The order N contribution to $-\langle out+|in-\rangle_{con}$ is given by an expression analogous to (2.13),

$$\begin{aligned}&\frac{(-i)^N}{N}\int dt_1\ldots dt_N\ e^{-\varepsilon(|t_1|+\ldots+|t_N|)}\ \cdot\\ &\cdot\int\left(\frac{dp_1}{2\pi}\right)^4\ldots\left(\frac{dp_N}{2\pi}\right)^4\ tr\Big[V(p_1,p_2)S_F(p_2,t_1,t_2)V(p_2,p_3)\ldots\\ &\qquad\ldots V(p_N,p_1)S_F(p_1,t_N,t_1)\Big]\end{aligned} \tag{2.18}$$

Finally, these results can be put together to give the effective action functional, $\Gamma(A)$, defined by (1.5),

$$\begin{aligned}\langle +|-\rangle\ e^{-\Gamma(A)} &= \langle A+|A-\rangle\\ &= \langle out+|in-\rangle\ e^{-(\hat{E}_++\hat{E}_-)/i\varepsilon}\\ &= \langle out+|in-\rangle\langle out+|in+\rangle^{-1/2}\ \langle out-|in-\rangle^{-1/2}\end{aligned}$$

or, in terms of connected vacuum graphs,

$$\Gamma(A) = -\langle out+|in-\rangle_{con} + \frac{1}{2}\ \langle out+|in+\rangle_{con} + \frac{1}{2}\ \langle out-|in-\rangle_{con} \tag{2.19}$$

The respective terms are to be computed using the propagators, S_F, S_+ and S_-. The limit $\varepsilon\to 0$ is understood. In fact, the auxiliary pieces, $\langle out\pm|in\pm\rangle_{con}$, are needed only to cancel the singularity in $\langle out+|in-\rangle_{con}$ as ε tends to zero. The effective action is obtained as the regular part of $-\langle out+|in-\rangle_{con}$.

3 Gauge transformations

Infinitesimal time-independent gauge transformations are generated by the operator

$$\begin{aligned}F_\theta &= \Sigma\ \psi(n)^\dagger\ \theta(n)\ \psi(n)\\ &= \int\left(\frac{dp}{2\pi}\right)^4\left(\frac{dq}{2\pi}\right)^4\ \psi(p)^\dagger\ \tilde{\theta}(p-q)\ \psi(q)\end{aligned} \tag{3.1}$$

where $\theta(n)$ is an hermitian matrix belonging to the algebra of the gauge group. It is slowly varying on the lattice and will be interpolated by a smooth function, $\theta(x)$, that defines the transformations of the background fields, $A\to A^\theta$, such that

$$e^{iF_\theta}\ H_\pm(A)\ e^{-iF_\theta} = H_\pm(A^\theta) \tag{3.2}$$

Since the ground states are non-degenerate, at least in perturbation theory, it follows that they must transform according to

$$e^{iF_\theta}|A\pm\rangle = |A^\theta\pm\rangle\ e^{i\Phi_\pm(\theta,A)} \tag{3.3}$$

where the angles $\Phi_\pm$ are real. These angles provide a representation of the group. Thus, if the product of two group elements is defined by

$$e^{i\theta_1}\, e^{i\theta_2} = e^{i\theta_{12}}$$

then it is easy to show that the corresponding composition rule for $\Phi_\pm$ is given by

$$\Phi(\theta_1, A^{\theta_2}) + \Phi(\theta_2, A) = \Phi(\theta_{12}, A) \tag{3.4}$$

Gauge transformations of the effective action, $\Gamma(A)$, are obtained by substituting (3.3) into the definition (1.5),

$$\Gamma(A^\theta) = \Gamma(A) - i\,\Phi_+(\theta, A) + i\,\Phi_-(\theta, A) \tag{3.5}$$

There is no reason to expect the difference, $\Phi_+ - \Phi_-$, to vanish in general but we should expect some simplifications to occur when the background fields are slowly varying. To see what happens it is necessary to compute these angles in perturbation theory.

To first order in θ the transformation rule (3.3) takes the form

$$\delta_\theta|A\pm\rangle = i(F_\theta - \Phi_\pm)|A\pm\rangle$$

which implies, for example,

$$\Phi_+ = \frac{\langle +|F_\theta|A+\rangle}{\langle +|A+\rangle} + i\,\delta_\theta\,\ell n\langle +|A+\rangle \tag{3.6}$$

and likewise for Φ_-. This can be calculated in time dependent perturbation theory using the adiabatic formula (2.8). Firstly,

$$\begin{aligned} i\,\delta_\theta\,\ell n\langle +|A+\rangle &= i\,\delta_\theta\,\ell n\langle +|in+\rangle - \frac{1}{\varepsilon}\,\delta_\theta\hat{E}_+ \\ &= \int_{-\infty}^{0} dt\,\frac{\langle +|T\left(\delta_\theta V(t)\, e^{-i\int_{-\infty}^{0} dt' V(t')}\right)|+\rangle}{\langle +|in+\rangle} - \frac{1}{\varepsilon}\,\delta_\theta\hat{E}_+ \end{aligned} \tag{3.7}$$

A suitable formula for $\delta_\theta V(t)$ can be extracted from (3.2),

$$\begin{aligned} \delta_\theta V(t) &= e^{itH_{0+}}\,\delta_\theta V\, e^{-itH_{0+}}\, e^{\varepsilon t} \\ &= e^{itH_{0+}}\, i[F_\theta, H_{0+} + V]\, e^{-itH_{0+}}\, e^{\varepsilon t} \\ &= e^{\varepsilon t}\, i[F_\theta(t), H_{0+}] + i[F_\theta(t), V(t)] \\ &= -e^{\varepsilon t}\,\partial_t\, F_\theta(t) + i[F_\theta(t), V(t)] \end{aligned}$$

which implies

$$\begin{aligned} &\langle +|T\left(\delta_\theta V(t)\, e^{-i\int_{-\infty}^{0} dt' V(t')}\right)|+\rangle = \\ &\quad = -e^{\varepsilon t}\,\partial_t\langle +|T\left(F_\theta(t)\, e^{-i\int_{-\infty}^{0} dt' V(t')}\right)|+\rangle \\ &\quad\quad -(e^{\varepsilon t} - 1)\langle +|T\left(i[F_\theta(t), V(t)]e^{-i\int_{-\infty}^{0} dt' V(t')}\right)|+\rangle \end{aligned} \tag{3.8}$$

For Abelian symmetries the result is relatively simple. In such cases the energy $E_+(A)$ is invariant in each order so that $\delta_\theta\hat{E}_+ = 0$. Also, the commutator $[F_\theta, V]$ vanishes and the second part of (3.8) is absent. It follows that

$$\begin{aligned} \Phi_+ &= \frac{\langle +|F_\theta|in+\rangle}{\langle +|in+\rangle} - \lim_{\varepsilon\to 0}\int_{-\infty}^{0} dt\, e^{\varepsilon t}\,\partial_t\,\frac{\langle +|T\left(F_\theta(t)e^{-i\int_{-\infty}^{0} dt' V(t')}\right)|+\rangle}{\langle +|in+\rangle} \\ &= \lim_{\varepsilon\to 0}\varepsilon\int_{-\infty}^{0} dt\, e^{\varepsilon t}\langle +|T\left(F_\theta(t)\, e^{-\int_{-\infty}^{0} dt' V(t')}\right)|+\rangle_{con} \end{aligned} \tag{3.9}$$

For non-Abelian symmetries the second part of (3.8) cannot be ignored. It contributes a singular term that cancels $\delta_\theta\hat{E}/\varepsilon$ from (3.7) and a regular term that must be retained. Hence we can write

$$\Phi_+ = reg\int_{-\infty}^{0} dt\langle +|T\left(\left\{\varepsilon\, e^{\varepsilon t}F_\theta(t) - (e^{\varepsilon t} - 1)\; i[F_\theta(t), V(t)]\right\} e^{-i\int_{-\infty}^{0} dt' V(t')}\right)|+\rangle \tag{3.10}$$

meaning the regular part at $\varepsilon = 0$. A comparison of this result with the corresponding angle obtained in the time-independent formalism using the Brillouin-Wigner phase convention is made in Appendix C where (3.10) is computed up to second order in $V(0)$.

4 Infrared behaviour

Having obtained general expressions for the perturbative contributions to the vacuum amplitude, $\langle out + |in-\rangle$, we now consider how to approximate them when the external fields are slowly varying.

In each order, the formula (2.18) corresponds to a 1-loop vacuum graph constructed from the vertices, $V(p,q)$, and propagators, $S_F(p,t,t')$. The vertex is expressible as an expansion in powers of the gauge field, $A_\mu(x)$, the first two terms of which are given by (A.12). In general it should include powers of the Higgs field, $\phi(x)$, as well, but we are simplifying the structure by choosing ϕ to be constant and incorporating it in the free propagator, S_F. This propagator, discussed in Appendix B, is given by (B.14). While its general structure is quite complicated, near $p = \phi = 0$ it simplifies to the form given by (B.15), which exhibits a pole corresponding to the propagation of light chiral fermions.

The loop integration in (2.18) ranges over the Brillouin zone, a cell of volume $(2\pi)^4$ in lattice units. There is, of course, no ultraviolet divergence. What interests us here is the possibility of infrared divergences which we expect to dominate the amplitude when the gauge fields are slowly varying and the Higgs field is small. To see that there is an infrared singularity it is sufficient to examine the integrand of (2.18) in the vicinity of the point, $p_1 = p_2 = \ldots = p_N = \phi = 0$. In this region the vertex (A.12) reduces to

$$V(p,q) \simeq -i\, \gamma_5 \gamma^\mu \tilde{A}_\mu(p-q) + \ldots \tag{4.1}$$

and the propagator is dominated by the pole term in (B.15),

$$S_F(p,t,t') \simeq \frac{1+\gamma_5 T_c}{2}\, \frac{1}{i\not{p} + \phi\cdot T}\, \gamma_5 \Lambda\, e^{-i\Lambda(|t|+|t'|)} + \ldots \tag{4.2}$$

Integration over the time coordinates, $t_1, \ldots, t_N$, is trivial in this approximation since

$$\Lambda \int_{-\infty}^{\infty} dt\, e^{-(\epsilon+2i\Lambda)|t|} = \frac{2\Lambda}{\varepsilon + 2i\Lambda} \to \frac{1}{i}$$

in the limit, $\varepsilon \to 0$. Hence the integrand of (2.18) reduces to

$$\begin{aligned} &\frac{i^N}{N}\, tr\left[\tilde{\not{A}}(p_1-p_2)\, \frac{1+\gamma_5 T_c}{2}\, \frac{1}{i\not{p}_2 + \phi\cdot T} \cdots \tilde{\not{A}}(p_N - p_1) \frac{1+\gamma_5 T_c}{2} \frac{1}{i\not{p}_1 + \phi\cdot T}\right] \\ &= \frac{i^N}{N}\, tr\left[\tilde{\not{A}}(k_1)\, \frac{1+\gamma_5 T_c}{2}\, \frac{1}{i(\not{p}-\not{k}_1) + \phi\cdot T} \cdots \tilde{\not{A}}(k_N) \frac{1+\gamma_5 T_c}{2} \frac{1}{i\not{p} + \phi\cdot T}\right] \end{aligned} \tag{4.3}$$

where $p_1 = p - k_1, p_2 = p - k_1 - k_2, \ldots$ and the external momenta are constrained to satisfy $k_1 + k_2 + \ldots + k_N = 0$. For $N \geq 4$ there is clearly an infrared singularity at $k_1 = \ldots = k_N = \phi = 0$, because the loop integration then diverges at $p = 0$. For $N \leq 3$, derivatives of order $3 - N$ with respect to external momenta also diverge. These singularities are of course threshold effects associated with the propagation of light chiral fermions near their mass shell. The expression (4.3) is what one would expect to find in a 4-dimensional continuum theory described by the (Euclidean) Lagrangian density

$$\mathcal{L} = \bar{\psi}(\not{\partial} - i\not{A} + \phi\cdot T)\, \frac{1+\gamma_5 T_c}{2}\, \psi \tag{4.4}$$

Our point is that (4.3) emerges as the dominant infrared effect in the lattice model. Subleading terms, down with respect to (4.3) by powers of $k_1, \ldots, k_N, \phi$ could be computed by improving the approximate formulae (4.1) and (4.2), but they would be lattice dependent and should be interpreted as scaling violations, irrelevant in the continuum approximation. In this sense the lattice overlap, or domain wall, prescription constitutes an ultraviolet regularization of the continuum system (4.4).

To summarize our approach: we seek to isolate the contributions that are singular in the infrared. The leading singularity is lattice-independent, sub-leading and non-singular quantities are sensitive to the lattice and should not be computed. Lattice dependent quantities are either not relevant to the continuum theory or they can be incorporated in counterterms. A detailed exposition of this approach as applied to the vacuum polarization tensor ($N = 2$) is given in Ref.[7].

An important qualification should be made. The expression (4.3) represents the dominant infrared contribution only if the propagator $S_F(p,t,t')$ has no other poles. The argument assumes that $p = 0$ is the only point in the Brillouin zone where the propagator is singular. It must be shown explicitly that there are no other such points, i.e. no doubling of fermions [19,20]. This matter is dealt with in Appendix B.

It may be remarked that, since the infrared dominant and lattice independent contributions (4.3) coincide exactly with the continuum theory formulae, they must also carry the expected chiral anomalies. For example, with $N = 3$ and, for simplicity, $\phi = 0$, the parity violating part of the amplitude is given by

$$\begin{aligned} \Gamma_5(A) = -\frac{i^3}{3} \int \left(\frac{dk_1}{2\pi}\right)^4 \left(\frac{dk_2}{2\pi}\right)^4 \left(\frac{dk_3}{2\pi}\right)^4 (2\pi)^4\, \delta_4(\Sigma k)\, \cdot \\ \cdot \int \left(\frac{dp}{2\pi}\right)^4 tr\left[\frac{\gamma_5 T_c}{2}\, \tilde{\not{A}}(k_1)\, \frac{1}{i(\not{p}-\not{k}_1)}\, \tilde{\not{A}}(k_2)\, \frac{1}{i(\not{p}-\not{k}_1-\not{k}_2)} \tilde{\not{A}}(k_3)\, \frac{1}{i\not{p}}\right] \end{aligned} \tag{4.5}$$

where the loop integration is understood to comprise a small region around $p = 0$. The asymptotically dominant contribution to Γ_5 when k_1, k_2, k_3 tend to zero is obtainable, however, by extending the range of p to the entire $\mathbb{R}^4$ since the resulting integral is, in fact, ultraviolet convergent. (There is no $SO(4)$ invariant, pseudoscalar local term of dimension 4 that could serve as a counterterm.) One may calculate this amplitude and verify that it satisfies

$$\delta_\theta\, \Gamma_5(A) = -\frac{i}{24\pi^2} \int d^4x\, \varepsilon^{\kappa\lambda\mu\nu}\, tr\Big(T_c\, \partial_\kappa\, A_\lambda\, \partial_\mu\, A_\nu\, \theta\Big) \tag{4.6}$$

to second order in A.

It is interesting to compare the result (4.6) with the general formula (3.5) or, to first order in θ,

$$\delta_\theta\, \Gamma(A) = -i\, \Phi_+(\theta, A) + i\, \Phi_-(\theta, A) \tag{4.7}$$

The angle Φ_+ is given by (3.10) which can be expanded in powers of the interaction, V. The result up to terms of second order is given by (C.14). In the time-dependent formalism the angles Φ_+ and Φ_- are computed using the Green's functions S_+ and S_-, respectively. These propagators, discussed in Appendix B, do not have a long range structure. They are regular at $p = 0$,

$$S_\pm(p, t-t') = \frac{1}{2}\Big(\varepsilon(t-t') \pm \gamma_5 T_c\Big)\, e^{-i\Lambda|t-t'|} + O\left(\frac{p}{\Lambda}\right) \tag{4.8}$$

This means that the functionals Φ_+ and Φ_- do not have any singularities. For slowly varying fields they are effectively local, i.e. expressible as integrals over 4-dimensional Euclidean spacetime of local functions of $A(x), \partial A(x), \ldots$ The coefficients in these local functions are, of course, lattice dependent. Indeed, they must scale with the barrier height, Λ, and lattice cutoff, a^{-1}, to a power given by their canonical dimension. In the continuum limit, coefficients with negative dimensionality will tend to zero and we may therefore restrict attention to those with non-negative dimensionality. For the gauge variation (4.7) only the pseudoscalar, $\Phi_+ - \Phi_-$, needs to be considered and this functional involves only one relevant quantity, the dimension zero coefficient of the integral in (4.6). However, this coefficient generally depends in a complicated way on Λa and other dimensionless lattice parameters. It is expressible as an integral over the Brillouin zone and it does not agree with the coefficient in (4.6). But one can show that agreement is recovered in the limit, $\Lambda a \to 0$. (This calculation was carried out in Ref.[5] where it was shown that the integral over the Brillouin zone develops an infrared singularity at $p = 0$ in the limit $\Lambda \to 0$.) This phenomenon seems to indicate a lack of "uniformity" in the continuum limit. The

gauge variation of the continuum limit of $\Gamma(A)$ does not coincide with the continuum limit of its gauge variation unless the secondary limit, $\Lambda \to 0$, is also taken. It should be interpreted as the lattice version of an effect that is familiar in continuum chiral theory. There, it is well known that the parity violating amplitude is ultraviolet convergent and unambiguous, as must be its gauge variation, the (consistent) chiral anomaly. On the other hand, this gauge variation is expressible as the difference of two formally identical, but ultraviolet divergent integrals, that have to be regulated carefully in order to obtain the correct anomaly. Ultraviolet convergent (divergent) integrals in continuum theory correspond to infrared singular (non-singular) integrals in lattice theory.

In obtaining the continuum limit of the overlap amplitude we have used the approximate formula, (4.2), the leading term in an expansion of S_F in powers of p/Λ and ap. The result (4.3) is presumably valid if $k_1, k_2, \ldots, \phi$ are all small compared to Λ and a^{-1}. No condition on the magnitude of Λa is involved. However, in view of the non-uniform response to gauge transformations outlined above, we suspect that it would be safer to choose Λa to be small.

5 Conclusions

In this paper we have provided a set of rules for computing an overlap amplitude, order by order, in weak field aproximation. We have shown that this amplitude can be interpreted as a lattice regularization of the vacuum amplitude for a 4- dimensional Euclidean continuum theory of chiral fermions coupled to background gauge and Higgs fields.

We find that the most efficient approach is through the use of time-dependent perturbation theory. The 4-dimensional Euclidean lattice is embedded in 4+1-dimensional Minkowski space with a continuous and unbounded time coordinate. This leads to an expansion of the overlap in terms of 1-loop vacuum graphs. We found it convenient in this work to use real time formalism but one could easily construct analogous formulae using imaginary time. When the background fields are slowly varying on the lattice scale and also on the scale of the inverse barrier height, the perturbative expressions simplify.

The infrared dominant term can be extracted in each order and, after integrating the time coordinates, put into correspondence with the continuum formula for that order.

Although the lattice amplitude may not be itself gauge invariant, it is guaranteed that the continuum limit, i.e. the infrared dominant part, will be gauge invariant up to chiral anomalies. These would have to be compensated in the standard way in order to recover a fully gauge invariant theory in the continuum.

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Appendix A: The model

The purpose of this Appendix is to specify the details of the lattice model including the functions $H_\pm(p), V(p,q)$ used in Sec.2 [5].

The dynamical variables comprise a set of Dirac fields, $\psi_i(n,t)$, $i = 1,\dots,N$ associated with the sites of a 4-dimensional integer lattice, $n \in \mathbb{Z}^4$. The time coordinate is continuous. The Hamiltonian is bilinear and time-dependent,

$$H(t) = \sum_{n,m} \psi(n,t)^\dagger\, H(n,m,t)\, \psi(m,t) \tag{A.1}$$

where the coefficients $H(n,m,t)$ are $4N \times 4N$ matrices acting in the product of Dirac and flavour spaces. They incorporate the couplings to external gauge and Higgs fields, $A_\mu(x)$ and $\phi(x)$, that are assumed to be smooth functions on $\mathbb{R}^4$, interpolating the lattice sites. The general structure is

$$H(n,m,t) = H_0(n-m)\; U(n,m|A) + \delta_{nm} M(n,t|\phi) \tag{A.2}$$

where H_0, U and M are matrices defined as follows.

Firstly, the gauge factor U is specified by a path ordered exponential where the path is chosen to be the straight line joining lattice sites n and m,

$$U(n,m|A) = P\left(\exp\left\{i\int_m^n dx^\mu\; A_\mu^\alpha(x) T_\alpha\right\}\right) \tag{A.3}$$

where $x^\mu(t) = tn^\mu + (1-t)m^\mu$, $0 \le t \le 1$. The $N \times N$ hermitian matrices T^α belong to the algebra of the gauge group.

Next, the mass term, M, includes the Higgs background and the barrier effect,

$$M(n,t|\phi) = \begin{cases} \gamma_5(\phi^i(n)T_i + \Lambda\; T_c), & t > 0 \\ \gamma_5(\phi^i(n)T_i - \Lambda\; T_c), & t < 0 \end{cases} \tag{A.4}$$

The $N \times N$ matrices T_i incorporate Yukawa coupling parameters and define the representation of the gauge group to which ϕ^i belongs, i.e.

$$[T_i, T_\alpha] = i(t_\alpha)_i{}^j\; T_j \tag{A.5}$$

The "chirality" matrix, T_c, is diagonal with eigenvalues ± 1 corresponding to right or left handed flavours in the continuum limit. This matrix is required to be gauge invariant,

$$[T_c, T_\alpha] = 0 \tag{A.6}$$

On the other hand, since the role of Higgs fields is to connect left with right handed fermions, the matrices T_i are required to anticommute with T_c,

$$\{T_c, T_i\} = 0 \tag{A.7}$$

Finally, to specify the hopping term $H_0(n-m)$, it is useful to employ Fourier series. Define the Fourier components, $\psi(p,t)$, by the lattice sum

$$\psi(p,t) = \sum_n \psi(n,t)\ e^{-ipn}$$

where $pn = p_\mu n^\mu$. These components are periodic in momenta with period 2π (in lattice units). The translation invariant hopping term is represented by the Fourier integral

$$H_0(n-m) = \int_{BZ} \left(\frac{dp}{2\pi}\right)^4 \gamma_5 (i\gamma^\mu\ C_\mu(p) + B(p)\ T_c)\ e^{ip(n-m)} \tag{A.8}$$

where the integral ranges over a Brillouin zone, a cell of volume $(2\pi)^4$ in lattice units. The functions C_μ and B are real and periodic. Their detailed structure is not important for us, except in the infrared. We require that they have no common zeroes, apart from the origin. Near $p = 0$ they must take the form,

$$C_\mu(p) \simeq p_\mu + \ldots, \quad B(p) \simeq r\ p^2 + \ldots \tag{A.9}$$

where $p^2 = g^{\mu\nu} p_\mu p_\nu$, and r is a constant (Wilson parameter) [19]. The metric tensor, $g^{\mu\nu}$, is Euclidean and we may suppose that it is invariant with respect to one of the 4-dimensional crystal groups. This tensor is involved also in the Dirac algebra,

$$\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu} \tag{A.10}$$

It is convenient to normalize the metric such that $\det g = 1$. In choosing a metric tensor we are essentially choosing a crystal structure for the lattice. The particular choice is

presumably not very important as regards infrared behaviour, although we should insist that the invariance group of C_μ and B be large enough to enforce the structure (A.9) near $p = 0$. We also assume that the crystal symmetry includes reflections with the barrier term transforming as a pseudoscalar. This will ensure that quantities such as $\Phi_+ - \Phi_-$ transform as pseudoscalars under space reflections of the background fields when $\Lambda^{-1} \gg a$ and lattice effects are ignored i.e., in the infrared regime.

Under gauge transformations,

$$\begin{aligned} A(x) \rightarrow A^\theta(x) &= e^{i\theta(x)}\ (A(x) + id)\ e^{-i\theta(x)} \\ \phi(x)\cdot T \rightarrow \phi^\theta(x)\cdot T &= e^{i\theta(x)}\ \phi(x)\cdot T\ e^{-i\theta(x)} \end{aligned}$$

the coefficient matrices (A.2) are clearly covariant,

$$H(n,m,t) \rightarrow e^{i\theta(n)}\ H(n,m,t)\ e^{-i\theta(m)}$$

since $\theta(x) = \theta^\alpha(x) T_\alpha$ commutes with T_c. This guarantees the formula (3.2). (In the text we have used the collective notation, A, to represent both gauge and Higgs fields.)

In developing the perturbative formulae of Sec.2 we have assumed that the Higgs field is constant and incorporated it into the free Hamiltonian. The free Hamiltonians, $H_{0\pm}$, are therefore defined by the 1-body expressions,

$$H_\pm(p) = \gamma_5 \big(i\ \gamma^\mu\ C(p) + \phi\cdot T + (B(p) \pm \Lambda) T_c\big) \tag{A.11}$$

The vertex functions are obtained by expanding (A.3) in powers of the gauge field. One obtains,

$$\begin{aligned} V(p,p) = & -\int \left(\frac{dk}{2\pi}\right)^4 \tilde{A}^\alpha_\mu(k)\ (2\pi)^4\ \delta_{2\pi}(-p+q+k) \int_0^1 dt\ H(p-tk)^{\cdot\mu}\ T_\alpha + \\ & + \frac{1}{2} \int \left(\frac{dk_1}{2\pi}\right)^4 \left(\frac{dk_2}{2\pi}\right)^4 \tilde{A}^\alpha_\mu(k_1)\ \tilde{A}^\beta_\nu(k_2)\ (2\pi)^4\ \delta_{2\pi}(-p+q+k_1+k_2)\cdot \\ & \cdot \int_0^1 dt_1 dt_2\ H(p - t_1 k_1 - t_2 k_2)^{\cdot\mu\nu} \big(\theta(t_1-t_2) T_\alpha T_\beta + \theta(t_2-t_1) T_\beta T_\alpha\big) \\ & + \ldots \end{aligned} \tag{A.12}$$

where $H(p)^{\cdot\mu} = \partial H_\pm(p)/\partial p_\mu$, etc. The periodic delta function is defined by the lattice sum,

$$(2\pi)^4\,\delta_{2\pi}(p) = \sum_n e^{ipn} = \sum_n (2\pi)^4\,\delta_4(p+2\pi n)$$

The momentum integrals in (A.12) are over $\mathbb{R}^4$ but, since $A_\mu(x)$ is assumed to be slowly varying, its Fourier transform $\tilde{A}_\mu(k)$ is concentrated around $k=0$.

The only part of (A.12) that is needed in the continuum limit is the first term near $p=q=0$,

$$V(p,q) = -i\,\gamma_5\gamma^\mu\,\tilde{A}^a_\mu(p-q)\,T_a + \dots \tag{A.13}$$

Appendix B: Free fermions

The purpose of this Appendix is to examine the spectrum of free fermion states and derive expressions for the propagators, S_F, S_+ and S_-.

The square of the 1-body Hamiltonian (A.11) is proportional to the unit Dirac matrix,

$$H_\pm(p)^2 = g^{\mu\nu}\,C_\mu C_\nu + (\phi\cdot T)^2 + (B\pm\Lambda)^2 \tag{B.1}$$

since $T_c^2=1$ and $\{T_i,T_c\}=0$. Choose a set of $2N$ orthonormal spinors $\chi(\sigma)$ such that [7]

$$\gamma_5 T_c\,\chi(\sigma) = \chi(\sigma), \qquad \sigma=1,\dots,2N$$
$$(\phi\cdot T)^2\,\chi(\sigma) = m_\sigma^2\,\chi(\sigma) \tag{B.2}$$

and define the eigenspinors of $H_\pm(p)$,

$$u_\pm(p,\sigma) = \frac{\omega_\pm + H_\pm(p)}{\sqrt{2\omega_\pm(\omega_\pm + B\pm\Lambda)}}\,\chi(\sigma)$$
$$v_\pm(p,\sigma). = \frac{\omega_\pm - H_\pm(p)}{\sqrt{2\omega_\pm(\omega_\pm - B\mp\Lambda)}}\,\chi(\sigma) \tag{B.3}$$

where $\omega_\pm(p,\sigma)$ is given by the positive square root,

$$\omega_\pm(p,\sigma) = \sqrt{C(p)^2 + m_\sigma^2 + (B(p)\pm\Lambda)^2} \tag{B.4}$$

There are no zero modes, even in the absence of Higgs fields ($m_\sigma=0$). The positive and negative energy eigenspinors, u_+, v_+ of $H_+(p)$ comprise a complete, orthonormal set. Likewise for u_- and v_-. The two sets are related by a unitary transformation,

$$u_- = u_+\cos\beta - v_+\sin\beta$$
$$v_- = u_+\sin\beta + v_+\cos\beta \tag{B.5}$$

where the angle $\beta(p,\sigma)$ can be chosen to lie in the interval $(0,\pi/2)$. It is given by

$$\cos\beta = \sqrt{\frac{\omega_+ - B - \Lambda}{2\omega_+}\,\frac{\omega_- - B + \Lambda}{2\omega_-}} + \sqrt{\frac{\omega_+ + B + \Lambda}{2\omega_+}\,\frac{\omega_- + B - \Lambda}{2\omega_-}} \tag{B.6}$$

where the roots are non-negative.

Near $p = 0$ the energies can be expanded,

$$\begin{aligned}\omega_\pm &= (\Lambda \pm B)\sqrt{1 + \frac{C^2 + m^2}{(\Lambda \pm B)^2}} \\ &= \Lambda \pm B + \frac{1}{2}\,\frac{C^2 + m^2}{\Lambda \pm B} + \ldots \\ &= \Lambda \pm r p^2 + \frac{p^2 + m^2}{2\Lambda} + \ldots \end{aligned} \tag{B.7}$$

for $p, m \ll \Lambda$. In this approximation the eigenspinors (B.3) become

$$\begin{aligned} u_+ &= \left(1 + \frac{\gamma_5(i\not{p} + \phi\cdot T)}{2\Lambda} - \frac{p^2+m^2}{8\Lambda^2} + \ldots\right)\chi \\ u_- &= \left(\frac{\gamma_5(i\not{p} + \phi\cdot T)}{\sqrt{p^2+m^2}} + \frac{\sqrt{p^2+m^2}}{2\Lambda} + \ldots\right)\chi \\ v_+ &= \left(-\frac{\gamma_5(i\not{p} + \phi\cdot T)}{\sqrt{p^2+m^2}} + \frac{\sqrt{p^2+m^2}}{2\Lambda} + \ldots\right)\chi \\ v_- &= \left(1 - \frac{\gamma_5(i\not{p} + \phi\cdot T)}{2\Lambda} - \frac{p^2+m^2}{8\Lambda^2} + \ldots\right)\chi \end{aligned} \tag{B.8}$$

and β approaches $\pi/2$,

$$\cos\beta = \frac{\sqrt{p^2+m^2}}{\Lambda} + \ldots \tag{B.9}$$

To this order the quantities (B.8), (B.9) depend on the barrier height, Λ, but not on detailed lattice structure such as the Wilson parameter, r. In the next order such details would begin to appear.

The long wavelength approximation to the propagators $S_\pm$ can be recovered from (B.7), (B.8)

$$\begin{aligned} S_\pm(p, t-t') &= \theta(t-t') \sum_\sigma u_\pm u_\pm^\dagger\, e^{-i\omega_\pm(t-t')} - \\ &\quad -\theta(t'-t) \sum_\sigma v_\pm v_\pm^\dagger\, e^{i\omega_\pm(t-t')} \\ &= e^{-i\Lambda|t-t'|}\left[\frac{1}{2}\,\varepsilon(t-t') \pm \frac{1}{2}\,\gamma_5 T_c + \frac{\gamma_5(i\not{p} + \phi\cdot T)}{2\Lambda} + \ldots\right] \end{aligned} \tag{B.10}$$

These functions are regular at $p = 0$.

The propagator S_F is not regular at $p = \phi = 0$. To obtain it one must consider the 3! possible orderings of the time coordinates, t, t' and 0,

$$\begin{aligned} T\left(\psi(p,t)\ \psi(q,t')^\dagger\right) &= \theta(t-t')\ \psi(p,t)\ \psi(q,t')^\dagger - \\ &\quad -\theta(t'-t)\ \psi(q,t')^\dagger\ \psi(p,t) \\ &= \theta(t-t')\ \theta(t')\ \psi_+(p,t)\ \psi_+(q,t')^\dagger \\ &\quad +\theta(t)\ \theta(-t')\ \psi_+(p,t)\ \psi_-(q,t')^\dagger \\ &\quad +\theta(-t)\ \theta(t-t')\ \psi_-(p,t)\ \psi_-(q,t')^\dagger \\ &\quad -\theta(t'-t)\ \theta(t)\ \psi_+(q,t')^\dagger\ \psi_+(p,t) \\ &\quad -\theta(t')\ \theta(-t)\ \psi_+(q,t')^\dagger\ \psi_-(p,t) \\ &\quad -\theta(-t')\ \theta(t'-t)\ \psi_-(q,t')^\dagger\ \psi_-(p,t) \end{aligned} \tag{B.11}$$

where

$$\psi_\pm(p,t) = e^{-itH_\pm(p)}\ \psi(p) \tag{B.12}$$

The matrix element of (B.11) between free fermion ground states involves the polarization sums,

$$\begin{aligned} \frac{\langle +|\psi(p)\ \psi(q)^\dagger|-\rangle}{\langle +|-\rangle} &= (2\pi)^4\ \delta_{2\pi}(p-q) \sum_\sigma \frac{u_+(p,\sigma)u_-(p,\sigma)^\dagger}{\cos\beta(p,\sigma)} \\ \frac{\langle +|\psi(q)^\dagger\ \psi(p)|-\rangle}{\langle +|-\rangle} &= (2\pi)^4\ \delta_{2\pi}(p-q) \sum_\sigma \frac{v_-(p,\sigma)v_+(p,\sigma)^\dagger}{\cos\beta(p,\sigma)} \end{aligned} \tag{B.13}$$

which are obtained by elementary considerations (see Appendix A of Ref.[7]) using the plane wave expansions,

$$\psi(p) = \sum_\sigma \left(b_\pm(p,\sigma)\ u_\pm(p,\sigma) + d_\pm^\dagger(p,q)\ v_\pm(p,q)\right)$$

together with canonical anticommutation rules and the ground state definitions

$$b_\pm(p,q)|\pm\rangle = d_\pm(p,\sigma)|\pm\rangle = 0$$

On substituting from (B.12) and (B.13) into the ground state matrix element of (B.11), and using (B.5) to make the time-dependence explicit, one obtains the expression

$$
\begin{aligned}
&S_F(p,t,t') = \\
&= \sum_{\sigma} \frac{1}{\cos\beta}\Big[\theta(t-t')\theta(t')\, e^{-it\omega_+} u_+ \left(u_+^\dagger \cos\beta\, e^{it'\omega_+} - v_+^\dagger \sin\beta\, e^{-it'\omega_+}\right) \\
&\quad + \theta(t)\theta(-t')\, e^{-it\omega_+}\, u_+ u_-^\dagger\, e^{it'\omega_-} \\
&\quad + \theta(-t)\theta(t-t') \left(e^{-it\omega_-} \cos\beta\, u_- + e^{it\omega_-} \sin\beta\, v_-\right) u_-^\dagger\, e^{it'\omega_-} \\
&\quad - \theta(t'-t)\theta(t) \left(e^{-it\omega_+} \sin\beta\, u_+ + e^{it\omega_+} \cos\beta\, v_+\right) v_+^\dagger\, e^{-it'\omega_+} \\
&\quad - \theta(t')\theta(-t)\, e^{it\omega_-}\, v_- v_+^\dagger\, e^{-it'\omega_+} \\
&\quad - \theta(-t')\theta(t'-t)\, e^{it\omega_-}\, v_- \left(-u_-^\dagger \sin\beta\, e^{it'\omega_-} + v_-^\dagger \cos\beta\, e^{-it'\omega_-}\right)\Big]
\end{aligned}
\tag{B.14}
$$

This function has the expected discontinuity at $t = t'$ but is continuous, as it should be, at $t = 0$ and $t' = 0$. Its low momentum behaviour is dominated by the pole at $\cos\beta = 0$. It occurs at $p = \phi = 0$ and we can expand around this point using the formulae (B.8), (B.9). The result is

$$
\begin{aligned}
S_F(p,t,t') \;=\; & \frac{1+\gamma_5 T_c}{2} \frac{1}{i\not p + \not\phi \cdot T} \gamma_5\, \Lambda\, e^{-i\Lambda(|t|+|t'|)} \\
& + \frac{1}{2}\, \varepsilon(t-t')\, e^{-i\Lambda|t-t'|} \\
& + i\gamma_5 T_c \Big\{ \left(\theta(t-t')\theta(t') + \theta(-t')\theta(t'-t)\right) e^{-i\Lambda|t|} \sin\Lambda t' \\
& \quad + \left(\theta(t'-t)\theta(t) + \theta(-t)\theta(t-t')\right) e^{-i\Lambda|t'|} \sin\Lambda t \Big\} \\
& + \text{terms of order } (p,\phi)
\end{aligned}
\tag{B.15}
$$

The poles of S_F are crucial to the continuum limit since they control the infrared singularities. It is necessary, therefore, to establish conditions under which the pole at $p = \phi = 0$ is unique. According to the general formula (B.14) the poles of S_F correspond to the zeroes of the function $\cos\beta$ defined by (B.6). In this formula the square roots are non-negative and both of them must vanish to give a zero of $\cos\beta$. There appear to be two possibilities,

$$
\omega_+ = B + \Lambda, \quad \omega_- = -B + \Lambda \tag{B.16}
$$

and

$$
\omega_+ = -B - \Lambda, \quad \omega_- = B - \Lambda \tag{B.17}
$$

The alternative (B.17) can be excluded immediately because it implies $\omega_+ + \omega_- = -2\Lambda$ which contradicts the positivity of ω_+ and ω_-. The alternative (B.16) is possible only if $\pm B + \Lambda > 0$, i.e. if $B^2 < \Lambda^2$. From (B.4) one sees that (B.16) implies $C^2 + m^2 = 0$. Hence, the zeroes of $\cos\beta$ occur at isolated points defined by

$$
C_\mu(p) = 0, \quad \phi = 0 \quad \text{and} \quad B(p)^2 < \Lambda^2 \tag{B.18}
$$

The origin, $p = 0$, is certainly one such point in view of the equations (A.9). The vector function, $C_\mu(p)$, certainly has other zeroes. This is implied by the Poincaré-Hopf theorem since C_μ is defined on a torus [20]. In order that $\cos\beta$ should not vanish at these other points we have only to ensure that $B(p)^2 > \Lambda^2$ at such points. In other words, the zero of $\cos\beta$ at $p = \phi = 0$ is unique if $B(p)$ and $C_\mu(p)$ are chosen so as to have no common zero, apart from the origin, and Λ is smaller than $|B(p)|$ at all the other zeroes of $C_\mu(p)$.

Appendix C: Time–independent formalism

In previous work on the overlap prescription we used time–independent perturbation theory and a different phase convention. The purpose of this appendix is to clarify the relation between the two formalisms.

With the time-independent approach one constructs the two ground states $|A+\rangle$ and $|A-\rangle$ directly by solving the Schroedinger equations (2.9),

$$H_\pm(A)|A\pm\rangle = |A\pm\rangle E_\pm(A) \tag{C.1}$$

or, rather, the equivalent integral equations

$$|A\pm\rangle = |\pm\rangle\alpha_\pm + G_\pm(V - E_\pm)|A\pm\rangle \tag{C.2}$$

where $|\pm\rangle$ denotes the free fermion ground states (2.5). The operators $G_\pm$ are defined by

$$G_\pm = -\frac{1 - |\pm\rangle\langle\pm|}{H_{0\pm}} \tag{C.3}$$

with the understanding that $G_\pm|\pm\rangle = 0$. Iteration of (C.2) leads to the formal solution

$$|A\pm\rangle = \alpha_\pm\Big(1 - G_\pm(V - E_\pm)\Big)^{-1}|\pm\rangle \tag{C.4}$$

where the numerical factors $\alpha_\pm = \langle\pm|A\pm\rangle$ are determined, up to a phase, by requiring that the states $|A\pm\rangle$ be normalized. The energies $E_\pm$ are determined self-consistently from (C.1),

$$\begin{aligned} E_\pm &= \frac{\langle\pm|H_\pm(A)|A\pm\rangle}{\langle\pm|A\pm\rangle} \\ &= \langle\pm|V\Big(1 - G_\pm(V - E_\pm)\Big)^{-1}|\pm\rangle \end{aligned} \tag{C.5}$$

The method sketched here is straightforward and practical, at least in the lowest orders. However, it is less efficient than the time-dependent method discussed in the text. This is mainly because it requires the calculation of the subsidiary quantities, $\alpha_\pm$ and $E_\pm$, as well as the quantity of interest, $\langle A+|A-\rangle$. In addition, it obscures the fact that $\Gamma(A)$ is expressible in terms of connected graphs.

A possible advantage of the time-independent method is the very simple phase convention it allows. The Brillouin-Wigner convention is expressed in the requirement that the numerical factors, $\alpha_\pm$, should be real and positive for all values of the external fields, i.e.

$$\langle\pm|A\pm\rangle_{BW} > 0 \tag{C.6}$$

This makes it easy to compute the angles $\Phi_\pm(\theta, A)$ induced by gauge transformations. For example, to first order in θ, the formula (3.6) reduces to

$$\begin{aligned} \Phi_+^{BW} &= \mathrm{Re}\left(\frac{\langle+|F_\theta|A+\rangle}{\langle+|A+\rangle}\right) \\ &= \mathrm{Re}\ \langle+|F_\theta\Big(1 - G_+(V - E_+)\Big)^{-1}|+\rangle \end{aligned} \tag{C.7}$$

This was used to compute the chiral anomaly in Refs.[5,11].

To find the relative phase between the B-W ground state and the one used in the main text one must consider the definitions (2.8). These imply, in particular,

$$\begin{aligned} \lim_{\varepsilon\to 0} \langle+|in+\rangle\ e^{-\hat{E}_+/i\varepsilon} &= \langle+|A+\rangle \\ &= \langle+|A+\rangle_{BW}\ e^{i\beta_+(A)} \end{aligned} \tag{C.8}$$

so that β_+ can be obtained from the regular part of the connected amplitude, in the limit $\varepsilon \to 0$,

$$\beta_+(A) = \lim_{\varepsilon\to 0} \mathrm{Im}\left(\langle+|in+\rangle_{con} - \frac{\hat{E}_+}{i\varepsilon}\right) \tag{C.9}$$

where $\hat{E}_+$ is itself given by the limit

$$\hat{E}_+(A) = \lim_{\varepsilon\to 0}\Big(i\varepsilon\ \langle+|in+\rangle_{con}\Big) \tag{C.10}$$

The contribution of order N to the connected amplitude is expressed as a 1–loop integral constructed with the propagator S_+, viz.

$$\begin{aligned} \langle+|in+\rangle_{con} &= \sum_N \frac{(-i)^N}{N!}\int_{-\infty}^0 dt_1\ldots dt_N\ \langle+|T\Big(V(t_1)\ldots V(t_N)\Big)|+\rangle_{con} \\ &= -\sum_N \frac{(-i)^N}{N}\int_{-\infty}^0 dt_1\ldots dt_N\ e^{\varepsilon(t_1+\ldots+t_N)}\,. \end{aligned}$$
$$\cdot\int\left(\frac{dp_1}{2\pi}\right)^4\ldots\left(\frac{dp_N}{2\pi}\right)^4\quad tr\Big(V(p_1,p_2)S_+(p_2,t_1-t_2)\ldots V(p_N,p_1)S_+(p_1,t_N-t_1)\Big) \tag{C.11}$$

It is straightforward to integrate the time coordinates in the terms of this series using the explicit formula for S_+,

$$S_+(p,t-t') = \sum_\sigma \Big(\theta(t-t')u_+(p,\sigma)u_+(p,\sigma)^\dagger\, e^{-i\omega_+(p)(t-t')} - \\ -\theta(t'-t)v_+(p,\sigma)v_+(p,\sigma)^\dagger\, e^{i\omega_+(p)(t-t')}\Big)$$

One finds, in the limit $\varepsilon \to 0$,

$$\hat{E}_+(A) = \langle +|V|+\rangle + \frac{1}{2}\langle +|VG_+V|+\rangle + \\ +\frac{1}{3}\langle +|VG_+VG_+V|+\rangle - \frac{1}{3}\langle +|V|+\rangle\langle +|VG_+^2V|+\rangle + \dots \quad \text{(C.12)}$$

$$\beta_+(A) = -\frac{1}{3}\,\mathrm{Im}\,\langle +|VG_+^2VG_+V|+\rangle + \dots \quad \text{(C.13)}$$

In the same fashion one can eliminate the time integrations from the terms of the series (3.10) for Φ_+ to obtain

$$\Phi_+(\theta,A) = \langle +|F_\theta|+\rangle + \frac{1}{2}\,\langle +|(F_\theta G_+V + VG_+F_\theta)|+\rangle + \\ +\frac{1}{3}\langle +|(F_\theta G_+VG_+V + VG_+F_\theta G_+V + VG_+VG_+F_\theta)|+\rangle \\ -\frac{1}{3}\langle +|V|+\rangle\langle +|(F_\theta G_+^2V + VG_+^2F_\theta)|+\rangle - \frac{1}{3}\langle +|F_\theta|+\rangle\langle +|VG_+^2V|+\rangle \\ +\frac{1}{6}\langle +|\big([F_\theta,V]G_+^2V - VG_+^2[F_\theta,V]\big)|+\rangle + \dots \quad \text{(C.14)}$$

To the same order the Brillouin-Wigner formula (C.7) gives

$$\Phi_+(\theta,A)^{BW} = \langle +|F_\theta|+\rangle + \frac{1}{2}\langle +|(F_\theta G_+V + VG_+F_\theta)|+\rangle + \\ +\frac{1}{2}\langle +|(F_\theta G_+VG_+V + VG_+VG_+F_\theta)|+\rangle - \\ -\frac{1}{2}\langle +|V|+\rangle\langle +|(F_\theta G_+^2V + VG_+^2F_\theta)|+\rangle + \dots \quad \text{(C.15)}$$

These phases are related by

$$\Phi_+(\theta,A) = \Phi_+(\theta,A)^{BW} - \delta_\theta\beta_+(A) \quad \text{(C.16)}$$

where β_+ is given by (C.13). This can be verified using the formula (3.2) or

$$\delta_\theta V = i[F_\theta, H_{0+} + V]$$

According to (C.16) the chiral anomalies are related by

$$\Phi_+ - \Phi_- = \Phi_+^{BW} - \Phi_-^{BW} - \delta_\theta(\beta_+ - \beta_-) \quad \text{(C.17)}$$

and we must consider what happens to the functional, $\beta_+ - \beta_-$, in the continuum limit. Like $\Phi_\pm$, the angles $\beta_\pm$ do not involve any infrared singularities. In the continuum approximation they must be local, i.e. expressibie as integrals over 4-dimensional spacetime of local functions of the gauge field and its derivatives, with lattice dependent coefficients. More particularly, the difference, $\beta_+ - \beta_-$, must involve a pseudoscalar density. For example,

$$\beta_+ - \beta_- \simeq \frac{1}{\Lambda^2}\int d^4x\; g^{\rho\sigma}\,\varepsilon^{\kappa\lambda\mu\nu}\,\partial_\kappa A_\lambda\,\partial_\rho A_\mu\,\partial_\sigma A_\nu + \dots$$

There is no candidate of dimension 4. This means that in the continuum limit, $k/\Lambda \to 0$, this functional becomes vanishingly small: the continuum theory chiral anomalies are unaffected by the phase conventions.

References

[1] D. Kaplan, Phys. Lett. B288 (1992) 342;
For a review see K. Jansen, hep-lat/9410108 DESY 94-188;
For an overview of more recent developments in chiral lattice theory see Y. Shamir, hep-lat/9509023.

[2] R. Narayanan and H. Neuberger, Nucl. Phys. B412 (1994)

[3] R. Narayanan and H. Neuberger, Nucl. Phys. B443 (1995) 305.

[4] R. Narayanan and H. Neuberger, Phys. Lett. B348 (1995) 549, hep-lat/9509047;
See also R. Narayanan, H. Neuberger and P. Vranas, Phys. Lett. B353 (1995) 507, hep-lat/9509046.

[5] S. Randjbar-Daemi and J. Strathdee, Nucl. Phys. B443 (1995) 386.

[6] S. Aoki and R. Levien, Phys. Rev. D51 (1995) 3790.

[7] S. Randjbar Daemi and J. Strathdee, Trieste preprint *IC/95/305*; hep-th/9510067, to appear in Nucl. Phys. B.

[8] D. Kaplan and M. Schmaltz, hep-th/9510197;
See also Per Ernström and Ansar Fayyazuddin, hep-lat/9511028.

[9] L. Alvarez-Gaumé, S. Della Pietra, V. Della Pietra, Phys. Lett. B166 (1986) 177.

[10] C. Fosco and S. Randjbar-Daemi, Phys. Lett. B354 (1995) 383;
C. Fosco, hep-th/9511221.

[11] S. Randjbar-Daemi and J. Strathdee, Phys. Lett. B348 (1995) 543; Phys. Rev. D51 (1995) 6617.

[12] T. Kawano and Y. Kikukawa, hep-lat/9501032.

[13] P.D. Swift, Phys. Lett. B 145 (1984) 256;
J. Smit, Nucl. Phys. B (Proc. Suppl.) 4 (1988) 451;
I. Montvay, Phys. Lett. B 199 (1987) 89; B 205 (1988) 315.

[14] A. Borelli, L. Maiani, G.C. Rossi, R. Sisto and M. Testo, Nucl. Phys. B 333 (1990) 335;
L. Maiani, in Nucl. Phys. 29 B,C (1992) 33 (Proc. Suppl.), ed. L. Maiani et al.

[15] J.L. Alonso, Ph. Boucaud, F. Lesmes and A.J. Van der Sijs, Nucl. Phys. (Proc. Suppl.) B 42 (1995) 595 and hep-lat/95-07003.

[16] A.A. Slavnov, Phys. Lett. B 319 (1993) 231; Nucl. Phys. B 348 (1995) 553;
S.A. Frolov and A.A. Slavnov, Nucl. Phys. B 411 (1994) 647.

[17] G. 't Hooft, Phys. Lett. B 349 (1995) 491;
A.S. Kronfeld, hep-lat/9504007;
S.D.H. Hsu, hep-th/9503058;
W. Bietenholz and V.-J. Wise, hep-lat/9503022;
I. Montvay, hep-lat/9505015;
P. Hernandez and R. Sundrum, hep-ph/9506331.

[18] S.S. Schweber, "An Introduction to Relativistic Quantum Field Theory", New York, Harper and Row, 1961.

[19] K. Wilson, in "New Phenomena in Subnuclear Physics", (Erice 1975), ed. A. Zichichi, Plenum, NY 1977.

[20] L.H. Karsten, Phys. Lett. 104B (1981) 315;
L.H. Karsten and J. Smit, Nucl. Phys. *B183 (1981) 103;*
H.B. Nielsen and M. Ninomya, Nucl. Phys. B185 (1981) 20; Nucl. Phys. B193 (1981) 173; Phys. Lett. 105B (1981) 219;
J. Smit, Acta Phys. Pol. B17 (1986) 531.
Y. Shamir, ref.[1]