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ON FRACTIONAL SPIN SYMMETRIES AND STATISTICAL PHYSICS

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ON FRACTIONAL SPIN SYMMETRIES AND STATISTICAL PHYSICS[1]

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ABSTRACT

The partition function Z and the quantum distribution $\langle n \rangle$ of systems Σ of identical particles of fractional spin $s = 1/k \bmod 1$, $k \geq 2$, generalizing the well-known Bose and Fermi ones, are derived. The generalized Sommerfeld development of the distribution $\langle n \rangle$ around $T = 0°K$ is given. The low temperature analysis of statistical systems Σ is made. Known results are recovered.

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1 Introduction

Recently, there has been some interest in the study of the theory of fractional spin particles [1]. The interest in these exotic particles is due to their very specific properties and because of the role they play in physics in lower dimensions [1,2]. The area of physical relevance of exotic statistics is expected in condensed matter physics where two–dimensional systems have become experimentally available [3].

However, as far as exotic statistics is concerned, there are two ways of doing things namely the Wilczek *et al.* [2,4,5] and the way using deformations of conformal field theories [1,6,7] as well as quantum group symmetries formulated as generalized supersymmetries [8,9]. In the Wilczek *et al.* method, the two–dimensional unconstrained statistics deals with all possible statistics in the two–dimensional space which interpolates between bosons and fermions [10]. These exotic statistics are characterized by a continuous angle parameter θ taking values in the interval $[0, 2\pi[$. This issue has been recently subject to considerable interest [11] and to a lot of studies in connection with anyon superconductivity and the fractional quantum Hall effect. For a review see [12]. From the point of view of quantum groups, deformations of two–dimensional conformal field theory and especially the $\phi_{1,3}$ integrable deformation of the $N = 0$ minimal series of central charge $c(m) = 1 - 6/m(m+1)$ with $m = 3, 4, \ldots$ [13,14], the exotic statistics are carried by Fractional Spin (FS) particle states transforming under representations of FS supersymmetric algebras satisfying among other things the following operator equation [9,15]

$$Q^k = P; \qquad k \geq 2 \,, \tag{1}$$

where P is the heterotic component of the two–dimensional energy momentum vector operator and Q is a FS supersymmetric charge operator. Note that for the leading value $k = 2$, one obtains the usual $D = 2$ supersymmetric, algebra which in the language of deformations of conformal theories is the off critical symmetry of the thermal deformation of the $c = 7/10$ tricritical Ising model [16]. For $k = 3$ see also [17]. In these kinds of theories exhibiting Eq.(1) as a symmetry, one has besides a bosonic state, $(k-1)$ FS one particle states of spin n/k with $n = 1, 2, \ldots (k-1)$ which are related among themselves by generalized FS supersymmetric transformations. FS particles are then particles of well defined spin and are expected to exhibit features generalizing the standard fermionic ones.

The aim of this paper is to explore a new area where FS particles are expected to show remarkable properties. This area deals with the study of quantum statistics and the features of gases Σ (FSP) of FS particles. Of special interest we mention the following:

(i) The computation of the partition function $Z(\beta,\mu)$ of Σ (FSP) where β is the inverse of temperature and μ is the chemical potential. (ii) The calculation of the quantum distribution of Σ (FSP) described effectively by the mean occupation number $\langle n_j\rangle$ of an individual FS particle quantum state of energy ε_j. (iii) Finally, we analyse some aspects of the low temperature behaviour of Σ (FSP).

Before setting the problem of FS particles, let us give some motivations that pushed us to be interested in this subject. When doing the quantum statistics of systems of many identical particles (bosons and fermions), one is attracted by certain remarkable details among which we mention the following: (i) In establishing the basic formulae of quantum Bose and Fermi distributions, the only reference to the space-time dimension is through the number of spin polarizations g (degree of degeneracy) which is involved in the formalism as a global multiplicative parameter [18]. Thus, in two dimensions where FS particles, other than fermions exist, the quantum Bose and Fermi statistics formulas are then expected to be particular cases of more general ones, namely the FS statistics identities. (ii) The Fermi nilpotency and the Bose condensation which play a crucial role in the derivation of the fundamental equalities of quantum statistical mechanics seem to be two limiting situations of a general phenomena where the usual antisymmetry and symmetry requirements are relaxed to incorporate general commutation relations [17,19]. (iii) Although having very different properties, Fermi and Bose gases have the same state equation namely $\mathcal{P}V = aE$ where $\mathcal{P}, V$ and E are respectively the pressure of the gas, the volume and the mean energy and where a is a constant equal to 1/3 or 2/3 depending on whether the particles of the gas are relativistic or not. The theory of quantum statistics of FS particles we want to study here will shed much light on the properties of quantum gases of particles in two dimensions and especially the very different behaviour of Fermi and Bose gases at low temperature [18,20]. It might be used also to model the interactions between the particles of Fermi and Bose gases ignored in the theory of quantum ideal gases [21].

This paper is organized as follows. First, we start by recalling some general features of the notion of spin in D-dimensions by using both the language of fields and the representions theory of Lie groups. The point is that contrary to $D \geq 3$ where the spin s is constrained to take either integer or half integer values, the situation for $D = 2$ is completely different since the spin can take any value. The example of one particle states of FS $s = n/k$ mod 1 with $n = 0, 1, 2, \ldots, k-1$ contained in the spectrum of the $\phi_{1,3}$ deformation of the $c(2k) = \left[1 - \frac{3}{k(2k+1)}\right]$ minimal models is given with some details. As a

sequence of this discussion we conclude that FS particles of spin $1/k$ $k \geq 2$ are the basic particles of the scalar representation of the fractional supersymmetric algebra Eq.(1) and are expected to carry features generalizing the fermionic ones. In Section 3, we introduce the notion of Pauli index η allowing to classify FS particles into Pauli classes of equivalence containing particles obeying the same generalized Pauli exclusion principle. Here we would like to note that inspired from quantum group ideas and especially type B representations of the $U_1(SL_2)$ [9,21], we postulate a generalization of the Pauli exclusion principle whose statement is: No more than $(k-1)$ identical particles of fractional spin $s = 1/k$ (mod 1) in the same individual quantum state of energy $\in_j$ can live altogether. This statement is motivated by recent results on fractional supersymmetric theories [8,9], in particular the superspace realization of Eq.(1) given by [9]:

$$\begin{aligned} P &\sim \partial/\partial z \\ Q &= \partial/\partial\theta + \theta^{k-1}\partial/\partial z, \quad k \geq 2\,. \end{aligned} \tag{2}$$

In this equation z is the usual complex coordinate of the two-dimensional complex plane and θ is a FS parameter of spin $1/k$ generalizing the well-known Grassmann variable and satisfying

$$\theta^k = 0, \quad z\theta = \theta z\,. \tag{3}$$

Thus, products involving more than $(k-1)$ θ variable vanish identically. This result is in perfect agreement with the generalized Pauli exclusion principle given above. Using this property, we compute in Section 4 the partition function $Z(\beta,\mu;\eta)$ and the distribution $\langle n_j\rangle$ of a quantum gas $\Sigma(\eta)$ of N identical FS particles of spin $s = 1/k$ (mod 1) enclosed within a container of volume V. Among the results we have obtained in this section, we mention the derivation of the generic form of the partition function of systems of identical particles of spin $1/k$ (mod 1):

$$Z(\beta,\mu;k) = \prod_{j\geq 0} \left[1 - e^{k\beta(\in_j-\mu)}\right] / \left[1 - e^{\beta(\in_j-\mu)}\right]\,. \tag{4}$$

Setting $k = 2$ or $k = \infty$, one obtains respectively the known Fermi and Bose partition functions. In Section 5, we study some low temperature properties of statistical systems $\Sigma(\eta)$ of FS particles. In particular, we derive the development of the quantum FS distribution in the vicinity of $T = 0°K$. This expansion, which is given by:

$$\begin{aligned} \frac{e^{k\beta(\in-\mu)} - k\, e^{\beta(\in-\mu)} + k - 1}{[e^{\beta(\in-\mu)} - 1][e^{k\beta(\in-\mu)}]} \simeq\ & (k-1)\, H(\mu-\in) - \frac{2\pi^2}{6}\left(\frac{k-1}{k}\right)\beta^{-2}\delta'(\mu-\in) \\ & - \frac{\pi^4}{45}\left(\frac{k^3-1}{k^3}\right)\beta^{-4}\delta'''(\mu-\in) + O(\beta^{-4})\,, \end{aligned} \tag{5}$$

where $H(x)$ is the Heaviside function, generalizes the Sommerfeld expansion of the Fermi distribution near the zero temperature and turns out to be very useful in the analysis of the low temperature behaviour of the thermodynamic functions of $\Sigma(\eta)$. In Section 6 we study the behaviour of the chemical potential of $\Sigma(\eta)$ near the zero absolute and in Section 7 we make a discussion and give an outlook.

2 Generalities on fractional spin particles

We start by recalling briefly the notion of the spin in D-dimensional space-time, a matter of precising that FS particles that we will consider in this study are specific for two dimensions and have no analogue in higher dimensions. Thus, in the language of fields, the spin is a quantum number that has played a central role in the success of quantum mechanics, particle physics and more generally in D-dimensional quantum field theories. It is also the main actor in many areas of applied physics. In the language of representation theory of Lie groups, the spin is nothing but the highest weight of (spinorial) representations of the $SO(D)$ orthogonal group. For the non relativistic $SO(3) \simeq SU(2)$ symmetry of the three-dimensional theories for instance, the quantized spin is constrained to take, up to a global multiplicative factor, either integer or half odd integer values $s = 0, 1/2, 1, 1/3, \dots$ In four-dimensional space-time theories invariant under the group $SO(1,3)$ or roughly speaking $SO(4) \simeq SU(2) \times SU(2)$, the relativistic spin is still constrained to take either integer or half odd inter values. The same constraints exist in higher dimensional space-time [22]. In two dimensions, however, the above mentioned constraints are no longer required since the spin s is allowed to take arbitrary values [23]. The most familiar example is the so-called conformal spin given by the difference $h - \bar{h} = s$ where h and $\bar{h}$ are, in general, free parameters known as the conformal weights of the Virasoro algebra. By the way the sum $h + \bar{h} = \delta$ defines the anomalous dimension of critical models [23]. Another interesting example of models involving FS one particle states, and to which we shall often refer to hereafter, is given by the $\phi_{1,3}$ deformation of the $c(p) = 1 - 6/p(p+1), p = 3, 4, 5 \dots$ minimal series [17]. In these deformed theories, one particle states are massive and have very specific fractional values of the spin. To see this property more closely, recall that after the $\phi_{1,3}$ deformation of the special $C(2k)$ conformal subseries of the $N = 0$ minimal series, the resulting theory looses the scale invariance but still has a residual symmetry generated, among other things, by operators Q and $\bar{Q}$ carrying fractional values of the

spin, namely $s = 1/k$ and satisfying the following generalized supersymmetric algebra:

$$\begin{aligned} Q^k &= P \qquad &(a) \\ \bar{Q}^k &= \bar{P} \qquad &(b) \\ Q\bar{Q} - q\bar{Q}Q &= \Delta\,, \qquad &(c) \end{aligned} \tag{6}$$

where $q = \exp 2i\pi s$ and where Δ is a topological charge. Note that the above FS symmetry is, in fact, a subalgebra of the off critical symmetry of the $\phi_{1,3}$ deformation of the $C(2k)$ models. For more details see [9,17]. For pedagogical reasons, let us fix our attention on Eq.(6.a) and look for linear solutions. The first point to note is that irreducible representations of this algebra are k-th dimensional and consist of $(k-1)$ FS particles in addition to a bosonic state exactly as for type B periodic representations of the $U_q(s\ell_2)$ algebra [21]. For the simplest scalar representation whose fundamental state is a scalar boson say $|\psi_0\rangle$, the other FS one particle states $|\psi_{n/k}\rangle$; $0 < n < k$ satisfying the eigenvalue equation:

$$L_0|\psi_{n/k}\rangle = \frac{n}{k}\,|\psi_{n/k}\rangle\,, \tag{7}$$

where L_0 is the two-dimensional spin ($SO(2)$), charge operator are built as:

$$Q^n|\psi_0\rangle = |\psi_{n/k}\rangle\,. \tag{8}$$

Note that this representation generalizes the usual two-dimensional supersymmetric one containing a bosonic state and fermionic one. The two kinds of particles, bosons and fermions, are known to be completely specified by their quantum properties [19,20]. Whereas, bosons condensate and are described by commuting statistics, fermions obey the Pauli exclusion principle and are described by anticommuting statistics that forbids the condensation of fermions in the same individual quantum state. Actually this is the main difference between bosons and fermions. There are other differences, for example, one can usually construct a bosonic state from fermionic condensates in agreement with the Pauli exclusion principle but the inverse is not possible. Lower spin particles i.e. fermions in the case of supersymmetric theories, seem therefore to be the most fundamental particles. Note, moreover, that for FS supersymmetric models invariant under Eq.(6.a), the same properties as those of the fermionic ones are expected to hold for the FS representations Eqs.(7-8). There, FS particles of the lowest value of the spin, namely $s = 1/k, k \geq 2$ are the more basic ones and are expected to carry generalizations of many features fulfilled by the fermions. They may eventually share some common properties with fermions like, for example, the state equation of a gas of particles contained in a container of volume V. As we shall see in Section 5, Eqs.(52-54), all particles of spin $1/k$ mod 1 with $k \geq 2$

have the same state equation

$$\mathcal{P}V = a\,E\,. \tag{9}$$

In the remainder of this section we would like to comment on what we mean by saying that the basic FS particles are expected to carry generalizations of fermion's features. The idea may be summarized as follows: (i) FS particles of spin $1/k$ mod 1 should satisfy a generalized statistics which reduce to the anticommuting statistics once the integer k is set to two and to the commuting one when k goes to infinity. (ii) FS particles of spin $s = 1/k$ mod 1 should obey a generalized Pauli exclusion principle which reduce to the usual one for $k = 2$ and coincide with the Bose condensation phenomenon when $k = \infty$. (iii) Systems Σ (FSP) of identical FS particles should be described by generalized wave functions ψ (FSP) which should reduce to the Slater determinant ψ (Fermi) and Slater permanent Ψ (Bose) for $k = 2$ and $k = \infty$, respectively. Moreover, the systems Σ (FSP) are expected to be described by distributions $\langle n_j(k)\rangle$, which should reduce to

$$\langle n_j(2)\rangle = 1/[e^{\beta(\epsilon_j-\mu)} + 1] \tag{10}$$

for fermions and

$$\langle n_j(\infty)\rangle = 1/[e^{\beta(\epsilon_j-\mu)} - 1] \tag{11}$$

for bosons. There are other properties that one may consider; some of them will be given throughout this study. At the end of this paragraph we would like to note that FS particles may be used for more than one purpose. First, besides their theoretical existence, they may be thought of as a link between two–dimensional fermions and bosons so that the Fermi nilpotency and the Bose condensation phenomena appear just as the two faces of the same coin. In other words, between the two opposite and seemingly unrelated Fermi and Bose phenomena, there exists, in fact, an infinite hierarchy of relaxed nilpotencies interpolating between the Fermi and the Bose phenomena. Such features might be examined in the ^3He Fermi liquid, the ^4He Bose liquid and their mixture ^3He+^4He by looking for hypothetic phases which could be modeled by liquids of particles of FS values. The second note we would like to make is that FS statistics might be used to modelate the ignored interactions between particles of the perfect quantum gases.

3 The Pauli index η

The Pauli index is a number η that plays an important role in the study of the quantum statistics of systems $\Sigma(\eta)$ of individual particles of fractional spin. As we shall see later on

this index takes the value $\eta = 2$ for systems of identical fermions and $\eta = \infty$ for systems of indistinguishable bosons. More generally for systems of FS particles of spin $1/k$ mod 1 the Pauli index η is equal to k, Besides of its role in classifying different categories of particles, the Pauli index admits another property. Instead of the spin s, η is the right parameter that is involved in the statement of the generalized Pauli exclusion principle (GPEP) Eq.(3). Note that we have used the terminology index in referring to the number η. This is because of a global property exhibited by particles whose spin differ only by multiples of unity i.e.:

$$s = s_0 + n \tag{12}$$

where n is a positive integer. To see this property more closely let us illustrate the idea with examples. The first example deals with the $SO(3) \simeq SU(2)$ fermions of half odd integer spin. Following Pauli fermions in the same individual quantum state cannot live together. This exclusion rule which by the way applies to any space–time fermions, is valid either for spin 1/2 particles, spin 3/2 particles and, in general, for any particle of half odd integer spin. Therefore, the Pauli exclusion principle does not distinguish between any two systems Σ_1 and Σ_2 of identical fermions of spin S_1 and S_2, respectively ($S_1 - S_2$ is an integer). In other words if one is represented by the following figure

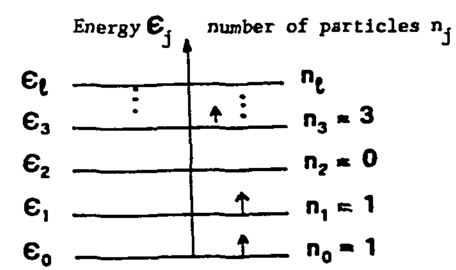


Fig.1: This figure represents a quantum state of a system Σ of identical polarized three–dimensional fermions. As far as the PEP in concerned; no indication on the nature of the considered fermions can be read from the figure.

a quantum state of a system Σ of polarized fermions, then as far as the Pauli exclusion principles is concerned, one can never predict which of the systems Σ_1 and Σ_2 is actually considered. The Pauli exclusion principle (PEP) does not make any distinction. This property becomes more transparent in the two–dimensional space where all fermions have the same degree g of degeneracy, namely $g = 2$. In the second example that we consider hereafter (Fig.2), both polarizations of the two-dimensional fermions are represented. As far as the (PEP) is solely concerned we still have no information on the exact value of the spin. Is it $1/2$, $3/2$ or another value of the infinite spectrum $(2r+1)/2; r = 0, 1, 2\ldots$? The PEP cannot answer this qeustion.

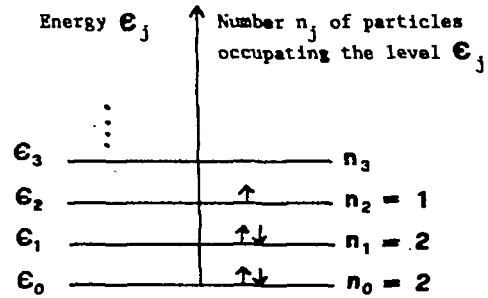


Fig.2: This is a quantum state of a system Σ, say a gas, of two-dimensional space-time fermions. In this figure, one learns that as far as the PEP is concerned, one cannot predict the real spin of the particles of the gas Σ. The Pauli index of this sytem Σ is $\eta = 1/s_0 = 2$.

Therefore, two fermions of spin s_1 and s_2 are said to belong to the same Pauli class if the following relation of equivalence holds:

$$s_1 - s_2 \in \mathbb{Z} . \tag{13}$$

These classes of equivalence are completely characterized by the Pauli index η defined by:

$$\eta = 1/s_0 , \tag{14}$$

where Eq.(12) has been used. Note that bosons for which $s_0 = 0$ and which are not distinguished by the PEP have a Pauli index $\eta = \infty$. Moreover, the notion of Pauli index which has been defined for fermions and bosons can be extended to the two-dimensional FS particles other than fermions. For the special type of particles that have been considered in this study, namely those having values of the spin s given by

$$s = 1/k + n; \quad n \in \mathbb{Z}^+ , \tag{15}$$

where $k \geq 2$, one may define η as in Eq.(14) so that η is equal to k. This is not only a natural extension, since it recovers the results of fermions and bosons, but it can be stated

in a hypothetic quantum theory of fields $\psi(z, \bar{z})$ of spin $1/k$ where consistency requires that ψ and its derivatives $\partial^n \psi$ should satisfy the same statistics. The less obvious but still natural hypothesis we would like to make regarding the particles of spin $s = 1/k$ mod 1 deals with the GPEP. Our proposal is that no more than $(k-1)$ identical FS particles Eq.(15) can live in the same individual quantum state. This generalized Pauli principle, which may be guessed from Eq.(3), recovers the Fermi antisymmetry and the Bose condensation as two limiting cases. In Fig.3 we give a quantum state of a system of polarized particles of spin $1/5$ mod 1. In this case no more than four particles can live on the same quantum state

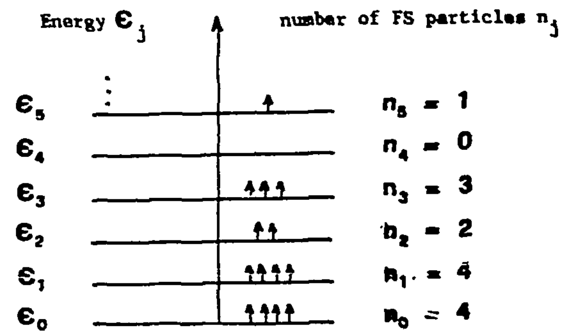


Fig.3: The wave function $\psi(k)$ describing systems $\Sigma(k)$ of N identical FS particles of spin $s = 1/k$ (mod 1) extending the usual Slater determinant $\psi(2)$ and Slater permanent $\psi(\infty)$ were constructed in [19,24].

4 The partition function $Z(\beta, \mu; \eta)$ and the FS quantum distribution

We start by considering a quantum gas $\Sigma(\eta)$ of identical FS particles of spin $1/k$ mod 1 enclosed within a container of volume V. This gas is assumed to be in equilibrium at a temperature T with a reservoir of FS particles of the same nature as those of $\Sigma(\eta)$. The quantum gas $\Sigma(\eta)$ is supposed to be a perfect gas. Denote by H and $\hat{N}$ respectively the Hamiltonian and the occupation operator number associated with the gas $\Sigma(\eta)$ of FS particles. These operators H and $\hat{N}$ can be expressed in statistical quantum field theory, often named as the grand canonical formalism, as:

$$\hat{N} = \sum_j \hat{N}_j, \quad H = \sum_j \in_j \hat{N}_j = \sum_j H_j \tag{16}$$

where the index j labels the possible quantum states of a single particle and $\in_j$ denotes the energy of a FS particle in the quantum state $|j\rangle$. $\hat{N}_j$ is the occupation operator number counting the number of FS particles of the gas $\Sigma(\eta)$ in the quantum state $|j\rangle$. At equilibrium, the partition function $Z(\beta,\mu,\eta)$ of the quantum gas is defined as:

$$Z(\beta,\mu;\eta) = Tr \exp -\beta[H - \mu\hat{N}] , \tag{17}$$

where the trace is taken on the Fock space of $\Sigma[\eta]$. Using the eigenvalue equations

$$\begin{aligned} \hat{N}_j|j\rangle &= n_j|j\rangle \\ H_j|j\rangle &= \in_j n_j|j\rangle \end{aligned} \tag{18}$$

together with the GPEP according to which the allowed eigenvalues n_j are only those given by

$$n_j = 0, 1, \ldots, k-1 , \tag{19}$$

Eq.(17) can be put in the following factorized form:

$$Z(\beta,\mu;\eta) = \prod_j Z_j(\beta,\mu;k) \tag{20}$$

where $Z_j(\beta,\mu;k)$ is given by

$$Z_j(\beta,\mu;k) = \sum_{n_j=0}^{k-1} \exp -[\beta(\in_j - \mu)n_j] \tag{21}$$

or equivalently

$$Z_j(\beta,\mu;k) = \frac{[1 - e^{-\beta k(\in_j-\mu)}]}{[1 - e^{-\beta(\in_j-\mu)}]} . \tag{22}$$

Having determined the partition function $Z(\beta,\mu;k)$ of the quantum gas $\Sigma[\eta]$ extending the well-known Fermi and Bose ones respectively given by

$$\begin{aligned} Z(\beta,\mu;2) &= \prod_{j\geq 0} \left[1 + e^{-\beta(\in_j-\mu)}\right] && \text{(a)} \\ Z(\beta,\mu;\infty) &= \prod_{j\geq 0} \left[1 + e^{-\beta(\in_j-\mu)}\right]^{-1} , && \text{(b)} \end{aligned} \tag{23}$$

one may calculate the thermodynamic potentials. For example, the grand potential $\Omega(\beta,\mu;k)$ of the quantum gas $\Sigma(\eta)$ of FS particles reads as:

$$\Omega(\beta,\mu;k) = \sum_{j\geq 0} \Omega_j(\beta,\mu;k) \tag{24}$$

where $\Omega_j(\beta,\mu;k)$ is given by

$$\Omega_j(\beta,\mu;k) = -\beta^{-1} \log\left[1 - e^{-\beta k(\in_j-\mu)}\right] + \beta^{-1} \log\left[1 - e^{-\beta(\in_j-\mu)}\right] . \tag{25}$$

Moreover, using Eq.(23.b), one sees that one may express the partition function of the quantum FS gas Eq.(22) and the related thermodynamic functions in terms of the corresponding ones of the Bose gas. We have:

$$Z[\beta,\mu;k] = Z[\beta,\mu;\infty]/Z(k\beta,\mu;\infty) \tag{26}$$

and

$$\Omega[\beta,\mu;k] = \Omega(\beta,\mu;\infty) - \Omega(k\beta,\mu;\infty) \tag{27}$$

Taking the limit $k = \infty$, Eqs.(27-28) imply:

$$\begin{aligned} Z(\infty,\mu;\infty) &= 1 = Z(\infty,\mu;k) \\ \Omega(\infty,\mu,\infty) &= 0 = \Omega(\infty,\mu;k) . \end{aligned} \tag{28}$$

The distribution of the perfect gas $\Sigma(\eta)$ of FS particles is effectively described by the mean value of the basic operator of the grand canonical formalism, namely the occupation operator number of the individual quantum state $|j\rangle$:

$$\langle\hat{N}_j\rangle = \mathrm{Tr}\left[\hat{N}_j\, e^{-\beta[H-\mu\hat{N}]}\right] / Z[\beta,\mu;\eta] . \tag{29}$$

Using Eqs.(22, 23), the above equation reads as:

$$\langle\hat{N}_j\rangle = -\partial \log Z_j[\beta,\mu;\eta]/\beta\partial\mu . \tag{30}$$

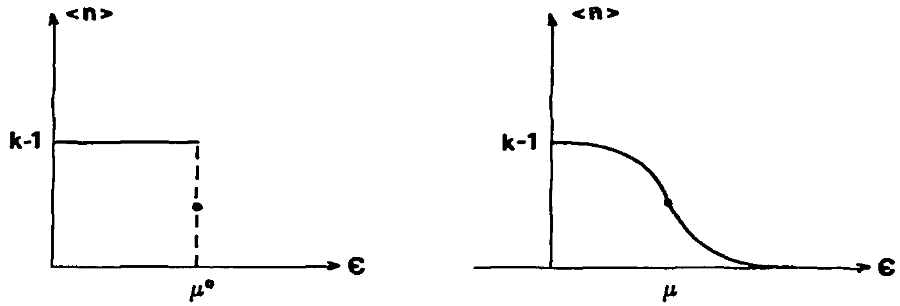
and consequently

$$\langle n_j(\beta,\mu;k)\rangle = \left[e^{\beta(\in_j-\mu)} - 1\right]^{-1} - k\left[e^{\beta k(\in_j-\mu)} - 1\right]^{-1} \tag{31}$$

where we have set $\langle\hat{N}_j\rangle = \langle n_j(\beta,\mu;k\rangle$. This is the distribution of the quantum gas $\Sigma(\eta)$ of identical FS particles of spin $1/k$ (mod 1). Eq.(31) exhibits a set of remarkable features. Taking $k = 2$ and $k = \infty$, one obtains, respectively, the Fermi and Bose distributions given by Eqs.(10,11). Moreover, as for the partition fuction $Z(\beta,\mu;\eta)$ and the free energy $F(\beta,\mu;\eta)$ of $\Sigma(\eta)$, Eq.(22) can also be expressed in terms of the Bose distribution $\langle n_j(\beta,\mu;\infty)\rangle$ as:

$$\langle n_j(\beta,\mu;k)\rangle = \langle n_j(\beta,\mu;\infty\rangle - k\langle n_j(k\beta,\mu;\infty)\rangle . \tag{32}$$

Furthermore, for a fixed temperature, say β^*, the variation of Eq.(31) with respect to the energy $\in$ is given by Fig.1



(a) Variation of $\langle n \rangle$ as a function of $\in$ at $1/\beta^{\bullet} = 0$
μ_0 is the chemical potential of $\Sigma(\eta)$ at $1/\beta^{\bullet} = 0$

(b) Variation of $\langle n \rangle$ as a function of $\in$ for$\beta^{\bullet} < \infty$
$\mu = \mu(\beta^{\bullet})$ is the chemical potential of Σ

At zero temperature, the mean occupation number $\langle n_j(\beta, \mu, k) \rangle$ is proportional to Heaviside function $H(x)$ taking the value one for $0 \leq x < 1$ and the value zero for $x > 1$, i.e.:

$$\langle n(\infty, \mu, k) \rangle = (k-1)\ H(\mu^0 - \in_j)\ ; \tag{33}$$

where the superscript carried by μ^0 refers to the value of the chemical potential at $T = 0$. Setting $k = 2$ in Eq.(25), one obtains the usual Fermi result according to which the mean occupation number of fermions, having energy level $\in_j$, is one if $\in_j < \mu_F^0$ and zero if $\in_j > \mu_F^0$. Putting $k = \infty$ in Eq.(25) and using the fact that the chemical potential μ_B^0 of Bose gasses is zero, one gets:

$$\langle n(\infty, 0, \infty) \rangle = \delta(\in)\ . \tag{34}$$

This equation shows that at $T = 0$, all bosons are condensed on the fundamental state $\in_0 = 0$. This phenomenon is known as the Bose condensation.

5 More on the FS quantum distribution

Recall that in the vicinity of the zero temperature, Fermi and Bose gases exhibit very different behaviours. This theoretical prediction had been observed experimentally on the two liquid helium isotopes ^3He and ^4He [25]. The phase diagrams of the ^3He Fermi liquid and the ^4He Bose one are very different. For a review see [26]. In the model of quantum

perfect gases, one of the basic formula playing a central role in the analysis of the low temperature behaviour of the perfect gas is given by the Sommerfeld formula, namely:

$$\langle n(\beta, \mu, 2) = H(\mu - \in) - \frac{\pi^2}{6}\beta^{-1}\delta'(\mu - \in) - \frac{7\pi^4}{360}\beta^{-4}\delta'''(\mu - \in) + 0(\beta^{-6}) \tag{35}$$

where $\delta^{(n)}$ is the n–th derivative of the Dirac δ function with respect to the energy parameter $\in$. For the bosonic distribution $\langle n(\beta, \mu, \infty) \rangle$ and to my knowledge, no low temperature expansion of type Eq.(27) has been written down. The derivation of the low temperature properties of the Bose gas is achieved by using a method different of that used for the Fermi gas. The study of the features of the FS gas near $T = 0$ will shed light on the methods used in the treatment of Fermi and Bose gases. To that purpose, one should derive the analogue of Eq.(27) for any k. This expansion whose three leading terms have been given in the introduction Eq.(5) may be obtained as follows: First, start from the integral

$$I[\gamma, \mu] = \int_0^\infty \frac{f(\in)}{e^{\gamma(\in - 1)} - 1}\, d\in\ , \tag{36}$$

where $f(\in)$ is a test function, a trick often used in the theory of distributions. For physical applications, the function $f(\in)$ can be thought of as

$$f(\in) = \in^{n/2};\quad n = 1, 3\ . \tag{37}$$

The parameter γ will be chosen either as $\gamma = \beta$ or $\gamma = k\beta$ as one may already wonder from Eq.(24). Second, make the change of variables $z = \gamma(\in - \mu)$ or equivalently $\in = \mu + \alpha z$ with $\alpha = \gamma^{-1}$ so that Eq.(28) becomes

$$I[\gamma, \mu] = \alpha \int_0^{\gamma\mu} \frac{f(\mu - dz)dz}{e^{-z} - 1} + \alpha \int_0^\infty \frac{f(\mu + \alpha z)}{e^z - 1}\, dz\ . \tag{38}$$

Here we would like to note that in order to compute the above integral near the zero absolute one should specify the sign of the chemical potential $\mu = \mu(\beta)$ and eventually its dependence on the spin of the FS particles of $\Sigma(\eta)$. Suppose for the moment that μ is larger than zero. In this case one may write Eq.(30) as

$$I[\gamma, \mu > 0] = -\int_0^\mu f(\in) d\in - \alpha \int_0^{\gamma\mu} \frac{f(\mu - dz)dz}{e^{-z} - 1} + \alpha \int_0^\infty \frac{f(\mu + \alpha z)dz}{e^z - 1} \tag{39}$$

where the following identity

$$1/(e^{-z} - 1) = -1 - 1/(e^z - 1)\ , \tag{40}$$

has been used. Moreover, in the vicinity of the zero absolute, i.e. γ goes to infinity, the upper bound $\gamma\mu$ of the second integral of the right–hand side of Eq.(31) can be replaced by plus infinity so that the integral $I[\gamma, \mu > 0]$ reads as:

$$I[\gamma, \mu > 0] = -\int_0^{\mu} f(\in)d\in + \sum_{n>0} \frac{(2\pi\alpha)^{2n}}{(2n+1)!2n} B_n\, f^{(2n-1)}(\mu)\,. \tag{41}$$

In deriving this equation we have used the identity

$$\int_0^{\infty} dz\; z^{2n-1}/(e^z - 1) = \frac{(2\pi)^{2n}}{4n} B_n; \quad n > 0\,, \tag{42}$$

where the B_n's are the Bernouilli numbers whose first values are given by

$$B_1 = 1/6,\; B_2 = 1/30,\; B_3 = 1/42,\; B_4 = 1/30\,. \tag{43}$$

Using now the relation Eq.(24), the low temperature expansion of the FS distribution generalizing the Sommerfeld development Eq.(27) may be read after solving the following equation

$$I[\beta, \mu > 0] - kI[k\beta, \mu > 0] = \int_0^{\infty} \langle n(\beta, \mu; k)\rangle \cdot f(\in)dt\,. \tag{44}$$

Straightforward calculations lead to

$$\langle n(\beta, \mu; k)\rangle = (k-1)H(\mu - \in) - \sum_{n>0} \frac{(2\pi/\beta)^{2n}}{(2n-1)!2n} B_m \frac{(k^{2n-1}-1)}{k^{2n-1}} \delta^{(2n-1)}(\mu - \in) \tag{45}$$

The first three leading terms of this expansion are given by Eq.(5). Note that the above generalized Sommerfeld development is invariant under the change β into $-\beta$. No odd integer powers of the temperature occur in Eq.(45).

On the other hand, in the case where the chemical potential μ is negative, say $\mu = -|\mu|$, no information can be read about the distribution $\langle m(\beta, \mu, k)\rangle$ since in the limit $\beta \to \infty$ Eq.(28) vanishes identically. However, for μ equal to zero or again when μ has a small enough value so that the product $\beta\mu$ remains very small when $\beta \to \infty$, Eq.(28) reduces to the following

$$I[\gamma, \mu \simeq 0] = \alpha \int_0^{\infty} dz\, \frac{f(\alpha z)}{e^z - 1}\,. \tag{46}$$

Using the identity

$$\int_0^{\infty} z^{n-1} dz/(e^z - 1) = \Gamma(n)\zeta(n); \quad n > 1 \tag{47}$$

where $\Gamma(n)$ and $\zeta(n)$ are respectively the Euler function and the Riemann ζ function given by

$$\begin{aligned} \Gamma(n) &= (n-1)!; \; n \in \mathbb{Z}^+ \\ \zeta(x) &= \sum_n 1/n^x \end{aligned} \tag{48}$$

one gets in the limit γ goes to infinity:

$$I[\gamma, \mu \simeq 0] = \sum_{n\geq 0} \alpha^{n+1}\, \zeta(n+1)\, f^{(n)}(0)\,. \tag{49}$$

Therefore, in the vicinity of the zero temperature and for a chemical potential small enough so that $\beta\mu$ remains very small when β goes to infinity, the low temperature of the distribution $\langle n(\beta, \mu \simeq 0, k)\rangle$ with $k >>> 2$ reads by help of

$$I[\beta, \mu \simeq 0] - kI[\beta k, \mu \simeq 0] = \int_0^{\infty} \langle n(\beta, \mu \simeq 0; k)\rangle f(\in)d\in\,, \tag{50}$$

as follows:

$$\langle n(\beta, \mu \simeq 0, k)\rangle = \sum_{n\geq 0} (-)^n [1 - k^{-n}] \beta^{-n-1}\, \zeta(n+1)\, \delta^{(n)}(\mu - \in)\,. \tag{51}$$

In the limit $k = \infty$, one obtains the low temperature expansions of the Bose distribution. At the end of this section we would like to show that systems $\Sigma[k]$ with $k \geq 2$ have the same state equation given by Eq.(10). The starting point is the grand potential Ω Eq.(26) which may be written in terms of the thermodynamic potentials $F(\beta, k)$ and $G(\beta, k)$ the Helmotz and Gibbs free energies as:

$$\Omega = F - G\,, \tag{52}$$

which is equal to $-\mathcal{P}V$ where $\mathcal{P}$ and V are respectively the pressure and the volume of the gas $\Sigma(k)$. The second step is to note that the grand canonical potential Ω is related to the mean energy E of the gas $\Sigma(k)$ in the same way for any $k \geq 2$, namely:

$$\Omega = -2/3\; E = \mathcal{P}V\,. \tag{53}$$

As mentioned in the Introduction, this state equation contains no explicit reference to the spin of the particles of the gas Σ. It can be proved in a straightforward way by going to the thermodynamic limit of Eq.(26) and

$$E = \sum_j n_j\; \in_j\,. \tag{54}$$

Details of calculations are omitted.

6 The chemical potential

So far we have seen that there is a profound difference between the Fermi and Bose quantum gases. This difference becomes more striking in the limit T goes to zero, when the quantum gas as a whole is in its lowest energy. Whereas the lowest energy E^0_{Bose} of the Bose gas is:

$$E^0_{Bose} = N \in_0 \tag{55}$$

where N is the total number of particles of $\Sigma(\infty)$, the energy E^0_{Fermi} of the Fermi gas (of polarized fermions for instance)

$$E^0_{Fermi} = \in_0 + \in_1 + \ldots + \in_n \ , \tag{56}$$

is considerably greater than E^0_{Bose}. This is obviously a direct consequence of the fermion's PEP. Another feature showing the very different behaviours of the Bose and the Fermi quantum gases is seen in the values μ^0_B and μ^0_F of the chemical potential $\mu(T)$ at zero temperature. Whereas μ^0 vanishes for $\Sigma(\infty)$ i.e.:

$$\mu^0_B = 0 \ , \tag{57}$$

it takes the value μ^0_F known as the Fermi energy [18]

$$\mu^0_F = \frac{h}{2m}\left(\frac{3d}{8\pi}\right)^{2/3} \ , \tag{58}$$

where h is the Planck constant, m is the mass of a fermion of the gas $\Sigma(2)$ and $d = N/V$ is the density of the gas. Note that Eq.(58) is valid for a Fermi gas taken in the nonrelativisitc limit. A relativistic formula extending Eq.(58) may be written down. This is not important for the present study and consequently our attention will be focussed in the remainder of this paper on nonrelativistic gases only. Moreover, for non zero values of the temperature, the chemical potential of both the Fermi and Bose gases is a function of T obeying the inequality

$$\mu(T) \le \mu(0) = \mu^0 \ . \tag{59}$$

For the Fermi gas for instance, the above inequality reads for temperature T smaller than the Fermi temperature $T_F = \mu^0_F/K$ where K is the Boltzmann constant, as

$$\mu_{Fermi}(T) = \mu^0_F[1 - \alpha T^2 + O(T^4)] \ , \tag{60}$$

where α is a positive constant. In Fig.4 we give the shape of the variation of the ratio μ_F/μ^0_F with respect T/T_F.

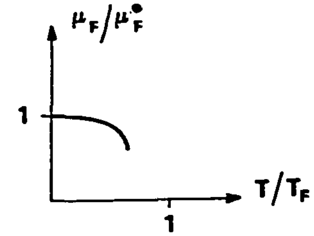


Fig.4

For the Bose gas, Eq.(5) reads for temperatures T smaller than the Bose critical temperature T_B as:

$$\mu_B(T) = -KT/N_0 - O(T^2) \le o, \quad T \le T_B \tag{61}$$

where $N_0 = \langle n_0(\beta, \mu, \infty)\rangle$ is the mean occupation number of the fundamental state of energy $\in_0 = 0$. For very low temperature $T \ll T_B$, N_0 is practically equal to the total number N of particles and consequently the chemical potential $\mu_B(T)$ may be set to zero. In Fig.5 we give the shape of the variation of μ_B with respect to T.

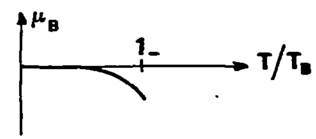


Fig.5

Note that the temperature T_B may be obtained by using the constraint equation

$$N = \sum_{j\ge0} 1/(e^{\beta(\in_j - \mu)} - 1) \ . \tag{62}$$

Straightforward calculations, see Eqs.(48) and (79) show that T_B is given by

$$T_B = \frac{h^2}{2mK}\,(2.16.d)^{2/3} \ . \tag{63}$$

For the bosonic ^4He isotope liquid helium described by the Bose gas model, the critical temperature T_B [20]

$$T_B(^4He) = 2.13°\text{Kelvin} \ . \tag{64}$$

We turn now to study the properties of quantum gases of FS particles of spin $s = 1/4$ mod 1 and examine the limit of the results when k goes to infinity in order to compare them with the known results for the Bose phase. We start first by noting that the lowest energy E^0_{FS} of the gas $\Sigma(k)$ extending Eqs.(55-56) reads as:

$$E^0_{FS} = (k-1) \sum_{j=0}^{m-1} \in_j + m' \in_m \tag{65}$$

where m and m' are related to the total number N of FS particles of the gas $\Sigma(k)$ as:

$$\begin{array}{rcl} N &=& m(k-1) + m' \\ m &=& [N/(k-1)], \quad 0 \le m' \le k-2 . \end{array} \tag{66}$$

For large values of k say k of the order N then m and m' are of order one. In this case almost all particle are condensed in the fundamental state of energy $\in_B$. Moreover, at zero temperature, the chemical potential μ^0_{FS} of the gas $\Sigma(k)$ is easily derived from Eq.(62) and Eq.(25). Indeed, using the thermodynamic limit by replacing the discrete series Eq.(62) by the following integral

$$N/V = \frac{2\pi(2m)^{3/2}}{h^3} \int_0^\infty d\in \in^{1/2} \langle n(\beta,\mu;k)\rangle , \tag{67}$$

the value of the chemical potential μ^0_{FS} may be written with the help of Eq.(55) as:

$$\mu^0_{FS} = (k-1)^{-2/3} \in^0_F , \tag{68}$$

where we have set $\mu^0_F = \in^0_F$. As expected, this equation shows that $\mu^0_{FS}(k)$ is bounded by the Bose and the Fermi values of the chemical potential at $T = 0$:

$$0 \le \mu^0_{FS} \le \in^0_F . \tag{69}$$

Note that when k goes to infinity, μ^0_{FS} runs to the bosonic value $\mu^0_B = 0$ as $k^{-2/3}$.
In the vicinity of the zero temperature, the inequalities Eq.(69) are still preserved i.e.

$$\mu_B(T) \le \mu_{FS}(T) \le \mu_F(T) . \tag{70}$$

Fig.6 shows the variations of the chemical potential $\mu = \mu(T)$ near the zero temperature

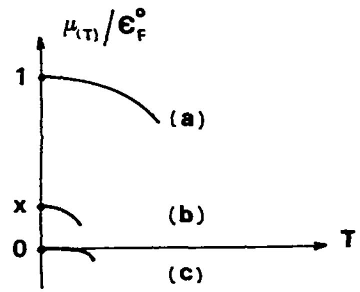


Fig.6: The curves (a), (b) and (c) show the shapes of the chemical potentials, in the vicinity of $T = 0$, of the quantum Fermi, FS and Bose gases. For large values of k, $\mu_{FS}(T)$ takes practically negative values, $\mu_{FS}(T) \lesssim 0$ almost as for $\mu_B(T)$. The temperature dependence of $\mu_{FS}(T)$ when $k \to \infty$ and $\mu_B(T)$ is not the same but still compatible near the zero absolute; see Eqs.(61) and (75-77). $x = (k-1)^{-2/3}$, $0 \le x \le 1$

To prove Eq.(70), it is sufficient to establish the inequality $\mu_{FS}(T) \le \mu_F(T)$. For that purpose one starts from Eq.(64) and solve the constraint equation

$$(k-1) \int_0^\infty d\in \in^{1/2} H(\mu \cdot \in) = \int_0^\infty d\in \in^{1/2} \langle n(\beta,\mu;k)\rangle , \tag{71}$$

stating that the density $d = N/V$ of the gas $\Sigma(k)$ is invariant under the small variations of temperature T. Replacing $\langle n(\beta,\mu;k)\rangle$ by its generalized Sommerfeld expansion Eqs.(5) and (41), one gets

$$\mu^{0\ 3/2}_{FS} = \mu^{3/2}_{FS} \left\{ 1 + \frac{\pi^2}{6k} (\beta\mu_{FS})^{-2} + O([\beta\mu_{FS}]^{-4}) \right\} . \tag{72}$$

Inverting this equation and using Eq.(65), one obtains finally:

$$\mu_{FS}(T) = \mu^0_{FS} \left\{ 1 - \frac{\pi^2}{6} \cdot \frac{(k-1)^{4/3}}{k} \cdot \left(\frac{T}{T_F}\right)^2 + O([T/T_F]^4) \right\} . \tag{73}$$

From this equation, one learns the following two points. First, near the zero absolute $T \gtrsim 0$, we have the obvious inequalities

$$\mu_{FS}(T) - \mu^0_{FS} \le 0, \quad \mu_{FS}(T) - \mu_F(T) \le 0 \tag{74}$$

giving the proof of Eqs.(70). The second feature we want to mention is that for large values of k, $\mu_{FS}(T)$ is almost everywhere negative as shown by the following equation

$$\mu_{FS}(T, k \to \infty) \sim -\frac{\pi^2}{6}\, k^{-1/3}\, \beta^{-2} + O(\beta^{-4}); \quad T < T_F\,. \tag{75}$$

As expected from the start, in the limit k goes to infinity the chemical potential is negative as required by the Bose distribution. However, this limit does not coincide exactly with $\mu_B(T)$, Eq.(61) since Eq.(75) does not contain any term in β^{-1} within the used approximations. This anomaly could be explained by the fact that near the zero temperature the expansion of the FS distribution namely

$$\langle n_j(\beta, \mu; k)\rangle = \frac{1}{(e^{\beta(\epsilon_j - \mu)} - 1)} - \frac{k}{(e^{k\beta(\epsilon_j - \mu)} - 1)}\,, \tag{76}$$

depends on the order followed when taking the limits $T \to 0$ and $k \to \infty$. If the limit $k \to \infty$ is taken first, the chemical potential $\mu_B(T)$ exhibits a linear dependence on T as in Eq.(61) since in this case the second term of the r.h.s. of Eq.(76) decouples completely. However, if one takes first the limit $\beta \to \infty$ and then $k \to \infty$ as well as taking into account the condensation phenomenon, one finds that the two linear T-terms coming from the expansions of the two pieces of the r.h.s. of Eq.(76) cancel exactly each other as shown herebelow:

$$N_0 = -(\beta\mu)^{-1} - k(-k\beta\mu)^{-1} + O(\beta^{-2}), \quad T < T_c \tag{77}$$

where N_0 was defined below Eq.(53).

7 Discussion and outlook

In this section we want to comment briefly on the low temperature properties of a perfect gas of FS particles of nonzero mass and a spin $s = 1/k$ mod 1, k much greater than two, within the approximation of negative $\mu_{FS}(T)$ chemical potential Eq.(75). The main thing to note is that for k very large the GPEP acts as a very weak constraint and then the low temperature properties of $\Sigma(k)$ are not very different from those of a Bose gas. There are many interesting thermodynamic quantities one can try to identify for the gas $\Sigma(k), k \gg 2$. May be the most relevant ones one should start with are the free energy $F(\beta, k)$, the specific heat $C(\beta, k)$ and the critical temperature T_s of $\Sigma(k)$. In what follows we shall focus our attention on the temperature T_s. Thus, starting from the identity

$$N = \sum_j \langle n_j(\beta, \mu; k)\rangle, \quad k \gg 2 \tag{78}$$

which reads in the thermodynamic limit as

$$N/V = A \int_0^\infty \epsilon^{1/2} \left[\frac{1}{(e^{\beta(\epsilon-\mu)} - 1)} - \frac{k}{(e^{k\beta(\epsilon-\mu)} - 1)}\right] d\epsilon\,, \tag{79}$$

where $A = 2\pi(2m/h^2)^{3/2}$ is a constant irrelevant for our calculations. Then varying the temperature of the gas $\Sigma(k)$ by keeping N, V and hence the particle density N/V constant, one sees that the r.h.s. of Eq.(79) should also remain invariant under temperature variations. Therefore if the r.h.s. of Eq.(79) has to remain constant as T is lowered, the chemical potential $\mu_{FS}(T)$ must vary so that to compensate the variation of T. For large values of k where the chemical potential $\mu_{FS}(T)$ is almost everywhere negative Eq.(75) and which reach the value zero more quickly than T, Eq.(79) defines a minimal temperature $T_s \neq 0$ such that for $T = T_s$ the chemical potential μ is practically zero. To calculate this temperature T_s, one may use the constraint equation

$$\lim_{k\to\infty} T_s = T_c\,, \tag{80}$$

where T_c is the critical temperature of the Bose gas given by Eq.(63). Eq.(80) means that the temperature T_s can be written as $T_s = T_c + \delta T$ where δT is a small variation of T_c which carries the effect of the FS particles and which vanishes when we set $k = \infty$. The variation δT can be computed by solving the following equation

$$N/V = A \int_0^\infty d\epsilon\, \epsilon^{1/2} \left[\frac{1}{(e^{\beta_s \epsilon} - 1)} - \frac{1}{(e^{k\beta_s \epsilon} - 1)}\right]\,, \tag{81}$$

where $\beta_s = 1/KT_s$. Straightforward calculations lead to the following k dependent expression of T_s

$$T_s = T_c/(1 - k^{-1/2}) \simeq \left(1 + k^{-1/2} + O(1/k)\right) T_c\,. \tag{82}$$

Moreover, using Eq.(81) once more, one finds that the condensation formula giving the ratio N_0/N, of the number N_0 of condensed particles of energy $\epsilon_0 = 0$ over the total number N of particles of the gas Σ, with respect to the temperature reads as

$$N_0/N = \left\{1 - \left[(1 - k^{-1/2})(T/T_s)\right]^{3/2}\right\}\,. \tag{83}$$

Putting the identity (82) back into this equation one gets the well-known Bose condensation formula.

In the end of this discussion, we would like to note that for intermediate values of the Pauli index k, the generalized Sommerfeld expansion we have derived in Section 5, Eq.(41) is expected to play a central role in the study of the low temperature behaviour of the thermodynamic quantities of the gas $\Sigma(k)$. It would be interesting to carry out this analysis.

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References

[1] C. Ahn, D. Bernard and A. LeClair, Nucl. Phys. **B346** (1990) 409;
D. Bernard and A. LeClair, Nucl. Phys. **B340** (1990) 721; Phys. Lett. **B247** (1990) 309; Commun. Math. Phys. **142** (1991) 99;
A. LeClair and C. Vafa, Nucl. Phys. **B401** (1993) 413;
A.B. Zamolodchikov, Int. J. Mod. Phys. **A4** (1989) 4235;
F.A. Smirnov, Int. J. Mod. Phys. **A4** (1989) 4213;
N.Y. Peshetikhin and F. Smirnov, Commun. Math. Phys. **131** (1990) 157.

[2] S.Y. Wu, Phys. Rev. Lett. **52** (1984) 2103;
G.S. Canright and S.M. Girvin, IJMP **B3** (1989) 1943;
D.H. Lee and M.P.A. Fisher, Phys. Rev. Lett. **63** (1989) 903;
T. Einarsson, Phys. Rev. Lett. **64** (1990) 1995;
S.B. Isakov, S. Mashlevich and S. Ouary, Nucl. Phys. **B448** [FS] (1995) 457.

[3] B.I. Halperin, Phys. Rev. Lett. **52** (1984) 1583;
Tong-Shi Wu, Phys. Rev. Lett. **24** (1984) 2103;
J. Fröhlich, U.M. Studer and E. Thiram, An ADE) deformation of minimal incompressible quantum Hall fluids, preprint cond-mat/94-6-9; proc. Conf. On Three levels, Lauven, 1993, eds. M. Fannes et al.;
A. Cappelli, C.A. Trugenberger and G.R. Zemba, Nucl. Phys. **B448** [FS] (1995) 470.

[4] F. Wilczek, Phys. Rev. Lett. **48** (1982) 1144; Phys. Rev. Lett. **49** (1982) 957;
F. Wilczek and A. Zee, Phys. Rev. Lett. **51** (1983) 2250;
D. Arovas, J.R. Schriffer and F. Wilczek, Phys. Rev. Lett. **53** (1984) 722;
J. Goldstone and F. Wilczek, Phys. Rev. Lett. **47** (1981) 986.

[5] J.M. Leinaas and J. Myrheim, Il Nuovo Cimento **37** (1977) 1;
B.I. Halperin, Phys. Rev. Lett. **52** 91984) 1583; Phys. Rev. Lett. **52** (1984) 2390;
R.B. Laughlin, Phys. Rev. Lett. **60** (1988) 2677;
V. Kalmeyer and R.B. Laughlin, Phys. Rev. Lett. **59** (1987) 2095;
A. Fetter, C. Hanna and R. Laughlin, The random phase approximation in the anyon gas, Stanford preprint.

[6] A.B. Zamolodchikov, JETP Lett. **43** (1986) 730; Sov. J. Nucl. Phys. **46** (1988) 1090;
A.W.W. Ludwig and J.L. Cardy, Nucl. Phys. **B285** [FS19] (1987) 687;

G. Sotkov and C.J. Zhu, Phys. Lett. **B151** (1985) 391;
C. Vafa and N. Warner, Phys. Lett. **B218** (1989) 51;
E. Abdalla, M.C.B. Abdalla, G. Sotkov and M. Stanishkov, IJMP **A10** (1995) 1717.

[7] E.H. Saidi and M. Zakkari, Phys. Rev. **D46** (1992) 777;
E.H. Saidi and M. Sedra, Mod. Phys. Lett. **A9** (1994) 3163.

[8] A. LeClair and C. Vafa, Nucl. Phys. **B401** (1993) 413;
J. Zerouaoni, These de 3^e Cycle, LMPHE, Rabat (1995).

[9] E.H. Saidi, M.B. Sedra and J. Zerouaoni, Class. Quant. Grav. **12** (1995) 1567;
E.H. Saidi, M.B. Sedra and J. Zerouaoni, ICTP, Trieste, preprint IC/94/131, to appear in Class. Quant. Grav. (1995).

[10] W. Lerche, C. Vafa and N. Warner, Nucl. Phys. **B324** (1989) 427.

[11] Y.H. Chen, F. Wilczek, E. Witten and B.I. Halperin, Int. J. Mod. Phys. B, Vol.3 *(1989) 1001 and references therein;*
S. Randjbar-Daemi, A. Salam and J. Strathdee, Nucl. Phys. **B340** (1990) 403.

[12] F. Wilczek, *Fractional Statistics and Anyon Superconductivity* (World Scientific, Singapore, 1990);
R.A. Prange and S.M. Girvin, *The Quantum Hall Effect* (Springer, Berlin, 1990).

[13] A. Zamolodchikov, Advanced Studies Pure math. **19** (1989) 641;
T. Eguchi and S.K. Yang, Phys. Lett. **B224** *(1989) 373*; **B235** *(1990) 282;*
I. Bakas, Int. J. Mod. Phys. **A9** (1994) 3443;
Q.H. Park, Phys. Lett. **B328** (1994) 329;
H.Q. Park and H.J.H. Shin, Phys. Lett. **B347** (1995) 73.

[14] G. Mussardo, Phys. rep. **218** (1992) 216;
A. Kadiri, These de 3^e Cycle, LMPHE, Rabat (1995);
C.D. Cabra and K.D. Rothe, Fermionic coset realization of primaries in critical statistical models, preprint HD–THEP95–6.

[15] E.H. Saidi, On the quantum parafermi gases I: The distribution, LMPHE preprint, Rabat (1995).

[16] P. Christe and G. Mussardo, Nucl. Phys. **B330** (1990) 465.

[17] A.B. Zamolodchikov, Moscow preprint (1989);
A. Kadiri, E.H. Saidi, M.B. Sedra and J. Zerouaoui, On exotic symmetries of the $\phi_{1,3}$ deformation of minimal models, ICTP, Trieste, preprint IC/94/216;
G. Sotkov and C.J. Zhu, Phys. Lett. **B229** (1989) 391.

[18] For a revie.. see L. Landau and E. Lifshitz, Physique Statistique, edition MIR (1984).

[19] E.H. Saidi and J. Zerouaoui, On the quantum parafermi gases II: The wave function, LMPHE preprint, Rabat (1995).

[20] F. Mandl, *Statistical Physics*, second edition (John Wiley & Sons, 1988);
E.M. Lifshitz and L.P. Pitaevski, *Statistical Physics*, Part 2 (Pergamon Press, Oxford, 1980).

[21] C. De Concini and V.G. Kac, Progress in Math. **92** (1990) 471;
D. Arnaudon, CERN preprint, TH6416/92 and 6730/92.

[22] J. Strathdee, Int. J. Mod. Phys. **A2** (1984) 273;
J. Jumphreys, *Introduction to Lie Algebras and Representations Theory* (Springer–Verlag, Berlin, 1972).

[23] A. Belavin, A. Polyakov and A. Zamolodchikov, Nucl. Phys. **B241** (1984) 333;
Freidan, Z. Qui and S. Shenker, Phys. Lett. **B151** (1985) 37.

[24] M. Rachidi, E.H. Saidi and J. Zerouaoui, On the quantum parafermi gases III: The Slater quasi determinant, LMPHE, Rabat (1995).

[25] H.N. Temperly and D.H. Trevena, *Liquids and their Properties*, Ellis Horwood Ltd. (John Wiley & Sons, New York, 1968).

[26] E.M. Lifshitz and L.P. Pitaevskii, *Statistical Physics*, Part 2 (Pergamon Press, Oxford, 1980).