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Elastic $\pi\pi$ scattering to two loops

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Abstract

We evaluate analytically the elastic $\pi\pi$ scattering amplitude to two loops in chiral perturbation theory and give numerical values for the two S -wave scattering lengths and for the phase shift difference $\delta_0^0 - \delta_1^1$.

1. In the framework of chiral perturbation theory (CHPT) [1, 2, 3], the elastic $\pi\pi$ scattering amplitude is evaluated by an expansion in powers of the external momenta and of the light quark masses,

$$A = A_2 + A_4 + A_6 + \dots \quad , \quad (1)$$

where A_n is of $O(p^n)$. The first two terms in this expansion have been extensively analyzed during the last three decades [4, 5, 6]. After a long period in which hardly any data have been collected, $\pi\pi$ scattering will receive in the near future interesting new input from the experimental side: i) It is expected that forthcoming precise data on K_{e4} decay at DAΦNE and at Brookhaven will allow one to determine the phase shift difference $\delta_{L=0}^{I=0} - \delta_1^1$ near threshold with considerably higher precision [8] than hitherto available [10]. ii) There are plans to measure the lifetime of the $\pi^+\pi^-$ -atom in the ground state [11], and to accurately determine in this manner the combination $a_0^0 - a_0^2$ of the two S -wave scattering lengths. In order to confront these data with precise theoretical predictions, it is necessary to go beyond the next-to-leading order term A_4 [5]. As has been pointed out in [12] and in [13], one may eventually obtain experimental information on the size of the quark-antiquark condensate in QCD in this manner.

In Ref. [13], the part of the amplitude A_6 containing branch points – required by unitarity – has been determined. A general crossing symmetric polynomial of $O(p^6)$, not fixed by unitarity, has been added by hand. In this letter, we present the analytical result for A_6 based on a full two-loop calculation in the framework of CHPT. We compare the results of the two approaches below.

2. The expansion (1) is most conveniently performed in the framework of an effective lagrangian [1, 2, 3]. Here we consider an expansion around the chiral limit $m_u = m_d = 0$, whereas the strange quark mass is kept at its physical value. We ignore isospin breaking effects and put $m_u = m_d = \hat{m}$. The effective lagrangian is expressed in terms of the pion field U and of the quark mass matrix χ ,

$$\mathcal{L}_{\text{eff}} = \mathcal{L}_2(U, \chi) + \hbar \mathcal{L}_4(U, \chi) + \hbar^2 \mathcal{L}_6(U, \chi) + \dots \quad ,$$

where \mathcal{L}_n contains m_1 derivatives of the pion field and m_2 powers of the quark mass matrix, with $m_1 + 2m_2 = n$. Given \mathcal{L}_{eff} , it is straightforward to expand the S -matrix elements in powers of \hbar . This procedure automatically generates the series (1), viz., $A = A_2 + \hbar A_4 + \hbar^2 A_6 + \dots$. The leading-order term A_2 has been evaluated by Weinberg [4], whereas the next-to-leading order correction A_4 was presented in [5]. The calculation of A_6 requires the evaluation of two-loop graphs with \mathcal{L}_2 , one-loop graphs with one vertex from \mathcal{L}_4 , and tree graphs generated by $\mathcal{L}_2 + \mathcal{L}_4 + \mathcal{L}_6$. Details of this calculation will be presented elsewhere. In particular, we refer the reader for the explicit expressions of $\mathcal{L}_{2,4}$ and of \mathcal{L}_6 to Ref. [2] and Ref. [14], respectively.

3. We use the notation

$$\begin{aligned} \langle \pi^d(p_4)\pi^c(p_3) \text{ out} | \pi^a(p_1)\pi^b(p_2) \text{ in} \rangle &= \langle \pi^d(p_4)\pi^c(p_3) \text{ in} | \pi^a(p_1)\pi^b(p_2) \text{ in} \rangle \\ &+ i(2\pi)^4 \delta^4(P_f - P_i) \left\{ \delta^{ab}\delta^{cd} A(s, t, u) + \text{permutations} \right\} \quad , \end{aligned}$$

where s, t, u are the usual Mandelstam variables, expressed in units of the physical pion mass squared M_π^2 ,

$$s = (p_1 + p_2)^2/M_\pi^2, \quad t = (p_3 - p_1)^2/M_\pi^2, \quad u = (p_4 - p_1)^2/M_\pi^2.$$

Using these dimensionless quantities, the momentum expansion of the amplitude amounts to a Taylor series in

$$x_2 = \frac{M_\pi^2}{F_\pi^2},$$

where F_π denotes the physical pion decay constant. We find

$$\begin{aligned} A(s, t, u) = & x_2 [s - 1] \\ & + x_2^2 [b_1 + b_2 s + b_3 s^2 + b_4 (t - u)^2] \\ & + x_2^2 [F^{(1)}(s) + G^{(1)}(s, t) + G^{(1)}(s, u)] \\ & + x_2^3 [b_5 s^3 + b_6 s (t - u)^2] \\ & + x_2^3 [F^{(2)}(s) + G^{(2)}(s, t) + G^{(2)}(s, u)] \\ & + O(x_2^4), \end{aligned} \quad (2)$$

with

$$\begin{aligned} F^{(1)}(s) &= \frac{1}{2} \bar{J}(s) (s^2 - 1), \\ G^{(1)}(s, t) &= \frac{1}{6} \bar{J}(t) (14 - 4s - 10t + st + 2t^2), \\ F^{(2)}(s) &= \bar{J}(s) \left\{ \frac{1}{16\pi^2} \left(\frac{503}{108} s^3 - \frac{929}{54} s^2 + \frac{887}{27} s - \frac{140}{9} \right) \right. \\ &+ b_1 (4s - 3) + b_2 (s^2 + 4s - 4) \\ &+ \left. \frac{b_3}{3} (8s^3 - 21s^2 + 48s - 32) + \frac{b_4}{3} (16s^3 - 71s^2 + 112s - 48) \right\} \\ &+ \frac{1}{18} K_1(s) \left\{ 20s^3 - 119s^2 + 210s - 135 - \frac{9}{16} \pi^2 (s - 4) \right\} \\ &+ \frac{1}{32} K_2(s) \{ s\pi^2 - 24 \} + \frac{1}{9} K_3(s) \{ 3s^2 - 17s + 9 \}, \\ G^{(2)}(s, t) &= \bar{J}(t) \left\{ \frac{1}{16\pi^2} \left[\frac{412}{27} - \frac{s}{54} (t^2 + 5t + 159) - t \left(\frac{267}{216} t^2 - \frac{727}{108} t + \frac{1571}{108} \right) \right] \right. \\ &+ b_1 (2 - t) + \frac{b_2}{3} (t - 4)(2t + s - 5) - \frac{b_3}{6} (t - 4)^2 (3t + 2s - 8) \\ &+ \left. \frac{b_4}{6} (2s(3t - 4)(t - 4) - 32t + 40t^2 - 11t^3) \right\} \\ &+ \frac{1}{36} K_1(t) \left\{ 174 + 8s - 10t^3 + 72t^2 - 185t - \frac{\pi^2}{16} (t - 4)(3s - 8) \right\} \end{aligned}$$

$$\begin{aligned}
& + \frac{1}{9} K_2(t) \left\{ 1 + 4s + \frac{\pi^2}{64} t(3s - 8) \right\} \\
& + \frac{1}{9} K_3(t) \left\{ 1 + 3st - s + 3t^2 - 9t \right\} + \frac{5}{3} K_4(t) \left\{ 4 - 2s - t \right\} . \quad (3)
\end{aligned}$$

The loop functions \bar{J} and K_i are displayed in appendix A, whereas the coefficients b_i in the polynomial part are given in appendix B.

4. We comment on the structure of the result.

- i) The amplitude $A(s, t, u)$ is expressed in terms of the external momenta, the physical pion mass, the physical pion decay constant, and the coefficients b_1, \dots, b_6 . To arrive at this result, one has to evaluate also M_π and F_π to two loops¹. Quantum field theory leads to the relations (B.1), that determine b_i in terms of

- chiral logarithms $L = \frac{1}{16\pi^2} \log \frac{M_\pi^2}{\mu^2}$,
- the low-energy couplings $l_i^r(\mu), \dots, l_4^r(\mu)$ from \mathcal{L}_4 ,
- the low-energy couplings $r_i^r(\mu), \dots, r_6^r(\mu)$ generated by \mathcal{L}_6 .

The scale dependent couplings l_i^r (r_i^r) are needed to remove the ultraviolet divergences at order p^4 (p^6), and to generate the most general solution of the Ward identities at these orders [2, 3]. The scale μ – introduced by the renormalization procedure – drops out in the full result (2).

- ii) The contributions proportional to x_2^n in (2) correspond to terms of $O(p^{2n})$. The terms proportional to x_2 in $G^{(2)}$ and $F^{(2)}$ generate contributions of $O(p^8)$ – these are beyond the accuracy we aim at here. In order to keep the formulae as simple as possible, we nevertheless retain them.
- iii) We compare the result (2) with the amplitude given in Ref. [13]. Identifying the low-energy couplings $\alpha, \beta, \lambda_1, \dots, \lambda_4$ introduced there with the relevant linear combinations of b_1, \dots, b_6 , the two expressions agree at $O(p^6)$. The S -matrix method used in [13] and the field theory calculation presented here therefore agree as far as the absorptive part of the amplitude and the general structure of the real part is concerned. Our use of an off-shell method provides the additional information Eq. (B.1). Together with an estimate of the new couplings at $O(p^6)$, this allows us to make predictions e.g. for the S -waves, see below. In Ref. [13], α and β are on the other hand treated as phenomenological parameters that appear in the expressions for the S -waves which remain undetermined. Once α and β have been pinned down, that approach will eventually allow one to determine the size of the quark-antiquark condensate, and to compare the result with the Gell-Mann-Oakes-Renner framework [16].

¹We are indebted to Urs Bürgi for providing us with the relevant expression for M_π prior to publication [15].

5. To proceed further, we need to know the values of the coefficients b_i . The constants l_i^r that occur in these have been determined from experimental data and using the Zweig rule some time ago [2] (for an update, see [17]). All these determinations are faced with the problem that the couplings l_i^r are quark mass independent, whereas the physical quantities, from where the l_i^r are evaluated, incorporate quark mass effects. Here we have, for the first time, a means to pin down these quark mass effects at leading order in an algebraically precise manner. In order to achieve this, we need an estimate of the couplings r_i^r . This is not an easy task, and we postpone a complete discussion to a later publication. Meanwhile, we use for l_i^r the values found in [2, 17],

$$\begin{aligned} l_1^r(M_\rho) &= -5.40 \cdot 10^{-3} , & l_2^r(M_\rho) &= 5.67 \cdot 10^{-3} , \\ l_3^r(M_\rho) &= 0.82 \cdot 10^{-3} , & l_4^r(M_\rho) &= 5.60 \cdot 10^{-3} ; & M_\rho &= 770 \text{ MeV} . \end{aligned}$$

As for r_i^r , we use a by now standard method to obtain an order of magnitude estimate: We incorporate in the above representation the contributions from the lowest heavy states and assume that these effects account for the bulk part in the low-energy couplings at $O(p^6)$. This procedure works very well at $O(p^4)$ [2, 18]. In particular, we include the contribution from vector and scalar exchange, using the couplings presented in Ref. [18]. In addition to these, kaons and etas also generate contributions of $O(p^6)$. To estimate those, we have taken from Ref. [13] (see also Ref. [19]) the elastic $\pi\pi$ scattering amplitude of $O(p^4)$ evaluated in the framework of $SU(3) \times SU(3)$ and restricted it to $SU(2) \times SU(2)$ by an expansion in inverse powers of the strange quark mass.

6. It is now straightforward to extract the scattering lengths a_i^f and slope parameters b_i^f . For the isospin zero S -wave, we obtain

$$\begin{aligned} a_0^0 &= \frac{7x_2}{32\pi} \left\{ 1 + \frac{x_2}{16\pi^2} \left[7 + \frac{16\pi^2}{7} (5b_1 + 12b_2 + 48b_3 + 32b_4) \right] \right. \\ &\quad \left. + \left(\frac{x_2}{16\pi^2} \right)^2 \left[\frac{7045}{63} + 16\pi^2 \left(10b_1 + 24b_2 + 96b_3 + 64b_4 \right. \right. \right. \\ &\quad \left. \left. \left. + \frac{3072\pi^2}{7} b_5 - \frac{215}{2016} \right) \right] \right\} , \end{aligned}$$

and similarly for the other threshold parameters. Numerically, we find² by keeping terms up to and including $O(x_2^3)$

$$\begin{aligned} a_0^0 &= 0.217 \text{ (0.215)} , \\ a_0^0 - a_2^0 &= 0.258 \text{ (0.256)} , \end{aligned} \tag{4}$$

where the numbers in brackets denote the values obtained by putting the couplings at $O(p^6)$ to zero at the scale $\mu = 1 \text{ GeV}$, $r_i^r(1 \text{ GeV}) = 0$. It is seen that for these threshold parameters the new couplings of $O(p^6)$ contribute a negligible amount, if their values are estimated in the manner described above. It is also worth emphasizing that there is essentially no scale dependence for $0.5 \text{ GeV} \leq \mu \leq 1 \text{ GeV}$. We

²For ease of comparison with earlier calculations, we use $F_\pi = 93.2 \text{ MeV}$ [27] and $M_\pi = 139.57 \text{ MeV}$. See also point 9. below.

comment on the theoretical uncertainties of these and other predictions below. The result (4) should be confronted with the data

$$\begin{aligned} a_0^0 &= 0.26 \pm 0.05 , \\ a_0^0 - a_0^2 &= 0.29 \pm 0.04 . \end{aligned}$$

The value for a_0^0 is from Ref. [20], whereas we have used the universal curve [21] to express a_0^2 through a_0^0 and to obtain the second relation. Finally, we recall the result for the one-loop approximation [5],

$$\begin{aligned} a_0^0 &= 0.201 \text{ (one-loop result) } , \\ a_0^0 - a_0^2 &= 0.242 \text{ (one-loop result) } . \end{aligned}$$

7. The chiral expansion of the S -wave threshold parameters contains chiral logarithms [22, 5]. At one loop, these are responsible for the bulk part of the correction to the tree-level result, if the scale in the logarithm is taken at 1 GeV [5]. At two loops, the expansion contains also squares of chiral logs [23], e.g.,

$$\begin{aligned} a_0^0 &= \frac{7x_2}{32\pi} \left\{ 1 - x_2 \left[\frac{9}{2}L + \text{analytic} \right] \right. \\ &\quad \left. + x_2^2 \left[\frac{58}{7}k_1 + \frac{96}{7}k_2 + 5k_3 + \frac{11}{2}k_4 + \frac{1697}{84} \frac{L}{16\pi^2} + \text{analytic} \right] \right. \\ &\quad \left. + O(x_2^3) \right\} , \end{aligned} \quad (5)$$

with

$$k_i(\mu) = (4I_i^r(\mu) - \gamma_i L)L ; \quad \gamma_1 = \frac{1}{3} , \quad \gamma_2 = \frac{2}{3} , \quad \gamma_3 = -\frac{1}{2} , \quad \gamma_4 = 2 . \quad (6)$$

The coefficients of the k_i 's in the threshold parameters, in particular in Eq. (5), have been evaluated earlier in Ref. [23] by means of renormalization group techniques [1]. Evaluating the expressions at the scale $\mu = 1$ GeV gives

$$\begin{aligned} a_0^0 &= \overbrace{0.156}^{\text{tree}} + \overbrace{0.039 + 0.005}^{\text{1 loop}} + \overbrace{0.013 + 0.003 + 0.001}^{\text{2 loops}} = \overbrace{0.217}^{\text{total}} , \\ &\quad \quad \quad L \quad \text{anal.} \quad k_i \quad L \quad \text{anal.} \\ a_0^0 - a_0^2 &= 0.201 + 0.036 + 0.006 + 0.012 + 0.003 + 0.001 = 0.258 . \\ &\quad \quad \quad L \quad \text{anal.} \quad k_i \quad L \quad \text{anal.} \end{aligned}$$

We conclude that the nonanalytic terms also dominate the two-loop corrections in this case.

8. In Fig. 1, we display the phase shift difference $\delta_0^0 - \delta_1^1$ (in degrees) as a function of the center of mass energy $E_{\pi\pi} = M_\pi \sqrt{s}$. The dotted (dash-dotted) line stands for the Born term (one-loop result), whereas the solid line shows the result at two-loop

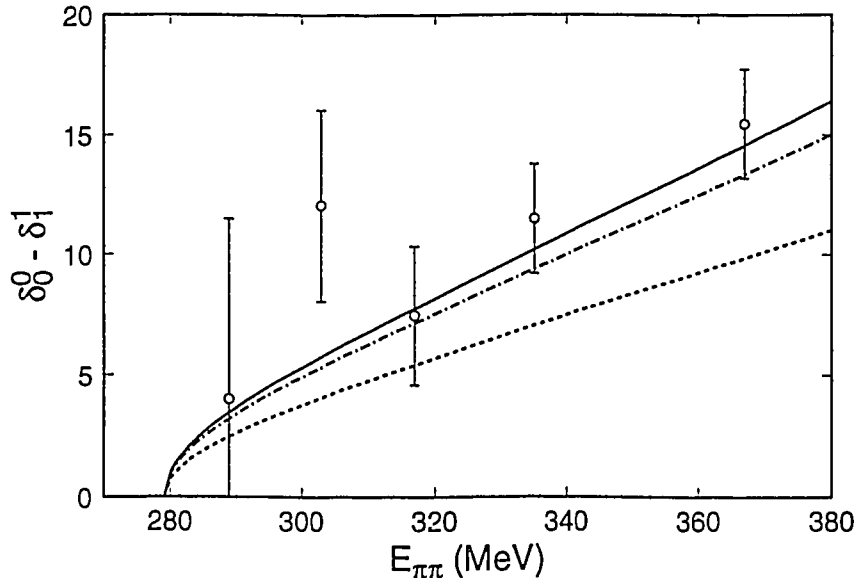


Figure 1: The phase shift difference $\delta_0^0 - \delta_1^1$ (in degrees) as a function of the center of mass energy of the two incoming pions. The dotted (dash-dotted) line displays the tree (one-loop) result, whereas the solid line denotes the two-loop approximation, evaluated with $r_i^r(1 \text{ GeV}) = 0$. The data are from Ref. [24].

accuracy, evaluated with $r_i^r(1 \text{ GeV}) = 0$. It is seen that the two-loop corrections are reasonably small also considerably above threshold. Using the couplings r_i^r estimated in the above described manner increases the two-loop result at 380 MeV by 0.4 degrees.

9. In summary, we have presented the analytic expression for the elastic $\pi\pi$ scattering amplitude to two-loop accuracy. In order to estimate numerically the size of the two-loop contributions, we have determined the new couplings that occur at this order by saturating them with the polynomial contributions to the amplitude generated by heavy states. In the case of the threshold parameters a_0^0 and a_0^2 , we then find that i) the new couplings are numerically negligible, ii) the bulk part of the correction is due to the presence of chiral logarithms, and iii) there is no manifestation of a strong enhancement of the two-loop contributions in this case.

A more reliable numerical exploitation of the representation (2), in particular the evaluation of the remaining uncertainties in the predictions, requires additional work: i) As we mentioned above, the quark mass effects in the determination of l_i^r must be investigated. ii) In view of the accuracy aimed at in future experiments [8, 11], isospin breaking effects cannot be neglected any further. For example, electroweak radiative corrections must be properly taken into account for extracting F_π [25]. To illustrate, using $F_\pi = 92.4 \text{ MeV}$ [26] instead of $F_\pi = 93.2 \text{ MeV}$ [27] increases the values (4) for a_0^0 ($a_0^0 - a_0^2$) by 0.005 (0.006). If we use the neutral pion mass instead of M_{π^+} , the effect goes in the opposite direction. In addition, real and virtual photon emission in the scattering process should be investigated. We defer these and related issues to a future publication.

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A Loop functions

Let

$$h(s) = \frac{1}{N\sqrt{z}} \log \frac{\sqrt{z}-1}{\sqrt{z}+1} \quad , \quad z = 1 - \frac{4}{s} \quad , \quad N = 16\pi^2 \quad .$$

Using matrix notation, the loop functions used in the text are given by

$$\begin{pmatrix} \bar{J} \\ K_1 \\ K_2 \\ K_3 \end{pmatrix} = \begin{pmatrix} 0 & 0 & z & -4N \\ 0 & z & 0 & 0 \\ 0 & z^2 & 0 & 8 \\ Nz s^{-1} & 0 & \pi^2(Ns)^{-1} & \pi^2 \end{pmatrix} \begin{pmatrix} h^3 \\ h^2 \\ h \\ -(2N^2)^{-1} \end{pmatrix} \quad ,$$

and

$$K_4 = \frac{1}{sz} \left(\frac{1}{2}K_1 + \frac{1}{3}K_3 + \frac{1}{N}\bar{J} + \frac{(\pi^2 - 6)s}{12N^2} \right) \quad .$$

The functions $s^{-1}\bar{J}$ and $s^{-1}K_i$ are analytic in the complex s -plane (cut along the positive real axis for $s \geq 4$), and they vanish as $|s|$ tends to infinity. Their real and imaginary parts are continuous at $s = 4$. The combination $NK_i(s)$ is denoted by $\bar{K}_i(s)$ in Ref. [13].

B The coefficients b_1, \dots, b_6

The quantities b_i in Eqs. (2) and (3) stand for

$$\begin{aligned} b_1 &= 8l_1^r + 2l_3^r - 2l_4^r + \frac{7}{6}L + \frac{1}{16\pi^2} \frac{13}{18} \\ &+ x_2 \left\{ \frac{1}{16\pi^2} \left[\frac{56}{9}l_1^r + \frac{80}{9}l_2^r + 15l_3^r + \frac{26}{9}l_4^r + \frac{47}{108}L - \frac{17}{216} + \frac{1}{16\pi^2} \frac{3509}{1296} \right] \right. \\ &\quad \left. + \frac{1}{6} [4k_1 + 28k_2 - 6k_3 + 13k_4] + [32l_1^r + 12l_3^r - 5l_4^r] l_4^r - 8l_3^r + r_1^r \right\} \quad , \\ b_2 &= -8l_1^r + 2l_4^r - \frac{2}{3}L - \frac{1}{16\pi^2} \frac{2}{9} \end{aligned}$$

$$\begin{aligned}
& + x_2 \left\{ \frac{1}{16\pi^2} \left[-24l_1^r - \frac{166}{9}l_2^r - 18l_3^r - \frac{8}{9}l_4^r - \frac{203}{54}L + \frac{317}{3456} - \frac{1}{16\pi^2} \frac{1789}{432} \right] \right. \\
& \quad \left. - \frac{1}{6} [54k_1 + 62k_2 + 15k_3 + 10k_4] - [32l_1^r + 4l_3^r - 5l_4^r] l_4^r + r_2^r \right\} , \\
b_3 & = 2l_1^r + \frac{1}{2}l_2^r - \frac{1}{2}L - \frac{1}{16\pi^2} \frac{7}{12} + x_2 \left\{ \frac{1}{16\pi^2} \left[\frac{178}{9}l_1^r + \frac{38}{3}l_2^r - \frac{7}{3}l_4^r - \frac{365}{216}L \right. \right. \\
& \quad \left. \left. - \frac{311}{6912} + \frac{1}{16\pi^2} \frac{7063}{864} \right] + 2[4l_1^r + l_2^r] l_4^r + \frac{1}{6} [38k_1 + 30k_2 - 3k_4] + r_3^r \right\} , \\
b_4 & = \frac{1}{2}l_2^r - \frac{1}{6}L - \frac{1}{16\pi^2} \frac{5}{36} + x_2 \left\{ \frac{1}{16\pi^2} \left[\frac{10}{9}l_1^r + \frac{4}{9}l_2^r - \frac{5}{9}l_4^r + \frac{47}{216}L \right. \right. \\
& \quad \left. \left. + \frac{17}{3456} + \frac{1}{16\pi^2} \frac{1655}{2592} \right] + 2l_2^r l_4^r - \frac{1}{6} [k_1 + 4k_2 + k_4] + r_4^r \right\} , \\
b_5 & = \frac{1}{16\pi^2} \left[-\frac{31}{6}l_1^r - \frac{145}{36}l_2^r + \frac{625}{288}L + \frac{7}{864} - \frac{1}{16\pi^2} \frac{66029}{20736} \right] - \frac{21}{16}k_1 - \frac{107}{96}k_2 + r_5^r , \\
b_6 & = \frac{1}{16\pi^2} \left[-\frac{7}{18}l_1^r - \frac{35}{36}l_2^r + \frac{257}{864}L + \frac{1}{432} - \frac{1}{16\pi^2} \frac{11375}{20736} \right] - \frac{5}{48}k_1 - \frac{25}{96}k_2 + r_6^r ,
\end{aligned} \tag{B.1}$$

with $L = \frac{1}{16\pi^2} \log \frac{M_\pi^2}{\mu^2}$, and where the k_i are defined in Eq. (6). We have denoted by l_i^r (r_i^r) the renormalized, quark mass independent couplings from \mathcal{L}_4 (\mathcal{L}_6), with [2] $\mu \frac{dl_i^r}{d\mu} = -\frac{\gamma_i}{16\pi^2}$. The scale dependences of r_i^r are fixed by the requirement $\mu \frac{dr_i^r}{d\mu} = 0$.

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