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ABOUT THE VELOCITY OPERATOR
FOR SPINNING PARTICLES
IN QUANTUM MECHANICS

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ABSTRACT – Starting from the formal expressions of the hydrodynamical (or "local") quantities employed in the applications of Clifford Algebras to quantum mechanics, we introduce —in terms of the ordinary tensorial framework— a new definition for the *field* of a generic quantity. By translating from Clifford into tensor algebra, we also propose a new (non-relativistic) velocity operator for a spin $\frac{1}{2}$ particle. This operator is the sum of the ordinary part $\vec{p}/m$ describing the *mean* motion (the motion of the center-of-mass), and of a second part associated with the so-called *zitterbewegung*, which is the spin "internal" motion observed in the center-of-mass frame. This spin component of the velocity operator is non-zero not only in the *Pauli* theoretical framework, i.e. in presence of external magnetic fields and spin precession, but also in the *Schrödinger* case, when the wave-function is a spin eigenstate. In the latter case, one gets a *decomposition* of the velocity field for the *Madelüng fluid* into two distinct parts: which constitutes the non-relativistic analogue of the Gordon decomposition for the Dirac current. We find furthermore that the zitterbewegung motion involves a velocity field which is solenoidal, and that the local angular velocity is parallel to the spin vector. In presence of a non-constant spin vector (Pauli case) we have, besides the component normal to spin present even in the Schrödinger theory, also a component of the local velocity which is parallel to the rotor of the spin vector.

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ABOUT THE VELOCITY OPERATOR FOR SPINNING PARTICLES IN QUANTUM MECHANICS*

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Abstract. Starting from the formal expressions of the hydrodynamical (or "local") quantities employed in the applications of Clifford Algebras to quantum mechanics, we introduce —in terms of the ordinary tensorial framework— a new definition for the *field* of a generic quantity. By translating from Clifford into tensor algebra, we also propose a new (non-relativistic) velocity operator for a spin $\frac{1}{2}$ particle. This operator is the sum of the ordinary part $\vec{p}/m$ describing the *mean* motion (the motion of the center-of-mass), and of a second part associated with the so-called *zitterbewegung*, which is the spin "internal" motion observed in the center-of-mass frame. This spin component of the velocity operator is non-zero not only in the *Pauli* theoretical framework, i.e. in presence of external magnetic fields and spin precession, but also in the *Schrödinger* case, when the wave-function is a spin eigenstate. In the latter case, one gets a *decomposition* of the velocity field for the *Madelüng fluid* into two distinct parts: which constitutes the non-relativistic analogue of the Gordon decomposition for the Dirac current. We find furthermore that the zitterbewegung motion involves a velocity field which is solenoidal, and that the local angular velocity is parallel to the spin vector. In presence of a non-constant spin vector (Pauli case) we have, besides the component normal to spin present even in the Schrödinger theory, also a component of the local velocity which is parallel to the rotor of the spin vector.

Key words: ICTE-1995, LaTeX, spin, zitterbewegung, velocity field, velocity operator, bilinear operators, Real Pauli Algebra, tensor algebra, Gordon decomposition, Madelüng fluid, hydrodynamical picture, Dirac current, Pauli current, Schroedinger theory, Asim O. Barut.

1. Hydrodynamical Observables in Quantum Theory

The *Multivector* or *Geometric* Algebras are essentially due to the work of great mathematicians of the nineteenth century as Hamilton (1805–1863), Grassmann (1809–1877) and mainly Clifford (1845–1879). More recently, starting from the sixties, some authors, and in particular Hestenes,[1-3] did sistematically study various interesting physical applications of such algebras, and especially of the Real Dirac Algebra $\mathbb{R}_{1,3}$, often renamed *Space-Time Algebra* (STA).[4-6] Rather interesting

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appear, in microphysics, the *applications* to space-time [O(3), Lorentz] transformations, to gauge [SU(2), SU(5), strong and electroweak isospin] transformations, to chiral [$SU(2)_L$] transformations, to the Maxwell equations, magnetic monopoles,[7] and so on. But the most rich and rigorous application is probably the formal and conceptual analysis of the geometrical, kinematical and hydrodynamical content of the Pauli and Dirac equations, performed by means of the Real Pauli and Real Dirac Algebras, respectively. We shall refer ourselves here to non-relativistic physics, and therefore adopt the Real Pauli Algebra, which is known to be *isomorphic to the ordinary tensorial algebra of the so-called Pauli matrices* [SU(2)]. In this paper, when ambiguities arise, the operators will be distinguished by a cap.

As well-known, in the usual *hydrodynamical picture of fluids*, every physical quantity depends not only on time, but also on the considered space point. In other words, every quantity is a *local* or *field* quantity:

$$G \equiv G(x) \qquad x = (t; \mathbf{x}) \,. \tag{1}$$

In the Pauli Algebra the *local* value of G may be expressed in the following form:

$$G(x) = \rho^{-1} \langle \Psi \hat{G} \Psi^{\dagger} \rangle_0 \,, \tag{2}$$

where $\langle\,\rangle_0$ indicates the *scalar part* of the *Clifford product* of the quantities appearing within the brackets. Let us translate eq.(2) into the ordinary tensorial language:

$$G(x) \;=\; \rho^{-1} \mathrm{Re}\,[\psi^{\dagger} \hat{G} \psi] \,. \tag{3}$$

It is easy to see, and remarkable, that this operator definition for $G(x)$ is equivalent to the *real part* of the so-called dual representation for *bilinear* operators, sometimes utilized in the literature.[8-10] In the ordinary approaches, such operators are commonly, even if *implicitly* only, employed for obtaining the probability densities of various quantities entering the Schrödinger, Klein–Gordon or Dirac wave-equations. In this sense, one can say that definition (3) agrees with the theoretical apparatus of ordinary wave-mechanics.

In connection with the first two mentioned wave-equations (and in the Dirac case, when confining ourselves to the translational–convective part of the well-known Gordon decomposition[11]), the energy density may be put into the following form:

$$\frac{i\hbar}{2} [\psi^{\star}(\partial_t \psi) - (\partial_t \psi^{\star})\psi] \;\equiv\; \frac{1}{2} \psi^{\star} i\hbar \overleftrightarrow{\partial_t} \psi \,, \tag{4}$$

as easily obtained from eq.(3) for the hamiltonian $\hat{G} \equiv H \equiv i\hbar\partial_t$. Analogously, for the current density one can write:

$$-\frac{i\hbar}{2m} [\psi^{\star}(\nabla \psi) - (\nabla \psi^{\star})\psi] \;\equiv\; \frac{1}{2m} \psi^{\star}(-i\hbar \overleftrightarrow{\nabla})\psi \,, \tag{5}$$

as required by eq.(3) if $G \equiv p/m$; $\hat{G} \equiv -i\hbar\nabla/m$. Therefore, the use of the bilinear operators $\overleftrightarrow{\partial_t}$ and $\overleftrightarrow{\nabla}$ does allow us to write the above densities in the form expected for quantum-mechanical densities:[9] namely, in the form $\psi^{\star} \hat{X} \psi$.

Let us notice that, even if $\widehat{G} \neq \widehat{G}^{\dagger}$ [non-hermiticity], quantity $G(x)$ computed by means of eq.(3) will be always *real*. The only difference with respect to the case of hermitian operators is that the mean value $< G >$ and the eigenvalues G_i will not be real; but this does *not* necessarily mean that G is unobservable. From the very definition of eigenstate in quantum mechanics, in fact, an eigenstate of $\widehat{G}$ from a "local point of view" is characterized by a function $G(x)$ *uniformly distributed (spatially homogeneous and constant in time)*: $G(x) = G_i$ for any x. Then, one can conclude[1] that necessary condition for the hermiticity of G, and the consequent existence of real eigenvalues G_i, is the possibility of creating and observing a *uniform distribution for quantity* G in corrispondence with the chosen eigenstates. The inverse does not hold: it is possible to have real and locally uniform quantities not corresponding to hermitian operators. A noticeable example of this occurrence is given by the non-hermitian *velocity operator* proposed below. In spite of its non-hermiticity, we shall see for *plane waves* [p =constant] that its non-hermitian part will give no contribution, so that the velocity field will be real, locally uniform and equal to p/m.

2. The new non-relativistic Velocity Operator endowed with Zitterbewegung

In the framework of the Pauli geometric algebra, the local velocity is obtained from the usual operator $\widehat{p}/m$, once it is "translated" into the new algebraic language. Thus we shall have, following the standard rules for that translation,

$$ih \longrightarrow \mathbf{i}2s\,, \tag{6}$$

where s represents the *spin vector* and $\mathbf{i}$ the Pauli algebra *pseudoscalar unity* (which corresponds to the matrix $\sigma_x\sigma_y\sigma_z$, so that $\mathbf{i}^2 = -\mathbb{1}$). Therefore, we can write for the velocity field:

$$v(x) = -\frac{1}{m}\,\langle \psi \nabla \mathbf{i}\sigma\psi^{\dagger}\rangle_0\,. \tag{7}$$

The "tensorial version" of this expression is the velocity operator:[#1]

$$\widehat{v} = \frac{1}{m}\sigma\,(\widehat{p}\cdot\sigma)\,, \tag{8}$$

where σ indicates the usual 2×2 Pauli matrices (hereafter we shall work in the tensorial formalism). Due to the mathematical identity

$$\sigma\,(a\cdot\sigma) \equiv a + i\,a\wedge\sigma\,, \tag{9}$$

a being a generic 3-vector, we shall finally get:

$$\widehat{v} = \frac{\widehat{p}}{m} + \frac{i}{m}\,(\widehat{p}\wedge\sigma) \equiv -\frac{ih}{m}\,\nabla + \frac{\hbar}{m}\,(\nabla\wedge\sigma)\,, \tag{10}$$

where $\widehat{p}$ and σ commute, making ininfluent the order in which they appear in the product.

[#1] Let us recall that, with regard to the *vectorial* basis $\vec{\sigma}^1, \vec{\sigma}^2, \vec{\sigma}^3$ of the Pauli multivector algebra, we have by definition: $\vec{p} \equiv \vec{\sigma}^i p_i$, indicating by p_i the i-th component of vector $\vec{p}$.

The above operator results to be composed by a hermitian part, $\widehat{p}/m$, and by a non-hermitian part, $i\,(\widehat{p} \wedge \sigma)/m$. The hermitian part reduces to the ordinary (but "incomplete": see below) non-relativistic operator for wave-mechanics, usually written as $i[\widehat{H}\,,\,\widehat{x}]/h \equiv i[\widehat{p}^2/2m\,,\,\widehat{x}]/\hbar$. The non-hermitian part is strictly related to the so-called *zitterbewegung* (zbw),[12-14] which is the spin motion, or "internal" motion —since it is observed in the CMF,— expected to exist for spinning particles. Such a motion of an internal "constituent" $\mathcal{Q}$ appears only for particles endowed with spin,[13, 14] and is to be added to the drift–translational, or "external", motion of the CM, p/m (which is the only one occurring for scalar particles). In the Dirac theory, indeed, the operators $\widehat{v}$ and $\widehat{p}$ are in general not parallel:

$$\widehat{v} \neq \widehat{p}/m\,. \tag{11}$$

Moreover, while $[\widehat{p}, \widehat{H}] = 0$ so that p is a conserved quantity, v is *not* a constant of the motion: $[\widehat{v}, \widehat{H}] \neq 0$ (quantity $\widehat{v} \equiv \alpha \equiv \gamma^0\gamma$ being the usual vector matrix of the Dirac theory). Let us notice that in case of zbw it is highly convenient[12-14] to split the motion variables as follows (the dot meaning derivation with respect to time):

$$\widehat{x} \equiv \widehat{\xi} + \widehat{X}\,; \quad \dot{\widehat{x}} \equiv \widehat{v} = \widehat{w} + \widehat{V}\,, \tag{12}$$

where $\widehat{\xi}$ and $\widehat{w} \equiv \dot{\widehat{\xi}}$ describe the motion of the CM, whilst $\widehat{X}$ and $\widehat{V} \equiv \dot{\widehat{X}}$ describe the zbw motion. From an electrodynamical point of view, the conserved electric current is associated with the trajectories of $\mathcal{Q}$ (i.e., with $\widehat{x}$), whilst the center of the particle Coulomb field —obtained[15] via a time average over the field produced by the quickly oscillating charge— coincides with the particle CM (i.e., with $\widehat{w}$) and therefore, for free particles, with the geometrical center of the helical trajectory.[#2]

As a consequence, it is $\mathcal{Q}$ which performs the *total motion*, while the CM undergoes the *mean motion* only. The resulting electron can be regarded as "extended-like",[13, 14] because of the existence in the CMF of an internal spin motion.

As required by eq.(11), one has to assume the existence of zbw also in the standard Dirac theory. In fact, the above decomposition for the total motion comes out in two well-known relativistic quantum–mechanical procedures: namely, in the *Gordon decomposition* of the Dirac current, and in the *decomposition of the Dirac velocity operator and Dirac position operator* proposed by Schrödinger in his pioneering works.[16]

The Gordon decomposition of the Dirac current reads (hereafter we shall choose units such that $c = 1$):

$$\bar{\psi}\gamma^\mu\psi = \frac{1}{2m}\,[\bar{\psi}\widehat{p}^\mu\psi - (\widehat{p}^\mu\bar{\psi})\psi] - \frac{i}{m}\widehat{p}_\nu\,(\bar{\psi}S^{\mu\nu}\psi)\,, \tag{13}$$

$\bar{\psi}$ being the "adjoint" spinor of ψ; quantity $\widehat{p}^\mu \equiv i\partial^\mu$ the 4-dimensional impulse operator; and $S^{\mu\nu} \equiv \frac{i}{4}\,(\gamma^\mu\gamma^\nu - \gamma^\nu\gamma^\mu)$ the spin-tensor operator. The ordinary

[#2] As shown in refs.[14], solutions of the Dirac equations exist for which the zbw disappears while spin is still present! Such solutions correspond however to the limiting cases of plane waves ($\vec{p}$-eigenfunctions). Moreover, even when the helical motion goes into a straight motion, the Frenet tedrad still rotates, denouncing an internal motion even in this limiting situation.

interpretation of eq.(13) is in total analogy with the decomposition given in eq.(10). The first term in the r.h.s. of eq.(13) results to be associated with the translational motion of the CM (the *scalar* part of the current, corrisponding to the traditional Klein–Gordon current). By contrast, the second term in the r.h.s. results related to the existence of spin, and describes the zbw motion.

In the above quoted papers, Schrödinger started from the Heisenberg equation for the time evolution of the acceleration operator in Dirac theory

$$\hat{a} \equiv \frac{\mathrm{d}\hat{v}}{\mathrm{d}t} = \frac{i}{\hbar}[\hat{H}, \hat{v}] = \frac{i}{\hbar} 2(\hat{H}\hat{v} - \hat{p}) ; \tag{14}$$

where $\hat{H}$ is equal as usual to $\hat{v}\cdot\hat{p} + \beta m$ (where $\hat{v} \equiv \alpha$). By integrating once this operator equation over time, he obtained:

$$\hat{v} = \hat{H}^{-1}\hat{p} + \hat{\eta}(0)\,\mathrm{e}^{-2i\hat{H}t/\hbar} \qquad (\hat{\eta} \equiv \hat{v} - \hat{H}^{-1}\hat{p}) . \tag{15}$$

After a little algebra, we may get a more interesting form for the velocity decomposition:

$$\hat{v} = \hat{H}^{-1}\hat{p} - \frac{1}{2} i\hbar\, \hat{H}^{-1}\hat{a} . \tag{16}$$

By integrating a second time, Schrödinger ended up also with the spatial part of the decomposition:

$$\hat{x} \equiv \hat{\xi} + \widehat{X} , \tag{17}$$

where we have

$$\hat{\xi} = \hat{r} + \hat{H}^{-1}\hat{p}t , \tag{18}$$

linked to the motion of the CM, and

$$\widehat{X} = \frac{1}{2} i\hbar\, \hat{\eta}\hat{H}^{-1} , \tag{19}$$

linked to the zbw motion.

We can therefore consider decomposition (10) of our velocity operator as the *non-relativistic analogue of decomposition* (16) *of the relativistic velocity operator*. It is not at all surprising that (besides spin and the related intrinsic magnetic moment) also another "spin effect", the zbw, does not vanish in the non-relativistic limit, i.e., for small velocities of the CM [$p \longrightarrow 0$]. Therefore also the Schrödinger electron, being endowed with a zbw motion, does actually show its spinning nature, and is not a "scalar" particle (as often assumed)! As a matter of fact, when constructing atoms, we have necessarily to introduce "by hand" the Pauli exclusion principle; and in the Schrödinger equation the Planck constant $\hbar$ implicitly denounces the presence of spin. In ref.[17] we have proved that the non-hermitian (zbw) part of our velocity field gives origin to the *quantum potential* of the Madelüng fluid, as well as to the related *zero-point energy* of the Schrödinger theory.

3. On the Velocity Field of the Madelüng Fluid

Even if the above operator $\hat{v}$ is not hermitian, the local velocity $v(x)$, as said above, will result to be always a real quantity. Let us see it by inserting eq.(10) into definition (3).

A spinning non-relativistic particle can be represented by means of a Pauli 2-components spinor:

$$\psi \equiv \sqrt{\rho}\,\Phi\,, \tag{20}$$

where, if we require $|\psi|^2 = \rho$, quantity Φ must obey the normalization constraint

$$\Phi^\dagger\Phi = 1\,. \tag{21}$$

Inserting the factorization (20) into definition (3), we have:

$$\rho s \equiv \mathrm{Re}\,\{\psi^\dagger \frac{\hbar}{2}\sigma\psi\} \equiv \rho\,\Phi^\dagger \frac{\hbar}{2}\sigma\Phi\,, \tag{22}$$

if G is the spin vector; and

$$\rho p \equiv \mathrm{Re}\,\{\psi^\dagger(-i\hbar\nabla)\psi\} = \frac{i\hbar\rho}{2m}[(\nabla\Phi^\dagger)\Phi - \Phi^\dagger\nabla\Phi]\,, \tag{23}$$

if G is the impulse.

In the most general case (*Pauli* generalization of Schrödinger theory), when it is present an external potential $\boldsymbol{A} \neq 0$, we have to replace in the translational term of expression (10), as a "minimal prescription", the canonic impulse $\widehat{\boldsymbol{p}}$ by the kinetic impulse $\widehat{\boldsymbol{p}} - e\boldsymbol{A}$; where e is the particle electric charge. Let us substitute in eq.(3) this "generalized" velocity operator for G, and factorization (20) for ψ: we finally get the following decomposition for the velocity field of the non-relativistic quantum fluid:

$$\boldsymbol{v} = \frac{\boldsymbol{p} - e\boldsymbol{A}}{m} + \frac{\nabla\wedge(\rho\boldsymbol{s})}{m\rho}\,. \tag{24}$$

This expression may be considered as the non-relativistic analogue of the Gordon decomposition (13) of the Dirac current.

We want to stress that decomposition (24), just now derived by means of definition (3), may be also obtained within standard wave-mechanics: and this will be a further test of the validiy of our operator. It is sufficient, in fact, to take the familiar expression of the Pauli current (i.e., the non-relativistic limit of the Gordon decomposition[18])

$$\boldsymbol{j} \equiv \rho\boldsymbol{v} = \frac{ih}{2m}[(\nabla\psi^\dagger)\psi - \psi^\dagger\nabla\psi] - \frac{e\boldsymbol{A}}{m}\psi^\dagger\psi + \frac{h}{2m}\nabla\wedge(\psi^\dagger\boldsymbol{\sigma}\psi)\,, \tag{25}$$

and to insert into it factorization (20) in place of ψ, for obtaining the velocity distribution given by eq.(24).

Let us single out in the total velocity field the zbw-component:

$$\boldsymbol{V} \equiv \frac{\nabla\wedge(\rho\boldsymbol{s})}{m\rho}\,. \tag{26}$$

It is furtherly decomposable in two distinct parts:

A) $\boldsymbol{V}_1 \equiv (\nabla\rho\wedge\boldsymbol{s})/m\rho$

due to the presence of the gradient of ρ, this term refers to local motions, in which

the constant density surfaces [$\nabla\rho = 0$] do rotate around the spin axis; and it vanishes identically for the plane waves [$\boldsymbol{p}$ = constant], for which $\nabla\rho = 0$;

B) $\boldsymbol{V}_2 \equiv (\mathrm{rot}\, \boldsymbol{s})/m$

such a term does not depend on the density ρ and is different from 0 only in presence of external magnetic fields (the spin vector precedes, and the wave-functions are *not* spin eigenstates).

The *Schrödinger theory* is of course a particular case in the present Pauli framework, corresponding to $\boldsymbol{A} = 0$ (and then to zero magnetic field) and to a *uniform local spin vector* with no precession. The wave-function is then a *spin eigenstate* and may be factorized as the product of a "scalar" part $\sqrt{\rho}e^{i\varphi}$ and of a "spin" part χ (a 2-components spinor):

$$\psi \equiv \sqrt{\rho}\, e^{i\frac{\varphi}{\hbar}}\chi\,, \tag{27}$$

quantity χ being *constant* in space and time. Let us underline that, even if $\boldsymbol{s} = \chi^{\dagger}\frac{\hbar}{2}\boldsymbol{\sigma}\chi$ = constant, *in the Schrödinger case the zbw does not vanish* —except for the unrealistic case of plane waves—, while the velocity $\boldsymbol{V}$ reduces to

$$\boldsymbol{V} = \boldsymbol{V}_1 \equiv \frac{(\nabla\rho \wedge \boldsymbol{s})}{m\rho}\,. \tag{28}$$

All this does actually contribute to remove some difficulties remaining in the classical representation of the particle motion, and in the interpretation of the particle energies, for some stationary solutions of the Schrödinger equation. For instance, let us refer ourselves to the stationary states of a particle inside a box or of the harmonic oscillator, or the $l = 0$ eigenstates of the hydrogen atom. In all these cases the wave-function results to be *real* (φ is uniform and equal to a constant which for "global gauge invariance" may be assumed equal to zero), and therefore *the velocity obtained from standard quantum mechanics* $\boldsymbol{p}/m \equiv \nabla\varphi/m$ *is zero* everywhere, at any time. As it was first remarked by Einstein and Perrin and by de Broglie,[19] such a result *seems to be really in contrast* from a cassical point of view *with the non-vanishing of the energy eigenvalue for those stationary states.* But we now know, from our previous analysis, that $\boldsymbol{p}/m$ is the *mean* velocity, describing only the motion of the CM, whilst the zbw-component $\boldsymbol{V}$ —which depends on the ρ-gradient, and not on the phase-gradient of ψ— does not identically vanish, thus implying an internal motion around the spin axis.

Let us finally analyse the velocity distribution (24) for $\boldsymbol{A} = 0$, and stress some interesting properties of its.

Since the rotor of a gradient is identically zero, we shall have $\mathrm{rot}\,\boldsymbol{p} = 0$; and since $\boldsymbol{s}$ is furthermore uniform, it will also be $\nabla\boldsymbol{s} = 0$. As a consequence, by employing the known property of the double vectorial product

$$\boldsymbol{a} \wedge (\boldsymbol{b} \wedge \boldsymbol{c}) = (\boldsymbol{a}\cdot\boldsymbol{c})\,\boldsymbol{b} - (\boldsymbol{a}\cdot\boldsymbol{b})\,\boldsymbol{c}\,, \tag{29}$$

we can show that the local rotational properties of the Madelüng fluid are actually given by the following expression:

$$\mathrm{rot}\, \boldsymbol{v} = \frac{1}{m}\left[\left(\frac{\nabla\rho}{\rho}\right)^2 - \frac{\Delta\rho}{\rho}\right]\boldsymbol{s}\,. \tag{30}$$

Moreover, the zbw field V results to be *solenoidal* (and this happens also in the most general case of the Pauli fluid with a non-uniform s):

$$\operatorname{div} V \equiv \operatorname{div}[\operatorname{rot}(\rho s)] = 0 . \tag{31}$$

The flux stream-lines will be *closed lines*, as in the magnetic field case: therefore, we expect *the zbw to be a limited, finite, periodical motion.* The explicit calculations performed by us for the known Barut–Zanghi model[13,14] did really lead in the CMF of the electron to closed periodical motions. The total motion of the system will be helical around the p-direction. We can see eventually that the *local angular velocity ω is parallel to the spin vector s*:

$$\omega = \frac{1}{2} \operatorname{rot} v = \frac{1}{2m} \left[\left(\frac{\nabla \rho}{\rho} \right)^2 - \frac{\Delta \rho}{\rho} \right] s . \tag{32}$$

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