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Bound States of Quarks Calculated With a Stochastic Integration of the Bethe-Salpeter Equation

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Abstract

We have computed the masses, wave functions and sea quark content of mesons in their ground state by integrating the Bethe-Salpeter equation with a stochastic algorithm. This method allows the inclusion of a large set of diagrams. Inspection of the kernel of the equation shows that q-$\bar{q}$ pairs with similar constituent masses in a singlet spin state exhibit a highly bound state which is not present in other pairs. The pion, kaon and eta belongs to this category.

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I. Introduction

Potential models of hadrons composed of constituent quarks[1] have been very successful in predicting many properties of mesons and baryons. This indicates that the concept of "physical" or constituent quarks is valid and useful. On the other hand, lattice calculations[2] with bare quarks as well as bag models[3] have also been successful in obtaining hadron properties, and represent alternative methods to determine the properties of QCD bound states.

A fourth possibility for obtaining these physical observable as well as wave functions is the integration of the Bethe-Salpeter[4] (BS) equation. This has also been realized[5-15], although analytical approaches have always been hampered by the complexity of the equation.

Recently we proposed[16] a novel method to solve the BS equation using a stochastic algorithm. Although lacking in the generality and accuracy of analytical solutions it has the advantage of allowing the inclusion of many diagrams, as well as providing a complete relativistic formulation for any value of the coupling constant.

In this paper we will describe in Sec. 2 the formalism mentioned above for the case of a bound quark-antiquark pair. In Sec. 3 we establish the parameters used in the evaluation, the general method of integration and the accuracy of the results. These are listed in Sec. 4. Conclusions and final remarks are presented in Sec. 5.

II. The Formalism

The BS equation can be used to express in a covariant formalism (in momentum space) the wave function of bound states of particles. In a compact form this equation reads:

$$F(a,b,...) = \sum \int K(a,b,...;a',b',...)F(a'b'...)\frac{da'}{(2\pi)^4}\frac{db'}{(2\pi)^4}... \qquad (1)$$

where $a, b, ...$ are the 4-momenta of quarks $a, b, ...$ and $F(a,b,...)$ is the Fock space wave function. To simplify the notation we are not indicating spin and color labels in F and K. The kernel $K(a,b,... ; a'b'...)$ of the equation is composed of the sum of the amplitudes for the different diagrams to be considered. The sum and integral include all 4-momenta, spins and color states involved. The bound system has a total 3-momentum $\mathbf{P} = \mathbf{a} + \mathbf{b} + ... = 0$ and its mass is $M = < a_0 + b_0 .... >$

In the past, this equation was solved only for simple cases i.e. the ladder approximation with and without spin, although it has spawned a large number of related equations and approximations. We found recently[16] an alternative method of integration using a simple stochastic method described in the next section. The diagrams whose amplitudes were included in K are shown in Fig. 1 and were chosen because they constitute the major contributions of all diagrams with one, two and three gluon exchanges.

Using the Feynman gauge and renormalized fields we have,

$$\delta^{kl} g_{\mu\nu}/q^2 \qquad (2)$$

for the gluon propagator, where k and l are color labels,

$$i\delta^{ij}/(\not p - m) \qquad (3)$$

for the quark propagator with constituent mass m and momentum p,

$$-ig\gamma^{\mu}(L^{a})_{,} \quad (4)$$

for the gluon-quark vertex, where L^{a} are the Gell-Mann matrices,

$$-gf_{abc}[(q_4 - q_5)_{\nu}g_{\lambda\mu} + (q_5 - q_6)_{\lambda}g_{\mu\nu} + (q_6 - q_4)_{\mu}g_{\nu\lambda}] \quad (5)$$

for the three gluon vertex.

From the above, we obtained in the usual fashion the amplitudes for the diagrams shown in Fig. 1. For instance, the first one has the following amplitude,

$$A1 = 4/3\frac{\alpha_s}{\pi}\frac{(m_a + a_0)(m_b + b_0) + Ca^2}{(a^2 - m_a^2)(b^2 - m_b^2)q^2} \quad (6)$$

where the parameter C has values 3 and 0 for the spin singlet and triplet cases respectively. In Eq. (2) we have changed the usual $-i$ factor with 1. With this change the Wick rotation is not necessary. If one compares the ground state binding energy as a function of the coupling constant α using the BS equation in the latter approximation using the Wick rotation or the phase change mentioned above, one obtains better agreement with the Dirac equation with the latter.

Several functions were tested for the momentum dependence of the running coupling constant $\alpha_s = g^2/4\pi$, including in particular the ones used in Ref. [1] and in Ref. [15], as well as a constant. We used α_s constant for the results presented here as no improvement in the fit was obtained with the other functions. Electromagnetic effects have been neglected, and diagram (c) in Fig. 1 is the sum over loops with up, down and strange quarks.

In Table I we show the input parameters used in the calculations, where the value of $\alpha_s = 0.3$ is used only for the pion.

III. Computation

A detailed description of the algorithm can be found in Ref. [16]. We have assumed spherical symmetry in the ground state wave functions, which reduces the number of independent components of all momenta. We defined in this computation the average $< q_0 >= \sum m_q - M$. This ansatz has the appropriate small coupling limit, the correct relativistic rise in the binding energy for QED and simplifies the integration. Initial values for F and the meson mass M were assumed and the integrand of Eq. (1) was evaluated using m trials in which random numbers with uniform distribution were assigned to the momenta. In this fashion an approximate value of the integral was obtained, which in turn produced a new F. The deviation from unity of the integral of F was then used to obtain a new value of M. This process was repeated n times, and the final F and M were the averages of these intermediate values.

To test the algorithm, its convergence speed and accuracy we used it for QED and compared it with the Dirac equation for several values of α_s (the last diagram of Fig. 1 was excluded and $m_a << m_b$). This comparison showed that the method required values of m = 20,000 and n =20 for an accuracy of M smaller than 5%. Triplet spin states were somewhat more accurate. Larger values of α_s required larger m and n as the fluctuations in F and M increased.

In the computation there are two implicit cut-offs. Small values of q are restricted by the smallest possible angle between **p** and **p'**, and the largest values of **p** are given by a parameter T (where $T > 4\ m_q\alpha_s$). We tested extensively the dependence of the final results with T, and within a large margin of values the results are independent of this parameter. The quality of the integration can be tested also by comparing results with different sets of random numbers, and by the final total average of $\sum F$ which has to be 1.

IV. Results

Using the parameters indicated in Table I we obtained the masses for the ground states of 18 mesons. The quark masses were found by optimizing the results in a search procedure, although a global fit was not performed. The values of the masses could be improved further with a global least square fit and larger values of m and n, as well as a larger set of diagrams. Apart from the masses we also obtained the wave functions and the sea quark contents of these states. The results are summarized in Table II, where the A terms are the integrated amplitudes of the diagram (c) for the up, down and strange quarks.

An unexpected result was the case in which both quarks had similar masses and were in a singlet spin state. Then the system supports a deeply bound state (Goldstone boson) with a value of α_s for the pion exactly half of the other states. In this case the binding energy is larger than m_q and smaller than $2m_q$. In Fig. 2 we show the unnormalized wave functions for the $\pi^{\pm}$ (140) and the $D^{\pm}$ (1869).

The statistical uncertainty in the calculation of the masses in Table II is 4%, and the uncertainty in the amplitudes of u or d sea quark pairs $A(u,d)$ as well as s quark pairs $A(s)$ is 8%. The statistical uncertainties of the F are 15%.

Using the $\pi^{\pm}$ wave function we calculated the rms radius and obtained $< r >=$ 0.63 fm in good agreement with experimental results[17] of $< r_\pi > = (0.65 \pm 0.03)$ fm.

V. Conclusions

We have presented an algorithm with which it is possible to integrate the covariant BS equation for a large number of variables, opening the possibility of including many diagrams and more than two particles. The physical interpretation of the method is akin to following the evolution of a system of fermions bound by the exchange of gluons.

The formalism was used to compute the masses of mesons using the five largest amplitudes involving the exchange of one, two and three gluons.

Within the accuracy of the computation the results obtained agree with the experimental values. The pion, kaon and eta wave function are obtained as deeply bound states allowed only for singlet spin states of similar masses. Sea quark pairs can be included and their relative amplitude calculated.

This method can be compared with other recently developed formalisms.[18,19] The first one finds analytic solutions to potential models in momentum space, while the second one uses a spectrum generating algebra based on general symmetries. Although both can determine excited state energies the algorithm described above can be used for more than two fermions and therefore could be used to evaluate baryons.

VI. Acknowledgement

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VII. REFERENCES

[1] S. Godfrey and N. Isgur, *Phys. Rev.* **D32**, *189 (1985)*.

[2] Lattice-90, Proc. of the Intern. Conf. on Lattice Field Theories, Nucl. Phys. B20 (Proc. Suppl.) May 1991.

[3] A. Chodos *et al.*, Phys. Rev. **D9**, 3471 (1974); and A.W. Thomas, Advances in Nuclear Physics, **13**, 1 (1982).

[4] E.E. Salpeter and H.A. Bethe, Phys. Rev. **84**, 1232 (1951).

[5] V.A. Karamanov, *Nucl. Phys.* **B166**, *378 (1980)*.

[6] C.R. Ji and R.J. Furnstahl, Phys. Lett. **167B**, 11 (1986).

[7] D.R. Yennie, "Quarks and Leptons", Cargese 1979, Plenum Press.

[8] E. van Faasen and J.A. Tjon, Phys. Rev. **C33**, 2105 (1986).

[9] R.T. Cahill *et al.*, Phys. Rev. **D36**, 2804 (1987).

[10] L.S. Celenza *et al.*, Phys. Rev. **D36**, 2506 (1987).

[11] H.J. Munczek and A.M. Nemirovsky, Phys. Rev. **D28**, 181 (1983).

[12] S.J. Brodsky and G.P. Lepage, "Perturbative QCD", Ed. by A.H. Mueller, World Scientific, Singapore, 1989.

[13] K.I. Aoki *et al.*, Kyoto preprints KUNS 1044 (1990).

[14] J.H. Connell, Phys. Rev. **D43**, 1393 (1991).

[15] M.G. Mitchard, Workshop on Strong Coupling Gauge Theories, Nagoya, Japan, July 1990.

[16] M. Salomon, Phys. Rev. **A39**, 4935 (1989).

[17] S.R. Amendolia *et al.*, Nuclear Physics **B277**, 168 (1986).

[18] F. Gross *et al.*, Phys. Rev. **D43**, 2401 (1991).

[19] F. Iachello, Yale Univ. Preprint YCTP-n7-91.

Table I Input parameters used in the computation

α_s	m_u (GeV)	m_d (GeV)	m_s (GeV)	m_c (GeV)
0.6(0.3)	.375	.385	.620	1.760

Table II Results of the computation

Meson	M(exp)	M(calc)	A(u,d)	A(s)
$\pi^\pm$	.140	.140	.30	.09
π^o	.135	.138	.30	.08
$\rho^\pm$	.768	.682	.26	.09
ρ^o	.770	.672	.26	.09
η^o	.549	.645	.15	.08
$K^\pm$	.494	.505	.19	.09
K^o	.498	.509	.19	.10
$K^{*\pm}$	.892	.828	.33	.19
K^{*o}	.896	.840	.32	.18
D^o	1.865	1.853	.23	.17
D^{*o}	2.007	1.893	.25	.17
$D^\pm$	1.869	1.836	.25	.16
D^*	2.010	1.902	.23	.17
$D_s^\pm$	1.969	2.014	.16	.14
D_s^*	2.110	2.118	.17	.14
ϕ^o	1.020	1.057	.27	.17
η_c^o	2.980	3.016	.07	.07
J/ψ	3.097	3.177	.06	.08

Figure Captions

1. The diagrams whose amplitudes were included in the kernel of Eq. (1). In diagram (c) we take the sum over up,down and strange for the pair c, c'.
2. The unnormalized wave functions F of the $\pi^\pm$ (140) and the $D^\pm$ (1869).

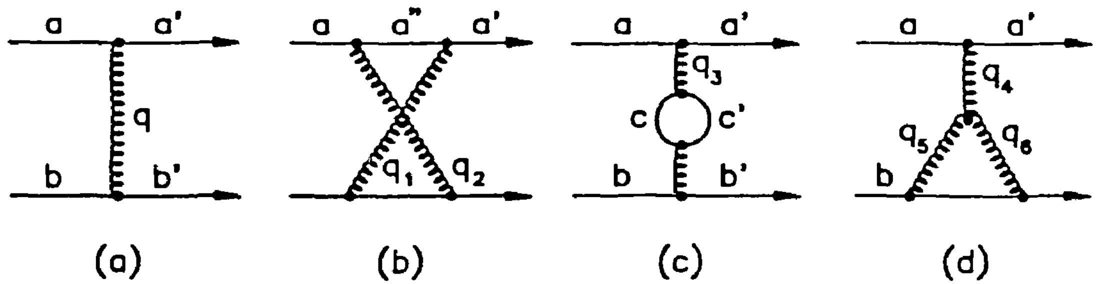
a
a'
q
b
b'
(a)
a
a"
a'
q_1
q_2
(b)
a
a'
q_3
c
c'
b
b'
(c)
a
a'
q_4
q_5
q_6
b
(d)

Fig. 1

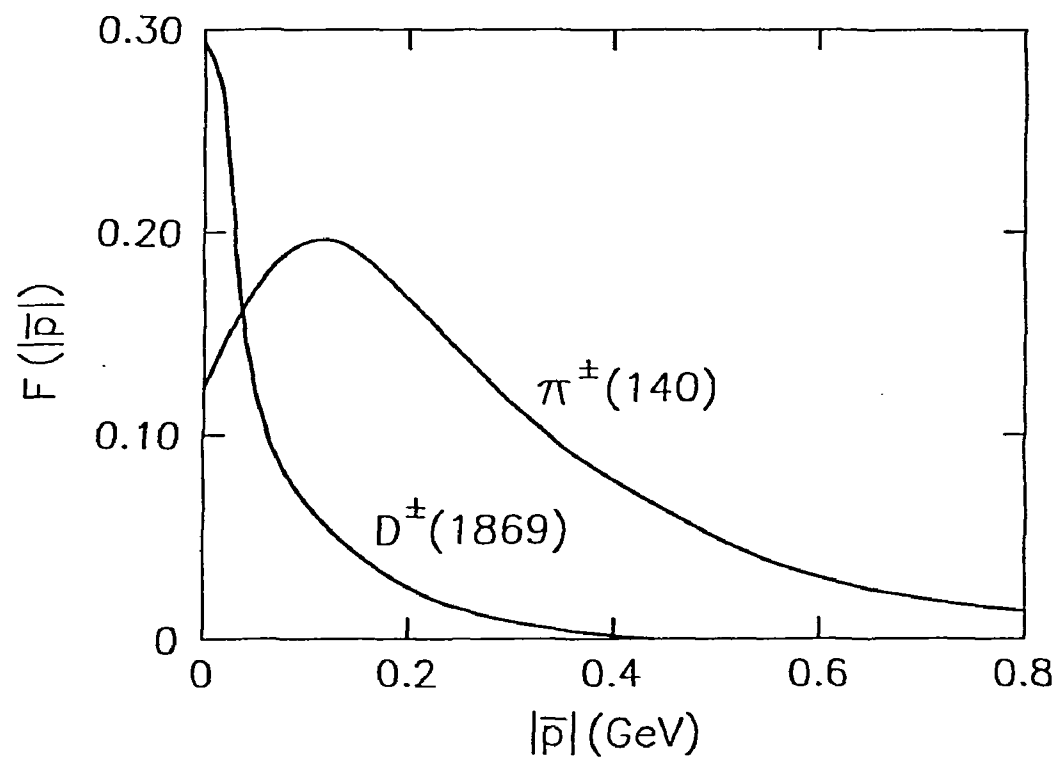
0.30
0.20
0.10
0
F (|p̄|)
$\pi^{\pm}(140)$
$D^{\pm}(1869)$
0
0.2
0.4
0.6
0.8
|p̄| (GeV)

Fig. 2

3.