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Joint Probabilities of Noncommuting Observables and the Einstein-Podolsky-Rosen Question in Wiener-Siegel Quantum Theory †

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Abstract

Ordinary quantum theory is a statistical theory without an underlying probability space. The Wiener-Siegel theory provides a probability space, defined in terms of the usual wave function and its "stochastic coordinates"; i.e., projections of its components onto differentials of complex Wiener processes. The usual probabilities of quantum theory emerge as measures of subspaces defined by inequalities on stochastic coordinates. Since each point α of the probability space is assigned values (or arbitrarily small intervals) of *all* observables, the theory gives a pseudo-classical or "hidden-variable" view in which normally forbidden concepts are allowed. Joint probabilities for values of noncommuting variables are well-defined. This paper gives a brief description of the theory, including a new generalization to incorporate spin, and reports the first concrete calculation of a joint probability for noncommuting components of spin of a single particle. Bohm's form of the Einstein-Podolsky-Rosen *Gedankenexperiment* is discussed along the lines of Carlen's paper at this Congress. It would seem that the "EPR Paradox" is avoided, since to each α the theory assigns opposite values for spin components of two particles in a singlet state, along any axis. In accordance with Bell's ideas, the price to pay for this attempt at greater theoretical detail is a disagreement with usual quantum predictions. The disagreement is computed and found to be large.

1 Introduction

The Wiener-Siegel interpretation of quantum theory was proposed in 1953 [1], and was studied and developed in a few papers during the following decade [2, 3, 4, 5, 6, 7, 8]. The state of the theory at the end of this period was summarized by Siegel in a chapter of Ref. [9], 1966. Since then there has been little activity. Among the few notices is a chapter in the book of Belinfante [10], some work of Bohm and Bub [11] adopting elements of the Wiener-Siegel viewpoint, a paper of Durdević *et al.* [12], a short review by Masani [13] in his biography of Wiener, and a recent look at the theory by Carlen [14] on the occasion of this Congress.

In my first year after graduate school in physics, 1959-60, I had the good fortune to work on this theory with Armand Siegel. It was exciting to learn about measure in function space, to imagine its future applications in physics, and to meet Prof. Wiener. As Irving Segal has told us [16], Wiener measure later played a pivotal role in defining the path integrals of quantum field theory, and a hint at that development was already present in the stochastic coordinate of the wave function introduced by Wiener and Siegel.

My work with Siegel led to a somewhat negative result [15], in that the original "dichotomic algorithm" for assigning values of observables to points in differential space was found to be untenable. This left the theory to depend on the "polychotomic algorithm", which is more satisfactory but does not give a unique assignment, except in the case of observables with discrete and finite spectra. A fully satisfactory definition of the theory will require a limiting form of the polychotomic algorithm, but the existence of the limit appears to be a difficult mathematical question involving path integrals over restricted paths.

Here I report on the calculation of a quantity that is meaningless in the accepted view of quantum theory, the joint probability for specified values of two noncommuting spin components of a single particle. The results depend only on parameters that are present in the usual theory; namely, the single probabilities for values of the two components. Of course, one does not know how to associate this joint probability with an experiment. On the other hand, in a system of two particles with spin, usual quantum theory does assign a value for a joint probability that is measurable in principle; namely, the probability for particle 1 to have a specified value for $S_1 \cdot e$ while particle 2 has a specified value of $S_2 \cdot f$, where S_i is the spin of particle i, and e and f are arbitrary unit vectors. This is possible because spin operators of particle 1 commute with those of particle 2. Since Wiener-Siegel theory assigns values of both $S_1 \cdot e$ and $S_2 \cdot f$ to each point α of the probability space, it makes its own statement about this joint probability in the usual manner of probability theory. The value it gives is different from that of the usual doctrine. As Carlen argues in Ref.[14], this situation is expected on the basis of Bell's analysis of hidden-variable theories. I try to sharpen the discussion by computing the amount of disagreement. If the angle between e and f is 120°, then for spin 1/2 particles quantum theory gives 3/8, while I find 0.2663... in Wiener-Siegel theory.

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2 Statistical postulate of ordinary quantum theory

We treat nonrelativistic quantum mechanics with spin, allowing only pure states described by a state vector ψ in a Hilbert space $\mathcal{H}$. (Masani [13] has raised the question of how to put mixed states into Wiener-Siegel theory; we avoid that topic for now.) With n particles, ψ may be realized as a wave function $\psi : \mathbb{R}^{3n} \times U^n \to \mathbb{C}$, where $U = \{1, -1\}$ is the domain of the spin index of a particle. Reference to the time dependence of ψ will be suppressed. For each spin state σ, $\psi(x,\sigma) \in \mathcal{L}^2(\mathbb{R}^{3n})$. The scalar product in $\mathcal{H}$ is

$$(\psi,\phi) = \int \sum_\sigma \bar{\psi}(x,\sigma)\phi(x,\sigma)dx \qquad (1)$$

Physical observables correspond to self-adjoint operators $\mathbf{R}$. We decompose the spectrum of $\mathbf{R}$ into disjoint sets $R_1, R_2, \ldots$, and let $P(R_i)$ be the projection operator onto the subspace of $\mathcal{H}$ corresponding to R_i. We have

$$\sum_i P(R_i) = 1 \ , \qquad P(R_i)P(R_j) = 0 \ , \ i \neq j \qquad (2)$$

The statistical postulate of quantum mechanics is this: If r is the result of a measurement of the observable corresponding to $\mathbf{R}$, then the probability that $r \in R_i$ is

$$p(r \in R_i) = (\psi, P(R_i)\psi) \ . \qquad (3)$$

The word *probability* refers to frequency of the observation $r \in R_i$ among a large number of observations on an ensemble of *identically prepared systems*. The state vector is the theoretical counterpart of the whole ensemble, not that of a single system.

We have a statistical statement of the usual kind, a prediction of a frequency, for many observations on a physical ensemble that can be prepared in the laboratory. On the other hand, the theory provides no description of this ensemble as a probability space in the mathematical sense. There is no theoretical counterpart of a single system, no candidate for a point in probability space! This striking omission provides the motivation for the Wiener-Siegel theory.

3 Stochastic coordinates of the wave function

The required probability space is derived from a complex Wiener process X on the *configuration space* $\mathbb{R}^{3n}$, one for each spin state σ. Note that X *does not depend on the time*, $X : \mathbb{R}^{3n} \times U^n \to \mathbb{C}$. Following Wiener's custom of labeling individual Brownian motions with a parameter $\alpha \in [0,1]$, we characterize the process $X(x,\sigma,\alpha) = X_1 + iX_2$ in terms of the expectation value $\mathcal{E}$ (integral over α) as follows:

$$X(0,\sigma,\alpha) = 0 \ , \qquad (4a)$$

$$\mathcal{E}[X_i(x,\sigma,\alpha)] = 0 \ , \qquad (4b)$$

$$\mathcal{E}\left[[X_i(x,\sigma,\alpha) - X_i(y,\sigma,\alpha)]\,[X_j(x,\tau,\alpha) - X_j(y,\tau,\alpha)]\right] = \delta_{ij}\delta_{\sigma\tau}m([x,y]) \ , \qquad (4c)$$

$$\mathcal{E}\left[[X_i(x,\sigma,\alpha) - X_i(y,\sigma,\alpha)]\,[X_j(u,\tau,\alpha) - X_j(v,\tau,\alpha)]\right] = 0 \qquad \text{if} \quad m([x,y] \cap [u,v]) = 0 \ , \qquad (4d)$$

where m is Lebesgue measure on $\mathbb{R}^{3n}$ and $[x,y]$ is the $3n$-dimensional interval $\{z|x_i \le z_i \le y_i\ ,\ i=1,\cdots,3n\}$; thus $m([x,y]) = \prod_{k=1}^{3n}(y_k - x_k)$. Increments of real and imaginary parts are independent, as are increments with different spin states, and general increments for non-overlapping intervals.

Wiener's stochastic integral, generalized to include spin, is written as

$$\sum_\sigma \int \bar{\psi}(x,\sigma) dX(x,\sigma,\alpha) = \langle\psi|\alpha\rangle \ . \tag{5}$$

As we shall see, the Dirac-style notation $\langle\psi|\alpha\rangle$, with $\langle\alpha|\psi\rangle = \overline{\langle\psi|\alpha\rangle}$ is convenient and heuristically suggestive. Henceforth we also use Dirac notation for the scalar product, $\langle\psi|\phi\rangle = (\psi,\phi)$. Since almost all X have unbounded variation, the integral is not a Stieltjes integral. It is defined as a limit in the mean through approximation of $\bar{\psi}$ by step functions. We divide $\mathbb{R}^{3n}$ into non-overlapping intervals $[x^i, x^{i+1}]$ by a Cartesian grid, and approximate $\psi(x,\sigma)$ by a constant $\psi_{i\sigma}$ on the i-th interval, taking $\psi_{i\sigma} = 0$ for $i > N$. For the approximate ψ, the integral is defined as

$$\sum_\sigma \sum_{i=1}^{N} \bar{\psi}_{i\sigma}[X(x^{i+1},\sigma,\alpha) - X(x^i,\sigma,\alpha)] \ , \tag{6}$$

and by (4b,4c)

$$\mathcal{E}(\langle\psi|\alpha\rangle\langle\alpha|\psi\rangle) = \sum_{i,\sigma,j,\tau} \bar{\psi}_{i\sigma}\psi_{j\tau}\mathcal{E}[[X(x^{i+1},\sigma,\alpha) - X(x^i,\sigma,\alpha)]\alpha[\bar{X}(x^{j+1},\tau,\alpha) - \bar{X}(x^j,\tau,\alpha)]] \tag{7}$$

$$= 2\sum_{i,\sigma}|\psi_{i\sigma}|^2 m([x^i,x^{i+1}]) = 2\int |\psi|^2 dx = 2\langle\psi|\psi\rangle \ . \tag{8}$$

Invoking the Riesz-Fischer theorem and the fact that the step functions are dense in $\mathcal{L}^2$, we conclude that the integral exists as a limit in the mean. Similarly,

$$\mathcal{E}(\langle\psi|\alpha\rangle\langle\alpha|\phi\rangle) = 2\langle\psi|\phi\rangle \ . \tag{9}$$

Now $\mathrm{Re}\langle\psi|\alpha\rangle$ and $\mathrm{Im}\langle\psi|\alpha\rangle$ are independent Gaussian variables, and each has unit variance if $\langle\psi|\psi\rangle = 1$. The following formulas can be used to set up the covariance matrix of a set of such variables:

$$\mathcal{E}(\mathrm{Re}\langle\psi|\alpha\rangle\mathrm{Re}\langle\phi|\alpha\rangle) = \mathcal{E}(\mathrm{Im}\langle\psi|\alpha\rangle\mathrm{Im}\langle\phi|\alpha\rangle) = \mathrm{Re}\langle\psi|\phi\rangle = \mathrm{Re}\langle\phi|\psi\rangle \ , \tag{10}$$

$$\mathcal{E}(\mathrm{Im}\langle\psi|\alpha\rangle\mathrm{Re}\langle\phi|\alpha\rangle) = \mathrm{Im}\langle\psi|\phi\rangle = -\mathrm{Im}\langle\phi|\psi\rangle \ . \tag{11}$$

Remarks: (1) The above treatment of spin seems to me an obvious choice, but it differs from that of [2]. As I recall, Siegel had rejected the method of [2] in 1959. His review in [9] avoids spin. (2) The random function $X(x,\sigma,\alpha)$ is sometimes called a *point α in differential space*; its differentials dX are the significant thing. (3) I propose the name *stochastic coordinate of the wave function* for $\langle\psi|\alpha\rangle$, since it may be thought of as the projection of $|\psi\rangle$ along the direction in differential space [9] labeled by α. (4) Siegel told me that Wiener resisted strongly the idea of making the Brownian motions independent of time. He had a vision of random motions proceeding in time, as had other creators of hidden-variable theories. In the theory finally proposed, time dependence enters only through the wave function. The time dependence can be transferred formally to the Brownian motion through the concept of *stochastic adjoint* [6].

4 Statistical postulate of Wiener-Siegel quantum theory

We decompose the spectrum of an observable $\mathbf{R}$ into a *finite* number of disjoint sets, $R_1, R_2, \cdots, R_p$. At least one R_i is unbounded if the spectrum extends to infinity, but since $\psi \in \mathcal{L}^2$ the required unbounded set can be chosen so that the quantum probability $\langle\psi|P(R_i)|\psi\rangle$ is arbitrarily small. For discrete spectra a bounded R_i would normally consist of just one point (for instance, an energy level of an atom), while for continuous spectra it would be a very small interval of an eigenvalue.

The *polychotomic algorithm* assigns a spectral set R_i to each point α in differential space. The set R_i is assigned to α if for all $j \ne i$,

$$\left|\frac{\langle\psi|P(R_i)|\alpha\rangle}{\langle\psi|P(R_i)|\psi\rangle}\right|^2 \le \left|\frac{\langle\psi|P(R_j)|\alpha\rangle}{\langle\psi|P(R_j)|\psi\rangle}\right|^2 \ . \tag{12}$$

To find the measure of the set of all α satisfying (12), define

$$x_k + iy_k = \langle\psi|P(R_k)|\alpha\rangle/a_k \ , \qquad a_k = \langle\psi|P(R_k)|\psi\rangle^{1/2} \tag{13}$$

$$r_k^2 = x_k^2 + y_k^2 \ , \qquad dx_k dy_k = r_k dr_k d\theta_k \ . \tag{14}$$

Since the x_k, y_k are Gaussian with unit variance, the required measure can be expressed through polar coordinates as the p-fold integral

$$\int_\Omega \exp(-\frac{1}{2}\sum_{k=1}^{p} r_k^2) \prod_{k=1}^{p} r_k dr_k \ , \tag{15}$$

where Ω consists of points for which $r_i^2/a_i^2 < r_j^2/a_j^2$, all $j \ne i$. This is easily computed to yield the desired value of ordinary quantum theory, $a_i^2 = \langle\psi|P(R_i)|\psi\rangle$.

We can apply the algorithm to any other observables $\mathbf{S}, \mathbf{T}, \cdots$, which need not commute with $\mathbf{R}$, and determine the spectral sets $R_i, S_i, T_i, \cdots$ that are assigned to a single α. Thus, we have a pseudo-classical description if we think of points α as corresponding to individual physical systems of an ensemble. A range Δx of position and a range Δp of momentum may be assigned to α, with $\Delta x \Delta p < \hbar$.

Unfortunately, one has to be satisfied with just one spectral decomposition, since assignments of a second decomposition would be incompatible with the first. The discovery of this situation [15] led us to abandon the *dichotomic algorithm*, which works by dividing the spectrum of $\mathbf{R}$ into just two sets. The intention was to perform a sequence of dichotomies, thereby narrowing down arbitrarily the assignment of observables to a point α. The random variables associated with a particular cut are dependent on those of a previous cut, and that turns out to spoil the argument.

A fully satisfactory and unique theory would seem to require the existence of a limit for successive polychotomies. In the case of continuous spectra the limit would have to result in a set of joint probability densities. The hoped-for limit can be expressed formally as a sort of path integral.

In the case of observables with a finite set of eigenvalues, such as spin coordinates, the polychotomic algorithm gives a unique assignment immediately. Examples are given in the following sections.

5 Joint probabilities

The joint probability for an eigenvalue of $\mathbf{R}$ to lie in R_i and an eigenvalue of $\mathbf{S}$ to lie in S_i is the measure of the set of all α such that both inequalities (12) and a similar set of inequalities for $\langle\psi|P(S_k)|\alpha\rangle$ hold. As the simplest example we take one electron with spin, with wave function $\psi(x,1) = \psi_1(x), \psi(x,-1) = \psi_2(x)$. The spin indicies ± 1 correspond to the eigenvalues $\pm\hbar/2$ of the third component of spin, S_z. The projection operators for $S_z = \pm\hbar/2, S_x = \pm\hbar/2$ will be written as $P(z\pm), P(x\pm)$. Then one finds by the usual Pauli spin algebra that

$$\langle\psi|P(z+)|\alpha\rangle = \langle\psi_1|\alpha\rangle \ , \qquad \langle\psi|P(z-)|\alpha\rangle = \langle\psi_2|\alpha\rangle \ , \tag{16}$$

$$\langle\psi|P(x+)|\alpha\rangle = \frac{1}{\sqrt{2}}[\langle\psi_1|\alpha\rangle + \langle\psi_2|\alpha\rangle] \ , \qquad \langle\psi|P(x-)|\alpha\rangle = \frac{1}{\sqrt{2}}[\langle\psi_1|\alpha\rangle - \langle\psi_2|\alpha\rangle] \ , \tag{17}$$

where the stochastic integrals on the right entail no spin sum. With $\langle\psi|\psi\rangle = \langle\psi_1|\psi_1\rangle + \langle\psi_2|\psi_2\rangle = 1$, we also have

$$\langle\psi|P(z+)|\psi\rangle = \langle\psi_1|\psi_1\rangle \ , \qquad \langle\psi|P(z-)|\psi\rangle = \langle\psi_2|\psi_2\rangle \ , \tag{18}$$

$$\langle\psi|P(x\pm)|\psi\rangle = \frac{1}{2} \pm \mathrm{Re}\langle\psi_1|\psi_2\rangle \ . \tag{19}$$

We compute the joint probability $p(S_z = \hbar/2 \,\cap\, S_x = \hbar/2)$. Define

$$x_k + iy_k = \langle\psi_k|\alpha\rangle/a_k \ , \qquad a_k = \langle\psi_k|\psi_k\rangle^{1/2} \ , \qquad b = 2\mathrm{Re}\langle\psi_1|\psi_2\rangle \ . \tag{20}$$

By Eqs. (12,16,17,18,19) the required probability is

$$p(S_z = \hbar/2 \,\cap\, S_x = \hbar/2) = M(\Omega) = \frac{1}{(2\pi)^2}\int_\Omega \exp(-\frac{1}{2}z^t z)dz \ , \tag{21}$$

$$z = (x_1, y_1, x_2, y_2) \ , \tag{22}$$

where Ω is the set in $\mathbb{R}^4$ on which the following inequalities hold:

$$\frac{x_1^2 + y_1^2}{a_1^2} < \frac{x_2^2 + y_2^2}{a_2^2} \ , \tag{23}$$

$$\frac{(a_1x_1 + a_2y_2)^2 + (a_1y_1 + a_2y_2)^2}{(1+b)^2} < \frac{(a_1x_1 - a_2y_2)^2 + (a_1y_1 - a_2y_2)^2}{(1-b)^2} \ . \tag{24}$$

To impose the inequalities, we use a Fourier representation of the unit step function,

$$\theta(x) = \lim_{\epsilon\to 0+} \frac{1}{2\pi i}\int_{-\infty}^{\infty} \frac{e^{i\lambda x}d\lambda}{\lambda - i\epsilon} \ , \tag{25}$$

choosing x so that $\theta = 1$ when an inequality is satisfied, otherwise $\theta = 0$. Taking limits outside the integral we obtain

$$M(\Omega) = -\frac{1}{(2\pi)^4}\lim_{\eta\to 0+}\lim_{\epsilon\to 0+}\int_{-\infty}^{\infty}\frac{d\mu}{\mu - i\eta}\int_{-\infty}^{\infty}\frac{d\lambda}{\lambda - i\epsilon}\int_{\mathbb{R}^4}\exp(-\frac{1}{2}z^t Az)dz \ . \tag{26}$$

Here A is a 4×4 matrix that can be read off from (23,24,25). To justify taking the limits outside the integral, note that the integral in (25) is equal to $\theta(x)\exp(-\epsilon x)$ and therefore approximates $\theta(x)$ uniformly on any finite interval for small ϵ. On the other hand, the real part of A is the unit matrix, so that the z integral converges uniformly in ϵ and η. For any prescribed accuracy, only a finite region of integration on z is relevant.

The z-integral equals $(2\pi)^2 \det^{-1/2} A$, and $\det^{1/2} A$ is a quadratic polynomial in λ with roots $\lambda_i(\mu)$. An analysis (not very simple) shows that, for all μ, only one of the roots is in the lower half λ-plane. Consequently, one can evaluate the λ-integral as a single residue, and find eventually the formula

$$M(\Omega) = \frac{a_1^2}{2} + \frac{(a_1a_2)^2}{4\pi}\int_0^\infty \frac{d\mu}{\mu}\mathrm{Im}[\frac{1}{\lambda_1(\mu) - \lambda_2(\mu)}\frac{1}{\lambda_2(\mu)}] \ . \tag{27}$$

The integrand is continuous and bounded at all μ, and falls off as μ^{-3} at infinity. For the rest we resort to numerical integration, taking care to use the correct branches of square roots in the complicated expression for λ_i.

The joint probability depends on two independent parameters, which we take to be $p(S_z = \hbar/2) = a_1^2$ and $p(S_x = \hbar/2) = (1+b)/2$. Figure 1 shows $p(S_z = \hbar/2 \,\cap\, S_x = \hbar/2)$ as a function of the latter, for a few choices of the former. The upper limit of $p(S_x = \hbar/2)$ for given $p(S_z = \hbar/2)$ is determined by Schwarz's inequality and the normalization $\langle\psi|\psi\rangle = a_1^2 + a_2^2 = 1$.

The curves look quite reasonable, showing that the joint probability is sometimes larger and sometimes smaller than the product of the individual probabilities, depending on the spin content of the wave function. A check on the computation comes from the case $p(S_z = \hbar/2) = 1/2$, which can be handled analytically. It yields $p(S_z = \hbar/2, S_x = \hbar/2) = p(S_z = \hbar/2)p(S_x = \hbar/2)$, in good agreement with the numerical results.

DISCLAIMER

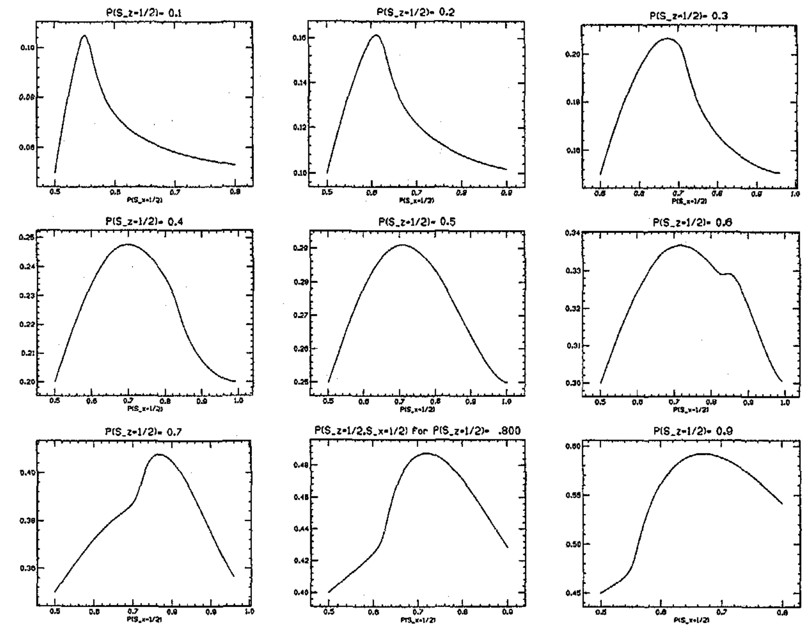

Figure 1: $p(S_z = \hbar/2 \,\cap\, S_x = \hbar/2)$ as a function of $p(S_x = \hbar/2)$ for various choices of $p(S_z = \hbar/2)$

6 The Einstein-Podolsky-Rosen-Bohm *Gedankenexperiment*

We consider two spin 1/2 particles in a singlet state (total spin 0). The state vector is a sum of direct products of vectors belonging to the separate particles:

$$|\psi\rangle = \frac{1}{\sqrt{2}}\left(|e_1(1)\rangle \otimes |e_2(-1)\rangle - |e_1(-1)\rangle \otimes |e_2(1)\rangle\right) , \tag{28}$$

where $|e_i(\pm 1)\rangle$ is an eigenvector of $S_i \cdot e$ with eigenvalue $\pm\hbar/2$. Here $S_i \cdot e$ is the scalar product of the spin operator for the i-th particle with a unit vector e. The singlet state is spherically symmetric, which means that the choice of e is arbitrary. The projection operator for $S_i \cdot e$ having eigenvalue $\pm\hbar/2$ is denoted by $P_i(e(\pm 1))$. We have

$$P_1(e(k)) = |e_1(k)\rangle\langle e_1(k)| \otimes 1_2 \ , \ P_2(e(k)) = 1_1 \otimes |e_2(k)\rangle\langle e_2(k)| \ , \tag{29}$$

$$[P_1(e(k)), P_2(f(l))] = 0 \ , \tag{30}$$

where 1_i is the unit operator for particle i in spin space.

Usual quantum theory gives the joint probability for $S_1 \cdot e = k\hbar/2$ and $S_2 \cdot f = l\hbar/2$ as

$$p(e_1(k) \cap f_2(l)) = \langle\psi|P_1(e(k))P_2(f(l))|\psi\rangle = \tag{31}$$

$$\frac{kl}{2}\langle e_1(k)|f_1(-l)\rangle\langle e_2(-k)|f_2(l)\rangle \ , \tag{32}$$

where the final equality is obtained by writing $\langle\psi|$ in terms of e and $|\psi\rangle$ in terms of f, then applying Eq.(29). This hypothesis is unambiguous, thanks to the commutativity of Eq.(30). The Wiener-Siegel theory of course gives the value (31) for the measure of the set of all α that are assigned to the projection $P_1(e(k)))P_2(f(l))$ by the polychotomic algorithm. In this sense the theory agrees with ordinary quantum theory, but the theory is rather inconsequential unless we ask questions about other projections. If the polychotomic algorithm may be applied to *any* set of orthogonal projections, as was indicated in Section 4, then we are forced to consider the measure of the set assigned to $P_1(e(k))$ (which is 1/2 in agreement with quantum theory), the measure of the set assigned to $P_2(f(l))$ (also 1/2), and the measure of the intersection of these sets. The probabilistic viewpoint demands that the latter be identified with $p(e_1(k) \cap f_2(l))$, but in fact it will not have the value of quantum theory as defined in Eq.(31).

We shall compute the exact value presently, but one can see from a nice argument of Faris [17] (as applied by Carlen [14]) that the value must disagree with (31). Faris describes his result as a version of "Bell's First Theorem" [18] Considering assignments of single-particle projections for the singlet state, first note that whenever the polychotomic algorithm assigns the value $S_1 \cdot e = k\hbar/2$ to α it assigns $S_2 \cdot e = -k\hbar/2$ to the same α. This follows because the denominators in Eq.(12) are all equal to 1/4 for both projections, and the random variables $\langle\psi|P_1(e(k))|\alpha\rangle$ and $\langle\psi|P_2(e(-k))|\alpha\rangle$ are equal, each being equal to $(k/\sqrt{2})(\langle e_1(k)| \otimes \langle e_2(-k)|)|\alpha\rangle$. This would seem to avoid the "EPR Paradox", since the property of having opposite spins is intrinsic to an "individual system" corresponding to a point α in differential space. Following Faris, we next consider a triad of unit vectors in a plane, denoted by e, f, g, with an angle of 120° between adjacent vectors. From the property of opposite spins just mentioned, it follows that

$$\begin{aligned} &p(e_1(1) \cap f_2(1)) + p(f_1(1) \cap g_2(1)) + p(g_1(1) \cap e_2(1)) = \\ &p(e_1(1) \cap f_1(-1)) + p(f_1(1) \cap g_1(-1)) + p(g_1(1) \cap e_1(-1)) \ . \end{aligned} \tag{33}$$

But the three probabilities on the right-hand side refer to mutually exclusive events, so that their sum is equal to the probability of the union event (the probability for at least one of the three events to occur), and the latter is at most 1. On the other hand, quantum theory assigns the value 3/8 to each of the probabilities on the left-hand side of (33); see Eq.(38) below. Thus, unbridled application of the polychotomic algorithm to build a probabilistic theory leads to a disagreement with quantum theory. Another way of putting the matter is to say that assigning the projection $P_1(e(k))P_2(f(l))$ to α is not the same thing as assigning both $P_1(e(k))$ and $P_2(f(l))$ to α.

By symmetry, each of the terms on the right-hand side of (33) must be less than or equal to 1/3. In fact, each must be strictly less than 1/3, since to the three events considered, two more may be added to make a list of five mutually exclusive events. The additional events are $e_1(1) \cap f_1(1) \cap g_1(1)$ and $e_1(-1) \cap f_1(-1) \cap g_1(-1)$.

Finally we compute the exact value of $p(e_1(1) \cap f_1(-1))$ by the method of the previous section. We now have eight Gaussian variables with unit variance, making up the components of a vector $z = (x_1, y_1, \cdots, x_4, y_4)$ in $\mathbb{R}^8$, where

$$x_1 + iy_1 = \sqrt{2}\langle\psi|P_1(e(1))|\alpha\rangle \ , \qquad x_2 + iy_2 = \sqrt{2}\langle\psi|P_1(e(-1))|\alpha\rangle \ , \tag{34}$$

$$x_3 + iy_3 = \sqrt{2}\langle\psi|P_1(f(-1))|\alpha\rangle \ , \qquad x_4 + iy_4 = \sqrt{2}\langle\psi|P_1(f(1))|\alpha\rangle \ . \tag{35}$$

The required probability is

$$p(e_1(1) \cap f_1(-1) = M(\Omega) = \frac{1}{(2\pi)^4}\frac{1}{(\det\Lambda)^{1/2}}\int_\Omega \exp(-\frac{1}{2}z^t\Lambda^{-1}z)dz \ , \tag{36}$$

where Λ is the covariance matrix, and Ω is the set in $\mathbb{R}^8$ on which the following inequalities hold:

$$x_1^2 + y_1^2 < x_2^2 + y_2^2 \ , \qquad x_3^2 + y_3^2 < x_4^2 + y_4^2 \ . \tag{37}$$

According to Eq.(11), Λ can be expressed in terms of the right-hand side of Eq.(32). Denoting the latter by M_{kl}, we have

$$M_{11} = M_{-1-1} = \frac{1}{2}\cos^2\frac{\theta}{2} \ , \qquad M_{1-1} = M_{-11} = \frac{1}{2}\sin^2\frac{\theta}{2} \ , \tag{38}$$

where θ is the angle between e and f. It is easy to derive the formulas (38) from the usual law of transformation of spinors under rotation. Tractable expressions for the inverse and the determinant of the resulting 8×8 matrix Λ can be derived. We can now proceed as in the previous section, enforcing the inequalities (37) by means of the integral representation of the step function (25). As before, the z-integral and one of the integrals from the step functions can be evaluated. For the case $\theta = 120°$ the result after those integrations is

$$M(\Omega) = \frac{1}{4} + \frac{1}{8\pi}\int_0^\infty \frac{1}{1+4\mu^2}\frac{1}{\left(1+(3\mu/2)^2(3+5\mu^2)\right)^{1/2}}d\mu \tag{39}$$

A numerical integration yields $M(\Omega) = 0.266300...$, in sharp disagreement with the value 3/8 of ordinary quantum theory.

No experiments have been made to measure directly the probability we have discussed. There have been several experiments designed to imitate the EPRB *Gedankenexperiment* as closely as possible, usually using photons from cascade decays of an atom [19]. A difficulty in principle is the ambiguity in the interpretation of missing particles. If particle 1 is observed passing through polarizer 1, particle 2 may be unobserved, for one of two reasons that cannot be distinguished: (i) the detector has very poor efficiency and simply fails to detect the particle or (ii) the particle is blocked by polarizer 2. Certain techniques and assumptions have been introduced to avoid this and other problems, and the general opinion seems to be that the experiments favor quantum theory over hidden-variable theories. A forthcoming experiment by E. Fry at Texas A & M University, which is to observe mercury atoms from dissociation of a mercury molecule, may give a clearer test.

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