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POLYNOMIAL REALIZATION OF THE $U_q(sl(3))$ GEL'FAND–(WEYL)–ZETLIN BASIS

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ABSTRACT

We give an explicit realization of the $\mathcal{U} \equiv U_q(sl(3))$ Gel'fand-(Weyl)-Zetlin (GWZ) basis as polynomial functions in three variables. This realization is obtained in two complementary ways. First we establish a 1-to-1 correspondence between the abstract GWZ basis and explicit polynomials in the quantum subgroup $\mathcal{U}^+$ of the raising generators. We then use an explicit construction of arbitrary lowest weight (holomorphic) representations of $\mathcal{U}$ in terms of three variables on which the generators of $\mathcal{U}$ are realized as q-difference operators. Applying the GWZ corresponding polynomials in this realization to the lowest weight vector (the function 1) produces one realization of our GWZ basis. Another realization of the GWZ polynomial basis is found by the explicit diagonalization of the operators of isospin $\hat{I}^2$, third component of isospin $\hat{I}_z$, and hypercharge $\hat{Y}$, in the same realization as q-difference operators. The result is that the eigenvectors can be written in terms of q-hypergeometric polynomials in our three variables. Finally we construct an explicit scalar product (adapting the Shapovalov form to our setting). Using it we prove the orthogonality of our GWZ polynomials for which we use both realizations. This provides a polynomial construction for the orthonormal GWZ basis. We work for generic q, leaving the root of unity case for a following paper. It seems that our results are new also in the classical situation ($q = 1$).

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1. Introduction

In 1950 Gel'fand and Zetlin [1] introduced a pattern to enumerate the basis vectors in the carrier spaces of the irreducible representations of $U(n)$, (or $u(n)$), and also of the irreducible (anti)holomorphic representations the general linear group $GL(n)$ or algebra $gl(n)$. The pattern is a triangular array of n rows of integers m_{ij} :

$$(\mathrm{m}) = \begin{pmatrix} m_{1n} & m_{2n} & \dots & \dots & \dots & \dots & m_{nn} \\ & m_{1,n-1} & m_{2,n-1} & \dots & m_{n-1,n-1} & & \\ & & & \dots & & & \\ & & & m_{11} & & & \end{pmatrix}$$

The integers satisfy the betweenness constraints:

$$m_{ij} \geq m_{i,j-1} \geq m_{i+1,j}$$

which are the expressions of the Weyl [2] branching law for the general linear group. To apply the pattern to $SU(n)$ or to the special linear group $SL(n)$ one just sets $m_{nn} = 0$. In the literature the pattern is called mostly Gel'fand-Zetlin pattern, sometimes simply Gel'fand pattern, but also Gel'fand-Weyl pattern (cf. [3], [4], and references therein). We shall use the connotation *Gel'fand-(Weyl)-Zetlin* (GWZ) for the pattern, state, or basis, as may be the case. The GWZ pattern is very useful in physical applications mainly for providing the explicit matrix elements of the irreps of $U(n)$, cf., e.g., [4], [5], [6], and many references therein. In the case of $SU(3)$ it is important that this basis diagonalizes the operators which (in some applications) are identified as those of isospin $\hat{I}^2$, third component of isospin $\hat{I}_z$, and hypercharge $\hat{Y}$. With the advent of quantum groups [7], [8], the GWZ pattern was adapted to the quantum groups $U_q(gl(n))$ and $U_q(sl(n))$, cf. [9], [10], [11]. The GWZ pattern was extended also to the inhomogeneous unitary group $IU(n)$ [12] and its q-deformation [13].

In spite of much development until now there was no explicit realization of the GWZ states as polynomial functions, even in the classical situation ($q = 1$). This is what we do in the present paper for $U_q(sl(3))$ for generic q, though the method works for q a root of 1, and for arbitrary $U_q(sl(n))$, cf. Outlook at the end.

To be more precise we do the following. For every GWZ state we give an explicit polynomial in three variables in terms of q-hypergeometric polynomials. Our starting point is the explicit construction of arbitrary lowest weight (holomorphic) representations of $U_q(sl(3))$ in terms of three variables [14], on which the generators of $U_q(sl(3))$ act as q-difference operators. (The lowest weight vector is the function 1.) Our first result is to write the GWZ states in terms of the polynomial basis of [14] in the finite-dimensional case. We prove Theorem 1 which contains two statements. First it gives a 1-to-1 correspondence between the abstract GWZ states and monomials in the q-deformed enveloping algebra $U_q(\mathcal{G}^-)$ of the lowering generators which monomials are not in the standard Poincaré–Birkhoff–Witt basis of $U_q(\mathcal{G}^-)$. These monomials produce polynomials in $U_q(\mathcal{G}^+)$ by acting on the monomial in $U_q(\mathcal{G}^+)$ which represents the highest weight vector. In a Corollary we substitute the explicit q-difference realization of the raising generators to produce the GWZ polynomial basis when acting on 1. Another realization of the GWZ polynomial basis is found by the explicit diagonalization of the operators $\hat{I}^2$, $\hat{I}_z$, $\hat{Y}$, in the same realization as q-difference operators. It turns out that the eigenvectors can be written in terms of q-hypergeometric polynomials in our three variables. Finally we construct an explicit scalar product (adapting the Shapovalov form to our setting). Using it we prove the orthogonality of our GWZ polynomials for which we use both realizations (Theorem 2). This provides a polynomial construction for the orthonormal GWZ basis.

The paper is organized as follows. In Section 2 we introduce the quantum group $U_q(sl(n))$, briefly summarize the needed results from [14], and obtain some consequences from these which are needed in the paper. In Section 3 we give the first GWZ basis in Theorem 1. In Section 4 we make the diagonalization of the operators $\hat{I}^2$, $\hat{I}_z$, $\hat{Y}$ and write explicitly their eigenvectors in terms of of q-hypergeometric polynomials. In Section 5 we construct the scalar product and prove the orthogonality in Theorem 2. In Appendix A we give the diagonalization of the operators $\hat{I}^2$, $\hat{I}_z$, $\hat{Y}$ in the case $q=1$ since it is also interesting. In Appendix B we give an alternative proof of the orthogonality.

2. Polynomial realization of $U_q(sl(3))$ representations

The quantum algebra $U_q(sl(n))$ is defined as the associative algebra over $\mathbb{C}$ with Chevalley generators $X_j^{\pm}$, H_j, $j=1,\ldots,n-1$, and with relations [7], [8] :

$$[H_j\,,\,H_k] = 0,\ \ [H_j\,,\,X_k^{\pm}] = \pm a_{jk}X_k^{\pm}\,,\ \ [X_j^+\,,\,X_k^-] \ = \ \delta_{jk}\,[H_j]_q\,, \qquad (1a)$$

$$\begin{aligned}(X_j^{\pm})^2 X_k^{\pm} - [2]_q X_j^{\pm}X_k^{\pm}X_j^{\pm} + X_k^{\pm}(X_j^{\pm})^2 \ &= \ 0\,,\ \ |j-k|=1 \\ [X_j^{\pm}, X_k^{\pm}] \ &= \ 0\,,\ \ |j-k|>1\end{aligned} \qquad (1b)$$

where $[x]_q = \frac{q^{x/2}-q^{-x/2}}{q^{1/2}-q^{-1/2}}$ is the basic q-number notation and it is used also for diagonal operators H replacing x. $(a_{jk}) = (2(\alpha_j,\alpha_k)/(\alpha_j,\alpha_j))$, $j,k=1,2,\ldots,n-1$, is the Cartan matrix of $sl(n)$; α_j are the simple roots; the non-zero products between the simple roots are: $(\alpha_j,\alpha_j)=2$, $j=1,\ldots,n-1$, $(\alpha_j,\alpha_k)=-1$ for $|j-k|=1$. Arbitrary positive roots α_{jk}, $1\leq j<k\leq n$ are given explicitly in terms of the simple roots as:

$$\begin{aligned}\alpha_{jk} \ &= \ \alpha_j \ + \ \cdots \ + \ \alpha_{k-1}\,,\ j<k-1 \\ &= \ \alpha_j\,,\ \ j=k-1\end{aligned} \qquad (2)$$

The elements H_j span the Cartan subalgebra $\mathcal{H}$, while the elements $X_j^{\pm}$ generate the subalgebras $\mathcal{G}^{\pm}$. To the positive roots α_{jk} correspond the raising Cartan–Weyl generators E_{jk}, $(j<k)$, while to the negative roots $-\alpha_{jk}$ correspond the lowering Cartan-Weyl generators E_{kj}, $(j<k)$. Thus, for the Chevalley generators we have:

$$X_j^+ \ = \ E_{j,j+1}\,, \qquad X_j^- \ = \ E_{j+1,j} \qquad (3)$$

The rest of the Cartan–Weyl generators can be defined recursively [8], [15] :

$$E_{jk} \ = \ E_{j,j+1}\,E_{j+1,k} \ - \ q^{1/2}\,E_{j+1,k}\,E_{j,j+1}\,, \qquad j<k-1 \qquad (4a)$$

$$E_{kj} \ = \ E_{k,j+1}\,E_{j+1,j} \ - \ q^{-1/2}\,E_{j+1,j}\,E_{k,j+1}\,, \qquad j<k-1 \qquad (4b)$$

We recall the representations of $U_q(sl(n))$ constructed in [14]. They are given in terms of $n(n-1)/2$ variables denoted in [14] by z_i^k, $2\leq k\leq n$, $1\leq i\leq k-1$. Next we introduce the number operator N_i^k for the coordinate z_i^k, i.e., $N_i^k\ z_j^m = \delta_{mk}\delta_{ij}z_j^m$ and the q - difference operators D_i^k, which admit a general definition on a larger domain than polynomials, but on polynomials are well defined as follows:

$$D_i^k \ = \ \frac{1}{z_i^k}[N_i^k]_q\,. \qquad (5)$$

In this construction the representations of $U_q(sl(n))$ are characterized by $n-1$ complex parameters $r_k\in\mathbb{C}$, $1\leq k\leq n-1$, [14].

Further we restrict to the case $n=3$. We set $N_i^3 = N_i$, $D_i^3 = D_i$, $z_1^3 = z$, $z_2^3 = y$, $r = r^2 = r_1 + r_2$. The explicit realization of $U_q(sl(3))$ is [14] :

$$\begin{aligned}
\Gamma_3(E_{12}) \ &= \ x[r_1-N_x]_q\,q^{\frac{1}{4}(N_z-N_y)} \ + \ zD_y q^{\frac{1}{4}(r_1-2N_x)}\,, & (6a)\\
\Gamma_3(E_{21}) \ &= \ D_x\,q^{\frac{1}{4}(N_z-N_y)} \ + \ yD_z q^{\frac{1}{4}(r_1-2N_x)}\,, & (6b)\\
\Gamma_3(H_1) \ &= \ 2N_x - r_1 + N_z - N_y\,, & (6c)\\
\Gamma_3(H_2) \ &= \ 2N_y - r_2 + N_z - N_x\,, & (6d)\\
\Gamma_3(E_{31}) \ &= \ D_z q^{\frac{1}{4}(r_1-N_x+2N_y)}\,, & (6e')\\
\Gamma_3(E_{32}) \ &= \ D_y q^{\frac{1}{4}N_x}\,, & (6e'')
\end{aligned}$$

$$\begin{aligned}
\Gamma_3(E_{13}) \ &= \ z[r-N_x-N_z-N_y]_q\,q^{\frac{1}{4}(N_x-r_1-2N_y)} - \\
&\quad - xy[r_1-N_x]_q\,q^{\frac{1}{4}(2r_2+N_x-N_z-3N_y+1)} & (6f')\\
\Gamma_3(E_{23}) \ &= \ y\,[r_2+N_x-N_z-N_y]_q\,q^{-\frac{1}{4}N_x} - \\
&\quad - z\,D_x\,q^{-\frac{1}{4}(2r-r_1+N_x-N_z-N_y+1)}\,. & (6f'')
\end{aligned}$$

The representation depends only on the parameters r_1, r_2, as in the classical case of holomorphic $sl(3)$ representations. It is straightforward to check (using also (3), (4)) that (6) satisfies (1).

Let us apply (6) to the function 1 :

$$\begin{aligned}
\Gamma_3(E_{12})\,1 \ &= \ x[r_1]_q\,, & (7a)\\
\Gamma_3(E_{21})\,1 \ &= \ 0\,, & (7b)\\
\Gamma_3(H_1)\,1 \ &= \ -r_1\,, & (7c)\\
\Gamma_3(H_2)\,1 \ &= \ -r_2\,, & (7d)\\
\Gamma_3(E_{31})\,1 \ &= \ 0\,, & (7e')\\
\Gamma_3(E_{32})\,1 \ &= \ 0\,, & (7e'')\\
\Gamma_3(E_{13})\,1 \ &= \ q^{-\frac{1}{4}r_1}z[r]_q \ - q^{\frac{1}{4}(2r_2+1)}yx[r_1]_q & (7f')\\
\Gamma_3(E_{23})\,1 \ &= \ y[r_2]_q\,, & (7f'')
\end{aligned}$$

Thus, we obtain a lowest weight module with lowest weight vector 1 and lowest weight Λ such that $\Lambda(H_k) \ = \ -r_k$. All states are given by powers of x, y, z , i.e., the basis is generated by $x^j z^k y^\ell$ with $j, k, \ell \in \mathbb{Z}_+$. The action of $U_q(sl(3))$ is given by [14]:

$$\Gamma_3(E_{12})\, x^j z^k y^\ell = [r_1 - j]_q\, q^{\frac{1}{4}(k-\ell)}\, x^{j+1} z^k y^\ell + [\ell]_q\, q^{\frac{1}{4}(r_1-2j)}\, x^j z^{k+1} y^{\ell-1}\,, \tag{8a}$$

$$\Gamma_3(E_{21})\, x^j z^k y^\ell = [j]_q\, q^{\frac{1}{4}(k-\ell)}\, x^{j-1} z^k y^\ell + [k]_q\, q^{\frac{1}{4}(r_1-2j)}\, x^j z^{k-1} y^{\ell+1}\,, \tag{8b}$$

$$\Gamma_3(H_1)\, x^j z^k y^\ell = (-r_1 + 2j - \ell + k)\, x^j z^k y^\ell\,, \tag{8c}$$

$$\Gamma_3(H_2)\, x^j z^k y^\ell = (-r_2 - j + 2\ell + k)\, x^j z^k y^\ell\,, \tag{8d}$$

$$\Gamma_3(E_{31})\, x^j z^k y^\ell = [k]_q\, q^{\frac{1}{4}(r_1-j+2\ell)}\, x^j z^{k-1} y^\ell\,, \tag{8c'}$$

$$\Gamma_3(E_{32})\, x^j z^k y^\ell = [\ell]_q\, q^{\frac{1}{4}j}\, x^j z^k y^{\ell-1}\,, \tag{8c''}$$

$$\begin{aligned}\Gamma_3(E_{13})\, x^j z^k y^\ell &= q^{\frac{1}{4}(j-r_1-2\ell)}[r - j - k - \ell]_q\, x^j z^{k+1} y^\ell - \\ &- q^{\frac{1}{4}(2r_2+j-k-3\ell+1)}[r_1 - j]_q\, x^{j+1} z^k y^{\ell+1}\,,\end{aligned} \tag{8f'}$$

$$\begin{aligned}\Gamma_3(E_{23})\, x^j z^k y^\ell &= q^{-\frac{1}{4}j}\, [r_2 + j - k - \ell]_q\, x^j z^k y^{\ell+1} - \\ &- [j]_q\, q^{-\frac{1}{4}(r_1+2r_2+j-k-\ell+1)}\, x^{j-1} z^{k+1} y^\ell\,.\end{aligned} \tag{8f''}$$

In [14] we have shown the following results which parallel the classical situation (cf. [15]):

1. If r_1, or r_2, or $r+1 \in \mathbb{Z}_+$ this representation is reducible. It contains an irreducible subrepresentation which is infinite-dimensional, except when both $r_1, r_2 \in \mathbb{Z}_+$;

2. If $r_1, r_2, r+1 \notin \mathbb{Z}_+$ this representation is irreducible and infinite dimensional.

In the present paper we consider the case when both $r_1, r_2 \in \mathbb{Z}_+$. The irreducible subrepresentation is finite-dimensional of dimension:

$$\hat{d}_{r_1,r_2} = \frac{1}{2}(r_1+1)(r_2+1)(r+2)\,. \tag{9}$$

This recovers the complete list of the finite dimensional holomorphic irreps of $U_q(sl(3))$ and $SL(3)$, and by default, also the complete list of the finite dimensional unitary irreps of $U_q(su(3))$ and $SU(3)$ (we have assumed that q is not a nontrivial root of 1).

Next we use the following general formula valid for arbitrary r_k :

$$v_{\ell kj} \equiv \Gamma_3(E_{23})^\ell\, \Gamma_3(E_{13})^k\, \Gamma_3(E_{12})^j\, 1 =$$

$$\begin{aligned} &= \sum_{s=0}^{k}\sum_{n=0}^{\ell}(-1)^{s-n} \binom{k}{s}_q \binom{\ell}{n}_q q^{\frac{1}{4}\{(j-r_1)k-\ell j+(s-n)(r_1+2r_2-k-\ell+2)\}} \times \\ &\times \frac{\Gamma_q(r_1+1)\Gamma_q(r-j-s+1)}{\Gamma_q(r_1-j-s+1)\Gamma_q(r-j-k+1)}\, \frac{\Gamma_q(r_2+j+s-k-n+1)}{\Gamma_q(r_2+j+s-k-\ell+1)}\, \frac{[j+s]_q!}{[j+s-n]_q!} \times \\ &\times x^{j+s-n}\, z^{k-s+n}\, y^{\ell+s-n}\,. \end{aligned} \tag{10}$$

Note that for $r_2 \in \mathbb{Z}$ the ratio $\Gamma_q(r_2+j+s-k-n+1)/\Gamma_q(r_2+j+s-k-\ell+1)$ means just $[r_2+j+s-k-n]_q[r_2+j+s-k-n-1]_q \ldots [r_2+j+s-k-l+1]_q$.

One of the main results of [14] is that a basis for the above representation space is given by $v_{\ell kj}$ iff $\ell+k+j \le r$, $0 \le j \le r_1$, $0 \le \ell \le r_2$. In the next Section we relate this basis to the standard Gel'fand–Weyl–Zetlin basis.

For later reference we note the special polynomial v_{0r0} which corresponds to the highest weight vector (as we shall see later) :

$$v_{0r0} \equiv \Gamma_3(E_{13})^r\, 1 = q^{-\frac{1}{4}rr_1}\, [r]_q! \sum_{s=0}^{r_1} (-1)^s \binom{r_1}{s}_q q^{\frac{1}{4}s(r-r_1+2)}\, x^s z^{r-s} y^s \tag{11}$$

Also for later reference we note the explicit value of $v_{\ell kj}$ for $z=0$ (given by the term $s=k$ and $n=0$) :

$$\begin{aligned} v_{\ell kj}\big|_{z=0} &= (-1)^k\, q^{\frac{1}{4}\{(j-r_1)k-\ell j+k(r_1+2r_2-k-\ell+2)\}} \times \\ &\times \frac{\Gamma_q(r_1+1)}{\Gamma_q(r_1-j-k+1)}\, \frac{\Gamma_q(r_2+j+1)}{\Gamma_q(r_2+j-\ell+1)}\, x^{j+k}\, y^{\ell+k} \end{aligned} \tag{12}$$

Note that the RHS of (12) is equal to zero when $j+k \ge r_1+1$ (because of the $\Gamma_q(r_1-j-k+1)$ in the denominator). In this case one applies $(D_z)^{j+k-r_1}$ to both sides of (10) and then sets $z=0$.

3. Correspondence with the GWZ basis

We would like to establish the correspondence between the representation parameters r_k and states $v_{\ell kj}$ (cf. (10) and [14]) for undeformed finite-dimensional irreducible representations and the $SU(3)$ Gel'fand–Weyl–Zetlin basis:

$$(\mathrm{m}) = \begin{pmatrix} m_{13} & & m_{23} & & m_{33} \\ & m_{12} & & m_{22} & \\ & & m_{11} & & \end{pmatrix} \tag{13}$$

In fact, the above is for $U(3)$, and we shall set $m_{33}=0$ to restrict to $SU(3)$. In this case, using the labels p, p', I, I_z, Y where p, p' denote the Young pattern $[p, p']$, I denotes the isospin and Y the hypercharge, the Gel'fand pattern becomes

$$(\mathrm{m}) = \begin{pmatrix} p & & p' & & 0 \\ & I+\frac{1}{2}Y+\frac{1}{3}(p+p') & & -I+\frac{1}{2}Y+\frac{1}{3}(p+p') & \\ & & I_z+\frac{1}{2}Y+\frac{1}{3}(p+p') & & \end{pmatrix} \tag{14}$$

Let us choose the operators corresponding to $\hat{I}_z$, $\hat{Y}$, $\hat{I}^2$:

$$\hat{I}_z = \tfrac{1}{2}H_1 \tag{15a}$$

$$\hat{Y} = \tfrac{1}{3}(H_1 + 2H_2) \tag{15b}$$

$$\hat{I}^2 = E_{21}E_{12} + [\tfrac{1}{2}H_1]_q[\tfrac{1}{2}H_1 + 1]_q \tag{15c}$$

Note that $\hat{I}^2$ is the Casimir of the $U_q(sl(2))$ quantum subgroup generated by E_{21}, E_{12}, H_1. It is very easy to see that, like the GWZ states, also the $v_{\ell kj}$ states are eigenvectors of $\hat{I}_z$ and $\hat{Y}$, but they are not eigenvectors of $\hat{I}^2$. In fact we have:

$$\Gamma_3(\tfrac{1}{2}H_1)\, v_{\ell kj} = \left(j + k - \tfrac{1}{2}(r_1 + \ell + k)\right) v_{\ell kj} \tag{16}$$

$$\Gamma_3\left(\frac{1}{3}(H_1+2H_2)\right) v_{\ell kj} \;=\; \left(r_1+k+\ell-\frac{2}{3}(r+r_1)\right) v_{\ell kj} \tag{17}$$

$$\begin{aligned}\Gamma_3(E_{21})\Gamma_3(E_{12})\, v_{\ell kj} \;=\; & \left((j+1)(r_1-j)+\ell(k+1)\right) v_{\ell kj} \;+\; k\, v_{\ell+1,k-1,j+1} \\ & +(r_1-j+1)\ell j\, v_{\ell-1,k+1,j-1}\end{aligned} \tag{18}$$

The last formula is given for $q = 1$ since it is only to illustrate our point. We shall diagonalize $\Gamma_3(E_{21})\Gamma_3(E_{12})$ in the next Section and find explicit polynomial eigenvectors. In this Section we find alternatively an explicit correspondence between $(\mathbf{m})$ and the appropriate linear combination of $v_{\ell kj}$'s. But first we shall to stick to the labels ℓ, k, j and place them in a Gel'fand-like pattern.

First of all, we must set up a correspondence between the two representations, namely between the labels $\{p, p'\}$ and $\{r, r_1\}$. This can be achieved but considering the lowest weight vector. For our representation this is the polynomial 1, which corresponds, as can be readily checked to the GWZ vector

$$\begin{pmatrix} p & & p' & & 0 \\ & p' & & 0 & \\ & & 0 & & \end{pmatrix} \tag{19}$$

This is a state with $I = -I_z = p'/2$ and $Y = -\frac{1}{3}(2p-p')$, whereas the lowest weight vector 1 is a state with $I = -I_z = r_1/2$ and $Y = -\frac{1}{3}(2r-r_1)$. Therefore we find $p = r$ and $p' = r_1$.

Remark: Notice that the conjugation of representation ($[p, p', 0] \to [p, p-p', 0]$) corresponds to the exchange of r_1 with r_2 and that the dimension of the representation $[p, p', 0]$, namely $\frac{1}{2}(p+2)(p'+1)(p-p'+1)$, matches (9).

To place the $v_{\ell,k,j}$ states in a Gel'fand-like pattern we split them as in [14] (cf. (66)) in two subsets depending whether $j+k \le r_1$ or $j+k > r_1$. In the first case the correspondence is given by :

$$v_{\ell,k,j} \;=\; \begin{pmatrix} r & & r_1 & & 0 \\ & r_1+\ell & & k & \\ & & j+k & & \end{pmatrix}, \qquad j+k \le r_1 \,. \tag{20}$$

In the second case the correspondence is given by :

$$v_{\ell,k,j} \;=\; \begin{pmatrix} r & & r_1 & & 0 \\ & j+k+\ell & & r_1-j & \\ & & j+k & & \end{pmatrix}, \qquad j+k > r_1 \,, \tag{21}$$

which is valid also for the boundary case $j+k = r_1$, when it coincides with (20).

Notice that the eigenvalues of $\hat{I}_z$ and $\hat{Y}$ match those of the Gel'fand pattern, namely $I_z = m_{11} - \frac{1}{2}(m_{12}+m_{22})$ and $Y = m_{12}+m_{22}-\frac{2}{3}(m_{13}+m_{23})$ with corresponding entries. The betweenness constraint

$$m_{ij} \;\ge m_{i,j-1} \;\ge m_{i+1,j}\,. \tag{22}$$

typical of the Gel'fand pattern applies also here and gives the constraints $0 \le j \le r_1$, $0 \le \ell \le r_2$ and $0 \le j+k+\ell \le r$ found above for the finite dimensional representations.

The actual correspondence may be proved in an arbitrary realization of the finite dimensional representations. Identifying the lowest weight states we can find explicitly a polynomial in $U_q(\mathcal{G}^+)$ which corresponds to $(\mathbf{m})$, namely, we have:

Theorem 1: Let us denote the unknown polynomial in $U_q(\mathcal{G}^+)$ corresponding to $(\mathbf{m})$ by $p_{(m)}$. Let us denote by $\hat{1}$ the lowest weight state of any realization of the $U_q(sl(3))$ finite dimensional representation with parameters r_1, r_2. Then we have (up to multiplicative normalization constant) :

$$p_{(\mathbf{m})}\,\hat{1} \;=\; (E_{21})^{m_{12}-m_{11}}\,\hat{C}^{r-m_{12}}\,(E_{32})^{r_1-m_{22}}\,(E_{13})^{r}\,\hat{1} \;= \tag{23a}$$

$$\begin{aligned} &= [m_{12}+m_{22}-r_1]_q! \sum_{t=0}^{r-m_{12}} \sum_{u\in\mathbb{Z}_+} (-1)^t \binom{m_{12}-m_{11}+t}{u}_q \times \\ &\times \frac{q^{\frac{1}{2}(m_{22}-t-r_1)(m_{12}+m_{22}-r_1)+\frac{u}{2}(u-2m_{22}-m_{12}+m_{11}+r_1+t)}}{[t]_q!\,[m_{22}-t]_q!\,[m_{12}-m_{22}+1+t]_q!} \times \\ &\times \frac{[m_{12}-m_{22}+1]_q!\,[t+r_1-m_{22}]_q!\,[m_{12}+m_{22}-m_{11}-u]_q!}{[m_{11}-m_{12}-m_{22}+r_1+u]_q!\,[m_{12}+m_{22}-r_1-u]_q!} \times \\ &\times (E_{23})^u\,(E_{13})^{m_{12}+m_{22}-r_1-u}\,(E_{12})^{m_{11}-m_{12}-m_{22}+r_1+u}\,\hat{1} \end{aligned} \tag{23b}$$

$$\begin{aligned}\hat{C} \;&\equiv\; E_{31}\,[H_1+1] \;+\; E_{21}\,E_{32}\,q^{-\frac{H_1+1}{2}} \;= \\ &=\; E_{32}\,E_{21}\,[H_1+1] \;-\; E_{21}\,E_{32}\,[H_1]\end{aligned} \tag{23c}$$

Proof: We start with (23*a*). **i.** First we note that $(E_{13})^r\,\hat{1}$ is the highest weight state with $I_z = I = \frac{r_2}{2}$ and $Y = \frac{1}{3}(r+r_1)$. This follows since $\hat{I}_z\,(E_{13})^r = (E_{13})^r\,(\hat{I}_z+\frac{1}{2}r)$, $\hat{Y}\,(E_{13})^r = (E_{13})^r\,(\hat{Y}+r) \Rightarrow m_{11}=r \Rightarrow m_{12}=r$ and $I = I_z = \frac{r_2}{2} \Rightarrow m_{22}=r_1$. Thus:

$$(E_{13})^r\,\hat{1} \;\longleftrightarrow\; \begin{pmatrix} r & & r_1 & & 0 \\ & r & & r_1 & \\ & & r & & \end{pmatrix} \tag{24}$$

ii. Next we note that $(E_{32})^\ell\,(E_{13})^r\,\hat{1}$, for $\ell \le r_1$, is an eigenstate of $\hat{I}^2$ with $I = I_z = \frac{\ell+r_2}{2}$. This follows since $\hat{I}_z\,(E_{32})^\ell\,(E_{13})^r = (E_{32})^\ell\,(E_{13})^r\,(\hat{I}_z+\frac{1}{2}(r+\ell))$, $\hat{Y}\,(E_{32})^\ell\,(E_{13})^r = (E_{32})^\ell\,(E_{13})^r\,(\hat{Y}+r-\ell) \Rightarrow m_{11}=r \Rightarrow m_{12}=r$ and $I = I_z = \frac{(\ell+r_2)}{2}$. Thus:

$$(E_{32})^\ell\,(E_{13})^r\,\hat{1} \;\longleftrightarrow\; \begin{pmatrix} r & & r_1 & & 0 \\ & r & & r_1-\ell & \\ & & r & & \end{pmatrix}, \qquad \ell \le r_1 \tag{25a}$$

$$(E_{32})^{r_1-m_{22}}\,(E_{13})^r\,\hat{1} \;\longleftrightarrow\; \begin{pmatrix} r & & r_1 & & 0 \\ & r & & m_{22} & \\ & & r & & \end{pmatrix} \tag{25b}$$

iii. Next we need:

Lemma: Let ψ be eigenstate of $\hat{I}^2$ and $\hat{I}_z$ with $I = I_z = \mu$ and $Y = \nu$, then

$\psi^- = \hat{C}\,\psi$ is an eigenstate of $\hat{I}^2$ and $\hat{I}_z$ with $I = I_z = \mu - \frac{1}{2}$ and $Y = \nu - 1$. The easiest way to check the Lemma is to notice that ψ is a $U_q(sl(2))$ highest weight ($I = I_z$) and this is the case if and only if $E_{21}\,\psi = 0$. Since also $E_{21}\,\psi^- = 0$ (as can be readily checked using the commutation rules) and ψ^- has $I_z = \mu - \frac{1}{2}$ (since $\hat{I}_z\,\hat{C} = \hat{C}\,(\hat{I}_z - \frac{1}{2})$). Next we use $\hat{Y}\,\hat{C} = \hat{C}\,(\hat{Y} - 1)$. This proves the Lemma. Thus:

$$(\hat{C})^s\,(E_{12})^{r_1 - m_{22}}\,(E_{13})^r\,\hat{1} \quad\longleftrightarrow\quad \begin{pmatrix} r & & r_1 & & 0 \\ & r-s & & m_{22} & \\ & & r-s & & \end{pmatrix}, \quad s \le r_2 \qquad (26a)$$

$$(\hat{C})^{r - m_{12}}\,(E_{32})^{r_1 - m_{22}}\,(E_{13})^r\,\hat{1} \quad\longleftrightarrow\quad \begin{pmatrix} r & & r_1 & & 0 \\ & m_{12} & & m_{22} & \\ & & m_{12} & & \end{pmatrix} \qquad (26b)$$

iv. Now remains the easy part, since it is known that the only effect of the operator E_{21} is to diminish the eigenvalue I_z by 1 while preserving I and Y. Thus:

$$(E_{21})^{m_{12} - m_{11}}\,\hat{C}^{r - m_{12}}\,(E_{32})^{r_1 - m_{22}}\,(E_{31})^r\,\hat{1} \quad\longleftrightarrow\quad \begin{pmatrix} r & & r_1 & & 0 \\ & m_{12} & & m_{22} & \\ & & m_{11} & & \end{pmatrix} \qquad (27)$$

This proves (23*a*). To prove (23*b*) one starts from (23*a*) and pulls all lowering generators to the right (using the commutation relations of $U_q(sl(3))$, cf. (1)) and that $E_{ij}\;\hat{1} = 0$ for $i > j$. After a tedious calculation we get (23*b*). This proves the Theorem. •

Remark: We would like to stress the peculiarity of (23*a*). We do get (in (23*b*)) a correspondence of the GWZ states with polynomials in $U_q(\mathcal{G}^+)$ but formula (23*a*) first gives us a 1-to-1 correspondence of the GWZ states with *monomials* in the q-deformed enveloping algebra $U_q(\mathcal{G}^-)$ of the *lowering* generators. Note that the latter monomials are not in the standard Poincaré-Birkhoff-Witt basis of $U_q(\mathcal{G}^-)$, namely, instead of the generator E_{31} we have the generator of the same weight $\hat{C}$. These monomials produce the polynomials of $U_q(\mathcal{G}^+)$ since they act on $(E^{13})^r$ which is in $U_q(\mathcal{G}^+)$ and $(E^{13})^r\;\hat{1}$ is the highest weight vector. Note also that (25) and (27) may be obtained by standard formulae in the literature (cf., e.g., [5], [13], [11]), however, we could not find in the literature formulae involving the operator $\hat{C}$. Finally, we note that there exists a similar description of this correspondence only in terms of raising generators, in particular, involving an analogue of $\hat{C}$ in $U_q(\mathcal{G}^+)$. However, the present description is simpler for our purposes here, while the other we give in [16], where it is more useful.

Corollary: A polynomial realization of the GWZ basis is given by the formula:

$$\varphi_{(\mathrm{m})} = \Gamma_3(p_{(\mathrm{m})})\,1 = [m_{12} + m_{22} - r_1]_q! \sum_{t=0}^{r - m_{12}} \sum_{u \in \mathbb{Z}_+} \binom{m_{12} - m_{11} + t}{u}_q \times$$

$$\times\;(-1)^t\,\frac{q^{\frac{1}{2}(m_{22} - t - r_1)(m_{12} + m_{22} - r_1) + \frac{u}{2}(u - 2m_{22} - m_{12} + m_{11} + r_1 + t)}}{[t]_q!\,[m_{22} - t]_q!\,[m_{12} - m_{22} + 1 + t]_q!} \times$$

$$\times\;\frac{[m_{12} - m_{22} + 1]_q!\,[t + r_1 - m_{22}]_q!\,[m_{12} + m_{22} - m_{11} - u]_q!}{[m_{11} - m_{12} - m_{22} + r_1 + u]_q!\,[m_{12} + m_{22} - r_1 - u]_q!} \times$$

$$\times\;v_{u,\,m_{12} + m_{22} - r_1 - u,\,m_{11} - m_{12} - m_{22} + r_1 + u} \qquad (28)$$

Proof: Straightforward using (23) and the definition of $v_{\ell kj}$.

For later reference we note the explicit value of $\varphi_{(\mathrm{m})}$ for $z = 0$ (using (12)) :

$$\varphi_{(\mathrm{m})}|_{z=0} = \frac{\mathcal{N}^+_{(\mathrm{m})}}{\Gamma_q(r_1 - m_{11} + 1)}\,x^{m_{11}}\,y^{m_{12} + m_{22} - r_1} \qquad (29a)$$

$$\mathcal{N}^+_{(\mathrm{m})} = [r_1]_q!\,[m_{12} + m_{22} - r_1]_q! \sum_{t=0}^{r - m_{12}} \sum_{u \in \mathbb{Z}_+} (-1)^{t + u + m_{12} + m_{22} + r_1} \times$$

$$\times\;\binom{m_{12} - m_{11} + t}{u}_q\;(r + m_{11} - m_{12} - m_{22} + 1)^q_u \times$$

$$\times\;\frac{q^{\frac{1}{2}\{(u - m_{12} - m_{22} + r_1)(m_{12} - r - 1 + t) + (m_{12} + m_{22} - r_1)(m_{11}/2 - r_1)\}}}{[t]_q!\,[m_{22} - t]_q!\,[m_{12} - m_{22} + 1 + t]_q!} \times$$

$$\times\;\frac{[m_{12} - m_{22} + 1]_q!\,[t + r_1 - m_{22}]_q!\,[m_{12} + m_{22} - m_{11} - u]_q!}{[m_{11} - m_{12} - m_{22} + r_1 + u]_q!\,[m_{12} + m_{22} - r_1 - u]_q!} \qquad (29b)$$

which is useful for $r_1 - m_{11} + 1 > 0$. Otherwise it is zero (due to the singled out factor $\Gamma_q(r_1 - m_{11} + 1)$), and to obtain a non-zero value one first has to differentiate w.r.t. z $m_{11} - r$ times.

We note also the expression for the lowest weight *state* obtained from (23), (28) for $m_{12} = r_1$, $m_{11} = m_{22} = 0$:

$$p_{\mathrm{lws}}\;\hat{1} = (E_{21})^{r_1}\,\hat{C}^{r - r_1}\,(E_{32})^{r_1}\,(E_{13})^r\,\hat{1} =$$

$$= \mathcal{N}^+_{\mathrm{lws}}\;\hat{1} = ([r_1]_q!)^3\;\hat{1} \qquad (30a)$$

$$\varphi_{\mathrm{lws}} = \Gamma_3(p_{\mathrm{lws}})\,1 = ([r_1]_q!)^3 \qquad (30b)$$

which of course differ from $\hat{1}$, 1, resp. by a constant - the corresponding value of $\mathcal{N}^+_{(\mathrm{m})}$.

4. q - hypergeometric realization of the GWZ basis

In the previous Section we exhibited the relation of the GWZ basis and the polynomial basis $v_{(k,j)}$. By formula (28) this provides also a polynomial realization of the GWZ basis in the same variables x, y, z. However, (28) is not very explicit, since it contains a quadruple sum (a double sum in (28) and a double sum in (10)). Instead of partially summing (28), in this Section we shall find a polynomial realization directly (not relying on the correspondence with $v_{(k,j)}$) using the fact that the GWZ states are eigenvectors of the operators $\hat I_z, \hat Y, \hat I^2$.

We shall proceed as follows. Let us denote (as in (28)) the unknown polynomial function corresponding to (m) by:

$$\psi = \psi(x,y,z) = \psi_{(\mathrm{m})}(x,y,z) \tag{31}$$

Naturally, $\psi_{(\mathrm{m})}$ can differ from $\phi_{(\mathrm{m})}$ in (28) only by a multiplicative constant which we shall fix later.

In order to use effectively the fact that ψ is an eigenfunction of $\hat I_z, \hat Y, \hat I^2$ we use their explicit q-difference realization (6). We write:

$$\hat I_z \equiv \tfrac12 \Gamma_3(H_1) = \tfrac12 (2N_x - r_1 + N_z - N_y) \tag{32a}$$

$$\hat Y \equiv \tfrac13 \Gamma_3 (H_1 + 2H_2) = N_y + N_z - \tfrac13 (r_1 + 2r_2) \tag{32b}$$

$$\begin{aligned} \hat I^2 &\equiv \Gamma_3(E_{21})\, \Gamma_3(E_{12}) + [\hat I_z]_q\, [\hat I_z + 1]_q = \\ &= [N_x+1]_q\, [r_1 - N_x]_q\, q^{\frac12 (N_z - N_y)} + [N_z+1]_q\, [N_y]_q\, q^{\frac12 (r_1 - 2N_x)} + \\ &+ \frac{z}{xy} [N_x]_q\, [N_y]_q\, q^{\frac14 (r_1 - 2N_x + N_z - N_y + 2)} + \\ &+ \frac{xy}{z} [N_z]_q\, [r_1 - N_x]_q\, q^{\frac14 (r_1 - 2N_x + N_z - N_y - 2)} + [\hat I_z]_q\, [\hat I_z + 1]_q \end{aligned} \tag{32c}$$

The eigenfunction conditions satisfied by ψ are:

$$\hat I_z\, \psi = I_z\, \psi = \left(m_{11} - \tfrac12 (m_{12} + m_{22})\right) \psi \tag{33a}$$

$$\hat Y\, \psi = Y\, \psi = \left(m_{12} + m_{22} - \tfrac23 (r + r_1)\right) \psi \tag{33b}$$

$$\hat I^2\, \psi = [I]_q\, [I+1]_q\, \psi = \left[\frac{m_{12} - m_{22}}{2}\right]_q \left[\frac{m_{12} - m_{22}}{2} + 1\right]_q \psi \tag{33c}$$

Next we consider the operators $\hat I_z + \frac12 \hat Y$, $\hat Y$, from which we obtain the following homogeneity conditions:

$$(N_x + N_z)\, \psi = \left(\hat I_z + \tfrac12 \hat Y + \tfrac13 (r + r_1)\right) \psi = m_{11}\, \psi \tag{34a}$$

$$(N_y + N_z)\, \psi = \left(\hat Y + \tfrac13 (r - r_1)\right) = \kappa\, \psi\, , \qquad \kappa \equiv m_{12} + m_{22} - r_1 \tag{34b}$$

From these homogeneity conditions and the explicit form of (33c) we are prompted to make the following change of variables:

$$x' = x\, , \qquad y' = y\, , \qquad \zeta = \frac{z}{xy} \tag{35}$$

from which follows:

$$N_x = N_{x'} - N_\zeta\, , \qquad N_y = N_{y'} - N_\zeta\, , \qquad N_z = N_\zeta \tag{36}$$

Thus, the homogeneity conditions (34) simplify to:

$$N_{x'}\, \psi = m_{11}\, \psi \tag{37a}$$

$$N_{y'}\, \psi = \kappa\, \psi \tag{37b}$$

i.e., our polynomials actually have the form:

$$\psi = \psi_{(\mathrm{m})} = x'^{m_{11}}\, y'^{\kappa}\, \tilde\psi(\zeta) \tag{38}$$

Actually from this expression we can deduce that $\tilde\psi$ is a polynomial in ζ of degree at most $n_0 \equiv \min(m_{11}, \kappa)$. Indeed, if $\tilde\psi$ is a polynomial in ζ of higher degree, then ψ would not be a polynomial in x or y or both, contradicting our starting assumption.

Substituting now (38) in (33c) and taking into account (37) we obtain the following equation for $\tilde\psi$:

$$\begin{aligned} \Big(& [m_{11} + 1 - N_\zeta]_q\, [r_1 - m_{11} + N_\zeta]_q\, q^{N_\zeta - \frac12 \kappa} + \\ & + [1 + N_\zeta]_q\, [\kappa - N_\zeta]_q\, q^{\frac12 r_1 - m_{11} + N_\zeta} + \\ & + \zeta\, [m_{11} - N_\zeta]_q\, [\kappa - N_\zeta]_q\, q^{\frac14 (r_1 - \kappa) + \frac12 (1 - m_{11}) + N_\zeta} + \\ & + \zeta^{-1}\, [N_\zeta]_q\, [r_1 - m_{11} + N_\zeta]_q\, q^{\frac14 (r_1 - \kappa) - \frac12 (1 + m_{11}) + N_\zeta} + \\ & + [m_{11} - m_{12}]_q\, [m_{11} - m_{22} + 1]_q \Big)\, \tilde\psi(\zeta) = 0 \end{aligned} \tag{39}$$

The unique (up to nonzero multiple) polynomial solution of (39) is given by q-Jacobi or, equivalently, by q-hypergeometric polynomials. In particular, if $\beta = r_1 - m_{11} + 1 \notin \mathbb{Z}_-$ then such a solution is:

$$\begin{aligned} \tilde\psi_1(\zeta) &= {}_1F_0^q(-m_{22};\ q^{\frac14 (m_{22} - m_{12} - 2)}\, \zeta\,) \times \\ &\times\ {}_2F_1^q(-m_{11},\ r_1 - m_{12};\ r_1 - m_{11} + 1;\ q^{\frac14 (r_1 + \kappa)}\, \zeta\,) \end{aligned} \tag{40}$$

where ${}_2F_1^q$ is a q-hypergeometric polynomial:

$${}_2F_1^q(a,b;c;\zeta) = \sum_{s \in \mathbb{Z}_+} \frac{(a)_s^q\, (b)_s^q}{[s]_q!\, (c)_s^q}\, \zeta^s\, , \qquad c \notin \mathbb{Z}_- \tag{41}$$

${}_1F_0^q$ is a degenerate q-hypergeometric polynomial:

$${}_1F_0^q(a;\zeta) = \sum_{s \in \mathbb{Z}_+} \frac{(a)_s^q}{[s]_q!}\, \zeta^s = {}_2F_1^q(a,b;b;\zeta) \tag{42}$$

and $(\nu)^q_s$ is the q-Pochhammer symbol:

$$(\nu)^q_s \doteq [\nu+s-1]_q\,[\nu+s-2]_q\ldots[\nu]_q = \frac{\Gamma_q(\nu+s)}{\Gamma_q(\nu)} \qquad (43)$$

Note that (43) ensures that (41), (42), are polynomials of degree $\min(-a,-b)$, $-a$, resp., when $a,b \in \mathbb{Z}_-$, as is in our case. Note that for $q=1$ (41) goes into the standard hypergeometric polynomial, while (42) becomes just the binomial $(1-\zeta)^{m_{22}}$.

If $\beta = r_1 - m_{11} + 1 \in \mathbb{Z}_-$ then the polynomial solution of (39) is given by:

$$\begin{aligned}\tilde{\varphi}_2(\zeta) &= \zeta^{m_{11}-r_1}\ {}_1F^q_0(-m_{22};\ q^{\frac{1}{4}(m_{22}-m_{12}-2)}\,\zeta\ ) \times \\ &\times\ {}_2F^q_1(-r_1,\ m_{11}-m_{12};\ m_{11}-r_1+1;\ q^{\frac{1}{4}(r_1+\kappa)}\,\zeta\ )\end{aligned} \qquad (44)$$

In order to relate (40) and (44) it is enough to replace in (40)

$$\begin{aligned}&{}_2F^q_1(-m_{11},\ r_1-m_{12};\ r_1-m_{11}+1;\ q^{\frac{1}{4}(r_1+\kappa)}\,\zeta\ ) \mapsto \\ &\mapsto \frac{1}{\Gamma_q(r_1-m_{11}+1)}\ {}_2F^q_1(-m_{11},\ r_1-m_{12};\ r_1-m_{11}+1;\ q^{\frac{1}{4}(r_1+\kappa)}\,\zeta\ )\end{aligned}$$

Then this expression is valid for arbitrary $r_1 - m_{11} + 1$, and up to some multiplicative constant is equal to (44) when $r_1 - m_{11} + 1 \in \mathbb{Z}_-$. Thus, finally we shall write the polynomial solution of (39) as:

$$\begin{aligned}\tilde{\varphi}(\zeta) &= \frac{1}{\Gamma_q(r_1-m_{11}+1)}\ {}_1F^q_0(-m_{22};\ q^{\frac{1}{4}(m_{22}-m_{12}-2)}\,\zeta\ ) \times \\ &\times\ {}_2F^q_1(-m_{11},\ r_1-m_{12};\ r_1-m_{11}+1;\ q^{\frac{1}{4}(r_1+\kappa)}\,\zeta\ )\end{aligned} \qquad (45)$$

For $q=1$ (45) can be expressed as $(1-\zeta)^{m_{22}}$ times a Jacobi polynomial, cf. Appendix A.

The normalization of (45) is chosen so that for the lowest weight state ($m_{12} = r_1$, $m_{11} = m_{22} = 0$) we get the function 1. For the highest weight state ($m_{12} = m_{11} = r$, $m_{22} = r_1$) we get:

$$\begin{aligned}\tilde{\varphi}_{\rm hws}(\zeta) &= q^{\frac{1}{4}(r^2-r_1^2)}\,\frac{[r]_q!\,[r-r_1]_q!}{[r_1]_q!}\,\zeta^{r-r_1}\ {}_1F^q_0(-r_1;\ q^{\frac{1}{4}(r_1-r-2)}\,\zeta\ ) \\ \tilde{\varphi}_{\rm hws}(x,y,z) &= q^{\frac{1}{4}(r^2-r_1^2)}\,\frac{[r]_q!\,[r-r_1]_q!}{[r_1]_q!}\,x^{r_1}\,y^{r_1}\,z^{r-r_1}\ {}_1F^q_0(-r_1;\ q^{\frac{1}{4}(r_1-r-2)}\,\frac{z}{xy}\ )\end{aligned} \qquad (46)$$

(From ${}_2F^q_1$ survives only the term $\zeta^{r-r_1} = \zeta^{r_2}$.)

We shall write down the relation between the expressions (28) and (38) (with (45)) as:

$$\phi_{(\mathrm{m})} = \mathcal{N}_{(\mathrm{m})}\,\psi_{(\mathrm{m})} \qquad (47)$$

For $r_1 - m_{11} + 1 \notin \mathbb{Z}_-$ we have $\mathcal{N}_{(\mathrm{m})} = \mathcal{N}^+_{(\mathrm{m})}$ which we find by comparing

$$\psi_{(\mathrm{m})}|_{z=0} = \frac{1}{\Gamma_q(r_1-m_{11}+1)}\,x^{m_{11}}\,y^{m_{12}+m_{22}-r_1}$$

with (29). Note that (30) is a partial case of (47). When $r_1 - m_{11} + 1 \in \mathbb{Z}_-$, (i.e., $m_{11} - r_1 \in \mathbb{N}$), one has first to differentiate $m_{11} - r_1$ times w.r.t. z both $\phi_{(\mathrm{m})}$ and $\psi_{(\mathrm{m})}$ and then to set $z = 0$. In particular, for the highest weight state we compare (11) since then $\phi_{\rm hws} = v_{0r0} = \Gamma_3(E_{13})^r\,1$, and (46). Rewriting (11) as:

$$\phi_{\rm hws} = (-1)^{r_1}\,q^{\frac{1}{4}r_1(2-r_1)}\,[r]_q!\,x^{r_1}\,y^{r_1}\,z^{r-r_1}\ {}_1F^q_0(-r_1;\ q^{\frac{1}{4}(r_1-r-2)}\,\frac{z}{xy}\ ) \qquad (48)$$

we get:

$$\phi_{\rm hws} = (-1)^{r_1}\,q^{\frac{1}{4}(2r_1-r^2)}\,\frac{[r_1]_q!}{[r-r_1]_q!}\,\psi_{\rm hws} \qquad (49)$$

5. Explicit orthogonality of the GWZ basis

For the orthogonality of the GWZ basis we shall use an adaptation of the so called Shapovalov form [17]. This is a bilinear $\mathbb{C}$-valued form on Verma modules. The Verma module V^Λ of lowest weight $\Lambda \in \mathcal{H}^*$ is the lowest weight module such that $V^\Lambda = U_q(\mathcal{G}^+) \otimes v_0$, where $\mathcal{G}^+$ is the subalgebra of the raising generators E_{jk}, $j < k$, and v_0 is the lowest vector such that:

$$\begin{aligned}E_{jk}\,v_0 &= 0\ , \quad j > k \\ H_k\,v_0 &= \Lambda(H_k)\,v_0\end{aligned} \qquad (50)$$

The states in a Verma module correspond to the monomials of the Poincaré-Birkhoff-Witt basis of $U_q(\mathcal{G}^+)$, namely:

$$\begin{aligned}u_{\ell kj} &\equiv p_{\ell kj} \otimes v_0\ , \\ p_{\ell kj} &\equiv (E_{23})^\ell\,(E_{13})^k\,(E_{12})^j\ , \qquad \ell,k,j \in \mathbb{Z}_+\end{aligned} \qquad (51)$$

i.e., this basis is 1-to-1 with the basis $v_{\ell kj}$ for general r_k. Further, for simplicity se shall omit the sign $\otimes$, i.e., we shall write: $u_{\ell kj} = p_{\ell kj}\,v_0$ or $u = p\,v_0$ for short. We need the involutive antiautomorphism of $U_q(\mathcal{G})$ such that:

$$\omega(H_k) = H_k\ , \qquad \omega(E_{jk}) = E_{kj}, \qquad \omega(q) = q^{-1} \qquad (52)$$

Using the above conjugation the Shapovalov form can be defined as follows:

$$\begin{aligned}&(\ u\ ,\ u'\ ) = (\ p\,v_0\ ,\ p'\,v_0\ ) \equiv (\ v_0\ ,\ \omega(p)\,p'\,v_0\ ) = (\ \omega(p')\,p\,v_0\ ,\ v_0\ )\ , \\ &u = p\,v_0\ ,\quad u' = p'\,v_0\ , \qquad p,p' \in U_q(\mathcal{G}^+),\quad u,u' \in V^\Lambda\end{aligned} \qquad (53)$$

supplemented by the normalization condition $(v_0, v_0) = 1$. More explicitly from (53) we have:

$$\begin{aligned}&(u_{\ell kj}\ ,\ u_{\ell'k'j'}) = (p_{\ell kj}\,v_0\ ,\ p_{\ell'k'j'}\,v_0) = \\ &= (v_0\ ,\ \omega(p_{\ell kj})\,p_{\ell'k'j'}\,v_0) = (\ \omega(p_{\ell'k'j'})\,p_{\ell kj}v_0\ ,\ v_0) = \qquad (54a) \\ &= \Big(v_0\ ,\ (E_{21})^j\,(E_{31})^k\,(E_{32})^\ell\,(E_{23})^{\ell'}\,(E_{13})^{k'}\,(E_{12})^{j'}\,v_0\Big) \qquad (54b) \\ &= \Big((E_{21})^{j'}\,(E_{31})^{k'}\,(E_{32})^{\ell'}\,(E_{23})^\ell\,(E_{13})^k\,(E_{12})^j\,v_0\ ,\ v_0\Big) \qquad (54c)\end{aligned}$$

Note that subspaces with different weights are orthogonal w.r.t. to this form:

$$(u_{\ell kj}\ ,\ u_{\ell' k' j'})\ \sim\ \delta_{k+\ell,k'+\ell'}\ \delta_{k+j,k'+j'} \tag{55}$$

To show (55) one uses (54b) when $k+\ell > k'+\ell'$ and/or $k+j > k'+j'$, while (54c) is used when $k+\ell < k'+\ell'$ and/or $k+j < k'+j'$.

We shall give a realization of the Shapovalov form in our setting in the following way. Using the 1-to-1 correspondence we replace $u_{\ell kj}$ by $v_{\ell kj}$ and the lowest weight vector v_0 by the lowest weight vector $\hat{1}$ of the abstract finite-dimensional irrep and by the function 1 in the polynomial realization. Namely, we shall use instead of (53) the following bilinear form:

$$(\ u\ ,\ u'\ )_f\ =\ (\ p\,\hat{1}\ ,\ p'\,\hat{1}\ )_f\ \equiv\ (\ \Gamma_3(\omega(p))\ \Gamma_3(p')\ 1)\,|_{x=y=z=0} \tag{56}$$

More explicitly, we have:

$$\begin{aligned}(u_{\ell kj}\ ,\ u_{\ell' k' j'})_f\ &=\ (p_{\ell kj}\,\hat{1}\ ,\ p_{\ell' k' j'}\,\hat{1})_f\ =\\ &=\ (\Gamma_3(\omega(p_{\ell kj}))\ \Gamma_3(p_{\ell' k' j'})\ 1)\,|_{x=y=z=0}\ =\ (\hat{p}_{\ell kj}\ v_{\ell' k' j'})\,|_{x=y=z=0}\ ,\\ \hat{p}_{\ell kj}\ &\equiv\ \Gamma_3(\omega(p_{\ell kj}))\ =\ (\Gamma_3(E_{21}))^j\ (\Gamma_3(E_{31}))^k\ (\Gamma_3(E_{32}))^\ell\end{aligned} \tag{57}$$

Clearly, when $k+\ell > k'+\ell'$ and/or $k+j > k'+j'$ we have $\hat{p}_{\ell kj}\ v_{\ell' k' j'}\ =\ 0$. When $k+\ell < k'+\ell'$ and $k+j < k'+j'$ the expression $\hat{p}_{\ell kj}\ v_{\ell' k' j'}$ is not zero but a homogeneous polynomial of x,y,z which vanishes after the substitution $x=y=z=0$. Finally, when $k+\ell = k'+\ell'$ and $k+j = k'+j'$ the expression $\hat{p}_{\ell kj}\ v_{\ell' k' j'}$ is a numerical one coinciding with $(u_{\ell kj}\ ,\ u_{\ell' k' j'})$ because of the automorphism.

We can further simplify (57) if we set $x=y=z=0$ in $\hat{p}_{\ell kj}$ from the very beginning, namely, we replace $\hat{p}_{\ell kj}$ by:

$$\begin{aligned}\check{p}_{\ell kj}\ &\equiv\ (\check{\Gamma}_3(E_{21}))^j\ (\check{\Gamma}_3(E_{31}))^k\ (\check{\Gamma}_3(E_{32}))^\ell\\ \check{\Gamma}_3(E_{21})\ &\equiv\ D_x\ q^{\frac{1}{4}(N_z-N_y)}\\ \check{\Gamma}_3(E_{32})\ &\equiv\ D_y q^{\frac{1}{4}N_x}\ =\ \Gamma_3(E_{32})\\ \check{\Gamma}_3(E_{31})\ &\equiv\ D_z q^{\frac{1}{4}(r_1-N_x+2N_y)}\ =\ \Gamma_3(E_{31})\end{aligned} \tag{58}$$

Note that this operation affects only $\Gamma_3(E_{21})$ and that it is easy to check that

$$(u_{\ell kj}\ ,\ u_{\ell' k' j'})_f\ \equiv\ (\check{p}_{\ell kj}\ v_{\ell' k' j'})\,|_{x=y=z=0} \tag{59}$$

Further we note that:

$$\check{p}_{\ell kj}\ =\ q^{\frac{1}{4}((\ell-k)N_x+(2k-j)N_y+jN_z+j\ell+k(r_1-j))}\ (D_x)^j\ (D_z)^k\ (D_y)^\ell \tag{60}$$

We shall use the above to prove the main result in this Section:

Theorem 2: Let $(\mathbf{m})$ and $(\mathbf{m}')$ be two different GWZ patterns. Then we have:

$$\left(\ p_{(\mathbf{m})}\,\hat{1}\ ,\ p_{(\mathbf{m}')}\,\hat{1}\ \right)_f\ =\ 0 \tag{61}$$

Proof: For the calculation we need first $\Gamma_3(\omega(p_{(\mathbf{m})}))$ for which we can use (23) to express it in terms of $\hat{p}_{\ell kj}$ and further, taking into account (58), in terms of $\check{p}_{\ell kj}$ writing directly (and using the notation $\kappa = m_{12}+m_{22}-r_1$):

$$\begin{aligned}\Gamma_3(\widetilde{\omega(p_{(\mathbf{m})})})\ =\ [\kappa]_q!\ &\sum_{t=0}^{r-m_{12}}\ \sum_{u\in\mathbb{Z}_+}\ (-1)^t \binom{m_{12}-m_{11}+t}{u}_q\ \times\\ &\times\ \frac{q^{\frac{1}{2}\kappa(t+m_{12}-\kappa)+\frac{u}{2}(2\kappa+r_1-m_{12}-m_{11}-t-u)}}{[t]_q!\ [m_{22}-t]_q!\ (m_{12}-m_{22}+2)^q_t}\ \times\\ &\times\ \frac{[t+m_{12}-\kappa]_q!\ [r_1+\kappa-m_{11}-u]_q!}{[m_{11}-\kappa+u]_q!\ [\kappa-u]_q!}\ \check{p}_{u,\kappa-u,m_{11}-\kappa+u}\end{aligned} \tag{62}$$

Further, we need $\Gamma_3(p_{(\mathbf{m}')})\,1\ =\ \phi_{(\mathbf{m}')}$ for which we shall use its relation (47) with our explicit solution $\psi_{(\mathbf{m}')}$ from (38) (with (45)) in terms of q-hypergeometric polynomial which we use in the original variables x,y,z. Thus we have to calculate:

$$\begin{aligned}\left(p_{(\mathbf{m})}\,\hat{1}\ ,\ p_{(\mathbf{m}')}\,\hat{1}\right)_f\ =\ \mathcal{N}_{(\mathbf{m}')}\ [\kappa]_q!\ &\sum_{t=0}^{r-m_{12}}\ \sum_{u\in\mathbb{Z}_+}\ (-1)^t \binom{m_{12}-m_{11}+t}{u}_q\ \times\\ &\times\ \frac{q^{\frac{1}{4}(u(r_1+\kappa-2t-2m_{12})\ +\ \kappa(r_1-\kappa+2t+2m_{12}-m_{11}))}}{[t]_q!\ [m_{22}-t]_q!\ (m_{12}-m_{22}+2)^q_t}\ \times\\ &\times\ \frac{[t+m_{12}-\kappa]_q!\ [r_1+\kappa-m_{11}-u]_q!}{[m_{11}-\kappa+u]_q!\ [\kappa-u]_q!}\ \times\\ &\times\ \Big(q^{\frac{1}{4}((2u-\kappa)N_x+(3(\kappa-u)-m_{11})N_y+(m_{11}-\kappa+u)N_z)}\ \times\\ &\times\ (D_x)^{m_{11}-\kappa+u}\ (D_z)^{\kappa-u}\ (D_y)^u\ \times\\ &\times\ \sum_{s=0}^{m'_{22}}\ \sum_{n\in\mathbb{Z}_+}\ (-1)^s\ q^{\frac{1}{4}s(m'_{22}-m'_{12}-2)}\binom{m'_{22}}{s}_q\ \times\\ &\times\ \frac{(m'_{22}-m'_{11})^q_n\ (r_1-m'_{12})^q_n}{[n]_q!\ \Gamma_q(r_1-m'_{11}+1+n)}\ \times\\ &\times\ q^{\frac{1}{4}n(r_1+\kappa')}\ x^{m'_{11}-s-n}\ y^{\kappa'-s-n}\ z^{s+n}\Big)\Big|_{x=y=z=0}\end{aligned} \tag{63}$$

In the next step we perform the action of the operators N_x, N_y, N_z, D_x, D_y, D_z and set $x=y=z=0$. Obviously nonzero contributions will come only from the terms in which the powers of D_x, D_y, D_z, resp., coincide with the powers of x,y,z, resp. From this follows, firstly, that the action of the operators N_x, N_y, N_z trivializes since the nonzero contributions have zero homogeneity, secondly, that the RHS of (63) is zero unless:

$$m_{11} = m'_{11}\,, \qquad \kappa = \kappa'\,, \qquad u+s+n = \kappa \tag{64}$$

The first two conditions are part of the orthogonality property which we aim to prove. In particular, from the second follows that $m_{12}+m_{22} = m'_{12}+m'_{22}$, i.e., $(\mathbf{m})$ and $(\mathbf{m}')$ already coincide up to one parameter. The last condition in (64) is fixing one of the summations, say, in u. Taking into account all this we have:

$$\begin{aligned}\left(p_{(\mathbf{m})}\,\hat{1}\,,\, p_{(\mathbf{m}')}\,\hat{1} \right)_f &= \delta_{m_{11},m'_{11}}\,\delta_{\kappa,\kappa'}\, q^{\frac{1}{4}\kappa(2r_1-m_{11})}\,\mathcal{N}_{(\mathbf{m}')}\,[\kappa]_q! \times \\ &\times \sum_{t=0}^{r-m_{12}}\ \sum_{n\in\mathbb{Z}_+}\ \sum_{s=0}^{m'_{22}} (-1)^{s+t}\, q^{\frac{1}{2}s(m_{12}-r_1-\kappa+t+m'_{22}-1)} \times \\ &\times \binom{m'_{22}}{s}_q \frac{[r_1-m_{11}+n+s]_q!}{[r_1-m_{11}-m_{22}+t+n+s]_q!} \times \\ &\times \frac{[t+m_{12}-\kappa]_q!\,[m_{12}-m_{11}+t]_q!}{[t]_q!\,[m_{22}-t]_q!\,(m_{12}-m_{22}+2)^q_t} \times \\ &\times q^{\frac{1}{2}n(m_{12}+t)}\,\frac{(m'_{22}-m_{11})^q_n\,(r_1-m'_{12})^q_n}{[n]_q!\,\Gamma_q(r_1-m_{11}+1+n)}\end{aligned} \tag{65}$$

In the next step we take the s summation using a formula generalizing the classical hypergeometric summation formula [18] (9.122) :

$${}_2F_1(a,b;c;1) = \frac{\Gamma(c)\,\Gamma(c-a-b)}{\Gamma(c-a)\,\Gamma(c-b)} \tag{66}$$

Or, equivalently for $a = -\nu$, $\nu\in\mathbb{Z}_+$:

$$\sum_{s=0}^{\nu} (-1)^s \binom{\nu}{s} \frac{(b)_s}{(c)_s} = \frac{(c-b)_\nu}{(c)_\nu} \tag{67}$$

For $q\neq 1$ there are actually two formulae:

$$\begin{aligned}{}_2F_1^q(-\nu,\ b;\ c;\ q^{\pm\frac{1}{2}(b-c+1-\nu)}) &= \sum_{s=0}^{\nu}(-1)^s\binom{\nu}{s}_q \frac{(b)^q_s}{(c)^q_s}\, q^{\pm\frac{1}{2}s(b-c+1-\nu)} = \\ &= \frac{(c-b)^q_\nu}{(c)^q_\nu}\, q^{\pm\frac{1}{2}b\nu}\end{aligned} \tag{68}$$

(cf. [19] for a partial case). Using (68) with $(\nu,b,c) = (m'_{22}, r_1-m_{11}+n+1, r_1-m_{11}+n+1-m_{22}+t)$ and sign minus we obtain:

$$\begin{aligned}\left(p_{(\mathbf{m})}\,\hat{1}\,,\, p_{(\mathbf{m}')}\,\hat{1} \right)_f &= \delta_{m_{11},m'_{11}}\,\delta_{\kappa,\kappa'}\,\mathcal{N}_{(\mathbf{m}')}\,[\kappa]_q!\, q^{\frac{1}{4}\kappa(2r_1-m_{11})}\, q^{\frac{1}{2}m'_{22}(m_{11}-r_1-1)} \times \\ &\times \sum_{t=0}^{r-m_{12}}\ \sum_{n\in\mathbb{Z}_+} (-1)^{t+m'_{22}}\,\frac{q^{\frac{1}{2}n(m_{12}+t-m'_{22})}}{[t]_q!\,(m_{12}-m_{22}+2)^q_t} \times \\ &\times \frac{[t+m_{12}-\kappa]_q!\,[m_{12}-m_{11}+t]_q!}{\Gamma_q(m_{22}-m'_{22}-t+1)} \times \\ &\times \frac{(m'_{22}-m_{11})^q_n\,(r_1-m'_{12})^q_n}{[n]_q!\,[r_1+m'_{22}-m_{11}-m_{22}+t+n]_q!}\end{aligned} \tag{69}$$

where we have used also the easy identity:

$$\frac{\Gamma_q(t+m'_{22}-m_{22})}{[m_{22}-t]_q!\,\Gamma_q(t-m_{22})} = \frac{(-1)^{m'_{22}}}{\Gamma_q(m_{22}-m'_{22}-t+1)} \tag{70}$$

In the next step we take the n summation using again (68) for sign minus and:

$$(\nu,b,c) = (m_{11}-m'_{22}, m'_{22}-\kappa, r_1+m'_{22}-m_{11}-m_{22}+t+1)$$

where we have taken into account that $r_1-m'_{12} = m'_{22}-\kappa$, (cf. (64)). Thus we obtain:

$$\begin{aligned}\left(p_{(\mathbf{m})}\,\hat{1}\,,\, p_{(\mathbf{m}')}\,\hat{1} \right)_f &= \delta_{m_{11},m'_{11}}\,\delta_{\kappa,\kappa'}\,\mathcal{N}_{(\mathbf{m}')}\,[\kappa]_q! \times \\ &\times q^{-\frac{1}{2}m'_{22}(m'_{12}+1)}\, q^{\frac{1}{4}\kappa(2r_1+m_{11})} \times \\ &\times \sum_{t=0}^{r-m_{12}} \frac{(-1)^{t+m'_{22}}}{[t]_q!\,(m_{12}-m_{22}+2)^q_t} \times \\ &\times \frac{\Gamma_q(m_{12}+t+1-m'_{22})}{\Gamma_q(m_{22}-m'_{22}-t+1)}\end{aligned} \tag{71}$$

Now the last sum is zero unless $m_{22}-m'_{22} \geq 0$ because of the $\Gamma_q(m_{22}-m'_{22}-t+1)$ in the denominator. (We note also that $m_{12}-m'_{22}\geq 0$.) With this supposition we take the last summation in t using again (68) for:

$$(\nu,b,c) = (m_{22}-m'_{22}, m_{12}+1-m'_{22}, m_{12}-m_{22}+2)$$

Actually here we have a particular case of (68) because $b-c+1-\nu = 0$. In this case and with $c(c+\nu)>0$ (as is our case) it becomes:

$${}_2F_1^q(-\nu,\ c+\nu-1;\ c;\ 1) = \frac{1}{(c)^q_\nu\,\Gamma_q(1-\nu)} = \delta_{\nu 0}\,, \qquad c(c+\nu)>0 \tag{72}$$

Thus, the sum in t is zero unless $\nu = m_{22}-m'_{22} = 0$, i.e., we have obtained the desired orthogonality. Thus we obtain (using (72)):

$$\begin{aligned}\left(p_{(\mathbf{m})}\,\hat{1}\,,\, p_{(\mathbf{m}')}\,\hat{1} \right)_f &= \delta_{m_{11},m'_{11}}\,\delta_{m_{12},m'_{12}}\,\delta_{m_{22},m'_{22}}\,(-1)^{m_{22}} \times \\ &\times q^{-\frac{1}{2}m_{22}(m_{12}+1)}\, q^{\frac{1}{4}\kappa(2r_1+m_{11})} \times \\ &\times \mathcal{N}_{(\mathbf{m})}\,[\kappa]_q!\,[m_{12}-m_{22}]_q!\end{aligned} \tag{73}$$

This proves the Theorem. •

We can use the form (54), (57) to define a scalar product if we consider our conjugation ω as antilinear. Then we actually restrict to the real form $U_q(su(3))$ and

q is restricted to be a phase $|q| = 1$ (cf. (52)). Then we define the scalar product of the functions $\dot\varphi_{(\mathbf{m})} = \Gamma_l(p_{(\mathbf{m})})\ 1$ or $\dot\psi_{(\mathbf{m})}$

$$\Big(\varphi_{(\mathbf{m})} \cdot \varphi_{(\mathbf{m}')} \Big)_p \ \equiv \ \Big(p_{(\mathbf{m})}\ \hat{1}\ ,\ p_{(\mathbf{m}')}\ \hat{1} \Big)_f \tag{74a}$$

$$\Big(\psi_{(\mathbf{m})} \cdot \psi_{(\mathbf{m}')} \Big) \ \equiv \ \frac{1}{|\mathcal{N}_{(\mathbf{m})}|^2} \Big(p_{(\mathbf{m})}\ \hat{1}\ ,\ p_{(\mathbf{m}')}\ \hat{1} \Big)_f \tag{74b}$$

We note two partial cases:

$$\begin{aligned} (\ \varphi_{\rm lws}\ ,\ \varphi_{\rm lws}\ )_p \ &= \ ([r_1]_q!)^4 \\ (\ \psi_{\rm lws}\ ,\ \psi_{\rm lws}\ ) \ &= \ \frac{1}{([r_1]_q!)^2} \end{aligned} \tag{75a}$$

$$\begin{aligned} (\ \varphi_{\rm hws}\ ,\ \varphi_{\rm hws}\ )_p \ &= \ [r]_q!\ [r_1]_q! \\ (\ \psi_{\rm hws}\ ,\ \psi_{\rm hws}\ ) \ &= \ \frac{[r]_q!\ ([r-r_1]_q!)^2}{[r_1]_q!} \end{aligned} \tag{75b}$$

Using this scalar product we can introduce orthonormal GWZ polynomials by:

$$\hat\psi_{(\mathbf{m})} \ \equiv \ \frac{\psi_{(\mathbf{m})}}{|\Big(\psi_{(\mathbf{m})}\ ,\ \psi_{(\mathbf{m})} \Big)|} \tag{76}$$

so that

$$\Big(\hat\psi_{(\mathbf{m})}\ ,\ \hat\psi_{(\mathbf{m}')} \Big) \ = \ \delta_{(\mathbf{m}),(\mathbf{m}')} \tag{77}$$

In particular, we have:

$$\hat\psi_{\rm lws}(x,y,z) \ = \ [r_1]_q! \tag{78a}$$

$$\begin{aligned} \hat\psi_{\rm hws}(x,y,z) \ &= \ \frac{1}{[r-r_1]_q!} \left(\frac{[r_1]_q!}{[r]_q!} \right)^{\frac{1}{2}} \psi_{\rm hws}(x,y,z) \ = \\ &= \ \left(\frac{[r]_q!}{[r_1]_q!} \right)^{\frac{1}{2}} q^{\frac{1}{4}(r^2-r_1^2)}\ x^{r_1}\ y^{r_1}\ z^{r-r_1}\ {}_1F_0^q(-r_1;\ q^{\frac{1}{4}(r_1-r-2)}\ \frac{z}{xy}\ ) \end{aligned} \tag{75b}$$

To obtain the explicit normalizations for $\hat\psi_{(\mathbf{m})}$ for arbitrary $(\mathbf{m})$ one can simply use that by Theorem 1 it is produced by the action of the operator:

$$(\Gamma_3(E_{21}))^{m_{12}-m_{11}}\ (\Gamma_3(\tilde{C}))^{r-m_{12}}\ (\Gamma_3(E_{32}))^{r_1-m_{22}}$$

on the highest weight state.

6. Outlook

In the present paper our considerations were for generic q since the representation theory of quantum groups is very different for q a root of unity, while our results here seem to be new also for the classical case with $q = 1$. In a follow-up paper [16] we give explicitly the GWZ description of the $U_q(sl(3))$ irregular irreps at roots of unity. (In [16] we give also references to related work in the roots of unity case which we omitted here.) Those are irreps fixed by the same parameters r_1, r_2 as here, yet they have dimensions smaller than the classical one (cf. (9)), the reason being that to obtain an irregular irrep we have to make factorization of an additional submodule. Though the dimension of the additional submodule to be factored out is known - it is of another $SU(3)$ irrep of smaller dimension (cf. [20]) - until now it was not clear which GWZ states decouple from the irrep (except for a couple of simple examples quoted in [16]). We give an explicit answer to this question in [16] moreover presented in an intuitively clear geometric picture.

Another interesting question is the extension of our approach to $U_q(sl(n))$ for $n > 3$. We have the starting tool, namely, the lowest weight representations [14], however, things become much more complicated since we have to deal with $n(n-1)/2$ variables. Work in this direction is in progress.

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Appendix A. Solving (39) for q=1

For $q = 1$ (39) is equivalent (looking for polynomial solutions) to:

$$\begin{aligned} &\Big(\zeta\,(1-\zeta)^2\,\partial_\zeta^2 \ + \ (1-\zeta)\,\big((m_{11}-\kappa-1)\zeta + r_1 - m_{11} + 1\big)\,\partial_\zeta \ - \\ &- \ \kappa\, m_{11}\,(1-\zeta) \ + \ m_{22}(1+m_{12})\Big)\ \tilde\psi(\zeta) \ = \ 0 \end{aligned} \tag{A.1}$$

We make the following substitution in (A.1): $\tilde\psi(\zeta) = (1-\zeta)^{m_{22}}\,\phi(\zeta)$. We thus obtain the following equation for ϕ :

$$\begin{aligned} &\Big(\zeta\,(1-\zeta)\,\partial_z^2 \ + \ \big((m_{11}+m_{12}-m_{22}-r_1-1)\zeta + r_1 - m_{11} + 1\big)\,\partial_\zeta \ + \\ &+ \ (m_{22}-m_{11})(m_{12}-r_1)\Big)\ \phi(\zeta) \ = \ 0 \end{aligned} \tag{A.2}$$

Further we make the change of variable: $\zeta = (1+\eta)/2$ after which (A.2) becomes:

$$\Big((1-\eta^2)\, \partial_\eta^2 + ((m_{12} - m_{22} + m_{11} - r_1 - 1)\eta + m_{12} - m_{22} - m_{11} + r_1 + 1)\, \partial_\eta + \\ + (m_{22} - m_{11})(m_{12} - r_1) \Big)\, \tilde{\phi}(\eta) = 0\,, \qquad \tilde{\phi}(\eta) = \tilde{\varphi}(\tfrac{1}{2}(1+\eta)) \tag{A.3}$$

The unique (up to nonzero multiple) polynomial solution of (A.3) is given by [18] (8.964):

$$\begin{aligned} \tilde{\phi}(\eta) &= P_n^{(m_{22}-m_{12}-1,r_1-m_{11})}(\eta) \\ n &= \min\,(m_{11} - m_{22}, m_{12} - r_1) \in \mathbb{Z}_+ \end{aligned} \tag{A.4}$$

where $P_n^{(a,b)}(\eta)$ is the the Jacobi polynomial [18] (8.960):

$$P_n^{(a,b)}(\eta) = \frac{1}{2^n} \sum_{s=0}^{n} \frac{\Gamma(n+a+1)}{s!\,\Gamma(n+a+1-s)}\, \frac{\Gamma(n+b+1)}{(n-s)!\,\Gamma(s+b+1)}\, (\eta-1)^{n-s}\, (\eta+1)^s \tag{A.5}$$

Thus the solution of (39) is

$$\tilde{\varphi}(\zeta) = (1-\zeta)^{m_{22}}\, P_n^{(a,b)}(2\zeta - 1) \tag{A.6}$$

When $a, b \in \mathbb{Z}$ (as is our case) we have:

$$\frac{\Gamma(n+a+1)}{s!\,\Gamma(n+a+1-s)} = \begin{cases} \binom{n+a}{s} & \text{for } s \le n+a \\ 0 & \text{for } s > n+a \ge 0 \\ (-1)^s \binom{s-n-a-1}{s} & \text{for } n+a < 0 \end{cases} \tag{A.7a}$$

$$\frac{\Gamma(n+b+1)}{(n-s)!\,\Gamma(s+b+1)} = \begin{cases} \binom{n+b}{n-s} & \text{for } s+b \ge 0 \\ 0 & \text{for } s+b < 0 \le n+b \\ (-1)^{n-s} \binom{-s-b-1}{n-s} & \text{for } n+b < 0 \end{cases} \tag{A.7b}$$

The middle terms of (A.7) indicate that some terms may be missing in the sums in (A.4), (A.5). In our case $n + a = \min\,(m_{11} - m_{12} - 1, m_{22} - r_1 - 1) < 0$, $n + b = \min\,(r_1 - m_{22}, m_{12} - m_{11}) \ge 0$, and thus some terms would be missing only when $b = r_1 - m_{11} < 0$, namely, these would be the terms with $s < m_{11} - r_1$. Taking this into account and the relation between the Jacobi polynomial and the hypergeometric function:

$$P_n^{(a,b)}(2\zeta - 1) = \frac{(-1)^n\, \Gamma(b+1+n)}{n!\,\Gamma(b+1)}\ {}_2F_1(-n,\ a+b+1+n,\ b+1;\ \zeta\,) \tag{A.8}$$

we see that (A.6) coincides with (45) for $q = 1$ up to the inessential factor $(-1)^n\,\Gamma(b+1+n)/n!$, while the important factor $1/\Gamma(b+1)$ was taken into account in (45).

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