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Temperature field due to time-dependent heat sources in a large rectangular grid – Derivation of analytical solution

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TEMPERATURE FIELD DUE TO TIME-DEPENDENT HEAT SOURCES IN A LARGE RECTANGULAR GRID

DERIVATION OF ANALYTICAL SOLUTION

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TEMPERATURE FIELD DUE TO TIME-DEPENDENT HEAT SOURCES IN A LARGE RECTANGULAR GRID.

I. Derivation of analytical solution.

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Keywords: Temperature field from heat sources, rectangular grid, time-dependent three-dimensional analytical solution, superposition, nuclear waste repository in rock, canister temperature.

Contents

Abstract

The temperature field in rock due to a large rectangular grid of heat-releasing canisters containing nuclear waste is studied. The solution is by superposition divided into different parts. There is a global temperature field due to the large rectangular canister area, while a local field accounts for the remaining heat source problem.

The global field is reduced to a single integral. The local field is also solved analytically using solutions for a finite line heat source and for an infinite grid of point sources. The local solution is reduced to three parts, each of which depends on two spacial coordinates only.

The temperatures at the envelope of a canisters are given by the global and local solutions. The local part is here represented by a single thermal resistance, which is given by an explicit formula.

The results are illustrated by a few numerical examples dealing with the KBS-3 concept for storage of nuclear waste.

Sammanfattning

Temperaturfältet i berg från ett stort rektangulärt fält av värmeavgivande kapslar, vilka innehåller radioaktivt avfall, studeras. Lösningen uppdelas genom superposition i olika delar. En global del avser värmeavgivning över en rektangulär area, medan ett lokalt fält tar hand om lösningen för det resterande värmekällsproblemet.

Det globala fältet kan reduceras till en enkelintegral. Det lokala fältet erhålls också analytiskt genom superponering av lösningar för en ändlig linjekälla och för ett oändligt rutnät av punktkällor. Den lokala lösningen består av tre delar, vilka är funktioner av enbart två rumskoordinater.

Temperaturen på kapslarnas yta ges av global och lokal lösning. Den lokala delen representeras här av ett enda termiskt motstånd, vilket ges av en explicit formel. Lösningen illustreras med några numeriska exempel med data från KBS-3-konceptet för lagring av kärnavfall.

1 Introduction

In the KBS-3 concept, nuclear waste from the Swedish nuclear power plants is put in some six thousand canisters, which are buried in solid rock at a depth H of 500 m below the ground surface. See Figure 1. The canisters with a height H_c of around 5 m and the diameter $2R_c \simeq 0.8$ m are placed in boreholes below parallel tunnels at a spacing D of some 6 m. The distance D' between the tunnels is some 25 m. The nuclear waste repository consists of canisters in a large, rectangular grid. The total area of the rectangle with the side lengths 2L and 2B is almost 1 km^2 ($6{\cdot}25 \cdot 6000$ m^2). Each canister lies at the center of a small rectangle with the side lengths D and D', Figure 1, right.

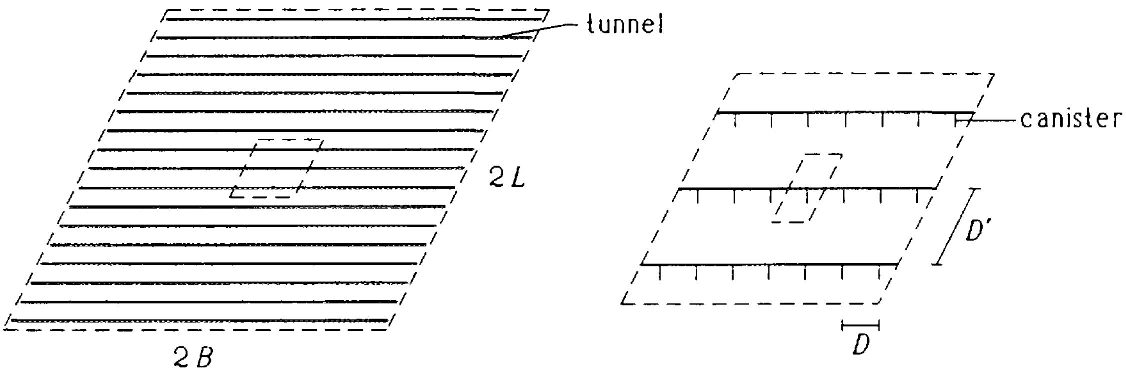

Figure 1: Rectangular grid of canisters according to the KBS-3 concept. The right-hand figure shows an internal part with tunnels and canisters in greater detail.

The canisters release heat due to radioactive decay in the nuclear waste. This heat release per canister, $Q_0(t)$ (W), decreases with time in a wellknown way. We will use a sum of a few exponentials with different decay times to represent the function $Q_0(t)$. The heat release from all canisters creates a complex three-dimensional, time-dependent temperature field in the ground in and around the repository.

The temperatures near canisters are a major design parameter. The maximum temperature is of particular interest. The large-scale field influences potential water movements in cracks and fissure zones in the rock. It also creates a thermal stress field in the rock. The temperature field around an internal canister has been studied numerically in Thunvik, Braester (1991). Thermal stresses and temperature fields are studied in Israelsson (1995). Here, the solution for point sources at the center of each canister is summed directly.

The aim of this study is to derive analytical solutions for the temperature field. The large-scale and long-time behaviour is one main interest. The analytical approach is used with the aim to gain a physical understanding of the various processes and their interactions. The second main interest is to obtain as simple formulas as possible for the canister temperatures. A third aim is to describe the local field around a single canister in an as lucid way as possible.

This paper deals mainly with derivation of the solution with its various components. The results will be applied to the KBS-3 case in a second report. The thermoelastic problem using the global temperature solution is studied in the parallel report, Claesson, Probert (1995). The corresponding problem of the thermoelastic response to a single, finite line heat source is studied in Claesson, Hellström (1995).

2 Thermal problem

The canisters lie in the rectangle $-L < x < L$, $-B < y < B$, $z = 0$, and the ground surface lies at $z = H$. There is an undisturbed ground temperature with an annual variation in the top few meters and a certain increase downwards due to the geothermal gradient. Our temperature field $T(x,y,z,t)$ due to the heat release from the canisters is the temperature *above* or superimposed on the undisturbed temperature field. The undisturbed temperature at the level of the repository is $T_{rep,0}$. The total temperature at $z = 0$ is then $T(x,y,0,t) + T_{rep,0}$.

The top layers near the ground surface is of no importance for our thermal process initiated at a depth of some 500 m below. We have by assumption a homogeneous ground $-\infty < z < H$ with the thermal conductivity λ (W/(m·K)) and volumetric heat capacity ρc (J/(m³·K)). The thermal diffusivity is $a = \lambda/(\rho c)$ (m²/s). The excess temperature satisfies the heat conduction equation:

$$\frac{1}{a}\frac{\partial T}{\partial t} = \frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2} + \frac{Q(x,y,z,t)}{\lambda} \tag{1}$$

Here, $Q(x,y,z,t)$ represents the heat sources at the canisters. The temperature at the ground surface is zero, since we consider the temperature above undisturbed conditions:

$$T(x,y,H,t) = 0 \tag{2}$$

The heat release from each canister is $Q_0(t)$ (W). The canisters lie in a rectangular grid:

$$\begin{aligned} &Q = Q_0(t) \quad \text{at} \quad (x,y,z) = (mD', nD, 0) \\ &-L < mD' < L \qquad -B < nD < B \end{aligned} \tag{3}$$

The grid points lie within the repository rectangle when the integers m and n satisfy the above inequalities. Actually, the heat release is distributed over the canister height H_c:

$$Q = Q_0(t)/H_c \quad \text{(W/m)} \quad \text{over} \quad -H_c/2 < z < H_c/2 \quad (x = mD', y = nD) \tag{4}$$

We will use this more precise heat source for the local temperature field near a canister, while all other canisters are represented by point heat sources $Q_0(t)$ at the canister grid points in the plane $z = 0$.

The heat release may be represented by a few exponentially decaying components. We will use two components:

$$Q_0(t) = Q_1 e^{-t/t_1} + Q_2 e^{-t/t_2} \quad \text{(W)} \tag{5}$$

We will in this study give a few numerical examples. The following data from the KBS-3 concept will be used:

$L = 500$ m	$B = 500$ m	$H = 500$ m	
$D = 6$ m	$D' = 25$ m	$T_{rep,0} = 15$ °C	
$\lambda = 3.5$ W/(m·K)	$\rho c = 2.16 \cdot 10^6$ J/(m³·K)	$a = 1.62 \cdot 10^{-6}$ m²/s	
$Q_1 = 750$ W	$t_1 = 46$ years	$H_c = 5$ m	(6)
$Q_2 = 250$ W	$t_2 = 780$ years	$R_c = 0.4$ m	

3 Superposition

Our temperature field is generated by a complex set of heat sources which contribute in different ways (depending on time and distance) to the total solution. A main tool of analysis is to use superposition. Our problem is a linear one, which means that solutions for different heat sources may be superimposed. Let $Q = Q_1 + Q_2$ be *any* division of the total heat source field in two components, and let T_1 and T_2 be the temperature fields caused by Q_1 and Q_2, respectively. Then the total field due to Q is the equal to the sum of T_1 and T_2:

$$T = T_1 + T_2 \tag{7}$$

The heat source $Q_0(t)$ varies in time. Let $T_{inst}(x, y, z, t)$ be the solution for the case when a unit amount of heat is released instantaneously at time $t = 0$ at all heat sources. The temperature field due to $Q_0(t)$ is obtained as an integral in time:

$$T(x, y, z, t) = \int_0^t Q_0(t')T_{inst}(x, y, z, t - t')dt' \tag{8}$$

This particular superposition is called Duhamel's theorem, Carslaw-Jaeger (1957). The amount $Q_0(t')dt'$ is released in the interval t' to $t' + dt'$. The response to this at time t is $Q_0(t')dt' \cdot T_{inst}(x, y, z, t - t')$. The integral over $0 < t' < t$ gives the total field.

Using Duhamel's superposition, we have to calculate the temperature field for unit, instantaneous heat sources released at $t = 0$. The final solution is then obtained from the above integral (8).

The distances D and D' are small compared to L and B in the KBS-3 concept. We may simplify the heat source distribution to a rectangular plane with the heat source strength $q_0(t)$ (W/m^2):

$$q_0(t) = \frac{Q_0(t)}{DD'} \quad \text{over} \quad -L < x < L, \quad -B < y < B, \quad z = 0 \tag{9}$$

Here, DD' is the small rectangle around each single canister. See figure 1, right. The rectangular heat source (9) gives the *global* temperature field valid at a certain distance from the repository region. We first neglect the boundary condition (2) at $z = H$ and consider the solution for an infinite surrounding $-\infty < z < \infty$. We will denote this solution by $T_{gl,inf}(x, y, z, t)$.

The boundary condition at $z = H$ is satisfied by adding a second rectangular heat source $-q_0(t)$ (W/m^2) over $-L < x < L$, $-B < y < B$, $z = 2H$. Then we obtain the global solution $T_{gl}(x, y, z, t)$, (16).

Our exact remaining problem is now to have the original canister heat sources and subtract the rectangular heat source (9). Now, the net heat release for each canister and its surrounding small rectangle with the area DD' is *zero*. We have a balanced heat source problem. See Figure 2.

The thermal range of this balanced heat source field will be limited to a distance, say, $2D'$ ($D' \geq D$) from the repository rectangle. Except for canisters near the rim of the repository rectangle, we may consider the canister grid with its balanced heat sources as infinite in x and in y. Our results will be valid for interior canisters. With this assumption, we now have an infinite grid of canister heat sources and a balancing surface source at $z = 0$:

$$Q_0(t) \quad \text{at} \quad x = mD', \quad y = nD \quad m = 0, \pm 1, \pm 2, \ldots \quad n = 0, \pm 1, \pm 2, \ldots$$
$$-q_0(t) \quad \text{at} \quad z = 0 \quad -\infty < x < \infty, \quad -\infty < y < \infty \tag{10}$$

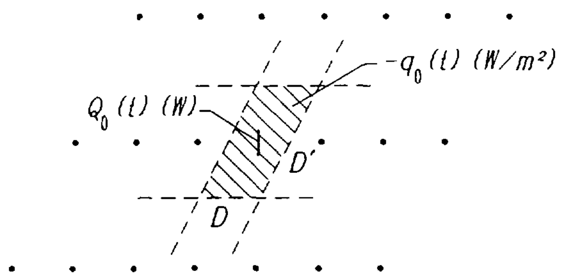

Figure 2: Canister heat source $Q_0(t)$ and a balancing plane source $-q_0(t)$, (9), over the rectangular area DD'.

The variation of $Q_0(t)$ with time is quite slow in the KBS-3 case, since the lowest time scale is $t_1 = 46$ years. The time-scale to attain steady-state conditions for constant $Q_0 = q_0 \cdot DD'$ in the balanced case (10) turns out to be a few years only. This will be discussed further in the coming publication, where the solution is applied to the KBS-3 case. The solution to (10) will be a quasi steady-state one. We will here use the steady-state solution derived for a constant heat source Q_0 at the canisters and a constant balancing plane source $-q_0$. Our slowly varying solution is obtained by using the actual $Q_0(t)$ (and $-q_0(t)$) in the steady-state solution.

Our balanced problem is reduced to a steady-state one. The next approximation is to replace all line heat sources except the one at $x = 0$, $y = 0$ by point heat sources at the plane $z = 0$. Our local grid problem now consists of a line heat source at the considered central canister, an infinite rectangular grid of point heat sources and a balancing plane heat sink at $z = 0$:

$$\text{Line heat source:} \quad Q_0/H_c \quad \text{(W/m)} \quad \text{at} \quad (0,0,z) \quad -H_c/2 < z < H_c/2 \tag{11}$$

$$\text{Infinite grid:} \quad Q_0 \quad \text{(W)} \quad \text{at} \quad (mD', nD, 0) \quad m,n = 0, \pm 1, \ldots \quad (m,n) \neq (0,0) \tag{12}$$

$$\text{Plane heat sink:} \quad -q_0 = -\frac{Q_0}{DD'} \quad (\text{W/m}^2) \quad \text{at} \quad z = 0 \quad -\infty < x,y < \infty \tag{13}$$

Here, we use the steady-state solutions for constant Q_0. The solution is then taken for $Q_0 = Q_0(t)$ at the considered time.

4 Global solution for rectangular heat source

The global solution $T_{gl}(x,y,z,t)$ gives the temperature field from the rectangular heat source (9) over the repository area:

$$q_0(t) = \frac{Q_0(t)}{DD'} \quad (\text{W/m}^2) \quad \text{over} \quad -L < x < L, \quad -B < y < B, \quad z = 0 \tag{14}$$

The temperature at the ground surface is zero:

$$T_{gl}(x, y, H, t) = 0 \tag{15}$$

The thermal problem for the global temperature is illustrated in Figure 3.

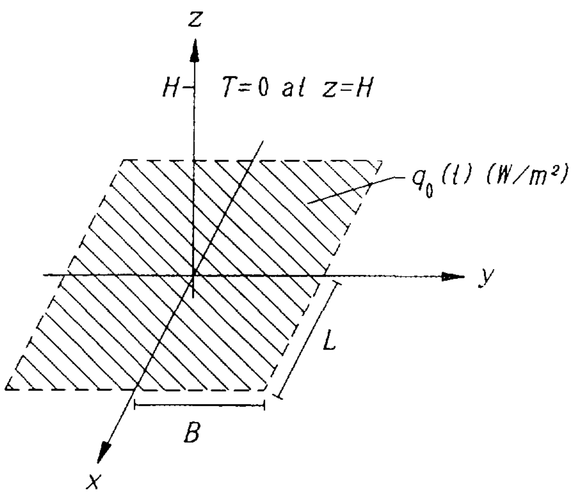

Figure 3: Thermal problem for the global temperature: Rectangular heat source, (14), over the repository plane and zero temperature at $z = H$.

4.1 Superposition

The superposition principle is used to reduce our problem to simpler ones. Let the temperature field for the rectangular heat source (14) in an infinite region $-\infty < z < \infty$ be denoted $T_{gl,inf}(x, y, z, t)$. The temperature shall tend to zero at infinity. The boundary condition (15) is fulfilled by adding a negative mirror heat source at $z = 2H$. Then we have:

$$T_{gl}(x, y, z, t) = T_{gl,inf}(x, y, z, t) - T_{gl,inf}(x, y, z - 2H, t) \tag{16}$$

Let $T_{gl,inf,inst}(x, y, z, t)$ denote the temperature field in an infinite surrounding, when a unit amount of heat is released instantaneously at $t = 0$. Using a Duhamel superposition, (8), we have:

$$T_{gl,inf}(x, y, z, t) = \int_0^t q_0(t') \cdot T_{gl,inf,inst}(x, y, z, t - t')dt' \tag{17}$$

4.2 Quadrantal solution

Our remaining problem is the following one. The heat +1 (J/m^2) is released at $t = 0$ over the rectangular surface $-L < x < L$, $-B < y < B$, $z = 0$. The temperature field due to this heat source involves the two lengths L and B. A similar problem without these lengths is to consider the heat source:

$$1 \cdot \text{sign}(x) \cdot \text{sign}(y) \quad \text{over} \quad z = 0 \quad \text{at} \quad t = 0 \tag{18}$$

The heat 1 (J/m^2) is released in the quadrants $x > 0$, $y > 0$, $z = 0$ and $x < 0$, $y < 0$, $z = 0$, while -1 (J/m^2) is released in the two other quadrants $x > 0$, $y < 0$, $z = 0$ and $x < 0$, $y > 0$, $z = 0$. We will call this heat source distribution a *quadrantal* heat source. Let $T_{quad}(x, y, z, t)$ be the temperature field from the quadrantal heat source (18) in an infinite surrounding.

We have, Carslaw-Jaeger, 1959:

$$T_{quad}(x,y,z,t) = \int_{-\infty}^{\infty} dx' \int_{-\infty}^{\infty} dy' \frac{\text{sign}(x')\text{sign}(y')}{\rho c(4\pi at)^{3/2}} \cdot e^{-\left[(x-x')^2+(y-y')^2+z^2\right]/(4at)} \tag{19}$$

or

$$T_{quad}(x,y,z,t) = \frac{1}{\rho c} \cdot \text{erf}\left(\frac{x}{\sqrt{4at}}\right) \text{erf}\left(\frac{y}{\sqrt{4at}}\right) \cdot \frac{1}{\sqrt{4\pi at}} \cdot e^{-z^2/(4at)} \tag{20}$$

Here, $\text{erf}(x)$ denotes the error function:

$$\text{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-s^2} ds \tag{21}$$

The temperature field from the rectangular heat source is obtained from superposition of four quadrantal solutions of the above type. This follows from the identity:

$$0.5\left[\text{sign}(L-x) + \text{sign}(L+x)\right] \cdot 0.5\left[\text{sign}(B-y) + \text{sign}(B+y)\right] =$$

$$= \begin{cases} 1 & |x| < L \quad \text{and} \quad |y| < B \\ 0 & \text{outside the rectangle} \end{cases} \tag{22}$$

The first factor in x is equal to $+1$ for $|x| < L$ and 0 for $|x| > L$. The other factor has the same behaviour in y. The product is equal to $+1$ for $|x| < L$ and $|y| < B$, and it is 0 elsewhere. We get four products of the type $\text{sign}(L \pm x) \cdot \text{sign}(B \pm y)$. Each of these corresponds to a solution T_{quad} with x replaced by $L \pm x$ and y by $B \pm y$.

The temperature field from the instantaneous rectangular heat source in an infinite surrounding is then:

$$\begin{aligned} T_{gl,inf,inst}(x,y,z,t) &= 0.25 \cdot [T_{quad}(L-x, B-y, z, t) + T_{quad}(L-x, B+y, z, t) \\ &+ T_{quad}(L+x, B-y, z, t) + T_{quad}(L+x, B+y, z, t)] \end{aligned} \tag{23}$$

Here, T_{quad} is given by (20).

4.3 General global solution

Combining (16), (17) and (23), we have the following integral for the global temperature:

$$T_{gl}(x,y,z,t) = \int_0^t \frac{q_0(t')}{4\rho c\sqrt{\pi}} \cdot \frac{1}{\sqrt{4a(t-t')}} \cdot \left[\text{erf}\left(\frac{L-x}{\sqrt{4a(t-t')}}\right) + \text{erf}\left(\frac{L+x}{\sqrt{4a(t-t')}}\right)\right]$$

$$\cdot \left[\text{erf}\left(\frac{B-y}{\sqrt{4a(t-t')}}\right) + \text{erf}\left(\frac{B+y}{\sqrt{4a(t-t')}}\right)\right] \cdot \left[e^{-z^2/[4a(t-t')]} - e^{-(z-2H)^2/[4a(t-t')]}\right] \cdot dt' \tag{24}$$

The last factor involving z ensures that T_{gl} is zero for $z = H$. The products of error functions in x and y correspond to the identity (22).

The remaining problem is to evaluate the above integral for any particular $q_0(t)$. Normally, this must be done numerically.

An alternative expression for the global temperature is obtained from the variable substitution:

$$s = \sqrt{\frac{t-t'}{t_0}} \quad t - t' = t_0 s^2 \quad dt' = -2t_0 s\, ds \tag{25}$$

Here, t_0 is any (positive) time. The integral (24) becomes:

$$T_{gl}(x,y,z,t) = \frac{1}{4\lambda}\sqrt{\frac{at_0}{\pi}} \cdot \int_0^{\sqrt{t/t_0}} q_0\left(t - t_0 s^2\right) \cdot \left[\mathrm{erf}\left(\frac{L-x}{s\sqrt{4at_0}}\right) + \mathrm{erf}\left(\frac{L+x}{s\sqrt{4at_0}}\right)\right] \cdot$$

$$\cdot \left[\mathrm{erf}\left(\frac{B-y}{s\sqrt{4at_0}}\right) + \mathrm{erf}\left(\frac{B+y}{s\sqrt{4at_0}}\right)\right] \cdot \left[e^{-z^2/(s^2 4at_0)} - e^{-(z-2H)^2/(s^2 4at_0)}\right] ds \tag{26}$$

4.4 One-dimensional solution

In the KBS-3 application, the heat source is a rectangular surface with large horizontal extensions. The temperature field within the rectangle above and below the repository is then essentially vertical during a first time period except for a region near the perimeter of the rectangle.

Consider the integral (24) at a certain time t. The arguments of the error functions determine the x- and y-dependence. A few values of the error functions are:

$$\mathrm{erf}(1) = 0.84 \quad \mathrm{erf}(1.5) = 0.97 \quad \mathrm{erf}(2) = 0.995 \quad \mathrm{erf}(2.5) = 0.9996 \tag{27}$$

The error functions may be replaced by +1 when the argument exceeds, say, 1.5. The smallest arguments occur for $t' = 0$. For example, we have:

$$\mathrm{erf}\left(\frac{L-x}{\sqrt{4a(t-t')}}\right) \simeq 1 \quad \text{for} \quad x < L - 3\cdot\sqrt{at} \quad (0 \le t' \le t) \tag{28}$$

For points x and y lying inside the rectangle with the margin $3\sqrt{at}$, we get from (24):

$$T_{gl}(x,y,z,t) = \int_0^t \frac{q_0(t')}{\rho c\sqrt{\pi}} \cdot \frac{1}{\sqrt{4a(t-t')}} \cdot \left[e^{-z^2/(4a(t-t'))} - e^{-(z-2H)^2/(4a(t-t'))}\right] dt'$$

$$|x| < L - 3\sqrt{at} \quad |y| < B - 3\sqrt{at} \tag{29}$$

This is actually the one-dimensional solution in z for a heat source $q_0(t)$ at $z = 0$ and a mirror source $-q_0(t)$ at $z = 2H$.

The mirror source may be neglected during a certain first period. The criterion for this is determined by the ratio between second and first exponential integrals in (29):

$$e^{-\left[(z-2H)^2 - z^2\right]/(4a(t-t'))} = e^{-H(H-z)/(a(t-t'))} \le e^{-H(H-z)/(4at)} \le 0.03 \tag{30}$$

The inequality is fulfilled for

$$H(H-z) > 3.5at \tag{31}$$

This means that the mirror source is negligible at or near $z = 0$ for times $H > 2\sqrt{at}$.

The global temperature is given by the plane heat source $q_0(t)$ at $z = 0$, when x, y, z and t satisfies the conditions:

$$|x| < L - 3\sqrt{at} \quad |y| < B + 3\sqrt{at} \quad H(H-z) > 3.5at \tag{32}$$

The global temperature is then from (29):

$$T_{gl}(x,y,z,t) = \int_0^t \frac{q_0(t')}{\rho c\sqrt{\pi}} \frac{1}{\sqrt{4a(t-t')}} e^{-z^2/(4a(t-t'))} dt' \tag{33}$$

In particular we have for $x = y = z = 0$ from (32-33):

$$T_{gl}(0,0,0,t) = \frac{1}{\rho c\sqrt{\pi}} \int_0^t \frac{q_0(t')}{\sqrt{4a(t-t')}} dt' \qquad t < \frac{L^2}{9a}, \frac{B^2}{9a}, \frac{H^2}{3.5a} \tag{34}$$

Using the substitution (25) or starting directly from (26), we have the alternative expression:

$$T_{gl}(0,0,0,t) = \frac{1}{\lambda}\sqrt{\frac{at_0}{\pi}} \int_0^{\sqrt{t/t_0}} q_0\left(t - t_0 s^2\right) ds \qquad t < \frac{L^2}{9a}, \frac{B^2}{9a}, \frac{H^2}{3.5a} \tag{35}$$

4.5 Exponential heat release

A major application is the case when the heat release $q_0(t)$ decreases exponentially, (5). We have for a single exponential component:

$$q_0(t) = q_1 \cdot e^{-t/t_1} \qquad q_1 = \frac{Q_1}{DD'} \tag{36}$$

We use the integral (26) with $t_0 = t_1$. We get:

$$T_{gl,1}(x,y,z,t) = \frac{q_1}{4\lambda} \cdot \sqrt{\frac{at_1}{\pi}} \int_0^{\sqrt{t/t_1}} e^{-t/t_1 + s^2} \left[\text{erf}\left(\frac{L-x}{s\sqrt{4at_1}}\right) + \text{erf}\left(\frac{L+x}{s\sqrt{4at_1}}\right)\right]$$

$$\cdot \left[\text{erf}\left(\frac{B-y}{s\sqrt{4at_1}}\right) + \text{erf}\left(\frac{B+y}{s\sqrt{4at_1}}\right)\right] \cdot \left[e^{-z^2/(s^2 4at_1)} - e^{-(z-2H)^2/(s^2 4at_1)}\right] ds \tag{37}$$

This integral must in general be evaluated numerically.

The heat source may consist of a sum of exponentials:

$$q_0(t) = \sum_i q_i e^{-t/ti} \qquad q_i = \frac{Q_i}{DD'} \tag{38}$$

Then the global temperature is a sum of integrals of the type (37):

$$T_{gl}(x,y,z,t) = \sum_i T_{gl,i}(x,y,z,t) \tag{39}$$

In the one-dimensional solution, the error functions of (37) are replaced by +1. We have for x and y within the heat source rectangle with the margin $3\sqrt{at}$:

$$T_{gl,1}(x,y,z,t) = \frac{q_1}{\lambda}\sqrt{\frac{at_1}{\pi}} \int_0^{\sqrt{t/t_1}} e^{-t/t_1+s^2} \cdot \left[e^{-z^2/(s^2 4at_1)} - e^{-(z-2H)^2/(s^2 4at_1)}\right] ds$$

$$|x| < L - 3\sqrt{at} \qquad |y| < B - 3\sqrt{at} \tag{40}$$

When the mirror source also is neglected we get:

$$T_{gl,1}(x,y,z,t) = \frac{q_1}{\lambda}\sqrt{\frac{at_1}{\pi}} \cdot e^{-t/t_1} \cdot \int_0^{\sqrt{t/t_1}} e^{s^2} \cdot e^{-z^2/(s^2 4at_1)} ds \tag{41}$$

$$|x| < L - 3\sqrt{at} \qquad |y| < B - 3\sqrt{at} \qquad 3.5at < H(H-z)$$

4.6 Global temperature at central canister

We are particularly interested in the highest canister temperatures. These temperatures occur at the center (0,0,0). The global temperature $T_{gl}(0,0,0,t)$ is of particular interest.

The general expression for any $q_0(t)$ is from (26):

$$\begin{aligned} T_{gl}(0,0,0,t) &= \frac{1}{\lambda}\sqrt{\frac{at_0}{\pi}} \int_0^{\sqrt{t/t_0}} q_0\left(t - t_0 s^2\right) \cdot \text{erf}\left(\frac{L}{s\sqrt{4at_0}}\right) \cdot \\ &\cdot \ \text{erf}\left(\frac{B}{s\sqrt{4at_0}}\right) \cdot \left[1 - e^{-H^2/(s^2 at_0)}\right] ds \end{aligned} \tag{42}$$

In the one-dimensional case, when the mirror source may be neglected, we have from (35):

$$T_{gl}(0,0,0,t) = \frac{1}{\lambda}\sqrt{\frac{at_0}{\pi}} \int_0^{\sqrt{t/t_0}} q_0\left(t - t_0 s^2\right) ds \qquad t < \frac{L^2}{9a}, \frac{B^2}{9a}, \frac{H^2}{3.5a} \tag{43}$$

The general formula for the exponential decay, (36), becomes by putting $t_0 = t_1$ in (42):

$$\begin{aligned} T_{gl,1}(0,0,0,t) &= \frac{q_1}{\lambda}\sqrt{\frac{at_1}{\pi}} \cdot e^{-t/t_1} \cdot \int_0^{\sqrt{t/t_1}} e^{s^2} \cdot \text{erf}\left(\frac{L}{s\sqrt{4at_1}}\right) \cdot \\ &\cdot \ \text{erf}\left(\frac{B}{s\sqrt{4at_1}}\right) \cdot \left[1 - e^{-H^2/(s^2 at_1)}\right] ds \end{aligned} \tag{44}$$

In the one-dimensional case without mirror source we get:

$$T_{gl,1}(0,0,0,t) = \frac{q_1}{\lambda}\sqrt{\frac{at_1}{\pi}} \cdot e^{-t/t_1} \cdot \int_0^{\sqrt{t/t_1}} e^{s^2} ds \qquad t < \frac{L^2}{9a}, \frac{B^2}{9a}, \frac{H^2}{3.5a} \tag{45}$$

The integral of $\exp(s^2)$ is given by Dawson's integral $F(\tau)$, Abramowitz, Stegun (1964):

$$F(\tau) = e^{-\tau^2} \cdot \int_0^{\tau} e^{s^2} ds \tag{46}$$

Tables for the integral is given in the above reference. The first three terms of the series expansion in τ is:

$$F(\tau) \simeq \tau - \frac{2}{3}\tau^3 + \frac{4}{15}\tau^3 \qquad (\tau < 0.8) \tag{47}$$

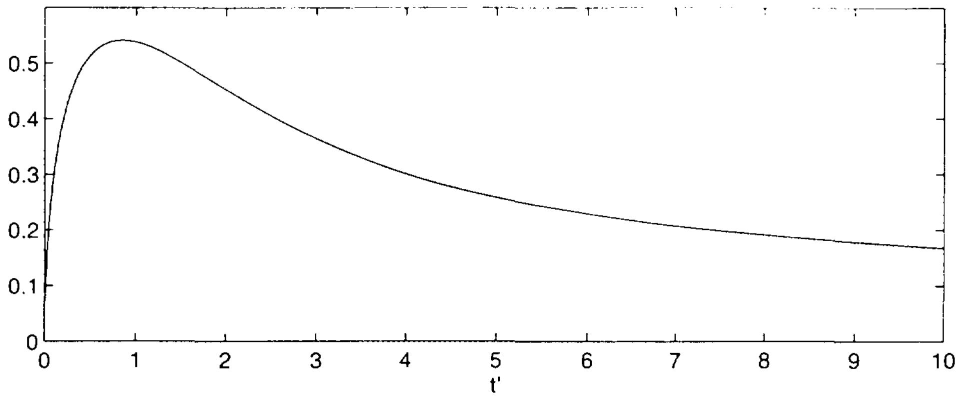

Figure 4: The Dawson function $F_D(t') = F(\sqrt{t'})$, (51).

These first three terms give the value $F(\tau)$ with an error below 3% for $0 < \tau < 0.8$. The function has a maximum:

$$F_{max} = 0.541 \quad \text{for} \quad \tau = 0.924 \tag{48}$$

For large τ, we have the asymptotic expansion:

$$F(\tau) \simeq \frac{1}{2\tau}\left(1 + \frac{1}{2\tau^2} + \frac{3}{4\tau^3}\right) \qquad \tau > 2 \tag{49}$$

The error for $\tau > 2$ is below 3%.

Actually we need $F_D(t') = F(\sqrt{t'})$. We have from (45) the basic formula:

$$T_{gl,1}(0,0,0,t) = \frac{q_1}{\lambda}\sqrt{\frac{at_1}{\pi}} \cdot F_D(t/t_1) \qquad t < \frac{L^2}{9a}, \frac{B^2}{9a}, \frac{H^2}{3.5a} \tag{50}$$

$$F_D(t') = F(\sqrt{t'}) = e^{-t'} \cdot \int_0^{\sqrt{t'}} e^{s^2} ds \tag{51}$$

The function $F_D(t')$ is shown in Figure 4. The maximum 0.541 occurs for $t' = 0.924^2 = 0.854$, (48). From (47) we have for small t':

$$F_D(t') \simeq \sqrt{t'}\left(1 - \frac{2t'}{3} + \frac{4(t')^2}{15}\right) \qquad (t' < 0.64) \tag{52}$$

When $q_0(t)$ consists of two exponential components, (5), we have from (50):

$$T_{gl}(0,0,0,t) = \frac{q_1}{\lambda}\sqrt{\frac{at_1}{\pi}} \cdot F_D(t/t_1) + \frac{q_2}{\lambda}\sqrt{\frac{at_2}{\pi}} F_D(t/t_2) \quad t < \frac{L^2}{9a}, \frac{B^2}{9a}, \frac{H^2}{3.5a} \tag{53}$$

For small t/t_1 ($t_2 > t_1$), we may use the expansion (52):

$$T_{gl}(0,0,0,t) = \sqrt{\frac{at}{\pi}}\left[\frac{q_1}{\lambda}\left(1 - \frac{2}{3}\frac{t}{t_1} + \frac{4}{15}\left(\frac{t}{t_1}\right)^2\right) + \frac{q_2}{\lambda}\left(1 - \frac{2}{3}\frac{t}{t_2} + \frac{4}{15}\left(\frac{t}{t_2}\right)^2\right)\right]$$

$$t < \frac{2t_1}{3}, \frac{2t_2}{3}, \frac{L^2}{9a}, \frac{B^2}{9a}, \frac{H^2}{3.5a} \tag{54}$$

This formula may be used to determine the maximum canister temperature, when the global and local temperature fields are added, provided that the maximum occurs within the time limits of (54).

5 Local solution for an infinite grid of heat sources

To the global solution we have to add the local solution due to the grid of heat sources in the rectangular plane of the repository. For internal canisters not to close to the rim of the repository rectangle, we can consider the grid of canisters as infinite in all directions. We get the heat sources (11) - (13).

The heat supply $Q_0(t)$ is slowly varying in time compared to the time-scale of the local transient process. We therefore solve the steady-state problems for a constant Q_0 (and q_0).

The plane heat sink (13) balances the heat sources of the grid, (11-12). The range of the local solution becomes quite limited. Therefore we can consider the surrounding ground as infinite upwards and disregard the boundary condition at $z = H$. The total local solution shall tend to zero for large z:

$$T_{\text{total local}} \rightarrow 0 \quad z \rightarrow \pm\infty, \tag{55}$$

Solutions for the plane sink at $z = 0$, (13), and the line heat source at the considered central canister, (11), are given in the first two subsections below, while subsequent sections deal with the infinite grid (12).

5.1 Plane heat sink

The solution to account for the plane heat sink $-q_0$ at $z = 0$, (13), is simple. The temperature increases linearly upwards and downwards from $z = 0$:

$$T_{plane}(z) = \frac{q_0}{2\lambda} \cdot |z| = \frac{Q_0}{2\lambda D D'}|z| \tag{56}$$

At $z = 0$, there is indeed a sink:

$$-\lambda \frac{dT_{plane}}{dz}\bigg|_{+0} + \lambda \frac{dT_{plane}}{dz}\bigg|_{-0} = -q_0 \tag{57}$$

The component $T_{plane}(z)$ of the local solution does not fulfill the boundary condition (55). But the solution from the infinite grid with the balancing heat sources, (12), will decrease as $-T_{plane}(z)$ for large $|z|$.

5.2 Line heat source from a canister

The canisters release heat over their height $-H_c/2 \leq z \leq H_c/2$. In (11-12), all canisters except the central one are approximated by point heat sources. But for the central canister at $(0, 0, z)$ we have the finite line heat source (11). Its strength is Q_c/H_c (W/m). Integration of point sources $dz' \cdot Q_c/H_c$ over the canister height gives:

$$T_{lcan}(x, y, z; t) = \int_{-H_c/2}^{H_c/2} \frac{Q_0}{4\pi\lambda H_c} \cdot \frac{1}{\sqrt{x^2 + y^2 + (z - z')^2}} dz' \tag{58}$$

or

$$T_{lcan}(x, y, z; t) = \frac{Q_0}{4\pi\lambda H_c} \cdot T'_{lcan}(2x/H_c, 2y/H_c, 2z/H_c) \tag{59}$$

$$T'_{lcan}(x',y',z') = \ln\left(\frac{\sqrt{(x')^2+(y')^2+(1+z')^2}+1+z'}{\sqrt{(x')^2+(y')^2+(1-z')^2}-1+z'}\right) \quad (60)$$

The dimensionless temperature field T'_{lcan} is rotationally symmetric around the z'-axis. It depends on $\sqrt{(x')^2+(y')^2}$ and z'. The isotherms are rotational ellipsoids. The temperature is infinite along the line heat source, $x' = y' = 0$, $-1 < z' < 1$. The temperature T_{lcan} tends to zero as $Q_0/(4\pi\lambda r)$ for large r.

The isotherms are cigar-shaped (prolate ellipsoids) near the line heat source. The canisters have a more or less constant temperature over their boundary, which is essentially a slim cylinder with the height H_c and the outer radius R_c. The best approximation between these two problems with slightly different boundary shapes is to take the ellipsoidal boundary for which the volume is equal to that of the cylinder. From this, Claesson, Bengtsson (1978), we get the formula:

$$T_{lcan}|_{boundary} = Q_0 \cdot \frac{1}{2\pi\lambda H_c} \cdot \ln\left(\frac{H_c}{R_c\sqrt{1.5}}\right) \quad (61)$$

The temperature far away from the line source is zero. The above formula gives the excess temperature on the canister boundary that must be imposed in order to obtain the heat flow Q_0. The second factor of (61) may be interpreted as the thermal resistance between the canister and the ground far away (for a single canister in an infinite surrounding):

$$T_{lcan}|_{boundary} - 0 = Q_0 \cdot R_{lcan}$$

$$R_{lcan} = \frac{1}{2\pi\lambda H_c}\ln\left(\frac{H_c}{R_c\sqrt{1.5}}\right) \qquad \text{(K/W)} \quad (62)$$

5.3 Point sources along a line

The last part of the heat sources (11)-(13) to be accounted for is the point heat sources in the infinite grid (12). We will first in this subsection consider an infinite line of point heat sources along the y-axis:

$$Q_0 \quad \text{at} \quad (0, nD, 0) \quad n = 0, \pm 1, \ldots \quad (63)$$

The central heat source at (0,0,0) is here included. The problem is rotationally symmetric around the y-axis. The distance to the y-axis is $\rho = \sqrt{x^2+z^2}$.

The temperature field $T_l(x,y,z)$ may be obtained as a sum of point heat sources:

$$T_l(x,y,z) = \frac{Q_0}{4\pi\lambda}\left[\frac{1}{\sqrt{x^2+y^2+z^2}} + \sum_{n=1}^{\infty}\left(\frac{1}{\sqrt{x^2+(y-nD)^2+z^2}} + \frac{1}{\sqrt{x^2+(y+nD)^2+z^2}} - \frac{2}{nD}\right)\right] \quad (64)$$

The first term is the contribution from the central point source, while the sum gives the point sources on the positive and negative y-axis. The last term $-2/(nD)$ is included in order to obtain a convergent series. The sum is zero for $(x,y,z) = (0,0,0)$. We may add any constant to the solution (64), but we choose the constant so that the solution behaves as $Q_0/(4\pi\lambda r) + 0$ near $r = 0$.

The temperature T_l from the infinite line of heat sources is periodic in y with the period D. We may seek the solution as a periodic Fourier series from separation of variables in $\rho = \sqrt{x^2+z^2}$ and y. The solution is:

$$T_l(x,y,z) = \frac{Q_0}{2\pi\lambda D}\left[\ln\left(\frac{2D}{\rho}\right) - \gamma + 2\sum_{n=1}^{\infty}\cos\left(\frac{2\pi n y}{D}\right)K_0\left(\frac{2\pi n\rho}{D}\right)\right]$$

$$\rho = \sqrt{x^2+z^2} \neq 0 \qquad (65)$$

The details of the derivation of this formula are not given here. They will be reported in a coming study, Claesson (1996). Here, $\gamma = 0.5772$ is Euler's constant, and $K_0(x)$ is a modified Bessel function, Abramowitz, Stegun (1964). The formula is not valid for $\rho = 0$, since $\ln(\rho)$ appears in the formula. There is an undefined constant in the series solution. But it has been chosen so that (65) behaves as $Q_0/(4\pi\lambda r) + 0$ near $r = 0$ as in (64). The expressions (64) and (65) represent the same solution T_l. The details of this are not given here. See Claesson (1996)

The modified Bessel function $K_0(x)$ behaves as $\ln(x)$ near $x = 0$, and it decreases exponentially as e^{-x} for large x.

The temperature field T_l may be written in dimensionless form:

$$T_l(x,y,z) = \frac{Q_0}{2\pi\lambda D}\cdot T_l'(\rho',y') \qquad (66)$$

Here, ρ' and y' are dimensionless coordinates:

$$\rho' = \sqrt{(x/D)^2+(z/D)^2} \quad y' = y/D \qquad (67)$$

The function $T_l'(\rho',z')$ is from (64) and (65) given by:

$$2\cdot T_l'(\rho',y') = \frac{1}{\sqrt{(\rho')^2+(y')^2}} + \sum_{n=1}^{\infty}\left(\frac{1}{\sqrt{(\rho')^2+(y'-n)^2}} + \frac{1}{\sqrt{(\rho')^2+(y'+n)^2}} - \frac{2}{n}\right) \qquad (68)$$

$$T_l'(\rho',y') = \ln(2/\rho') - \gamma + 2\sum_{n=1}^{\infty}\cos(2\pi n y')K_0(2\pi n\rho') \quad (\rho' > 0) \qquad (69)$$

The second expression is divergent for $\rho' = 0$.

The function $T_l'(\rho',y')$ is periodic in y' with the period +1. From (68) we have near $r' = 0$:

$$\left[T_l'(\rho',y') - \frac{1}{2r'}\right]_{r'=0} = 0 \qquad r' = \sqrt{(\rho')^2+(y')^2} \qquad (70)$$

The series in n in (69) converges rapidly for large ρ', since K_0 tends to zero exponentially for large arguments. A few values of K_0 are: $K_0(\pi) = 0.030$, $K_0(2\pi) = 0.00092$, $K_0(3\pi) = 0.000033$. We have with very good accuracy the following approximate expressions:

$$T_l' \simeq \ln(2/\rho') - \gamma \quad \rho' > 1.5 \qquad (71)$$

$$T_l' \simeq \ln(2/\rho') - \gamma + 2\cos(2\pi y')K_0(2\pi\rho') \quad 0.5 \leq \rho' \leq 1.5 \qquad (72)$$

5.4 Infinite grid of point sources

In the next step of our analysis we consider an infinite rectangular grid of point sources in the plane $z = 0$. See Figure 5. The central point (0,0,0) is included.

Q_0 (W) at $(mD', nD, 0)$

$m, n = 0, \pm 1, ...$

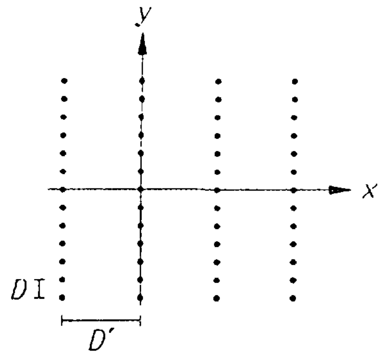

Figure 5. Infinite rectangular grid of point heat sources in the plane $z = 0$.

The solution is obtained as a sum of lines of point heat sources. We have:

$$T_{grid}(x,y,z) = T_l(x,y,z) + \sum_{m=1}^{\infty} [T_l(x - mD', y, z) + T_l(x + mD', y, z) - 2T_l(mD', 0, 0)] \quad (73)$$

The sum in m is the contribution from points along the lines $x = \pm mD'$. The value at $T_l(\pm mD', 0, 0)$ is subtracted for each line in order to ensure convergence.

In (73), we may insert (64) or (65) for T_l. The second expression has much better convergence properties, since $K_0(x)$ decreases at least exponentially with x. Using (65), we have:

$$T_{grid}(x,y,z) = T_l(x,y,z) + T_{2l,sur}(x,z) + T_{grid,K} \quad (74)$$

Here, the sum for T_{grid} is divided into three parts. The first part T_l is the contribution from the line of heat sources along the y-axis. The second part is the sum of logarithms in (65) from the surrounding lines $m = \pm 1, \pm 2, ..$. The third part is the double sum in m and n involving the Bessel functions K_0. We have:

$$T_{2l,sur} = \frac{Q_0}{2\pi\lambda D} \cdot \sum_{m=1}^{\infty} \ln \left(\frac{mD'}{\sqrt{(x - mD')^2 + z^2}} \cdot \frac{mD'}{\sqrt{(x + mD')^2 + z^2}} \right) \quad (75)$$

$$T_{grid,K} = \frac{Q_0}{2\pi\lambda D} \cdot 2 \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \cos\left(\frac{2\pi n y}{D}\right) \left[K_0\left(\frac{2\pi n\sqrt{(x - mD')^2 + z^2}}{D}\right) + \right.$$

$$\left. + K_0\left(\frac{2\pi n\sqrt{(x + mD')^2 + z^2}}{D}\right) - 2K_0\left(\frac{2\pi n m D'}{D}\right) \right]$$

$$\sqrt{(x - mD')^2 + z^2} \neq 0 \quad (m = \pm 1, \pm 2, \ldots) \quad (76)$$

The double sum involving K_0 converges very rapidly. The smallest argument in K_0 is obtained for $x = D'/2$ (for $-D'/2 \le x \le D'/2$), $z = 0$, $m = 1$ and $n = 1$. Then we get $K_0(\pi D'/D)$. We choose $D' \ge D$, which means that the smallest argument of K_0 is π. In the KBS-3 case, (6), the smallest argument becomes $\pi \cdot 25/6 = 13.1$ and $K_0(13.0) \simeq 0.0000014$. Then the double series is *completely* negligible. It is a very good approximation in our applications to neglect $T_{grid,K}$:

$$T_{grid,K} \simeq 0$$

$$T_{grid} \simeq T_l + T_{2l,sur} \qquad (D' > 3D) \tag{77}$$

The second term of (74) involves the sum (75) of logarithmic terms. These logarithms are independent of y. They represent two-dimensional line sources in the (x, z)-plane. This part is analysed in the next section.

5.5 Line of two-dimensional point sources

An infinite line of point sources Q_0 (W) with a spacing D may at a certain distance from the line be approximated by a continuous line source with the strength Q_0/D (W/m) along the line. The corresponding temperature becomes

$$T = \frac{Q_0}{2\pi\lambda D} \cdot \ln\left(\frac{1}{\rho}\right) + \text{constant} \tag{78}$$

Here, $\rho = \sqrt{x^2 + z^2}$ is the distance to the line. This approximation is valid for T_l with very good accuracy for $\rho' = \rho/D > 1.5$, (71) and (66).

Let us now consider an infinite row of two-dimensional point sources along the x-axis in the (x, z)-plane. See Figure 6. This means that we replace the tunnels with heat sources at a spacing D by line sources with the strength Q_0/D (W/m) along the tunnel.

Q_0/D W/m at $x = mD', z = 0$

$m = 0, \pm 1, \pm 2, ..$

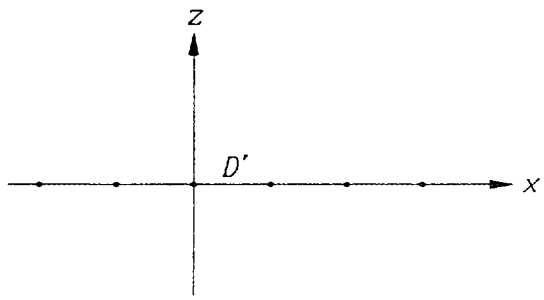

Figure 6. Infinite row of two-dimensional point sources in the (x, z)-plane.

The temperature field from this two-dimensional problem becomes:

$$\begin{aligned} T_{2l}(x,z) \;=\; & \frac{Q_0}{2\pi\lambda D}\left\{ \ln\left(\frac{1}{\sqrt{x^2+z^2}}\right) + \sum_{m=1}^{\infty}\left[\ln\left(\frac{1}{\sqrt{(x-mD')^2+z^2}}\right)\right.\right. \\ & + \left.\left. \ln\left(\frac{1}{\sqrt{(x+mD')^2+z^2}}\right) - 2\cdot\ln\left(\frac{1}{mD'}\right)\right] + A_1 \right\} \end{aligned} \tag{79}$$

The sum in m gives the contributions from the line heat sources surrounding the central one. The values from these at $x = 0$, $z = 0$ are subtracted in order to assure convergence. This problem, Figure 6, is a fairly wellknown one in electrostatics and in the theory of analytical functions. An alternative, closed-form expression exists:

$$T_{2l}(x,z) = -\frac{Q_0}{2\pi\lambda D} \cdot \ln\left[\sqrt{\cosh\left(\frac{2\pi z}{D'}\right) - \cos\left(\frac{2\pi x}{D'}\right)}\right] \tag{80}$$

The constant A_1 is determined by considering the limit $(x,z) \rightarrow (0,0)$ in (79) and (80):

$$-\frac{1}{2}\ln\left(x^2+z^2\right) + 0 + A_1 = -\frac{1}{2}\ln\left(1 + \frac{1}{2}\left(\frac{2\pi z}{D'}\right)^2 + \ldots - 1 + \frac{1}{2}\left(\frac{2\pi x}{D'}\right)^2 + \ldots\right) \tag{81}$$

or

$$A_1 = -\frac{1}{2}\ln\left(\frac{1}{2}\cdot\left(\frac{2\pi}{D'}\right)^2\right) \tag{82}$$

From (79-80) and (82), we get an expression for the sum of logarithms in (75). We have:

$$\sum_{m=1}^{\infty} \ln\left[\frac{mD'}{\sqrt{(x-mD')^2+z^2}} \cdot \frac{mD'}{\sqrt{x+mD')^2+z^2}}\right] =$$

$$\ln\left[\frac{2\pi\sqrt{x^2+z^2}}{D'\sqrt{2(\cosh(2\pi z/D') - \cos(2\pi x/D')}}\right] \tag{83}$$

From (83) and (75) we get:

$$T_{2l,sur}(x,z) = \frac{Q_0}{2\pi\lambda D} \cdot \ln\left(\frac{2\pi\sqrt{x^2+z^2}}{D'\sqrt{2(\cosh(2\pi z/D') - \cos(2\pi x/D'))}}\right) \tag{84}$$

The final expression for T_{grid} for the infinite rectangular grid of heat sources, Figure 5, is given by (77), (65) and (84). In this approximation, the line of point sources at $x = 0$ is retained, giving $T_l(x,y,z)$, while all other lines for $x = \pm mD'$ are replaced by continuous line sources parallel to the y-axis with the strength Q_0/D. The error involving this approximation is very small for $D' > 3D$. It is completely negligible in the KBS-3 case with $D'/D = 25/6$.

5.6 Total local solution

The total local solution shall account for the balanced heat sources (11)-(13). It shall also satisfy the boundary condition (55). Adding the solutions for the line heat source (11) of the central canister, the infinite grid (12) and the plane heat source (13) we have:

$$T_{loc} = T_{lcan} + T_{grid} - \frac{Q_0}{4\pi\lambda r} + T_{plane} + A_{loc} \tag{85}$$

The solution T_{grid} included the central point source at (0,0,0), which is to be excluded, since T_{lcan} gives a more precise value for the actual line source along the canister. The temperature from the infinite grid is given by the two components (77):

$$T_{grid} = T_l + T_{2l,sur} \tag{86}$$

Thus we have:

$$T_{loc} = T_{lcan} + T_l - \frac{Q_0}{4\pi\lambda r} + T_{2l,sur} + T_{plane} + A_{loc} \tag{87}$$

The four temperature fields are given by (59-60), (65), (84), and (56), respectively. We get the following explicit expression for $T_{loc}(x,y,z)$:

$$T_{loc} = \frac{Q_0}{4\pi\lambda H_c} \cdot \ln\left[\frac{\sqrt{x^2+y^2+(H_c/2+z)^2}+H_c/2+z}{\sqrt{x^2+y^2+(H_c/2-z)^2}-H_c/2+z}\right] - \frac{Q_0}{4\pi\lambda\sqrt{x^2+y^2+z^2}}$$

$$+\frac{Q_0}{2\pi\lambda D}\left[\ln\left(\frac{2D}{\sqrt{x^2+z^2}}\right) - \gamma + 2\sum_{n=1}^{\infty}\cos\left(\frac{2\pi ny}{D}\right)K_0\left(\frac{2\pi n\sqrt{x^2+z^2}}{D}\right)\right.$$

$$\left. + \ln\left(\frac{2\pi}{D'}\cdot\frac{\sqrt{x^2+z^2}}{\sqrt{2\left[\cosh\left(2\pi z/D'\right)-\cos(2\pi x/D')\right]}}\right) + \frac{\pi|z|}{D'}\right] + A_{loc} \tag{88}$$

The constant A_{loc} is determined by the boundary condition (55). We have for $z \to \pm\infty$:

$$T_{loc} \to 0 + \frac{Q_0}{2\pi\lambda D}\left[\ln(2D) - \gamma + 0 + \ln\left(\frac{2\pi}{D'}\right) + 0\right] + A_{loc} = 0 \tag{89}$$

The term $\ln(\sqrt{x^2+z^2})$ in (88) cancels. The term involving $|z|$ balances the term involving $\ln\sqrt{2\left[\cosh ...\right]}$ in the considered limit. The constant A_{loc} becomes:

$$A_{loc} = \frac{Q_0}{2\pi\lambda D}\left\{\gamma + \ln\left(\frac{D'}{4\pi D}\right)\right\} \tag{90}$$

The largest term in the square root involving $\cosh(2\pi z/D')$ is the exponential of $2\pi|z|/D'$. The logarithm of this term balances the term involving $|z|$ in (88). Using this and (90), the final most compact expression for T_{loc}, (88), is:

$$T_{loc} = \frac{Q_0}{4\pi\lambda H_c} \cdot \ln\left(\frac{\sqrt{x^2+y^2+(H_c/2+z)^2}+H_c/2+z}{\sqrt{x^2+y^2+(H_c/2-z)^2}-H_c/2+z}\right) +$$

$$\frac{Q_0}{2\pi\lambda D}\left\{2\sum_{n=1}^{\infty}\left[\cos\left(\frac{2\pi ny}{D}\right)K_0\left(\frac{2\pi n\sqrt{x^2+z^2}}{D}\right)\right] - \frac{D}{2\sqrt{x^2+y^2+z^2}}\right.$$

$$\left. -\frac{1}{2}\ln\left[1 - 2e^{-2\pi|z|/D'}\cdot\cos\left(\frac{2\pi x}{D'}\right) + e^{-4\pi|z|/D'}\right]\right\} \tag{91}$$

5.7 Canister temperature

The temperature at the canister boundary or envelope is of particular interest. The total local solution is given by (87) or (88). The contribution from the line source at the envelope of the considered central canister is given by (61). The next two terms in (87) represent the line of sources along the y-axis with the central source subtracted. This part is by construction zero at the center (0,0,0). See (64) and (70). The contribution from the adjacent lines, $T_{2l,sur}$, is also by construction zero at (0,0,0). See (75). The linear part T_{plane}, (56), is zero at $z = 0$. The temperature at the envelope of the canister is given by the constant A_{loc} and the expression (61). We have the important formula:

$$T_{loc}\Big|_{\text{canister envelope}} = \frac{Q_0}{2\pi\lambda H_c} \cdot \ln\left(\frac{H_c}{R_c\sqrt{1.5}}\right) + \frac{Q_0}{2\pi\lambda D}\left\{\gamma + \ln\left(\frac{D'}{4\pi D}\right)\right\} \tag{92}$$

6 Total temperature field

The total temperature field (above the undisturbed ground temperature) due to all heat sources is equal to the sum of the global temperature (26) and the local one (87-88):

$$T(x, y, z, t) = T_{gl}(x, y, z, t) + T_{loc}(x, y, z; t) \tag{93}$$

Here, T_{gl} is given by the integral (26):

$$T_{gl}(x, y, z, t) = \frac{\sqrt{at_0/\pi}}{4\lambda D D'} \cdot \int_0^{\sqrt{t/t_0}} Q_0\left(t - t_0 s^2\right) \cdot \left[\operatorname{erf}\left(\frac{L-x}{s\sqrt{4at_0}}\right) + \operatorname{erf}\left(\frac{L+x}{s\sqrt{4at_0}}\right)\right] \cdot$$

$$\cdot \left[\operatorname{erf}\left(\frac{B-y}{s\sqrt{4at_0}}\right) + \operatorname{erf}\left(\frac{B+y}{s\sqrt{4at_0}}\right)\right] \cdot \left[e^{-z^2/(s^2 4at_0)} - e^{-(z-2H)^2/(s^2 4at_0)}\right] ds \tag{94}$$

Here, t_0 is any time. The heat release per canister, Q_0, is used in the integral. In the case of exponential heat release, (36), (38), equations (37) and (39) are to be used.

The local temperature T_{loc} is given in Section 5.6. Using the most compact expression (91), we have:

$$T_{loc} = \frac{Q_0}{4\pi\lambda H_c} \cdot \ln\left(\frac{\sqrt{x^2 + y^2 + (H_c/2 + z)^2} + H_c/2 + z}{\sqrt{x^2 + y^2 + (H_c/2 - z)^2} - H_c/2 + z}\right) +$$

$$+ \frac{Q_0}{2\pi\lambda D}\left\{2\sum_{n=1}^{\infty}\left[\cos\left(\frac{2\pi n y}{D}\right) K_0\left(\frac{2\pi n\sqrt{x^2+z^2}}{D}\right)\right] - \frac{D}{2\sqrt{x^2+y^2+z^2}}\right.$$

$$\left. - \frac{1}{2}\ln\left[1 - 2e^{-2\pi|z|/D'} \cdot \cos(2\pi x/D') + e^{-4\pi|z|/D'}\right]\right\} \tag{95}$$

The whole field is proportional to the heat release $Q_0(t)$. The remaining part is time-independent in our quasi steady-state analysis. This part $T_{loc}/Q_0(t)$ consist of three fields (and the simple $1/r$ term). Each of these fields depends on *two* dimensionless spacial variables only. The first field is essentially the ellipsoidal or cigar-shaped field T'_{lcan}, (60), from the line heat source of the central canister. The second field involving the Bessel function K_0 is due to the line of heat sources along the y-axis, (69). The third field accounts for the line sources of adjacent tunnels. It is essentially the sum of $T_{2l,sur}$ and T_{plane}.

7 Formula for canister temperature

The largest temperatures at the canisters occur for canisters at the center of the repository. We consider the central canister at (0,0,0). The total temperature at the envelope of the canister is the sum of global and local temperatures. The undisturbed ground temperature $T_{rep,0}$ is also added:

$$T_{can}(t) = T_{gl}(0,0,0,t) + Q_0(t) \cdot R_{loc} + T_{rep,0} \tag{96}$$

Here, T_{gl} is given by (42):

$$\begin{aligned} T_{gl}(0,0,0,t) &= \frac{\sqrt{at_0/\pi}}{\lambda D D'} \int_0^{\sqrt{t/t_0}} Q_0\left(t - t_0 s^2\right) \cdot \text{erf}\left(\frac{L}{s\sqrt{4at_0}}\right) \cdot \\ &\cdot \text{erf}\left(\frac{B}{s\sqrt{4at_0}}\right) \cdot \left[1 - e^{-H^2/(s^2 a t_0)}\right] ds \end{aligned} \tag{97}$$

In the one-dimensional case, when the mirror source may be neglected, we have from (43):

$$T_{gl}(0,0,0,t) = \frac{\sqrt{at_0/\pi}}{\lambda D D'} \int_0^{\sqrt{t/t_0}} Q_0\left(t - t_0 s^2\right) ds \qquad t < \frac{L^2}{9a}, \frac{B^2}{9a}, \frac{H^2}{3.5a} \tag{98}$$

When $Q_0(t)$ consists of two exponential components, (5), we have, (53):

$$T_{gl}(0,0,0,t) = \frac{Q_1}{\lambda D D'}\sqrt{\frac{at_1}{\pi}} \cdot F_D(t/t_1) + \frac{Q_2}{\lambda D D'}\sqrt{\frac{at_2}{\pi}} F_D(t/t_2) \quad t < \frac{L^2}{9a}, \frac{B^2}{9a}, \frac{H^2}{3.5a} \tag{99}$$

Here, the function F_D is given by (51):

$$F_D(t') = e^{-t'} \cdot \int_0^{\sqrt{t'}} e^{s^2} ds \tag{100}$$

The constant R_{loc} in (96) is given by (92):

$$R_{loc} = \frac{1}{2\pi\lambda H_c} \cdot \ln\left(\frac{H_c}{R_c\sqrt{1.5}}\right) + \frac{1}{2\pi\lambda D}\left[\gamma + \ln\left(\frac{D'}{4\pi D}\right)\right] \qquad \gamma = 0.577 \tag{101}$$

The factor R_{loc} (K/W) is multiplied by Q_0. It may be interpreted as the thermal resistance between the canister envelope and the rectangle $z = 0$, $-D'/2 < x < D'/2$, $-D/2 < y < D/2$, where the heat Q_0 is absorbed by the heat sink $q_0 = Q_0/(DD')$. See Figure 2.

8 A numerical example

We will here illustrate the derived solution with a few numerical examples. A more systematic use of the solution for the KBS-3 application is deferred to another study.

The data (6) is used. The heat release from each canister is given by two exponentials:

$$Q_0(t) = 750 \cdot e^{-t/t_1} + 250 \cdot e^{-t/t_2} \quad \text{(W)} \qquad t_1 = 46 \text{ y} \quad t_2 = 780 \text{ y} \tag{102}$$

The initial effect $Q_0(0)$ is 1 kW.

The time-scales for a one-dimensional solution without mirror heat source are given by (32). At the center $x = y = z = 0$, the upper limits on t become:

$$\frac{L^2}{9a} = \frac{B^2}{9a} = \frac{500^2}{9 \cdot 1.62 \cdot 10^{-6}} \text{ s} = 540 \text{ y} \qquad \frac{H^2}{3.5a} = \frac{500^2}{3.5 \cdot 1.62 \cdot 10^{-6}} \text{ s} = 1400 \text{ y} \tag{103}$$

8.1 Global temperature

Formula (53) for $T_{gl}(0,0,0,t)$ is valid during the first 540 years, (103). We have:

$$T_{gl}(0,0,0,t) = \frac{750}{3.5 \cdot 6 \cdot 25}\sqrt{\frac{1.62 \cdot 10^{-6} \cdot 46 \cdot 365 \cdot 24 \cdot 3600}{\pi}} \cdot F_D(t/t_1) +$$
$$+\frac{250}{3.5 \cdot 6 \cdot 25}\sqrt{\frac{1.62 \cdot 10^{-6} \cdot 780 \cdot 365 \cdot 24 \cdot 3600}{\pi}} \cdot F_D(t/t_2)$$

or

$$T_{gl}(0,0,0,t) = 39.07 \cdot F_D(t/t_1) + 53.63 \cdot F_D(t/t_2) \qquad t < 540 \text{ years} \qquad (104)$$

The function $F_D(t')$ is defined by (100) and shown in Figure 4. This global temperature at the center of the repository is shown in Figure 7. For times larger than 540 years, the more complex integral (44) is evaluated numerically for the two decay times t_1 and t_2.

The temperature increases rapidly during the first ten years, Figure 7, top left. The maximum temperature 34.8 °C occurs after 82 years. There is a second maximum of 34.4 °C at $t = 380$ years, and an intermediate minimum 34.0 at $t = 206$ years. See Figure 12 which shows the region of highest temperatures with higher resolution. Actually, the temperature lies between 34 and 35 °C for $51 < t < 530$ years. It lies above 30 °C for $25 < t < 930$ years. The temperature decreases quite slowly for large times. The temperature is 28.5, 16.0, 4.6 and 0.8 °C for $t = 1000$, 2000, 4000 and 8000 years, respectively. A few values of $T_{gl}(0,0,0,t)$ for the data (6) are:

t (years)	0	10	20	25	30	50	82
$T_{gl}(0,0,0,t)$ (°C)	0	22.3	28.4	30.0	31.5	31.1	34.8
t (years)	206	390	930	1000	2000	4000	8000
$T_{gl}(0,0,0,t)$ (°C)	34.0	34.4	30.0	28.5	16.0	4.6	0.8

Figure 8 shows the global temperature along the z-axis, i.e. along the vertical line through the center of the repository. The profiles show the thermal range at increasing times.

Figure 9 shows the temperature field in a vertical cross-section $y = 0$ at a few times. The region around the edge $x = 500$ and $z = 0$ is shown in greater detail in Figure 13 for $t = 100$ years. Temperature profiles along the x-axis are shown in Figure 10. The temperature field in a horizontal cross-section at the level $z = 0$ of the repository is shown in Figure 11.

8.2 Canister temperature

The resistance R_{loc}, (101), becomes:

$$R_{loc} = \frac{1}{2\pi \cdot 3.5 \cdot 5} \ln\left(\frac{5}{0.4\sqrt{1.5}}\right) + \frac{1}{2\pi \cdot 3.5 \cdot 6}\left[0.577 + \ln\left(\frac{25}{4\pi \cdot 6}\right)\right] =$$
$$= 0.02113 - 0.00399 = 0.01714 \quad \text{K/W} \qquad (105)$$

The total canister temperature (96) becomes:

$$T_{can}(t) = T_{gl}(0,0,0,t) + 17.14 \cdot \left(0.75e^{-t/t_1} + 0.25e^{-t/t_2}\right) + 15 \quad {}^{\circ}\text{C} \qquad (106)$$

This temperature is shown in Figure 14. The largest temperature 58 °C occurs for $t = 43$ years. The maximum is very flat.

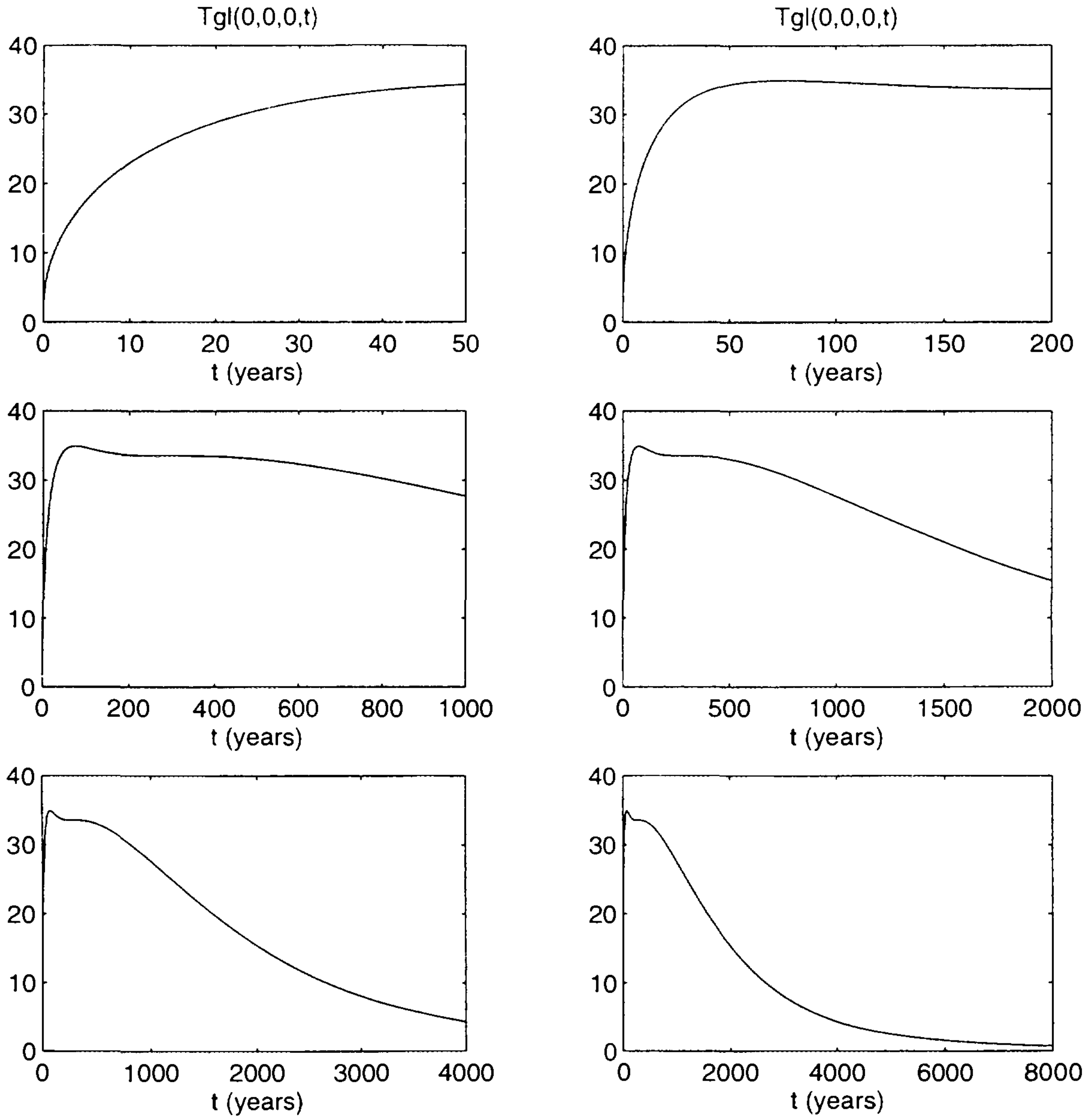

Figure 7. Global temperature at the center of the repository, (104), (99) and (44), for time spans from 50 to 8000 years.

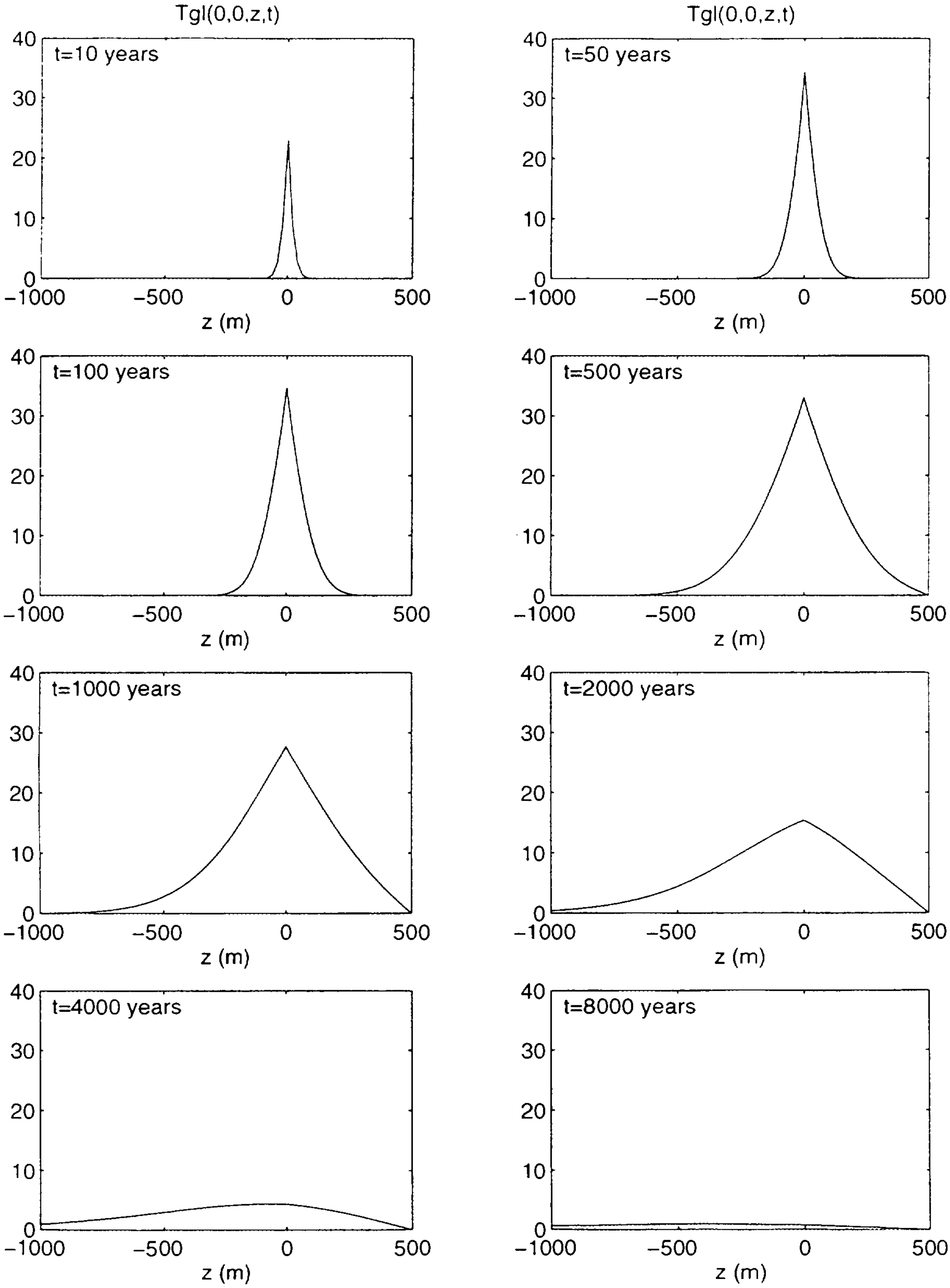

Figure 8. Global temperature, (44), along the z-axis at different times.

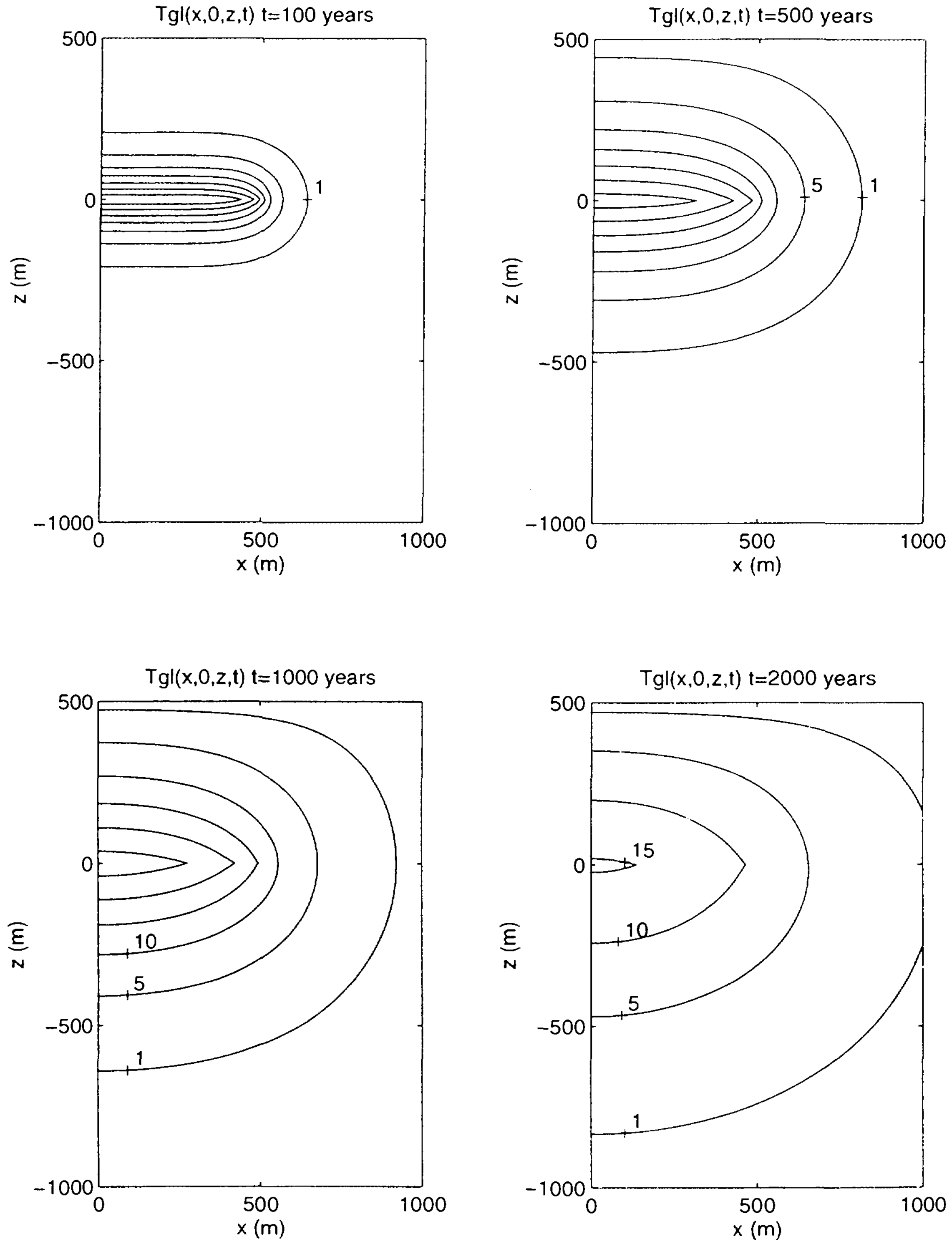

Figure 9. Global temperature, (44), in a vertical cross-section, $y = 0$, at different times. The isotherms are (counted from right to left) 1, 5, 10, 15, 20, 25 and 30 °C.

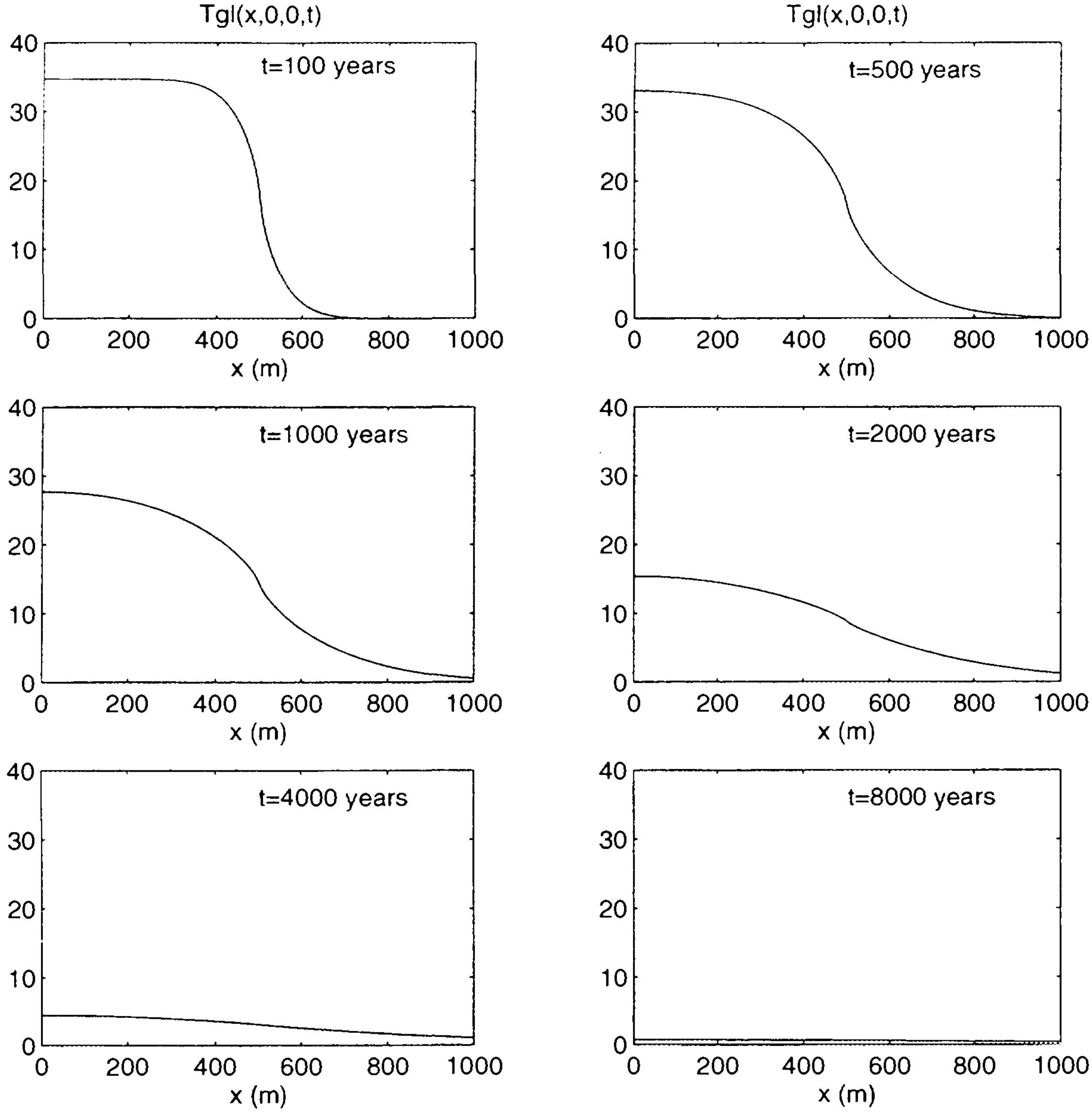

Figure 10. Global temperature, (44), along the x-axis at different times.

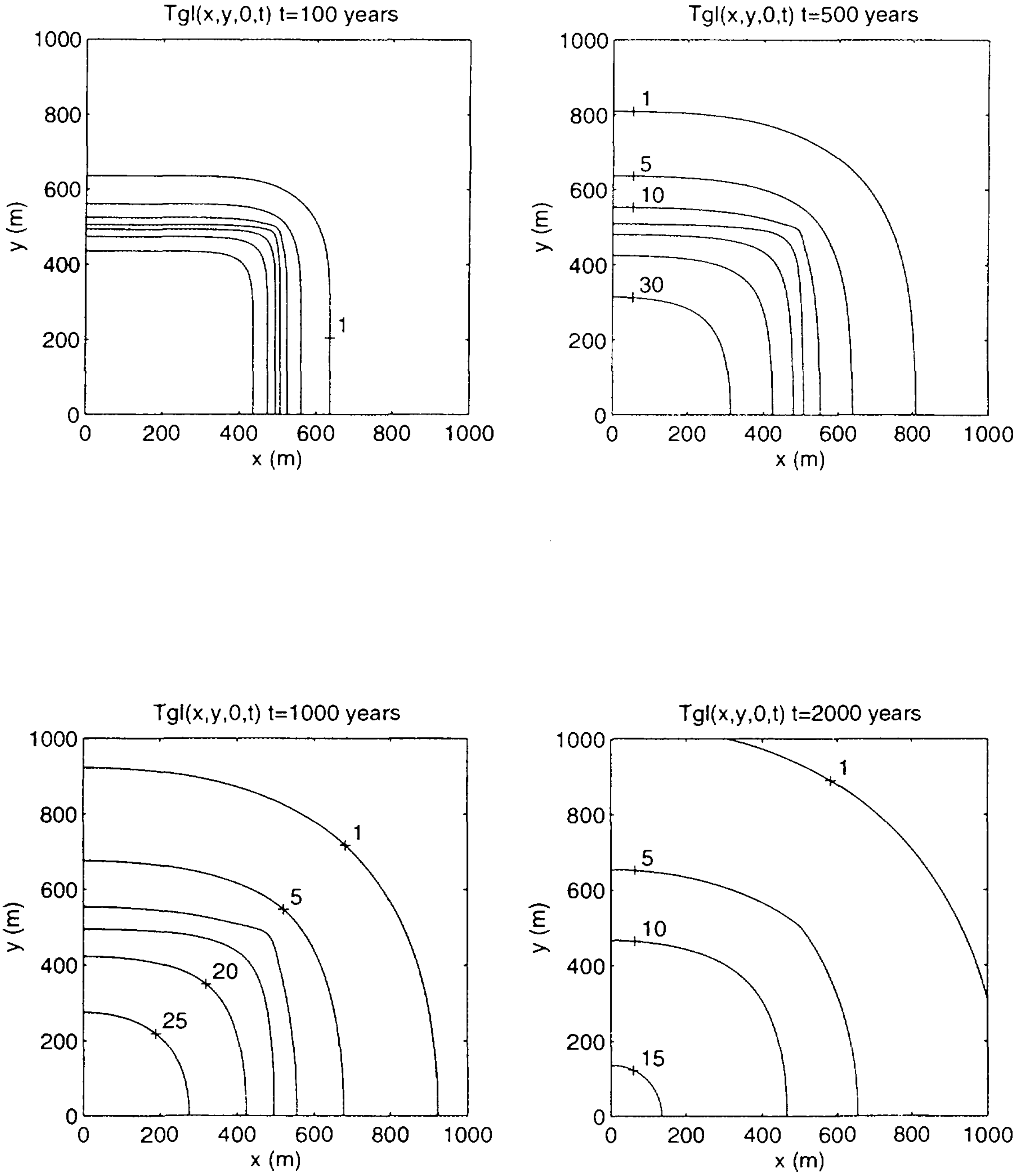

Figure 11. Global temperature, (44), in a horizontal cross-section $z = 0$ at different times. The isotherms are (counted from right to left) 1, 5, 10, 15, 20, 25 and 30 °C.

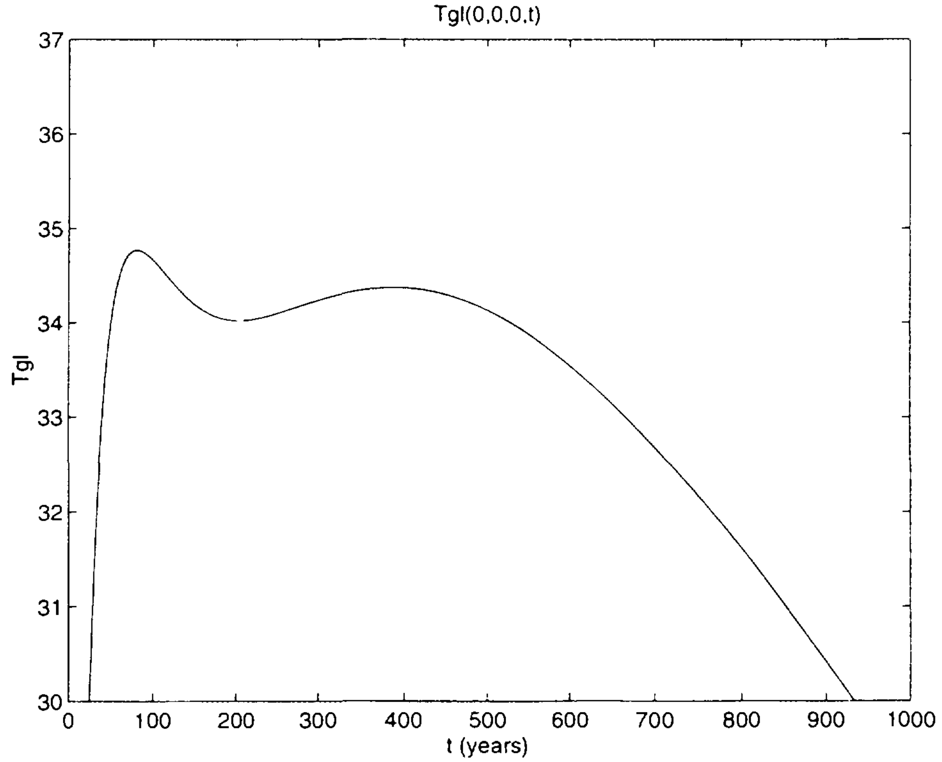

Figure 12. Global temperature, (44), at the center of the repository during the time of highest temperatures.

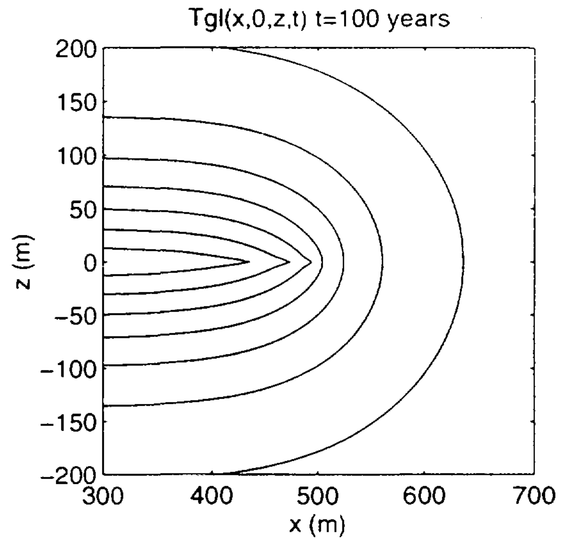

Figure 13. Global temperature, (44), in a vertical cross-section for $t = 100$ years. The region around the edge of the repository in Figure 9, top left, is shown in greater detail. The isotherms are (counted from right to left) 1, 5, 10, 15, 20, 25 and 30 °C.

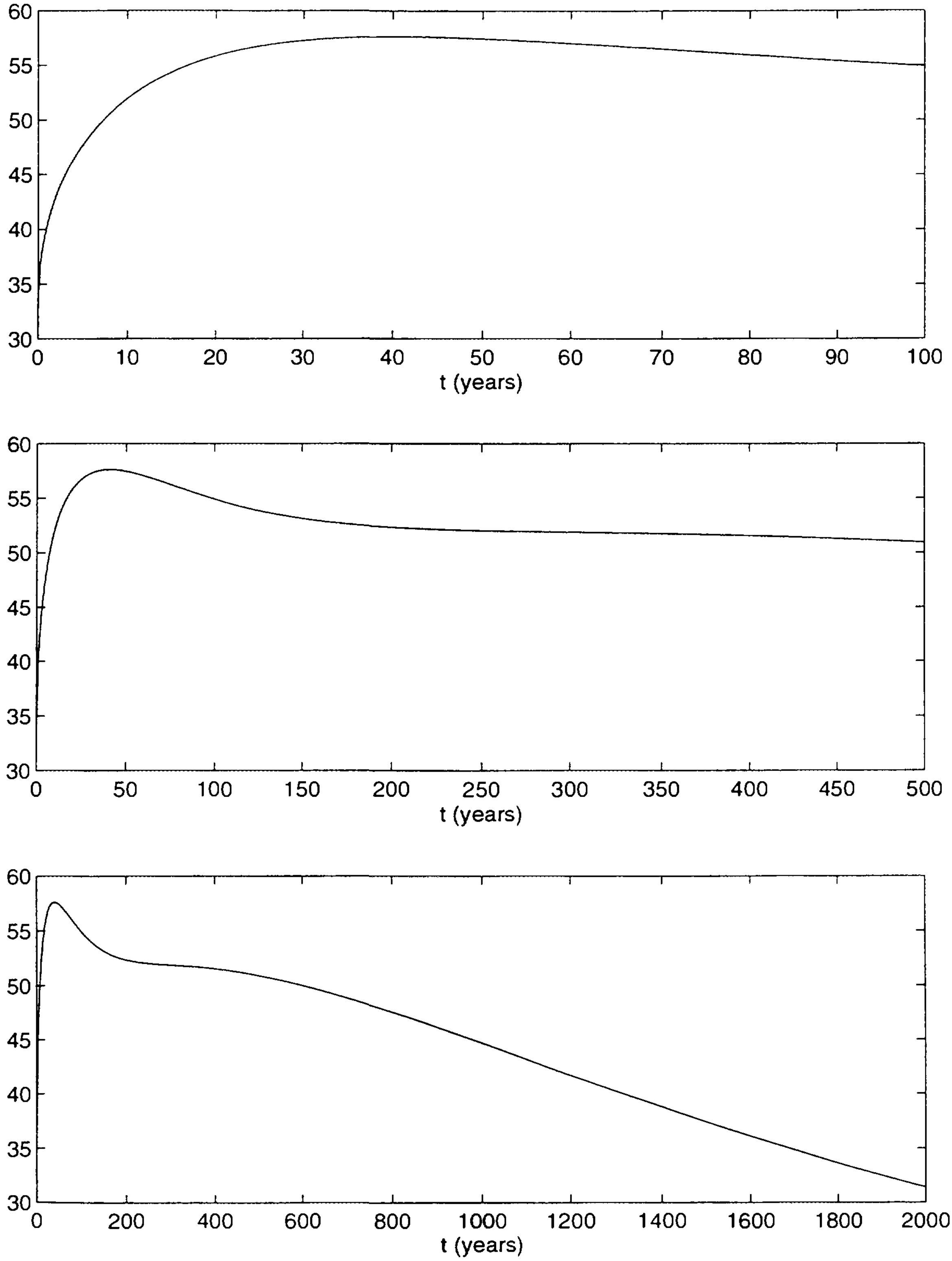

Figure 14. Canister temperature, (106), for different time spans (in °C including the undisturbed temperature $T_{rep,0} = 15$ °C).

9 References

Abramowitz, Stegun (1964). Handbook of Mathematical Functions. National Bureau of Standards, Washington DC (also available in Dover Paperback).
Carslaw, Jaeger (1957). Heat Conduction in Solids. Oxford University Press.
Claesson (1996). Temperature Fields due to Point Sources in an Infinite Row and in an Infinite Rectangular Grid. Dept of Building Physics, Lund University, Sweden. (To appear.)
Claesson, Bengtsson (1978). Stationary Heat Losses from a Spherical or Ellipsoidal Heat Storage Volume in the Ground. Dept of Mathematical Physics, Lund University, Sweden.
Claesson, Hellström (September 1995). Thermoelastic Stress Due to an Instantaneous Finite Line Heat Source in an Infinite Medium. Dept of Mathematical Physics, Lund University, Sweden. (To appear as SKB-report.)
Claesson, Probert (December 1995). Thermoelastic Stress Due to a Rectangular Heat Source in a Semi-Infinite Medium. I Derivation of Analytical Solution. Dept of Mathematical Physics, Lund University, Sweden. (To appear as SKB-report.)
Israelsson (June 1995). Global Thermo-Mechanical Effects from a KBS-3 Type Repository, Phase 1; Elastic Analyses. Itasca Geomekanik AB, Borlänge, Sweden.
Thunvik, Braester (1991). Heat Propagation from a Radioactive Waste Repository. SKB. Working Report, 91-17, Stockholm.

List of SKB reports

Annual Reports

1977-78
TR 121
KBS Technical Reports 1 – 120
Summaries
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Including Summaries of Technical Reports Issued during 1994.
Stockholm, May 1995

1995

TR 95-37

SKB Annual Report 1995

Including Summaries of Technical Reports Issued during 1995.
Stockholm, May 1996

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[1] AECL, Whiteshell Laboratories, Pinawa, Manitoba, Canada
[2] University of Göteborg, Department of General and Marine Microbiology, Göteborg, Sweden
[3] Guigues Recherche Appliquée en Microbiologie (GRAM), Aix-en-Provence, France
1996

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[2] SKB
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