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CAPACITANCE OF THE CIRCULAR PATCH RESONATOR

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Abstract

In this paper the capacitance of the circular microstrip patch resonator is computed. It is shown that the electrostatic problem can be formulated as a system of dual integral equations, and the most interesting techniques of solutions of these systems are reviewed. Some useful approximated formulas for the capacitance are derived and plots of the capacitance are finally given in a wide range of dielectric constants.

1. Introduction

The practical advantages of microstrip structures have been discussed in many papers and [1,2] are now too well known to be repeated here. The relatively simple geometry of microstrip structures and the theoretical possibility to obtain very well suited mathematical models certainly contributed to their popularity. Indeed, every analytical technique commonly used in electromagnetism has been applied to microstrip, giving rise to a large number of different and apparently unrelated approaches.

The developments of numerical methods for solving integral equations in electromagnetic theory have been the subject of intensive research for more than twenty years. During these years, careful analysis has paved the way for the development of efficient and effective numerical methods and, of equal importance, has provided a solid foundation for a thorough understanding of the available techniques. Progress in understanding and procedures has been so great that a lot of integral equations can be solved handily by undergraduate engineering students.

In this article we mainly deal with the evaluation of the capacitance of a circular microstrip disk, a problem particularly interesting for applications in microwave integrated circuits [1-3]. The determination of this capacitance has been a subject of investigation from the middle of sixties. Itoh and Mittra [4] proposed a Galerkin technique in the spectral domain; their general theory gives numerical results only with a first-order approximation. Besides the coefficients of the Galerkin matrix involve infinite integral of Bessel functions, requiring a considerable computer time. The more accurate calculations of the present analysis confirm, however, that the one-term result for capacitance given in [4] are sometimes remarkably accurate. Borkar and Yang [5] present a method based on dual integral equations, adapting some classical results discussed by Tranter. Their numerical results exhibit a disparity of more than 10 percent compared with the ones given in [4]; therefore some authors doubted the validity of the application of the theory of such systems of integral equations. Coen and Gladwell [6] develop a solution expanding the charge density in terms of modified Legendre polynomials, taking into account an adequate factor to provide the correct singularity at the edge of the disk. Chew and Kong [7] considered the effects of fringing fields when the substrate thickness is small (compared with the radius of the patch); they obtained an asymptotic expansion for the capacitance and, using a seminumerical approach, a simple approximate formula to yield accurate results with the simple use of a calculator.

Our aim is to demonstrate that the electrostatics of the circular microstrip disk can be formulated as a system of dual integral equations, and that all the theoretical techniques developed can be obtained from the general theory of these systems. In particular we will show how such a system can be rewritten as a single Fredholm integral equation of the second kind with a continuous kernel; besides we will discuss a general procedure to reduce this new integral equation to an algebraic system of linear equation. Though a vast amount of literature has been published on the numerical computation of the capacitance of the circular patch resonator, a few have noted the importance of the edge singularity effect, and in some cases, the simplicity of the stratified media Green's function in spectral domain representation. We will obtain the charge distribution density in a factorized form with the edge singularity in evidence, and a plot of the capacitance of the system. In this way we will review all the classical method of solution for such kind of problem, emphasising advantages and disadvantages of each technique of solution, and the range of validity of obtained approximations.

Dual integral equations system occur in general in the mixed boundary values problems where the metallic region is finite (or in the dual problem), such as holes in metallic screens, drift tube in accelerators [8], cylindrical antennas [9]. These coupled integral equations are obtained imposing the boundary conditions on the electric field and on the charge density distribution, as described in the following. An extensive survey of the historical developments, almost up to 1966, and of the methods of numerical treatment of the dual integral equations, such as to a Fredholm type of equation, or reduction to a system of algebraic equations, is well summarised by I.N. Sneddon [10]. Recently the solution of a general type of dual integral equations has been proposed by Eswaran [11]; he showed under what conditions a solution can exist, is unique and can be expressed as a Neumann series.

The basic idea to solve a system of dual integral equations is to find an adequate representation of the unknown satisfying automatically one of the two equations and transforming the other one into an expression, easy to manage and/or to treat numerically. What we define 'a numerical manageable expression' will be a well conditioned system of algebraic equations. There are many ways to conceive this transformation; one can transform the system into a single Fredholm integral equation of the second kind with a continuous kernel, and then sample the continuous variable in order to obtain a system of equations; one can expand the unknown as a series of Bessel functions, obtaining a Neumann series, and, by means of adequate projections in functional spaces, the problem is converted into a set of linear simultaneous equations with a symmetric coefficients matrix, whose solution can be obtained in a wide range of frequencies. Matrix inversion is easily accomplished with digital computer so that solutions of high order are feasible.

In our case the problem is formulated as a system of dual integral equations of Bessel type in section 2 and a first approximation (the patch very close to the ground plane) for the capacitance of the system is found. In section 3 a general way to transform Bessel functions is explained, and a first Fredholm integral equation of the second kind is derived; unfortunately this integral equation is defined on a non-compact support, and therefore can be used to deduce other approximate formulas. In section 4 the system is transformed into a single Fredholm integral equation with a continuous kernel; this equation is rewritten by means of a Neumann series as a system of linear algebraic equations in section 5. A more general Neumann expansion is proposed in section 6. Finally the numerical results are given in section 7, whereas conclusions and perspectives of the proposed methods are examined in section 8. Numerical details of each method are reported, if necessary, in appendix.

We hope that our various approaches to the derivation of approximate formulas for the microstrip disk resonator will be of interest to readers who are interested in the analysis of microwave integrated circuits.

2. Formulation of the problem

Consider the geometrical configuration shown in Figure 1 of a microstrip circular disk resonator of radius a, separated from a ground plane by a dielectric material of permittivity

$\varepsilon = \varepsilon_0\,\varepsilon_r$. The substrate and the ground plane have infinite transverse dimensions. Theoretical developments are given here for a single-layer substrate. Modifications needed to account for multiple layers will be mentioned in an other paper. The disk is charged to the potential V.

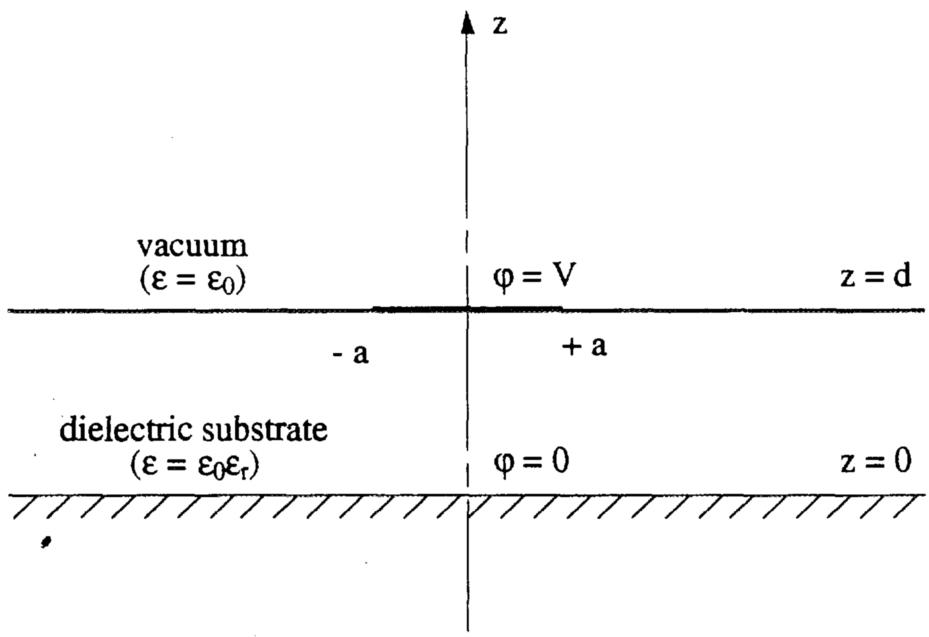

Figure 1 – schematic representation of the electromagnetic system.

The potential functions are considered to be defined in the two different regions

$$\varphi(r,z) = \begin{cases} \varphi_1(r,z)\,, & z \in [0,d]\,, \\ \varphi_2(r,z)\,, & z \in [d,\infty[\,. \end{cases} \tag{2.1}$$

The first step is to write Laplace equation for the potentials in each region (k=1,2)

$$\nabla^2\varphi_k = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\varphi_k}{\partial r}\right) + \frac{\partial^2\varphi_k}{\partial z^2} = 0\,, \tag{2.2}$$

in which the azimuthal terms vanished because of the circular symmetry.
Let us now introduce the Hankel transform of the order zero for the potential

$$\begin{cases} \Phi_k(w,z) = \displaystyle\int_0^\infty r\,J_0(wr)\,\varphi_k(r,z)\,dr & \text{(Hankel transform)}\,, \\ \varphi_k(r,z) = \displaystyle\int_0^\infty w\,J_0(wr)\,\Phi_k(w,z)\,dw & \text{(Hankel inverse-transform)}\,. \end{cases} \tag{2.3}$$

Applying Hankel inverse-transform to equation (2.2), we obtain

$$\nabla^2 \varphi_k = \int_0^\infty w\, J_0(wr) \left[\frac{\partial^2 \Phi_k(w,z)}{\partial z^2} - w^2 \Phi_k(w,z) \right] dw = 0\,, \tag{2.4}$$

or equivalently the partial differential equations (k=1,2)

$$\frac{\partial^2 \Phi_k(w,z)}{\partial z^2} - w^2 \Phi_k(w,z) = 0\,. \tag{2.5}$$

The general solution of equations (2.5) is

$$\begin{cases} \Phi_1(w,z) = A(w)\, \mathrm{sh}(wz) + C(w)\, \mathrm{ch}(wz)\,, & z \in [0,d]\,, \\ \Phi_2(w,z) = B(w)\, \exp(-wz) + D(w)\, \exp(wz)\,, & z \in [d,\infty[\,, \end{cases} \tag{2.6}$$

and satisfying the boundary conditions

$$\begin{cases} \varphi_1(r,z=0) = 0 \\ \varphi_2(r,z=\infty) = 0 \end{cases} \quad \forall r \tag{2.7}$$

we have the simplified forms

$$\begin{aligned} &\Phi_1(w,z) = A(w)\, \mathrm{sh}(wz)\,, && z \in [0,d]\,, \\ &\Phi_2(w,z) = B(w)\, \exp(-wz)\,, && z \in [d,\infty[\,. \end{aligned} \tag{2.8}$$

The continuity of the potential on the separation surface z=d, namely

$$\varphi_1(r,z=d^-) = \varphi_2(r,z=d^+) \qquad \forall r\,, \tag{2.9}$$

implies the following relation between the two unknowns $A(w)$ and $B(w)$

$$B(w) = A(w)\, \exp(wd)\, \mathrm{sh}(wd)\,. \tag{2.10}$$

We have to specify the boundary conditions on the plane z=d. The metallic patch is charged to the potential V, and therefore

$$\varphi_1(r,z=d) = \varphi_2(r,z=d) = V\,, \qquad r \in [0,a]\,, \tag{2.11}$$

namely the following integral equation

$$\int_0^\infty A(w)\, J_0(wr)\, w\, \mathrm{sh}(wd)\, dw = V\,, \qquad r \in [0,a]\,. \tag{2.12}$$

Besides the continuity of normal component of the field **D** implies

$$\varepsilon_0\, \varepsilon_r \left[\frac{\partial \varphi_1}{\partial z}\right]_{z=d^-} = \varepsilon_0 \left[\frac{\partial \varphi_2}{\partial z}\right]_{z=d^+}, \tag{2.13}$$

that is equivalent to the integral equation

$$\int_0^\infty A(w)\, J_0(wr)\, w^2\, [\varepsilon_r\, ch(wd) + sh(wd)]\, dw = 0\,, \qquad r \in [a,\infty[\,. \tag{2.14}$$

Summarising equations (2.12) and (2.14), we obtain the following system of integral equations

$$\begin{cases} \displaystyle\int_0^\infty A(w)\, J_0(wr)\, w\, sh(wd)\, dw = V\,, & r \in [0,a]\,, \\ \displaystyle\int_0^\infty A(w)\, J_0(wr)\, w^2\, [\varepsilon_r\, ch(wd) + sh(wd)]\, dw = 0\,, & r \in [a,\infty[\,. \end{cases} \tag{2.15}$$

System (2.15) represents a system of dual integral equations and it describes a boundary-value problem in which the boundary conditions are mixed in the sense that the unknown function satisfies different types of boundary conditions over distinct portions of the same boundary. The first equation states that the metallic circular plate is to the potential V, whereas the second one represents the continuity of the normal component of the field **D** outside the patch.

Finally we can evaluate the relations linking the unknown A(w) and the charge distribution density on the metallic patch, and, in order to compute the capacitance of the system, the total charge. We have for the density

$$\sigma(r) = \varepsilon_0\, \varepsilon_r \left[\frac{\partial \varphi_1}{\partial z}\right]_{z=d^-} - \varepsilon_0 \left[\frac{\partial \varphi_2}{\partial z}\right]_{z=d^+} = \varepsilon_0 \int_0^\infty A(w)\, w^2\, [\varepsilon_r\, ch(wd) + sh(wd)]\, J_0(wr)\, dw\,, \tag{2.16}$$

and consequently, because [12]

$$\int_0^a r\, J_0(wr)\, dw = \frac{a}{w}\, J_1(wa)\,,$$

the capacitance is

$$C = \frac{Q}{V} = \frac{2\pi}{V} \int_0^a r\, \sigma(r)\, dr = \frac{2\pi}{V}\, \varepsilon_0\, a \int_0^\infty A(w)\, w\, [\varepsilon_r\, ch(wd) + sh(wd)]\, J_1(wa)\, dw\,. \tag{2.17}$$

It is helpful to find an approximate solution of the system (2.15); this solution can be useful to test the numerical solutions we shall find in the next sections. A solution in a closed form can be obtained for small values of d (d«a); in this case we can say that the system (2.15) can be approximated as

$$\begin{cases} \displaystyle\int_0^\infty A(w)\, J_0(wr)\, w^2\, dw = \frac{V}{d}, & r \in [0,a]\,, \\ \displaystyle\int_0^\infty A(w)\, J_0(wr)\, w^2\, dw = 0\,, & r \in [a,\infty[\,. \end{cases} \tag{2.18}$$

The system (2.18) can be read as a single Hankel transform, and therefore the solution is [12]

$$w\, A(w) = \frac{V}{d} \int_0^a r\, J_0(wr)\, dr = \frac{a\,V}{d\,w} J_1(wa)\,, \qquad d \to 0\,. \tag{2.19}$$

From equation (2.16), we can conclude that the charge distribution density does not show the correct divergence in the vicinity of the edge ($r \to a^-$), being [12]

$$\sigma(r) = \varepsilon_0\, \varepsilon_r \frac{a\,V}{d} \int_0^\infty J_1(wa)\, J_0(wr)\, dw = \varepsilon_0\, \varepsilon_r \frac{V}{d} [u(r) - u(r-a)]\,, \qquad d \to 0\,, \tag{2.20}$$

where u(r) represents the unit step function. From a physical point of view this means that it is essential to take into account the fringing field to have a solution exhibiting the right divergence on the edge. Besides equation (2.17) gives for the capacitance [12]

$$C = C_0 = \frac{2\,\pi}{d} \varepsilon_0\, \varepsilon_r\, a^2 \int_0^\infty \frac{J_1^2(wa)}{w} dw = \pi\, a^2 \frac{\varepsilon_0\, \varepsilon_r}{d}\,, \qquad d \to 0\,, \tag{2.21}$$

which is the value obtained when the fringing fields are neglected. This capacitance will be indicated with C_0 in the following.

3. Reduction to a trigonometric kernel

In this section we shall show a general procedure to transform the Bessel functions of the system (2.15) into trigonometric functions (sine and cosine) in order to find a form suitable for the applications of the methods we shall discuss in the following sections. Our object is to change the orders of the Bessel functions appearing in (2.15) from 0 to 1/2 (trigonometric functions). In order to perform this change we shall follow the scheme described by Lewin and Schmeltzer [13].

We begin by multiplying each side of the first equation of (2.15), namely equation (2.12), by $x/\sqrt{r^2 - x^2}$, and integrating from 0 to r. If it is assumed that the orders of integration of the resulting double integral can be interchanged, we can use a result due to Sonine [12], namely

$$\int_0^r x\, J_0(wx) \frac{dx}{\sqrt{r^2 - x^2}} = \frac{\sin(wr)}{w}\,,$$

to obtain

$$\int_0^\infty A(w)\,\sin(wr)\,\mathrm{sh}(wd)\,dw = V\,r\,, \qquad r \in [0,a]\ . \tag{3.1}$$

Differentiating by each side of equation (3.1), we obtain

$$\int_0^\infty A(w)\,\cos(wr)\,w\,\mathrm{sh}(wd)\,dw = V\,, \qquad r \in [0,a]\ . \tag{3.2}$$

Similarly, we continue by multiplying each side of the second equation of the system (2.15), namely equation (2.14), by $x/\sqrt{x^2 - r^2}$, integrating from r and infinity. We can use the result [12]

$$\int_r^\infty x\,J_0(wx)\,\frac{dx}{\sqrt{x^2 - r^2}} = \frac{\cos(wr)}{w}\,,$$

to find

$$\int_0^\infty A(w)\,\cos(wr)\,w\,[\varepsilon_r\,\mathrm{ch}(wd) + \mathrm{sh}(wd)]\,dw = 0\,, \quad r \in [a,\infty[\ . \tag{3.3}$$

In order to write in a standard form equations (3.2) and (3.3), we introduce a new unknown F(w), related to the old A(w) by the relation

$$A(w) = \frac{(1+\varepsilon_r)V}{w\,[\varepsilon_r\,\mathrm{ch}(wd) + \mathrm{sh}(wd)]}\,F(w)\ . \tag{3.4}$$

Summarising equations (3.2-3) and relation (3.4), we can finally write the new system of dual integral equations

$$\begin{cases} \displaystyle\int_0^\infty F(w)\,\cos(wr)\,[1 - G(w)]\,dw = 1\,, & r \in [0,a]\,, \\ \displaystyle\int_0^\infty F(w)\,\cos(wr)\,dw = 0\,, & r \in [a,\infty[\,, \end{cases} \tag{3.5}$$

where we called for shortness

$$G(w) = 1 - \frac{(1+\varepsilon_r)\,\mathrm{sh}(wd)}{\varepsilon_r\,\mathrm{ch}(wd) + \mathrm{sh}(wd)} = \varepsilon_r\,\frac{1 - \mathrm{th}(wd)}{\varepsilon_r + \mathrm{th}(wd)} = \frac{2\,\varepsilon_r}{1+\varepsilon_r}\,\frac{\exp(-2wd)}{1 + \frac{\varepsilon_r - 1}{\varepsilon_r + 1}\exp(-2wd)}\,, \tag{3.6}$$

and a plot of G(w) is given in Figure 2. In this way we transformed Bessel functions of system (2.15) into trigonometric ones (3.6).

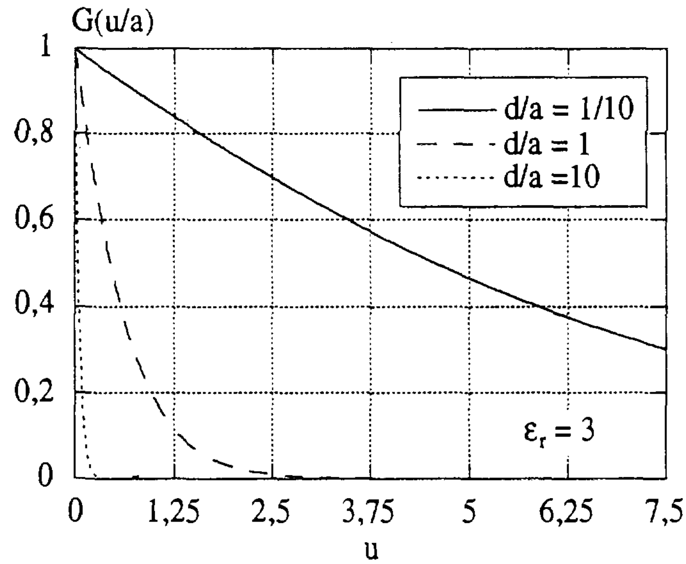

Figure 2 – the kernel G(w) for different values of the dimensionless ratio d/a.

Finally we aim to underline that formula (2.16), giving the charge distribution density on the metallic patch, can be simplified as

$$\sigma(r) = \varepsilon_0 (1 + \varepsilon_r) V \int_0^\infty F(w) J_0(wr) w \, dw , \tag{3.7}$$

whereas the capacitance (2.17) of the system becomes

$$C = 2 \pi \varepsilon_0 (1 + \varepsilon_r) a \int_0^\infty F(w) J_1(wa) \, dw . \tag{3.8}$$

It is worth noting that equation (3.7) states that the Hankel transform of F(w) is proportional to the charge density σ(r).

The usefulness of this new formulation appears evident when we write instead of the system (3.5) a single integral equation as follows

$$\int_0^\infty F(w) \cos(wr) \, dw = \left[1 + \int_0^\infty F(w) \cos(wr) G(w) \, dw \right] u(r-a) , \quad r \in [0,\infty[. \tag{3.9}$$

Taking the inverse Fourier cosine transform of equation (3.9), we have simply [14]

$$F(w) = \frac{2}{\pi}\frac{\sin(wa)}{w} + \frac{1}{\pi}\int_0^\infty G(u)\left\{\frac{\sin[(w-u)a]}{w-u} + \frac{\sin[(w+u)a]}{w+u}\right\} F(u)\, du\,, \tag{3.10}$$

namely an integral equation of Fredholm of the second kind. Unfortunately it is impossible to carry out an effective numerical procedure to solve this equation because of the non-compact support. But it is advantageous to find an approximate solution helpful to test the numerical solutions we shall find in the next sections. Now, as Figure 2 shows, from definition (3.6) follows that, for high values of d, $G(w) \approx 0$, we can say that the free term of equation (3.10) represents asymptotically (d»a) a good approximation of the solution, namely

$$F(w) \cong \frac{2}{\pi}\frac{\sin(wa)}{w}, \qquad \text{if } d \to \infty\,. \tag{3.11}$$

From equations (3.4) and (3.9), we can also conclude that the unknown A(w) shows a simple pole for w=0 since

$$A(w) \cong \frac{4\,V}{\pi}\frac{\sin(wa)}{w^2}\exp(-wd)\,, \qquad d \to \infty\,. \tag{3.12}$$

The asymptotic evaluation of the charge density distribution follows from relation (3.7) and is given by [12]

$$\sigma(r) \cong \varepsilon_0\,(1+\varepsilon_r)\frac{2V}{\pi}\int_0^\infty \sin(wa)\,J_0(wr)\,dw = \frac{2V\,\varepsilon_0\,(1+\varepsilon_r)\,u(a-r)}{\pi\sqrt{r^2-a^2}}\,, \qquad d \to \infty\,, \tag{3.13}$$

and exhibits the correct divergence at the edge, whereas the capacitance descends form formula (3.8) and can be written as [12]

$$C = C_\infty \approx 4\,\varepsilon_0\,(1+\varepsilon_r)\,a\int_0^\infty \sin(wa)\frac{J_1(wa)}{w}\,dw = 4\,\varepsilon_0\,(1+\varepsilon_r)\,a, \qquad d \to \infty\,. \tag{3.14}$$

Therefore the asymptotic solution (3.11-12) can be considered as a physical solution of the problem. For brevity capacitance (3.14) will be indicated with C_∞ in the following.

4. Fredholm integral equation

Aim of this section is to show how the system of dual integral equations (3.5) can be transformed into a single Fredholm integral equation of the second kind with continuous kernel and defined on a compact support.

We begin by introducing an auxiliary function to find an opportune representation of the unknown function F(w), so that one of the two integral equations of the system is automatically satisfied. The problem is not so easy as it appears to be at first sight, because, as the reader will presently realise, it has to deal with functions which must not only behave in a prescribed manner as the variable tends to $\pm\infty$, but must also satisfy an intricate system of integral equations. Following the analysis pointed out by Sneddon, the solution of the system (3.5) can be sought in the form [10]

$$F(w) = \frac{2}{\pi} \int_0^a p(t) \cos(wt)\, dt , \tag{4.1}$$

where p(t) is some unknown functions, continuous, together with its first derivative, in the closed interval (0,a). If we substitute expansion (4.1) of F(w) into the second equation of the system (3.5), this equation is automatically satisfied, because of the relevant Fourier transform [14]

$$\int_0^\infty \cos(wr) \cos(wt)\, dw = \frac{\pi}{2}\left[\delta(t+r) + \delta(t-r)\right] .$$

The first equation of the system (3.5), instead, can be written as the integral equation for the auxiliary p(t)

$$\frac{2}{\pi} \int_0^a p(t) \left\{ \int_0^\infty \cos(wr) \cos(wt) [1 - G(w)]\, dw \right\} dt = 1 , \qquad r \in [0,a] , \tag{4.2}$$

This is a Fredholm integral equation of the second kind defined on a compact, with a symmetric and continuous kernel, because, using the previous Fourier transform, it is

$$p(r) = 1 + \int_0^a p(t) [N(t+r) + N(t-r)]\, dt, \qquad r \in [0,a] , \tag{4.3}$$

where the kernel N(x) is the cosine Fourier transform

$$N(x) = \frac{1}{\pi} \int_0^\infty G(w) \cos(wx)\, dw , \qquad x \in [0.\infty[. \tag{4.4}$$

In order to investigate a numerical solution of this equation, we need a fast and accurate evaluation of the kernel N(x) given by the integral (4.4). A possible evaluation of N(x) can be easily found expanding G(w) as the following geometrical series (3.6)

$$G(w) = \frac{2\,\varepsilon_r}{1 - \varepsilon_r} \sum_{m=1}^{\infty} \left(\frac{1 - \varepsilon_r}{1 + \varepsilon_r}\right)^m \exp(-2wdm) . \tag{4.5}$$

Substituting representation (4.5) of G(w) into definition (4.4) of N(x), we have [12]

$$N(x) = \frac{2}{\pi} \frac{\varepsilon_r}{1 - \varepsilon_r} \sum_{m=1}^{\infty} \left(\frac{1 - \varepsilon_r}{1 + \varepsilon_r}\right)^m \frac{2dm}{(2dm)^2 + x^2} , \qquad (d > 0) \tag{4.6}$$

which reduces to only one term in absence of dielectric

$$N(x) = \frac{2d}{\pi} \frac{1}{(2d)^2 + x^2} \qquad (\varepsilon_r = 1) \,. \tag{4.7}$$

For large values of d/a the sum of series (5.7) gives the approximate asymptotic formula [12]

$$N(x) \approx \frac{2}{\pi} \frac{\varepsilon_r}{1 - \varepsilon_r} \sum_{m=1}^{\infty} \left(\frac{1 - \varepsilon_r}{1 + \varepsilon_r}\right)^m \frac{1}{2dm} = \frac{\varepsilon_r}{\pi\, d\, (1 - \varepsilon_r)} \ln\!\left(\frac{2\, \varepsilon_r}{1 + \varepsilon_r}\right) .$$

Unfortunately representation (4.6) is a slowly convergent series, and the convergence becomes more and more problematic when d approaches small values. Besides, because (4.5)

$$\lim_{d \to 0} G(w) = 1 \qquad \forall \varepsilon_r ,$$

the kernel N(x) reveals the presence of a Dirac δ-function, in the previous limit. Therefore, in order to solve equation (4.4) for small values of d/a, we need a procedure to accelerate the convergence of expansion (4.6) in terms of polylogarithm functions. In Appendix A we discuss a possible acceleration based on a generalisation of the previous approximate formula.

The charge distribution density can be related directly to the auxiliary function p(r). By means of an integration by parts, transformation (4.1) can be also written in the equivalent form

$$F(w) = \frac{2}{\pi\, w} \left[p(a) \sin(wa) - \int_0^a \dot{p}(t) \sin(wt)\, dt \right] , \tag{4.8}$$

where $\dot{p}(t)$ is the first derivative of the auxiliary function p(t). Substituting, in fact, expression (4.8) in relation (3.7), we have

$$\sigma(r) = \frac{2V}{\pi} \varepsilon_0 (1 + \varepsilon_r) \left[\frac{p(a)}{\sqrt{a^2 - r^2}} - \int_r^a \dot{p}(t) \frac{dt}{\sqrt{t^2 - r^2}} \right] . \tag{4.9}$$

Expansion (4.9) of σ(r) exhibits the right singularity at the edge of the patch [15]; the divergence is not the result of a numerical procedure, but the divergent term is factorized. Besides from (4.1) and (3.8) we can obtain the capacitance, given simply by

$$C = 4\, \varepsilon_0 (1 + \varepsilon_r) \int_0^a p(t)\, dt = 2\pi\, \varepsilon_0 (1+\varepsilon_r) F(0) \,. \tag{4.10}$$

Fredholm integral equation (4.3) can give in principle satisfactory numerical solution all over the range of values of d/a. In practice, we can use this equation only if d/a is not too small. We arrived up to d/a = 0,01 with our Alpha-station (for all ε_r).

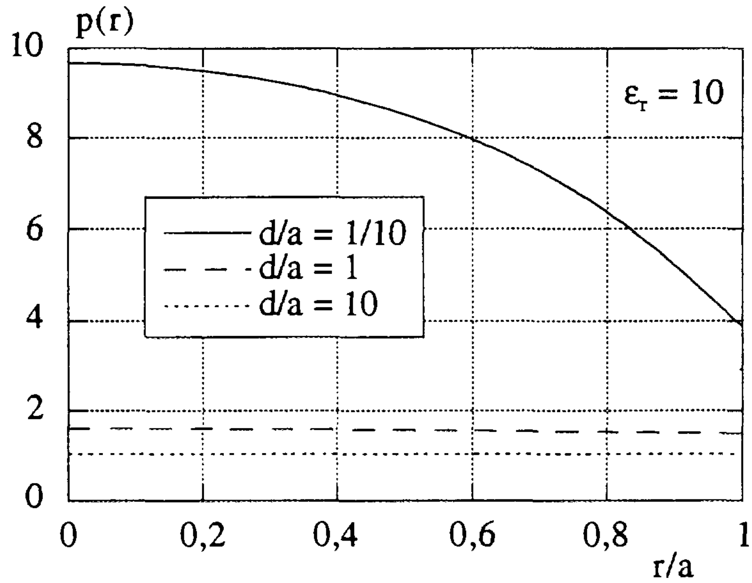

Figure 3 – example of the auxiliary function.

The free term of this integral, in fact, equation represents a first order solution of the problem for high values of d/a, and we can state that the asymptotic solution

$$p(r) \equiv 1\,, \qquad \frac{d}{a} \to \infty\,, \tag{4.11}$$

gives again the value (3.14) of C_∞ evaluated in the previous section. If we use the previous asymptotic expansion (or the more complete given in Appendix A) for the kernel

$$N(x) \approx \frac{\varepsilon_r}{\pi\, d\, (\varepsilon_r - 1)} \ln\left(\frac{2\varepsilon_r}{1 + \varepsilon_r}\right), \tag{4.12}$$

it is not difficult to get from equation (4.5) the approximate representation of the auxiliary function

$$p(r) \approx \frac{1}{1 - \dfrac{2\,\varepsilon_r\, a}{\pi\, d\, (\varepsilon_r - 1)} \ln\left(\dfrac{2\,\varepsilon_r}{1 + \varepsilon_r}\right)} \approx 1 + \frac{2\,\varepsilon_r\, a}{\pi\, d\, (\varepsilon_r - 1)} \ln\left(\frac{2\,\varepsilon_r}{1 + \varepsilon_r}\right), \tag{4.13}$$

and, as a consequence, the capacitance C at the first order in a/d

$$C \approx 4\, a\, \varepsilon_0\, (1 + \varepsilon_r) \left[1 + \frac{2\, a\, \varepsilon_r}{\pi\, d\, (\varepsilon_r - 1)} \ln\left(\frac{2\,\varepsilon_r}{1 + \varepsilon_r}\right)\right]. \tag{4.14}$$

In the particular case $\varepsilon_r = 1$, the previous asymptotic formula simplifies as

$$C \approx 8\, a\, \varepsilon_0 \left(1 + \frac{a}{\pi\, d}\right).$$

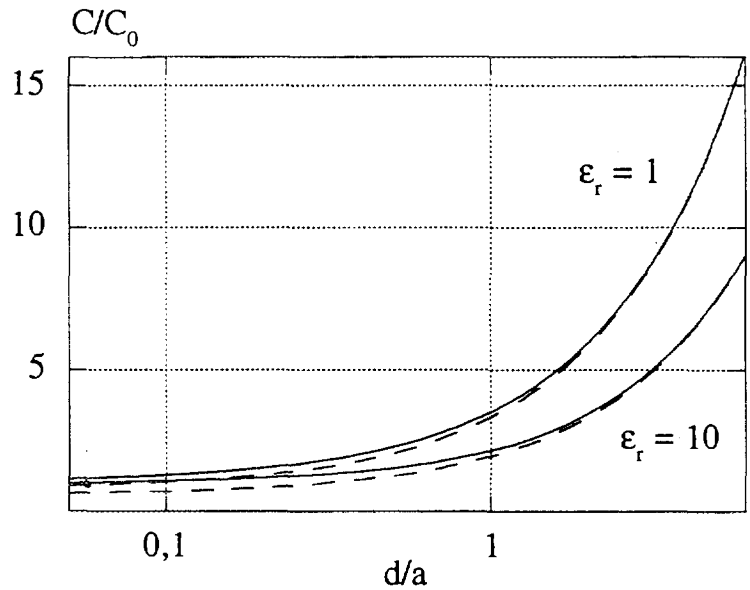

Figure 4 – validation of the asymptotic formula (4.14). The solid lines represent the computed values, whereas the dashed ones are the asymptotic approximation.

Asymptotic formula (4.14) is very useful because it is valid from d/a> 3 for all values of ε_r. Besides the procedure shown does not know breakdown, and more complicated asymptotic expansions for the capacitance can be obtained (Appendix B).

Finally using definition (4.1) and the limit expansion (2.19), we find that

$$p(r) \cong \frac{a\,\varepsilon_r}{d\,(1+\varepsilon_r)} \sqrt{1-\left(\frac{r}{a}\right)^2}\,, \qquad \frac{d}{a} \to 0\,. \tag{4.15}$$

It is worth noting that approximation (4.15) is not a correct solution of our problem because, as already observed for C_0 (the capacitance obtained neglecting the fringing field) the value for r=a is zero, namely p(a) = 0, and this eliminate the discontinuity in $\sigma(r)$ (4.9).

Finally, in order to study the capacitance all over the range of values of d/a, we introduce the dimensionless quantity

$$C_{norm} = \frac{C}{C_0\sqrt{1+\left(\frac{d}{a}\right)^2}} = \frac{4\,d\,(1+\varepsilon_r)}{\pi\,\varepsilon_r\,a\,\sqrt{a^2+d^2}} \int_0^a p(t)\,dt\,. \tag{4.16}$$

This normalization has been introduced because, as it can be easily verified,

$$1 \leq C_{norm} \leq \frac{4\,(1+\varepsilon_r)}{\pi\,\varepsilon_r} \leq \frac{8}{\pi}\,.$$

5. Solution of the integral equation

Equation (4.4) is a Fredholm integral equation of the second kind, whose kernel is a continuous function. In the case of small values of the parameter a/d, the solution of this equation can be represented by an expansion into powers of that parameter. In the general case it is necessary to use numerical methods; this requires in the first place the evaluation of the function N(x) in the interval [0,2a]. Furthermore replacing in (4.4) the value of the integral by a quadrature formula (we used a quadrature gaussian formula), we reduce the problem of determining the auxiliary function p(r) to that of solving a system of linear equations, which makes it possible to set up a plot of numerical values of this function with a necessary degree of accuracy. Therefore, having the numerical values of p(r), we will be able without difficulties to compute the corresponding values of the capacitance C (4.10), or the dimensionless quantity C_{norm} (4.16).

There is another way to reduce system (4.4) to a set of linear equations. Following the scheme already explained by an example in reference [9], we start with the expansion of the unknown p(r) in the functional space $L^2(0,a)$

$$p(r) = \sum_{n=1}^{\infty} (-1)^{n-1} a_n P_{2n-2}\left(\frac{r}{a}\right), \tag{5.1}$$

where a_n are unknown expansion coefficients, and $P_{2m}(x)$ are the subset of the even Legendre polynomials, a complete and orthogonal set, defined by

$$\int_0^1 P_{2n}(x)\, P_{2m}(x)\, dx = \begin{cases} 0, & m \neq n, \\ 1/(4n+1), & m = n. \end{cases} \tag{5.2}$$

In order to determine the unknown a_n we first plug representation (5.1) into the integral equation (4.4), obtaining

$$\sum_{n=1}^{\infty} (-1)^{n-1} a_n P_{2n-2}\left(\frac{r}{a}\right) = 1 + \sum_{n=1}^{\infty} (-1)^{n-1} a_n \int_0^a [N(t+r) + N(t-r)]\, P_{2n-2}\left(\frac{t}{a}\right) dt. \tag{5.3}$$

Multiplying each side of (5.3) for $P_{2m-2}(r/a)$ and using the orthogonality property (5.2), it is

$$\frac{a_m}{4m-3} = \delta_{1,m} + \frac{1}{a} \sum_{n=1}^{\infty} (-1)^{n-1} a_n \int_0^a P_{2m-2}\left(\frac{r}{a}\right) \left\{ \int_0^a [N(t+r) + N(t-r)]\, P_{2n-2}\left(\frac{t}{a}\right) dt \right\} dr, \tag{5.4}$$

where $\delta_{1,m}$ is the Kronecker's symbol. Now, substituting integral definition (4.5) of N(x) into (5.4), and using the relevant integral [12]

$$\int_0^a \cos(wr)\, P_{2m-2}\left(\frac{r}{a}\right) dr = (-1)^{m-1} \sqrt{\frac{\pi a}{2}}\, \frac{J_{2m-3/2}(wa)}{\sqrt{w}},$$

we obtain finally the system of linear equations ($m \in \mathbf{N}$)

$$\frac{a_m}{4m-3} = \delta_{1,m} + \sum_{n=1}^{\infty} a_n B_{n,m} , \tag{5.5}$$

where the symmetric matrix **B** is defined by

$$B_{n,m} = B_{m,n} = \int_0^{\infty} G(w) \frac{J_{2m-3/2}(wa)}{\sqrt{w}} \frac{J_{2n-3/2}(wa)}{\sqrt{w}} dw . \tag{5.6}$$

The properties of convergence and numerical stability of the linear system of algebraic equations (5.5) will be discussed in the next section. Here we want only to underline that from expansion (5.1) one can easily obtain the charge distribution density (showing the correct edge behaviour), and a very simple formula for the capacitance where only the first expansion coefficient is involved.

Substituting expansion (5.1) into integral relation (4.1), we can find a direct relation between the unknown coefficients a_n and F(w), and the resulting expansion will enable us to compute directly F(w) without the intermediate step for p(r), that is [12]

$$F(w) = \sqrt{\frac{2a}{\pi w}} \sum_{n=1}^{\infty} a_n J_{2n-3/2}(wa) . \tag{5.7}$$

This new expansion represents a Neumann series [16], whose general properties and characteristic will be discussed in the next section. We want only to underline that, as a consequence of the this expansion, the charge distribution density, given by (3.8), exhibits the correct edge divergence

$$\sigma(r) = \frac{2\,\varepsilon_0\,(1+\varepsilon_r)\,V}{\sqrt{\pi}} \frac{a^3}{\sqrt{a^2-r^2}} \sum_{m=1}^{\infty} a_m \frac{\Gamma(m)}{\Gamma(m-1/2)} P_{2m-2}\left(\sqrt{1-\frac{r^2}{a^2}}\right), \tag{5.8}$$

where we used the relevant integral [16]

$$\frac{a^{s-2}\,\Gamma(m)}{2^{s-1}\,\Gamma(m-1+s)} \left(1-\frac{r^2}{a^2}\right)^{1-s} \int_0^{\infty} w\, J_{2m-2+s}(aw)\, J_0(rw) \frac{dw}{w^s} = \begin{cases} P_{m-1}^{(0,s-1)}\left(1-\frac{2r^2}{a^2}\right) & r \in [0,a] , \\ 0 & r \in [a,\infty[, \end{cases} \tag{5.9}$$

in the particular case s=1/2. Expansion (5.8) represents the one used by Coen and Gladwell [6]. It is worth noting that the Jacobi polynomials [17]

$$P_{m-1}^{(0,-1/2)}\left(1-\frac{2r^2}{a^2}\right) = C_{2m-2}^{1/2}\left(\sqrt{1-\frac{r^2}{a^2}}\right) = P_{2m-2}\left(\sqrt{1-\frac{r^2}{a^2}}\right)$$

degenerate into the Legendre polynomials if s=1/2. Finally the capacitance follows immediately (4.10)

$$C = 2\pi a\, a_1\, \varepsilon_0\, (1+\varepsilon_r) , \tag{5.10}$$

and only the first expansion coefficient is in effect necessary to compute the capacitance. The choice of basis function clearly eliminates the need for tedious integration to compute C. Any numerical computation then is restricted to evaluation integrals (5.6) and the matrix inversion (5.5).

6. A more general solution

Aim of this section is to find a more general solution of the system of dual integral equations (3.5), adopting an expansion directly for the unknown F(w), without reformulate the problem as a single Fredholm integral equation, as shown in Section 3.

In order to find a solution of the system of integral equations (3.5) we adopt an expansion of the complex unknown in series of analytical functions whose generic term has one or more Bessel functions, or other functions related to these ones. Any series of the type [11]

$$f(z) = \frac{1}{z^s} \sum_{n=0}^{\infty} a_n J_{n+s}(\alpha z) \tag{6.1}$$

is called Neumann series, although in fact Neumann considered only the special type of series for which α and s are parameters depending upon the problem one is considering; the investigation of the more general series is due to Gegenbauer [16]. The possibility of expanding an arbitrary function into a Neumann series is discussed in the Watson's monumental treatise on Bessel functions. Recently Eswaran solved the question of the expansion; he demonstrated that any function whose Fourier transform is of a compact support can be developed in a Neumann series.

The second equation of system (3.5) states that the Fourier transform of the unknown F(w), namely the charge distribution density, is a function of a compact support (the function $\sigma(r) = 0$ for $r > a$). Thus F(w) can be expanded in a Neumann series defined as

$$F(w) = \frac{a}{\Gamma(1+s)\,(2aw)^s} \sum_{n=1}^{\infty} a_n J_{2n-2+s}(wa)\ , \tag{6.2}$$

where a_n are the unknown expansion coefficients, and we selected $\alpha = a$ because we need a function of compact support exactly a. The parameter s is real and such that $s \in (0,1]$, being

$$\lim_{w \to \infty} [1 - G(w)] = 1\ .$$

Expansion (6.2) has been already used by Borkar and Yang [5] for particular values of the index s; unfortunately there is an incorrect use of Neumann expansion since they claim to perform an acceleration procedure also in the particular case s=0. As it will be shown in the following, this acceleration procedure cannot be used for this particular case.

It is immediate to verify that the second equation of the system (3.5) is automatically satisfied, because of the following Fourier cosine transforms ($n \in \mathbb{N}$)

$$\int_0^{+\infty} \cos(wr)\,\frac{J_{2n-2+s}(wa)}{w^s}\,dw = \begin{cases} \dfrac{(-1)^{n-1}2^s(2n-2)!\Gamma(s)}{2a^s\Gamma(2n-2+s)}\left(a^2-r^2\right)^{s-1/2}C^s_{2n-2}\left(\frac{r}{a}\right), & r \in [0,a]\,, \\ 0\,, & r \in\,]a,\infty[\,, \end{cases} \tag{6.3}$$

where $C^s_m(x)$ are Gegenbauer polynomials [17]. Besides the first equation of our system (3.5) becomes

$$\frac{a}{(2a)^s\,\Gamma(1+s)} \sum_{n=1}^{\infty} a_n \int_0^{\infty} [1 - G(w)]\,\frac{J_{2n-2+s}(wa)}{w^s}\cos(wr)\,dw = 1\,, \qquad r \in [0,a]\,. \tag{6.4}$$

The last equation could be already used to compute the expansion coefficients; but an opportune projection will transform it in a system of algebraic equations with a symmetric matrix of the coefficients. In order to realise this, it is necessary to transform the cosine cos(wr) in a Bessel function. Gegenbauer polynomials can help us and, taking the inverse of the Fourier transforms (6.3) ($m \in N$), we find

$$\int_0^{a} \left(a^2-r^2\right)^{s-1/2} C^s_{2m-2}\left(\frac{r}{a}\right) \cos(wr)\,dr = (-1)^{m-1}\,\pi \left(\frac{a}{2}\right)^s \frac{\Gamma(2m-2+2s)}{(2m-2)!\,\Gamma(s)}\,\frac{J_{2m-2+s}(wa)}{w^s}\,, \tag{6.5}$$

equation (6.4) can be formally rewritten as the following algebraic system of linear and complex equations

$$\sum_{n=1}^{\infty} A_{m,n}\,a_n = \begin{cases} 1\,, & m = 1\,, \\ 0\,, & m = 2, 3, \cdots, \end{cases} \tag{6.6}$$

whose matrix coefficients is defined by

$$A_{m,n} = A_{n,m} = \frac{1}{a^{(2s-1)}} \int_0^{\infty} [1 - G(w)]\,\frac{J_{2n-2+s}(wa)}{w^s}\,\frac{J_{2m-2+s}(wa)}{w^s}\,dw\,. \tag{6.7}$$

Performing the first integration [12], it is

$$A_{n,m} = \frac{2^{-2s}\,\Gamma(2s)\,\Gamma(n+m-1.5)}{\Gamma(n-m+s+0.5)\,\Gamma(n+m+2s-1.5)\,\Gamma(m-n+s+0.5)} - \int_0^{\infty} G(u/a)\,\frac{J_{2n-2+s}(u)}{u^s}\,\frac{J_{2m-2+s}(u)}{u^s}\,du\,.$$

In the particular case $s = 1/2$ we find again system (5.5) of the previous section, whereas the first accelerating integral cannot be performed in the case $s=0$.

The solution of system (6.6) requires the inversion of the symmetric and infinite matrix **A**. However, it is necessary to truncate the system of equations to finite order and invert the finite matrix $\mathbf{A}_N$ using a digital computer. We have verified, by numerical experiments, that the inversion procedure of the matrix is well-behaved and stable[1]. This property holds because the matrix is strongly diagonal.

As a consequence of expansion (6.2), the charge distribution density, given by (3.8), exhibits the correct edge divergence

$$\sigma(r) = \frac{\varepsilon_0\,(1+\varepsilon_r)\,V\,a^{5-4s}}{2\,\Gamma(1+s)\,(a^2-r^2)^{1-s}} \sum_{m=1}^{\infty} a_m \frac{\Gamma(m)}{\Gamma(m-1+s)}\, P_{m-1}^{(0,s-1)}\left(1-\frac{2r^2}{a^2}\right), \tag{6.8}$$

where we used the relevant integral (5.9). Chew and Kong [7] already obtained charge distribution (6.8). Finally the capacitance follows immediately (4.10)

$$C = \frac{\pi\, a\, \varepsilon_0\,(1+\varepsilon_r)}{2^{2s-1}\,\Gamma^2(1+s)}\, a_1\,, \tag{6.9}$$

and, as in the previous case, only the first expansion coefficient is in effect necessary to compute the capacitance.

7. Numerical results

It is evident that much of the theory developed has been constructed expressly for the purpose of facilitating numerical computations. To the mathematician such computations are of less interest and importance than the construction of the theories which make them possible. But to physicists and engineers, particularly interested to electromagnetic problem, numerical results have a significance which formulas do not convey. That is why, in this numerical section, we have reproduced some classical results given in the literature as examples to demonstrate the accuracy of the proposed method.

A few words about the numerical algorithms used [18].

- An adaptive gaussian quadrature routine has been used to perform numerical integration (relative error less than 10^{-9}).
- The inverse of the coefficients matrix has been computed by triangular factorisation with row interchanges.
- Bessel functions of large order have been computed by the Miller's algorithm, while Bessel functions of order 0 and 1 have been approximated by rational functions or truncated Chebyshev series.

In order to find the capacitance of our system, we used the results of the previous section in the particular case s=1, using the following formula

$$\frac{C}{C_0} = \frac{\pi\,(1+\varepsilon_r)\,d}{2\,\pi\,\varepsilon_r\,a}\, a_1\,, \tag{7.1}$$

---

[1] We shall say that a stable Nth order inverse has been found 'if the elements of an Nth order matrix formed by truncating the inverse matrix of an (N+M)th order solution does not change appreciably as M is increased'.

We started by doing graphs of the coefficient C/C_0 versus the order of the solution (N) for different values of d/a with the aim to study the convergence of the proposed method and to find the minimum order to invert efficiently the matrix **A**.

The dimension (N) necessary to stabilise this coefficient is practically independent of the ratio d/a, and it is very small (N≈10) as demonstrated in Figure 5 where we represented the error defined by

$$\text{Error}\ (\%) = 100\left(1 - \frac{C_N}{C}\right) \tag{7.2}$$

as a function of the rank of the matrix used (C is the true value, C_N is the approximation of Nth order).

Once found the minimum order N, we can produce a plot of the normalized values of the capacitance of the cylindrical patch resonator as a function of d/a; Figure 6 shows such a plot for different dielectrics.

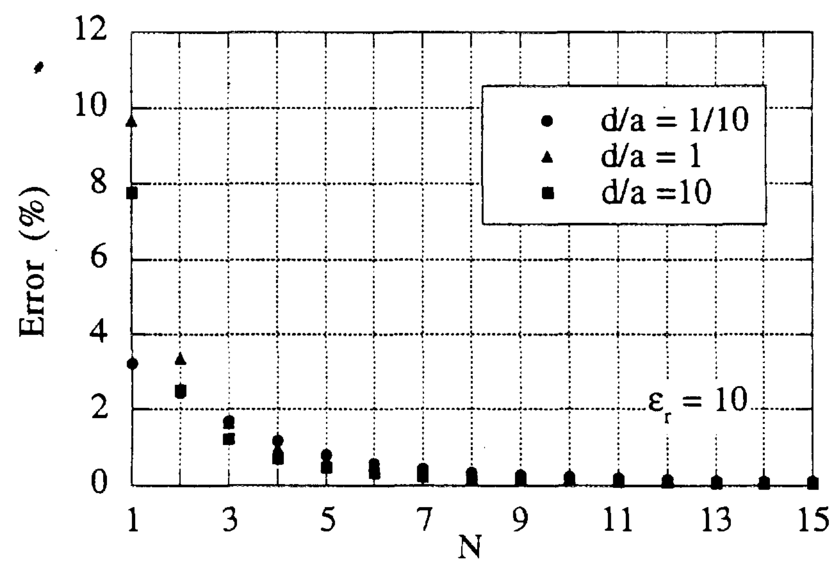

Figure 5 – error versus the order of solution.

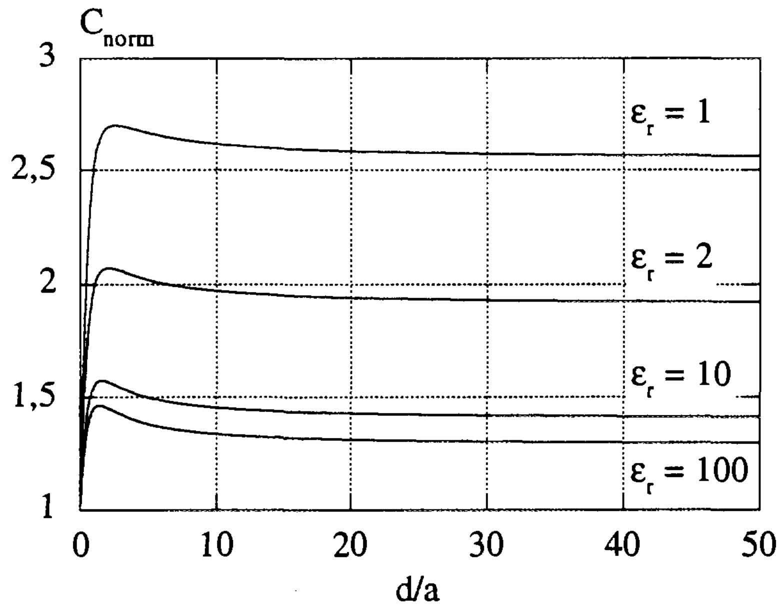

Figure 6 – normalized values of the capacitance for different values of the permittivity ε_r.

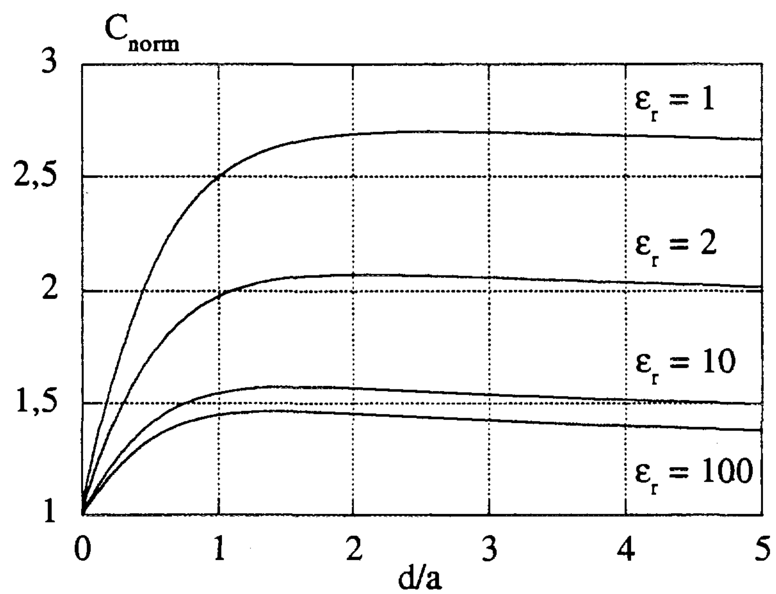

Figure 7 – normalized values of the capacitance for small values of d/a.

Figure 7 shows the region of small values of d/a; it is evident that the minimum value is reached for decreasing values of d/a, if ε_r goes up. It is worth noting that the tangent to these functions is finite in the origin. Chew and Kong [7] obtained a complicate formula valid for small values of d/a, and the simple seminumerical approximation

$$\frac{C}{C_0} \approx 1 + \frac{2\,d}{\pi\, a\, \varepsilon_r}\left[\ln\left(\frac{a}{2\,d}\right) + 1{,}41\ \varepsilon_r + 1{,}77 + \frac{d}{a}\,(0{,}268\ \varepsilon_r + 1{,}65)\right], \tag{7.3}$$

valid in the range d/a ≤ 0,5, as is shown in Figure 8. This ends our discussion on the capacitance of the circular patch resonator because we are able to cover the whole range of d/a: approximate formula (7.3) gives a simple approximation for small values of d/a, whereas formula (4.14), or the more general series (B.8), can represent the capacitance in the complementary range.

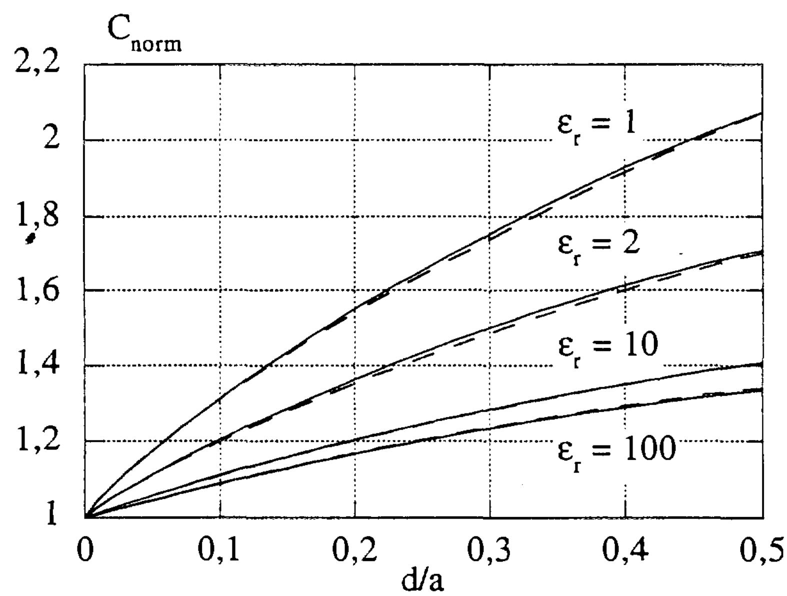

Figure 8 – comparison between computed (solid lines) and seminumerical values (dashed lines).

8. Conclusions and perspectives

We have shown that the electrostatics of a microstrip disk resonator can be described by a system of dual integral equations and that the application of the two methods of solutions of such a system can give numerical values of the capacitance in a wide range of the aspect ratio d/a, and for a large class of dielectric substrates. An approximate formula has been derived in the asymptotic case d/a»1. We have also demonstrated that all the analytical-numerical techniques of solution proposed in literature for this problem can be related to the general theory of dual integral equations theory.

We hope in a near future to apply the same techniques to the problem of a microstrip in order to produce numerical values of the capacitance.

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Appendix A

A way to accelerate the convergence is the use of the polylogarithm functions, defined as [19]

$$\mathrm{poln}_\nu(x) = -\sum_{j=1}^{\infty} \frac{(1-x)^j}{j^\nu} \tag{A.1}$$

where ν is not necessarily an integer; we need only integer index for our purposes. If ν=1, definition (A.1) coincides with the usual logarithmic function, whereas we have the so called dilogarithmic and trilogarithmic function for ν=2,3. Formally we indicate

$$\mathrm{poln}_1(x) = \ln(x)\,, \quad \mathrm{poln}_2(x) = \mathrm{diln}(x)\,, \quad \mathrm{poln}_3(x) = \mathrm{triln}(x)\,.$$

The recurrence relation linking two polylogarithmic functions can be easily deduced by definition (A.1), being

$$\frac{d}{dx}\,\mathrm{poln}_j(x) = \frac{\mathrm{poln}_{j-1}(x)}{x-1}, \qquad j = 2, 3, \cdots\,. \tag{A.2}$$

The first polylogarithmic functions are represented in Figure 9.

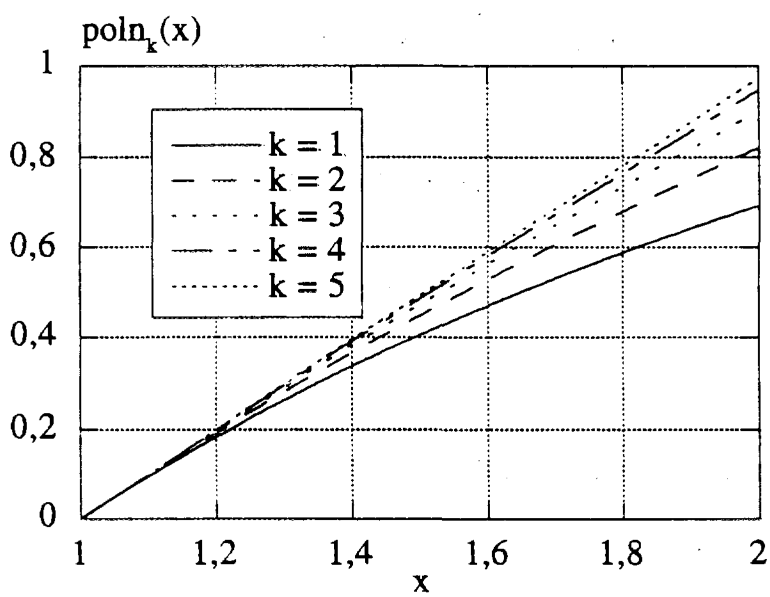

Figure 9 – : polylogarithmic functions of integer order.

In order to show how to employ these functions, we start by the relevant identity

$$\frac{1}{1+\left(\frac{x}{2dm}\right)^2} = \sum_{k=0}^{N} (-1)^k \left(\frac{x}{2dm}\right)^{2k} + \frac{1-(-1)^{N+1}\left(\frac{x}{2dm}\right)^{2N+2}}{1+\left(\frac{x}{2dm}\right)^2} . \qquad (A.3)$$

Substituting expansion (A.3) into series (4.7), and using definitions (A.1), it is not difficult to conclude that

$$N(x) = \frac{\varepsilon_r}{\pi\, d\, (\varepsilon_r - 1)} \sum_{k=0}^{N} (-1)^k \left(\frac{x}{2d}\right)^{2k} \mathrm{poln}_{2k+1}\left(\frac{2\,\varepsilon_r}{1+\varepsilon_r}\right) + R_N(x) , \qquad (A.4)$$

where we called for brevity with $R_N(x)$ the following series

$$R_N(x) = \frac{\varepsilon_r}{\pi\, d\, (1-\varepsilon_r)} (-1)^{N+1} \left(\frac{x}{2d}\right)^{2N+2} \sum_{m=1}^{\infty} \left(\frac{1-\varepsilon_r}{1+\varepsilon_r}\right)^m \frac{1}{m^{2N+1}\left[m^2 + \left(\frac{x}{2d}\right)^2\right]} . \qquad (A.5)$$

It is evident that the series in expansion (A.5) converges more rapidly that the one in (4.7). For numerical calculations we selected N=3. But, if $d/a > 1/2$, identity (A.3) can be used also in the limit $N = \infty$, and therefore we can write the Taylor expansion of the function N(x)

$$N(x) = \frac{\varepsilon_r}{\pi\, d\, (\varepsilon_r - 1)} \sum_{k=0}^{\infty} (-1)^k \left(\frac{x}{2d}\right)^{2k} \mathrm{poln}_{2k+1}\left(\frac{2\,\varepsilon_r}{1+\varepsilon_r}\right), \qquad d/a > 1/2 . \qquad (A.6)$$

Appendix B

Aim of this appendix is to find an expansion of the capacitance as a function of the aspect ratio a/d, whose first two terms are derived in section 4 and are given by formula (4.14).

In order to write integral equation (4.3) in a dimensionless form, we start by introduce the new variables

$$\rho = \frac{r}{a}, \quad \tau = \frac{t}{a}, \quad \xi = \frac{a}{2d},$$

and the new functions

$$p(\rho a) \to \omega(\rho,\xi), \qquad aN(a\rho) \to N(\rho,\xi) = \frac{2\,\varepsilon_r\,\xi}{\pi\,(\varepsilon_r - 1)} \sum_{k=0}^{\infty} (-1)^k \mathrm{poln}_{2k+1}\left(\frac{2\,\varepsilon_r}{1+\varepsilon_r}\right) \xi^{2k} \rho^{2k}.$$

Therefore integral equation (4.3) can be written as

$$\begin{cases} \omega(\rho,\xi) = 1 + \displaystyle\int_0^1 \omega(\tau,\xi)\left[N(\rho+\tau,\xi) + N(\rho-\tau,\xi)\right] d\tau, \\ \rho \in [0,1],\ \xi \in]1/2,\infty[\ . \end{cases} \tag{B.1}$$

Let us assume a Taylor expansion for the unknown

$$\omega(\rho,\xi) = 1 + \sum_{n=1}^{\infty} \omega_n(\rho)\,\xi^n, \tag{B.2}$$

and for the kernel

$$N(\rho,\xi) = \sum_{n=1}^{\infty} N_n(\rho)\,\xi^n, \tag{B.3}$$

where obviously we called

$$\omega_n(\rho) = \frac{1}{n!}\left[\frac{\partial^n \omega(\rho,\xi)}{\partial \xi^n}\right]_{\xi=0}, \qquad N_n(\rho) = \frac{1}{n!}\left[\frac{\partial^n N(\rho,\xi)}{\partial \xi^n}\right]_{\xi=0}. \tag{B.4}$$

Expansion coefficient (B.4) for the kernel derives directly from (A.6) and is defined by

$$N_n(\rho) = \frac{2\,\varepsilon_r}{\pi\,(\varepsilon_r - 1)} \frac{1-(-1)^n}{2} (-1)^{(n-1)/2}\, \mathrm{poln}_n\left(\frac{2\,\varepsilon_r}{1+\varepsilon_r}\right) \rho^{n-1}, \quad n \in N.$$

Thus integral equation (B.1) becomes

$$\begin{cases} \omega_n(\rho) = \displaystyle\int_0^1 \omega_{n-k}(\tau)\left[N_k(\rho+\tau) + N_k(\rho-\tau)\right] d\tau, \\ \rho \in [0,1]\ . \end{cases} \tag{B.5}$$

Because $\omega_0(\rho) = 1$, we can easily obtain the expansion functions $\omega_n(\rho)$ and discover that they are polynomials, defined by

$$\omega_n(\rho) = \sum_{i=0}^{n} \alpha_i(n)\, \rho^i \quad (n \geq 1)\,, \tag{B.6}$$

where, as usual, the generic $\alpha_i(n)$ represents a Taylor expansion coefficient

$$\alpha_i(n) = \frac{1}{i!}\left[\frac{d^i\omega_n(\rho)}{d\rho^i}\right]_{\rho=0} ;$$

and the recurrence equation among them is

$$\alpha_i(n) = \sum_{k=i+1}^{n} \binom{k-1}{i}\left[1 - (-1)^{k-i-1}\right] N_k(1) \left[\sum_{s=0}^{n-k-1} \frac{\alpha_s(n-k)}{k-i+s}\right]. \tag{B.7}$$

Finally it is possible to derive directly an expansion formula for the capacitance. Substituting, in fact, relations (B.1) and (B.6) into formula (4.10), we have

$$\frac{C}{C_0} = \frac{4\,d}{a}\,\frac{1+\varepsilon_r}{\varepsilon_r}\left\{1 + \sum_{n=1}^{\infty}\left[\sum_{i=0}^{n-1}\frac{\alpha_i(n)}{i+1}\right]\left(\frac{a}{2\,d}\right)^n\right\}. \tag{B.8}$$