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INTERACTING FIELDS OF ARBITRARY SPIN AND $N > 4$ SUPERSYMMETRIC SELF-DUAL YANG-MILLS EQUATIONS

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ABSTRACT

We show that the self-dual Yang-Mills equations afford supersymmetrisation to systems of equations invariant under global N-extended super-Poincaré transformations for arbitrary values of N, without the limitation ($N \leq 4$) applicable to standard non-self-dual Yang-Mills theories. These systems of equations provide novel classically consistent interactions for vector supermultiplets containing fields of spin up to $\frac{N-2}{2}$. The equations of motion for the component fields of spin greater than $\frac{1}{2}$ are interacting variants of the first-order Dirac–Fierz equations for zero rest-mass fields of arbitrary spin. The interactions are governed by conserved currents which are constructed by an iterative procedure. In (arbitrarily extended) chiral superspace, the equations of motion for the (arbitrarily large) self-dual supermultiplet are shown to be completely equivalent to the set of algebraic supercurvature constraints defining the self-dual superconnection.

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1 Introduction

1.1 It is widely believed that massless fields of spin greater than two cannot be consistently coupled to lower spin fields. This belief is based on the paucity of appropriate conserved charges. In the case of the free Maxwell equations for a set of spin one fields, $\partial^\mu F^a_{\mu\nu} = 0, a = 1...n$, conceivable interactions are introduced via source currents,

$$\partial^\mu F^a_{\mu\nu} = j^a_\nu \, . \tag{1}$$

Consistency then requires the conservation of these currents,

$$\partial^\nu j^a_\nu = 0 \, , \tag{2}$$

since $\partial^\mu \partial^\nu F^a_{\mu\nu} = 0$ in virtue of the antisymmetry of $F^a_{\mu\nu}$. These conservation laws are related to the gauge invariance of the Maxwell equations [1]. Further analysis shows that for massless vector fields the only consistent equations for interacting spin-one fields without higher derivatives are the Yang-Mills equations [2]. Similarly, for a massless symmetric tensor field [3], the only classically consistent theory [4] is the Einstein theory of gravity, with a conserved stress tensor as the source of interactions. Consistent coupling of massless spin 2 fields with spin $\frac{3}{2}$ fields yields the supergravity equations, with conserved spin-vector currents as sources of gravitino fields. There are not many examples of classically consistent relativistic equations for interacting massless fields of spin greater than two. Consistent light-cone frame interacting theories of higher spin fields with arbitrarily extended supersymmetry do however exist [5]. Furthermore, consistent systems of infinitely many fields of every (half-) interger spin are known from string theories and otherwise (e.g. [6]).

We shall show in the present paper that the possibilities for coupling higher spin fields to lower spin ones are dramatically changed in spaces of signature (4,0) or (2,2) (or in complex space) provided that the gauge field is taken to be self-dual. In the theory of interactions amongst higher spin fields, this is a hitherto unexplored possibility, which yields the unexpected result that the higher spin fields satisfy classically consistent interacting forms of the first-order zero rest-mass Dirac–Fierz equations [7, 8]. The absence of conjugation between dotted and undotted spinor indices in these spaces weakens the compatibility conditions, which in Minkowski space turn out to be forbiddingly strong [8, 3].

1.2 In two-spinor language, with dotted and undotted indices raised and lowered by the skew-symmetric symplectic invariants $\epsilon_{\alpha\beta}, \epsilon_{\dot\alpha\dot\beta}, \epsilon^{\alpha\beta}, \epsilon^{\dot\alpha\dot\beta}$, Maxwell's equations take the form

$$\epsilon^{\dot\alpha\dot\gamma}\partial_{\alpha\dot\gamma} f_{\dot\alpha\dot\beta} + \epsilon^{\beta\gamma}\partial_{\gamma\dot\beta} f_{\alpha\beta} = 2 j_{\alpha\dot\beta} \, , \tag{3}$$

where the symmetric tensor $f_{\dot\alpha\dot\beta}$ describes the helicity $+1$ component and its Minkowski space conjugate $f_{\alpha\beta}$ the helicity -1 component of the Maxwell field. The Bianchi identity $\partial_{[\mu}F_{\nu\rho]}=0$ however takes the form

$$\epsilon^{\dot\alpha\dot\gamma}\partial_{\alpha\dot\gamma}f_{\dot\alpha\dot\beta}-\epsilon^{\beta\gamma}\partial_{\gamma\dot\beta}f_{\alpha\beta}=0, \tag{4}$$

so the complete Maxwell equations may be compactly written in either the form

$$\partial_\alpha{}^{\dot\gamma}f_{\dot\gamma\dot\beta}=j_{\alpha\dot\beta} \tag{5}$$

or the conjugate form

$$\partial^\beta{}_{\dot\beta}f_{\alpha\beta}=j_{\alpha\dot\beta}\ , \tag{6}$$

provided the current is real. Generalising (5), a massless field of helicity s may be described by a symmetric rank $n=2s$ spinor $\varphi_{\dot\alpha_1\ldots\dot\alpha_n}$ satisfying

$$\partial_\alpha^{\dot\alpha_n}\varphi_{\dot\alpha_1\ldots\dot\alpha_n}=j_{\alpha\dot\alpha_1\ldots\dot\alpha_{n-1}}\ . \tag{7}$$

Now differentiating (7) and using the identity $\partial^{\alpha\dot\alpha}\partial_\alpha^{\dot\beta}=\frac{1}{2}\epsilon^{\dot\alpha\dot\beta}\partial_\gamma^{\alpha}\partial_\alpha^{\dot\gamma}$, we see that antisymmetry of ϵ and symmetry of φ in the dotted indices requires for consistency, that the current on the right is divergence free, viz.

$$\partial^{\alpha\dot\alpha_1}j_{\alpha\dot\alpha_1\ldots\dot\alpha_{n-1}}=0\ . \tag{8}$$

This generalises the argument for current conservation (2) in the Maxwell case above. In fact the investigation of source–free versions of (7), the 'zero rest-mass' equations of Dirac and Fierz,

$$\partial_\gamma^{\dot\alpha}\varphi_{\dot\alpha\dot\beta\ldots\dot\delta}=\ 0\ , \tag{9}$$

has a long history (see e.g. [7, 8, 9]). In particular, Fierz [8] discussed the problems of consistently coupling such fields to an external electromagnetic field. He noticed that the minimal substitution of $\partial_{\alpha\dot\alpha}$ in (9) by the gauge covariant derivative $\nabla_{\alpha\dot\alpha}=\partial_{\alpha\dot\alpha}+A_{\alpha\dot\alpha}$ requires the satisfaction of $f_{\dot\alpha\dot\beta}=0$ for consistency. This allows only a pure gauge spin 1 coupling in Minkowski space, where fields with dotted and undotted indices are related by complex conjugation, and eq. (7) is equivalent to the conjugate equation for the helicity +s field,

$$\partial_{\dot\alpha}^{\alpha_n}\varphi_{\alpha_1\ldots\alpha_n}=j_{\dot\alpha\alpha_1\ldots\alpha_n}\ . \tag{10}$$

However, in spaces of signature (4,0) or (2,2) the conjugation between dotted and undotted spinors is lifted, opening new horizons. Eqs. (7) and (10) can then be considered independently. In particular, there is then actually no problem in consistently coupling an external *self-dual* gauge field to such spinors, for the requirement $f_{\dot\alpha\dot\beta}=0$ is precisely the self-duality equation. Now, if a non-zero current J is present on the right-hand side of (9), consistency of such a minimal coupling, i.e. of the first-order equation

$$\nabla^{\dot{\alpha}}_{\gamma}\varphi_{\dot{\alpha}\dot{\beta}\dots\dot{\delta}} = J_{\gamma\dot{\beta}\dots\dot{\delta}} \, , \tag{11}$$

further requires the covariant constancy of this current,

$$\nabla^{\gamma\dot{\beta}} J_{\gamma\dot{\beta}\dots\dot{\delta}} = 0 \, . \tag{12}$$

Remarkably, such conserved currents are provided by the higher-N supersymmetric couplings of self-dual Yang-Mills fields.

1.3 It is noteworthy that higher-spin fields satisfying the interacting Dirac–Fierz equations (11) are precisely the ones which are needed in order to supersymmetrise the self-dual Yang-Mills equations beyond the traditionally 'maximal' $N = 4$ extension. Consistent lower-spin equations of motion recursively give rise to consistent higher-spin equations of motion, so that the self-duality equations may be extended to systems of equations invariant under the N–extended super-Poincaré algebra for *any* choice of N. The self-dual vector supermultiplet may therefore be made as large as one desires, to contain a spectrum of fields up to any given spin $\frac{N-2}{2}$. Classical consistency of the equations of motion does not set any limit on the extension N. This is unlike the situation for the full non-self-dual Yang-Mills equations which have $N = 4$ as the 'maximal' extension [10]. We have already announced the existence of these higher–N supersymmetric systems in [11], where we explicitly displayed the equations of motion for the first two unconventional extensions, $N = 5$ and $N = 6$. The purpose of this paper is to provide the complete proof of the consistency of these super self-duality equations for arbitrarily large N in a manifestly supersymmetric superspace setting. These super self-dual systems are manifestly covariant four dimensional globally supersymmetric realisations of $N > 4$ super-Poincaré algebras. They also provide classically consistent manifestly covariant equations of motion for interacting massless fields of spin greater than two, which are moreover possibly the only consistently coupled generalisations of the zero-rest mass Dirac-Fierz equations (9).

Up to and including $N = 3$ the super self-dual Yang-Mills systems [12] are reductions of the corresponding full non-self-dual super Yang-Mills equations. The former are constrained versions of the latter, with spectra consisting of precisely half the fields. For $N = 3$, for instance, the spectrum of the full super Yang-Mills theory consists of the following two irreducible representations of the super-Poincaré algebra

$$\{f_{\alpha\beta}, \quad \lambda_{i\alpha}, \quad W_{ij}, \quad \chi_{\dot{\alpha}}\} \qquad \text{and} \qquad \{f_{\dot{\alpha}\dot{\beta}}, \quad \lambda^{i}_{\dot{\alpha}}, \quad \overline{W}^{ij}, \quad \chi_{\alpha}\},$$

where the $f_{\alpha\beta}$ (resp. $f_{\dot{\alpha}\dot{\beta}}$) are the (anti-) self-dual parts of the Yang-Mills field-strength

$$[\nabla_{\alpha\dot{\alpha}}, \nabla_{\beta\dot{\beta}}] := \epsilon_{\dot{\alpha}\dot{\beta}} f_{\alpha\beta} + \epsilon_{\alpha\beta} f_{\dot{\alpha}\dot{\beta}} \, ,$$

the λ's and χ's are spin one-half fields, and the W's are scalar fields; all fields taking values in the Lie algebra of the gauge group. In Minkowski space these two supermultiplets are conjugate to each other, but in spaces of signature (4,0) or (2,2), or complexified space, they are independent; and the super self-duality restriction is precisely the condition that the anti-self-dual right-hand multiplet above is zero. Constraining the full super Yang-Mills equations in this fashion yields the super self-duality equations. Thus the equations of motion for the full N=3 theory [13]

$$\begin{aligned}
\epsilon^{\dot{\alpha}\dot{\gamma}}\nabla_{\alpha\dot{\gamma}}f_{\dot{\alpha}\dot{\beta}} \;+\; \epsilon^{\beta\gamma}\nabla_{\gamma\dot{\beta}}f_{\alpha\beta} &= \{\lambda_{\alpha i},\lambda^{i}_{\dot{\beta}}\} + \{\chi_{\alpha},\chi_{\dot{\beta}}\} + [W_i,\nabla_{\alpha\dot{\beta}}W^i] + [W^i,\nabla_{\alpha\dot{\beta}}W_i]\\
\epsilon^{\dot{\gamma}\dot{\alpha}}\nabla_{\alpha\dot{\gamma}}\lambda^{i}_{\dot{\alpha}} &= -\epsilon^{ijk}[\lambda_{j\alpha},W_k] + [\chi_{\alpha},W^i]\\
\epsilon^{\gamma\beta}\nabla_{\gamma\dot{\beta}}\lambda_{i\beta} &= -\epsilon_{ijk}[\lambda^{j}_{\dot{\beta}},W^k] + [\chi_{\dot{\beta}},W_i]\\
\epsilon^{\dot{\gamma}\dot{\alpha}}\nabla_{\alpha\dot{\gamma}}\chi_{\dot{\alpha}} &= -[\lambda_{k\alpha},W^k]\\
\epsilon^{\gamma\beta}\nabla_{\gamma\dot{\beta}}\chi_{\beta} &= -[\lambda^{k}_{\dot{\beta}},W_k]\\
\nabla_{\alpha\dot{\beta}}\nabla^{\alpha\dot{\beta}}W^i &= -2[[W_j,W^i],W^j] + [[W_j,W^j],W^i] + \tfrac{1}{2}\epsilon^{ijk}\{\lambda^{\alpha}_{j},\lambda_{k\alpha}\} + \{\lambda^{i\dot{\alpha}},\chi_{\dot{\alpha}}\}\\
\nabla_{\alpha\dot{\beta}}\nabla^{\alpha\dot{\beta}}W_i &= -2[[W^j,W_i],W_j] + [[W^j,W_j],W_i] + \tfrac{1}{2}\epsilon_{ijk}\{\lambda^{j\dot{\alpha}},\lambda^{k}_{\dot{\alpha}}\} + \{\lambda^{\alpha}_{i},\chi_{\alpha}\}
\end{aligned}$$

reduce to the $N = 3$ super self-duality equations, which we write in N-independent fashion using $\chi_{ijk\dot{\alpha}} = \epsilon_{ijk}\chi_{\dot{\alpha}}$ and $W_{ij} = \epsilon_{ijk}W^k$,

$$\begin{aligned}
\epsilon^{\beta\gamma}\nabla_{\gamma\dot{\beta}}f_{\alpha\beta} &= 0\\
\epsilon^{\gamma\beta}\nabla_{\gamma\dot{\beta}}\lambda_{i\beta} &= 0\\
\epsilon^{\dot{\alpha}\dot{\gamma}}\nabla_{\alpha\dot{\gamma}}\chi_{ijk\dot{\alpha}} &= [\lambda_{[i\alpha},W_{jk]}]\\
\nabla_{\alpha\dot{\beta}}\nabla^{\alpha\dot{\beta}}W_{ij} &= \{\lambda^{\alpha}_{i},\lambda_{j\alpha}\}.
\end{aligned} \tag{13}$$

The first equation is identically satisfied in virtue of the Bianchi identity and the condition $f_{\dot{\alpha}\dot{\beta}} = 0$ which is just the usual (N=0) first order self-duality equation for the vector potential, $F_{\mu\nu} = \frac{1}{2}\epsilon_{\mu\nu\rho\sigma}F^{\rho\sigma}$. For $N = 4$, however, there is no similar correspondence between the standard maximally supersymmetric super Yang-Mills equations [10] and the $N = 4$ supersymmetrisation of the self-duality condition $f_{\dot{\alpha}\dot{\beta}} = 0$ [14]. Being irreducible, the standard $N = 4$ Yang-Mills multiplet

$$\{f_{\alpha\beta},\quad \lambda_{i\alpha},\quad W_{ij},\quad \chi^{i}_{\dot{\alpha}},\quad f_{\dot{\alpha}\dot{\beta}}\}$$

does not admit the the self-duality constraint of the above type. In this case the so-called reality constraint for the scalar superfield $W_{ij} = \frac{1}{2}\epsilon_{ijkl}\overline{W}^{kl}$, which imposes Minkowski space self-conjugacy of the $N = 4$ multiplet, needs to be lifted. Thus having doubled the representation, the self-duality constraint $\overline{W}^{kl} = 0$ may now be imposed instead of the reality constraint in order to reduce the multiplet to the irreducible self-dual one [14]. Therefore, not only are the $N = 4$ super self-duality equations not restrictions of the full non-self-dual equations, but the spectrum of the former is in no sense a restriction of the latter and for $N > 4$ there do not even exist non-self-dual equations corresponding to the $N > 4$ extended super self-duality equations which we present in this paper. For the $N = 4$ case, the above $N = 3$ equations remain unchanged and are merely enhanced by an equation of the form

$$\epsilon^{\dot{\alpha}\dot{\beta}}\nabla_{\alpha\dot{\beta}}g_{ijkl\dot{\alpha}\dot{\gamma}} = J_{ijkl\alpha\dot{\gamma}} , \tag{14}$$

for the additional spin 1 field $g_{ijkl\dot{\alpha}\dot{\gamma}}$, where the current on the right satisfies (12) in virtue of the lower spin equations (13). This pattern actually repeats itself for higher spin fields, yielding first-order equations of the form

$$\nabla_{\alpha}^{\dot{\alpha}_n}\phi_{i_{n+2}\dots i_1\dot{\alpha}_1\dots\dot{\alpha}_n} = J_{i_{n+2}\dots i_1\alpha\dot{\alpha}_1\dots\dot{\alpha}_{n-1}} , \tag{15}$$

for arbitrary $n \geq 2$ up to $n = N-2$, essentially because the $(N-1)$–extended system, which contains fields of spin up to $\frac{(N-3)}{2}$, nestles within the N–extended system completely intact, and provides a conserved source current for a new spin $\frac{(N-2)}{2}$ field. The $N \geq 4$ systems may therefore be seen to be further consequences of the *matreoshka phenomenon* [12] of super self-dual systems; and the *self-dual matreoshka* can even be taken to have infinitely many layers.

The spectra of the N-extended super self-dual systems consist of the Yang-Mills vector potential $A_{\alpha\dot{\beta}}$ having self-dual field-strength $f_{\alpha\beta}$, a $(\frac{1}{2},0)$ spinor $\lambda_{i\alpha}$, and spin $\frac{n}{2}$ fields $\{\phi_{i_{n+2}\dots i_1\dot{\alpha}_1\dots\dot{\alpha}_n};\ 0 \leq n \leq (N-2)\}$ transforming according the totally symmetric $(0,\frac{n}{2})$ representations of the rotation group and according to skew-symmetric representations of the internal $SL(N)$ automorphism group of the N-extended supersymmetry algebra. The $N = 6$ theory, for instance, has the following spectrum transforming according to an irreducible representation of the $N = 6$ super-Poincaré algebra

$$A_{\alpha\dot{\beta}} \qquad \lambda_{i\alpha} \qquad W_{ij} \qquad \chi_{ijk\dot{\alpha}} \qquad g_{ijkl\dot{\alpha}\dot{\beta}} \qquad \psi_{ijklm\dot{\alpha}\dot{\beta}\dot{\gamma}} \qquad C_{ijklmn\dot{\alpha}\dot{\beta}\dot{\gamma}\dot{\delta}}$$

where $i,j = 1,\dots,N$ are internal $sl(N)$ indices which we always write as subscripts, so for instance, the spin 2 field $C_{ijklmn\dot{\alpha}\dot{\beta}\dot{\gamma}\dot{\delta}}$ above is an $sl(6)$ singlet and the spin $\frac{3}{2}$ field $\psi_{ijklm\dot{\alpha}\dot{\beta}\dot{\gamma}}$ is an $sl(6)$ vector, and these may be more conveniently denoted if $N = 6$ thus: $C_{\dot{\alpha}\dot{\beta}\dot{\gamma}\dot{\delta}} = \frac{1}{6!}\epsilon^{ijklmn}C_{ijklmn\dot{\alpha}\dot{\beta}\dot{\gamma}\dot{\delta}}$, $\psi^{i}_{\dot{\alpha}\dot{\beta}\dot{\gamma}} = \frac{1}{6!}\epsilon^{ijklmn}\psi_{ijklm\dot{\alpha}\dot{\beta}\dot{\gamma}}$. However, our notation has the advantage of being N–independent. All these fields take values in the Lie algebra of the gauge group and are linear in the Yang-Mills coupling constant, which we absorb into the definition of these fields. There is no other coupling constant. This means that unlike conventional field theories, where bosonic fields have dimension -1 and fermionic ones dimension $-\frac{3}{2}$, the fields in these self-dual Yang-Mills multiplets have dimensions which decrease with spin. So although the fields $A_{\alpha\dot{\beta}}, \lambda_{i\alpha}, W_{ij}, \chi_{ijk\dot{\alpha}}$ have conventional dimensionalities, the further fields$\phi_{i_{n+2}\dots i_1\dot{\alpha}_1\dots\dot{\alpha}_n}$ of spin $\frac{n}{2}$ have dimension $-\frac{(n+2)}{2}$. This, together with the fact that there is no coupling constant apart from the (dimensionless) Yang-Mills one, renders it impossible to write dimensionless action functionals for the $N > 4$ theories. The vector potential transforms in the usual inhomogeneous fashion, whereas all other fields transform covariantly under gauge transformations,

$$
\begin{aligned}
\delta A_{\alpha\dot{\beta}} &= -\partial_{\alpha\dot{\beta}}\tau(x) - [A_{\alpha\dot{\beta}}, \tau(x)] , \\
\delta\lambda_{i\alpha} &= [\tau(x), \lambda_{i\alpha}] , \\
&\vdots \\
\delta\phi_{i_{n+2}\ldots i_1\dot{\alpha}_1\ldots\dot{\alpha}_n} &= \left[\tau(x), \phi_{i_{n+2}\ldots i_1\dot{\alpha}_1\ldots\dot{\alpha}_n}\right] .
\end{aligned} \tag{16}
$$

These are the only gauge–transformations of these fields; there are no higher–spin gauge–invariances. The latter not being required since all the fields apart from the vector potential transform according to irreducible representations of the rotation group and therefore have no redundant degrees of freedom. We recall that inhomogeneous gauge transformation of a field can be understood as the condition for the field to describe a degree of freedom of unique spin [4]. It is in fact precisely these features of having only one coupling constant and one type of gauge invariance which render traditional theorems forbidding higher–spin couplings inapplicable to our systems.

The arbitrary N supersymmetry transformations take the form

$$
\begin{aligned}
\delta A_{\alpha\dot{\beta}} &= -\bar{\eta}^i_{\dot{\beta}}\lambda_{i\alpha} \\
\delta\lambda_{j\alpha} &= \eta^\beta_j f_{\alpha\beta} + 2\bar{\eta}^{i\dot{\beta}}\nabla_{\alpha\dot{\beta}}W_{ij} \\
\delta W_{jk} &= \eta^\alpha_{[j}\lambda_{k]\alpha} + \bar{\eta}^{i\dot{\beta}}\chi_{ijk\dot{\beta}} \\
\delta\chi_{jkl\dot{\alpha}} &= \eta^\alpha_{[j}\nabla_{\alpha\dot{\alpha}}W_{kl]} + \bar{\eta}^{i\dot{\beta}}\left(g_{ijkl\dot{\alpha}\dot{\beta}} + \epsilon_{\dot{\alpha}\dot{\beta}}[W_{i[j}, W_{kl]}]\right) \\
\delta g_{jklm\dot{\alpha}\dot{\beta}} &= \eta^\alpha_{[j}\nabla_{\alpha(\dot{\alpha}}\chi_{klm]\dot{\beta})} \\
&\quad +\bar{\eta}^{i\dot{\gamma}}\left(\psi_{ijklm\dot{\alpha}\dot{\beta}\dot{\gamma}} + \epsilon_{\dot{\gamma}(\dot{\alpha}}\left(\tfrac{2}{3}[W_{i[j}, \chi_{klm]\dot{\beta})}] - \tfrac{1}{3}[W_{[jk}, \chi_{lm]i\dot{\beta})}]\right)\right) \\
&\vdots \\
\delta\phi_{i_{n+2}\ldots i_1\dot{\alpha}_1\ldots\dot{\alpha}_n} &= \eta^\alpha_{[i_n}\nabla_{\alpha(\dot{\alpha}_n}\phi_{i_{n+1}\ldots i_1\dot{\alpha}_1\ldots\dot{\alpha}_{n-1})} \\
&\quad +\bar{\eta}^{i_{n+3}\dot{\alpha}_{n+1}}\left(\phi_{i_{n+3}\ldots i_1\dot{\alpha}_1\ldots\dot{\alpha}_{n+1}} + \epsilon_{\dot{\alpha}_{n+1}(\dot{\alpha}_n}\Xi_{i_{n+3}\ldots i_1\dot{\alpha}_1\ldots\dot{\alpha}_{n-1})}\right) ,
\end{aligned} \tag{17}
$$

where $\Xi_{i_{n+3}\ldots i_1\dot{\alpha}_1\ldots\dot{\alpha}_{n-1}}$ is a functional of fields of spin less than $\frac{n}{2}$.

1.4 The $N = 4$ self-dual theory was considered by Siegel [14]. The appearance of the additional spin 1 field $g_{ijkl\dot{\alpha}\dot{\beta}}$, which for $N = 4$ is equal to $\epsilon_{ijkl}g_{\dot{\alpha}\dot{\beta}}$, is particularly noteworthy. This possibilty of nontrivially coupling two mutually independent spin 1 fields is a peculiarity of self-dual theories. There is in fact no analogue in standard (non-self-dual) gauge theories, for which the conserved vector current which acts as a source for the Yang-Mills field, provides all consistent spin 1 couplings (including self-couplings) [2]. In Minkowski space the two helicities of the gauge field are complex conjugates of each other. However, for self-dual theories (in spaces of signature (4,0) or (2,2)), the Yang-Mills current acts as a source for only the self-dual $(1,0)$ half of the gauge field. This leaves room for a vector current which acts a source for a $(0,1)$ field; and the $N=4$ theory opens the way for such an additional spin-one field. The source vector current (14) for this new spin-one field is actually the Noether current

$$j_{\alpha\beta ijkl} = -[A_{\alpha}^{\dot\alpha}, g_{ijkl\dot\alpha\beta}] + \{\lambda_{[i\alpha}, \chi_{jkl]\beta}\} - [W_{[ij}, \nabla_{\alpha\beta} W_{kl]}] \tag{18}$$

corresponding to global gauge invariance of the action functional for the $N=4$ theory [14]

$$S = \int d^4x \ \mathrm{Tr}\ \epsilon^{ijkl} \left(f^{\dot\alpha\dot\beta} g_{ijkl\dot\alpha\dot\beta} + \chi^{\dot\alpha}_{[ijk} \nabla_{\alpha\dot\alpha} \lambda^{\alpha}_{l]} + W_{[ij} \Box W_{kl]} - W_{[ij} \{\lambda^{\alpha}_{k}, \lambda_{l]\alpha}\} \right). \tag{19}$$

Repeated supersymmetry transformations of this spin-one current yield source currents for succesively higher-spin fields, which make up successively higher-N supermultiplets.

The action (19) is a component version of the light-cone chiral superspace action [14] based on the Lagrangian due to [15]

$$L = \frac{1}{2} V^{--} \Box V^{--} - \frac{1}{3} V^{--} [\partial^{\alpha+} V^{--}, \partial^{+}_{\alpha} V^{--}],$$

which is an N-independent Lagrangian for the arbitrarily extended super self-duality equations (see also [16]). An alternative harmonic superspace action has been presented in [17].

The existence of the invariant action functional (19) for the $N=4$ theory also gives rise to a conserved gauge-invariant stress tensor for this theory, yielding a possible source term for Einstein's equations. This is a conserved tensor for all $N \geq 4$ theories, which therefore allow a non-trivial coupling to gravity, whereas for $N \leq 3$ the only possible gravitational source term is the standard Yang-Mills stress tensor

$$T_{\alpha\dot\alpha,\beta\dot\beta} = \mathrm{Tr}\ f_{\dot\alpha\dot\beta} f_{\alpha\beta},$$

which vanishes identically for self-dual theories.

1.5 Conventionally, manifestly supersymmetric forms of extended supersymmetric systems take the form of superspace supercurvature constraints. In fact, the standard natural set of supercurvature constraints describes our arbitrarily-extended systems as well.

In N-extended superspace with coordinates $\{x^{\alpha\dot\alpha}, \bar\vartheta^{i\dot\alpha}, \vartheta^{\alpha}_{i}\}$, where $\{\bar\vartheta^{i\dot\alpha}, \vartheta^{\alpha}_{i}; i = 1, \dots N\}$ are odd coordinates and $x^{\alpha\dot\alpha}$ are standard coordinates on $\mathcal{M}^4$, on which the *component fields* above depend. A *self-dual superconnection* is subject to the following constraints (e.g. [18])

$$\{\widehat{\nabla}^i_\alpha, \widehat{\nabla}^j_\beta\} = 0 \tag{20a}$$

$$[\widehat{\nabla}^i_\alpha, \widehat{\nabla}_{\beta\dot\beta}] = 0 \tag{20b}$$

$$\{\widehat{\nabla}^i_\alpha, \widehat{\nabla}_{j\dot\beta}\} = 2\delta^i_j \widehat{\nabla}_{\alpha\dot\beta} \tag{20c}$$

$$\{\widehat{\nabla}_{i(\dot\alpha}, \widehat{\nabla}_{j\dot\beta)}\} = 0 \tag{20d}$$

$$[\widehat{\nabla}_{i(\dot\alpha}, \widehat{\nabla}_{\beta\dot\beta)}] = 0 \tag{20e}$$

$$[\widehat{\nabla}_{\alpha(\dot\alpha}, \widehat{\nabla}_{\beta\dot\beta)}] = 0 \tag{20f}$$

The first three conditions allow the choice of a *chiral basis* in which the covariant derivatives take the form

$$\begin{aligned} \widehat{\nabla}^i_\alpha &= \frac{\partial}{\partial \vartheta^\alpha_i} \\ \widehat{\nabla}_{i\dot\alpha} &= \nabla_{i\dot\alpha} + 2\vartheta^\alpha_i \nabla_{\alpha\dot\alpha} \\ \widehat{\nabla}_{\alpha\dot\alpha} &= \nabla_{\alpha\dot\alpha}, \end{aligned}$$

where $(\nabla_{i\dot\alpha}, \nabla_{\alpha\dot\alpha})$ are covariant derivatives in the *chiral subspace* independent of the ϑ^α_i coordinates. In this basis the single constraint (20d), equivalently written in the form

$$\{\widehat{\nabla}_{i\dot\alpha}, \widehat{\nabla}_{j\dot\beta}\} = \epsilon_{\dot\alpha\dot\beta} \hat{f}_{ij} ,$$

encapsulates the content of all the other constraints and the odd component of the superconnection

$$\hat{A}_{i\dot\alpha}(x, \bar\vartheta, \vartheta) = A_{i\dot\alpha}(x, \bar\vartheta) + 2\vartheta^\alpha_i A_{\alpha\dot\alpha}(x, \bar\vartheta) ,$$

describes the entire self-dual supermultiplet in the form of the curvature component $\hat{f}_{ij}$, which has a quadratic ϑ-expansion in terms of chiral superfields of the form

$$\hat{f}_{ij}(x, \bar\vartheta, \vartheta) = f_{ij}(x, \bar\vartheta) + 2\vartheta^\alpha_{[i}\varphi_{j]\alpha}(x, \bar\vartheta) + 4\vartheta^\alpha_i \vartheta^\beta_j f_{\alpha\beta}(x, \bar\vartheta) . \tag{21}$$

As we shall see, the $\bar\vartheta$-expansion of f_{ij} yields all the higher spin fields $\chi_{ijk\dot\alpha}, g_{ijkl\dot\alpha\dot\beta}$, $\psi_{ijklm\dot\alpha\dot\beta\dot\gamma}$, $C_{ijklmn\dot\alpha\dot\beta\dot\gamma\dot\delta}, \ldots$ etc.

1.6 The plan of this paper is as follows. In section 2, we shall show that the self-dual supercurvature constraints in chiral superspace yield a spectrum of chiral superfields having the leading component fields described above, as well as their dynamical equations. This formulation of super self-duality in chiral superspace is the starting point for the establishment of the supertwistor correspondence for these systems. We have previously described [12, 19] a harmonic space formulation of this correspondence, which, being N–independent, holds in the arbitrary N case too. This supertwistor correspondence yields a complete characterisation of the solution space only if the relation between the Yang-Mills superconnection satisfying the supercurvature constraints and the set of component fields satisfying the component super self-duality equations is one-to-one. Machinery for establishing such equivalences was developed in [13], where the conventional $N = 3$ superconnection constraints were proven to be completely equivalent to the full super Yang-Mills equations of motion. The method was later applied to the ten dimensional case too [20]. This method, which we apply to the super self-duality conditions in sections 3 and 4, also yields a very effective mechanism for extracting component information from superfield data. The new $N = 4$ stress tensor is actually a member of a supermultiplet of conserved tensors. In fact $N > 4$ generalisations of these conserved tensors also exist and are presented in section 5.

2 Self-duality constraints for the superconnection and the superfield equations of motion

The N-extended super self-duality equations in four dimensional space are most economically written in N-extended *chiral superspace*, $\mathcal{M}^{4|2N}$, with coordinates $\{x^{\alpha\dot{\alpha}}, \bar{\vartheta}^{i\dot{\alpha}}\}$, where $\{\bar{\vartheta}^{i\dot{\alpha}}; i = 1, \dots N\}$ are odd coordinates and $x^{\alpha\dot{\alpha}}$ are standard coordinates on which the *component fields* on $\mathcal{M}^4$ depend. For generality, we shall work in the complexified setting. Reality conditions appropriate to a (4,0) or (2,2) signature may always be imposed. We shall take the extension N to be arbitrary. Gauge-covariant derivatives in chiral superspace $\mathcal{M}^{4|2N}$ take the form

$$\begin{aligned} \nabla_{i\dot{\alpha}} &= \partial_{i\dot{\alpha}} + A_{i\dot{\alpha}} \\ \nabla_{\alpha\dot{\alpha}} &= \partial_{\alpha\dot{\alpha}} + A_{\alpha\dot{\alpha}} \ , \end{aligned}$$

where the partial derivatives $\partial_{i\dot{\alpha}} \equiv \frac{\partial}{\partial\bar{\vartheta}^{i\dot{\alpha}}}$, $\partial_{\alpha\dot{\alpha}} \equiv \frac{\partial}{\partial x^{\alpha\dot{\alpha}}}$ provide a holonomic basis for $\mathcal{M}^{4|2N}$; chiral superspace being torsion-free. The components of the superconnection $(A_{i\dot{\alpha}}, A_{\alpha\dot{\alpha}})$ take values in the Lie algebra of the gauge group, their transformations being parametrised by Lie algebra-valued sections on $\mathcal{M}^{4|2N}$ (c.f. (16)):

$$\delta A_{i\dot{\alpha}} = -\partial_{i\dot{\alpha}}\tau(x,\bar{\vartheta}) - [A_{i\dot{\alpha}}, \tau(x,\bar{\vartheta})] \tag{22a}$$

$$\delta A_{\alpha\dot{\beta}} = -\partial_{\alpha\dot{\beta}}\tau(x,\bar{\vartheta}) - [A_{\alpha\dot{\beta}}, \tau(x,\bar{\vartheta})] \ . \tag{22b}$$

On $\mathcal{M}^{4|2N}$, the super self-duality conditions take the form of the following supercurvature constraints

$$\{\nabla_{i(\dot{\alpha}}, \nabla_{j\dot{\beta})}\} = 0 \tag{23a}$$

$$[\nabla_{i(\dot{\alpha}}, \nabla_{\beta\dot{\beta})}] = 0 \tag{23b}$$

$$[\nabla_{\alpha(\dot{\alpha}}, \nabla_{\beta\dot{\beta})}] = 0 \ , \tag{23c}$$

or equivalently

$$\{\nabla_{i\dot\alpha}, \nabla_{j\dot\beta}\} = \epsilon_{\dot\alpha\dot\beta} f_{ij} \tag{24a}$$

$$[\nabla_{i\dot\alpha}, \nabla_{\beta\dot\beta}] = \epsilon_{\dot\alpha\dot\beta} f_{i\beta} \tag{24b}$$

$$[\nabla_{\alpha\dot\alpha}, \nabla_{\beta\dot\beta}] = \epsilon_{\dot\alpha\dot\beta} f_{\alpha\beta} \ , \tag{24c}$$

where $f_{ij} = f_{ij}(x, \bar\vartheta)$ is skew–symmetric and $f_{\alpha\beta} = f_{\alpha\beta}(x, \bar\vartheta)$ is symmetric and has the corresponding $\mathcal{M}^4$ Yang-Mills field-strength $\mathring{f}_{\alpha\beta}(x)$ as its leading component in a $\bar\vartheta$–expansion. Henceforth all fields are superfields depending on both $x^{\alpha\dot\alpha}$ and $\bar\vartheta^{i\dot\alpha}$ and we distinguish superfields from their leading components in a $\bar\vartheta$-expansion (i.e. ordinary fields on $\mathcal{M}^4$) by placing a circle over the latter. We shall henceforth thus write all fields of section 1.3, which are leading components of corresponding superfields; and we shall denote the latter by the same letter as the former. Thus $\lambda_{i\alpha}$, for instance, will be used to denote the superfield containing $\mathring\lambda_{i\alpha}$ as its $\bar\vartheta$-independent part.

The superfield curvatures (24) are not independent; they are related by super-Jacobi identities. Firstly, the dimension -3 Jacobi identity implies, in virtue of the constraint (23c), the Yang-Mills equation

$$\nabla^{\alpha}_{\dot\beta} f_{\alpha\beta} = 0. \tag{25}$$

for the *superfield* $f_{\alpha\beta}$. Next, in virtue of the constraints (23b) and (23c), the dimension $-2\frac{1}{2}$ Jacobi identity yields the relationship

$$\nabla_{i\dot\alpha} f_{\alpha\beta} = \frac{1}{2}\nabla_{(\alpha\dot\alpha} f_{i\beta)}. \tag{26}$$

Multiplying both sides by $\epsilon^{\alpha\beta}$ yields a dynamical equation for the dimension $-\frac{3}{2}$ curvature, which allows its identification with a spinor superfield

$$\lambda_{i\alpha} := f_{i\alpha} \ ,$$

having equation of motion

$$\nabla^{\alpha}_{\dot\alpha} \lambda_{i\alpha} = 0 \ . \tag{27}$$

Now the dimension -2 Jacobi identity says that

$$\nabla_{\alpha\dot\alpha} f_{ij} = \nabla_{i\dot\alpha} \lambda_{j\alpha} \ . \tag{28}$$

Defining a scalar superfield

$$W_{ij} := \frac{1}{2} f_{ij} \ , \tag{29}$$

where the rescaling merely serves to bring our notation into correspondence with that in the literature, we obtain the equation of motion

$$\Box W_{ij} = \frac{1}{2}\{\lambda_i^{\alpha}, \lambda_{j\alpha}\} \ , \tag{30}$$

where the covariant d'Alembertian is defined by $\Box = \frac{1}{2}\nabla^{\alpha\dot\beta}\nabla_{\alpha\dot\beta}$. The curvature constraints (23), combined with the definitions (24), may therefore also be written

$$\begin{aligned} \{\nabla_{i\dot\alpha}, \nabla_{j\dot\beta}\} &= 2\epsilon_{\dot\alpha\dot\beta}W_{ij} \\ [\nabla_{i\dot\alpha}, \nabla_{\beta\dot\beta}] &= \epsilon_{\dot\alpha\dot\beta}\lambda_{i\beta} \\ [\nabla_{\alpha\dot\alpha}, \nabla_{\beta\dot\beta}] &= \epsilon_{\dot\alpha\dot\beta}f_{\alpha\beta}\,. \end{aligned} \tag{31}$$

Now the dimension $-\frac{3}{2}$ Jacobi identity tells us that the spinorial derivative of the scalar superfield, $\nabla_{i\dot\alpha}W_{jk}$, is a superfield, totally skew-symmetric in ijk:

$$\nabla_{i\dot\alpha}W_{jk} = \chi_{ijk\dot\alpha}\,. \tag{32}$$

Acting on both sides by $\nabla_\alpha^{\dot\alpha}$ yields

$$\begin{aligned} \nabla_\alpha^{\dot\alpha}\chi_{ijk\dot\alpha} &= \nabla_\alpha^{\dot\alpha}\nabla_{i\dot\alpha}W_{jk} \\ &= 2[\lambda_{i\alpha}, W_{jk}] + \tfrac{1}{2}\nabla_{i\dot\alpha}\nabla_j^{\dot\alpha}\lambda_{k\alpha} \quad \text{using (24b) and (28).} \end{aligned}$$

Now the second term on the right is equal to

$$\begin{aligned} &-2[W_{ij}, \lambda_{k\alpha}] - \tfrac{1}{2}\nabla_j^{\dot\alpha}\nabla_{i\dot\alpha}\lambda_{k\alpha} \\ &= 2[\lambda_{k\alpha}, W_{ij}] - \nabla_j^{\dot\alpha}\nabla_{\alpha\dot\alpha}W_{ik} \quad \text{using (28),} \\ &= 2[\lambda_{k\alpha}, W_{ij}] - 2[\lambda_{j\alpha}, W_{ik}] - \nabla_\alpha^{\dot\alpha}\chi_{ijk\dot\alpha} \quad \text{using (31),(32).} \end{aligned}$$

We therefore have the equation of motion for the superfield [2] $x_{ijk\dot\alpha}$,

$$\nabla_\alpha^{\dot\alpha}\chi_{ijk\dot\alpha} = [\lambda_{[i\alpha}, W_{jk]}]. \tag{33}$$

Action of the covariant derivative $\nabla_{\dot\beta}^{\alpha}$ yields the wave equation

$$\Box\chi_{ijk\dot\alpha} = [\lambda^\alpha_{[i}, \nabla_{\alpha\dot\alpha}W_{jk]}], \tag{34}$$

since $\nabla_{\dot\beta}^{\gamma}\nabla_\gamma^{\dot\alpha} = -\delta_{\dot\beta}^{\dot\alpha}\Box$ in virtue of the curvature constraint (23c). Now using (32) we have

$$3\nabla_{i\dot\alpha}\chi_{jkl\dot\beta} = \nabla_{i\dot\alpha}(\nabla_{[j\dot\beta}W_{kl]}) = 2\epsilon_{\dot\alpha\dot\beta}[W_{i[j}, W_{kl]}] - \nabla_{[j\dot\beta}\chi_{kl]i\dot\alpha}\,. \tag{35}$$

So

$$4\nabla_{i\dot\alpha}\chi_{jkl\dot\beta} = -\nabla_{[j\dot\beta}\chi_{kl]i\dot\alpha} + \nabla_{i\dot\alpha}\chi_{jkl\dot\beta} + 2\epsilon_{\dot\alpha\dot\beta}[W_{i[j}, W_{kl]}]\,. \tag{36}$$

Symmetrising in $\dot\alpha, \dot\beta$ yields the definition of a rank-4 spin-one superfield $g_{ijkl\dot\alpha\dot\beta}$,

$$4\nabla_{i(\dot\alpha}\chi_{jkl\dot\beta)} = -\nabla_{[j(\dot\alpha}\chi_{kli]\dot\beta)} = 8g_{ijkl\dot\alpha\dot\beta}\,. \tag{37}$$

[2] All our (skew-)symmetrisations are with weight one. For instance, $[\lambda_{[i\alpha}, W_{jk]}] \equiv [\lambda_{i\alpha}, W_{jk}] + [\lambda_{j\alpha}, W_{ki}] + [\lambda_{k\alpha}, W_{ij}]$.

Eq. (35) also imples that

$$2\nabla_{i\dot\alpha}\chi_{jkl\dot\beta} = -\nabla_{[j\dot\beta}\chi_{kl]i\dot\alpha} - \nabla_{i\dot\alpha}\chi_{jkl\dot\beta} + 2\epsilon_{\dot\alpha\dot\beta}[W_{i[j}, W_{kl]}]$$

and tracing over the spinor indices yields

$$2\nabla_i^{\dot\beta}\chi_{jkl\dot\beta} = \nabla_{[j}^{\dot\beta}\chi_{kli]\dot\beta} + 4[W_{i[j}, W_{kl]}] \ .$$

This implies that $\nabla_{[j}^{\dot\beta}\chi_{kli]\dot\beta} = 0$, identically, and we therefore have the superspace relationship between $g_{ijkl\dot\alpha\dot\beta}$ and lower-spin fields,

$$\begin{aligned} \nabla_{i\dot\alpha}\chi_{jkl\dot\beta} &\equiv \tfrac{1}{2}\nabla_{i(\dot\alpha}\chi_{jkl\dot\beta)} + \tfrac{1}{2}\epsilon_{\dot\alpha\dot\beta}\nabla_i^{\dot\gamma}\chi_{jkl\dot\gamma} \\ &= g_{ijkl\dot\alpha\dot\beta} + \epsilon_{\dot\alpha\dot\beta}\Xi_{ijkl} \ , \text{ where } \quad \Xi_{ijkl} = [W_{i[j}, W_{kl]}] \end{aligned} \tag{38}$$

Now, the action of $\nabla_\alpha^{\dot\beta}$ on both sides and use of (28) and (31)-(33) yields

$$\begin{aligned} \nabla_\alpha^{\dot\beta} g_{ijkl\dot\alpha\dot\beta} &= \{[\nabla_\alpha^{\dot\beta}, \nabla_{i\dot\alpha}], \chi_{jkl\dot\beta}\} + \{\nabla_{i\dot\alpha}, \nabla_\alpha^{\dot\beta}\chi_{jkl\dot\beta}\} - \nabla_{\alpha\dot\alpha}[W_{i[j}, W_{kl]}] \\ &= \{\lambda_{i\alpha}, \chi_{jkl\dot\alpha}\} + \{\nabla_{i\dot\alpha}, [\lambda_{[j\alpha}, W_{kl]}] - [\nabla_{\alpha\dot\alpha}W_{i[j}, W_{kl]}] - [W_{i[j}, \nabla_{\alpha\dot\alpha}W_{kl]}] \\ &= \{\lambda_{i\alpha}, \chi_{jkl\dot\alpha}\} - \{\lambda_{[j\alpha}, \chi_{kl]n\dot\alpha}\} + [\nabla_{\alpha\dot\alpha}W_{i[j}, W_{kl]}] - [W_{i[j}, \nabla_{\alpha\dot\alpha}W_{kl]}] \\ &= \{\lambda_{[i\alpha}, \chi_{jkl]\dot\alpha}\} + [\nabla_{\alpha\dot\alpha}W_{[ij}, W_{kl]}] \ . \end{aligned}$$

So the equation of motion for the spin-one superfield $g_{ijkl\dot\alpha\dot\beta}$ follows;

$$\nabla_\alpha^{\dot\beta} g_{ijkl\dot\alpha\dot\beta} = J_{ijkl\alpha\dot\alpha} \ , \tag{39}$$

where the current

$$J_{ijkl\alpha\dot\alpha} = \{\lambda_{[i\alpha}, \chi_{jkl]\dot\alpha}\} - [W_{[ij}, \nabla_{\alpha\dot\alpha}W_{kl]}]. \tag{40}$$

is covariantly conserved,

$$\nabla^{\alpha\dot\beta} J_{ijkl\alpha\dot\beta} = 0 \ . \tag{41}$$

Proof. In virtue of (27),

$$\nabla^{\alpha\dot\beta} J_{ijkl\alpha\dot\beta} = \{\lambda_{[i\alpha}, \nabla^{\alpha\dot\beta}\chi_{jkl]\dot\beta}\} - [\nabla^{\alpha\dot\beta}W_{[ij}, \nabla_{\alpha\dot\beta}W_{kl]}] - 2[W_{[ij}, \Box W_{kl]}]$$

The second term is identically zero and using the equations of motion (30) and (33) we see that the rest of the right-hand side also vanishes in virtue of the Jacobi identities. □

Now, action of the covariant derivative $\nabla_{\dot\gamma}^{\alpha}$ on (39) yields

$$\nabla_{\dot\gamma}^{\alpha}\nabla_\alpha^{\dot\beta} g_{ijkl\dot\alpha\dot\beta} = \nabla_{\dot\gamma}^{\alpha} J_{ijkl\alpha\dot\alpha} = \frac{1}{2}\epsilon_{\dot\gamma\dot\alpha}\nabla^{\alpha\dot\beta} J_{ijkl\alpha\dot\beta} + \frac{1}{2}\nabla_{(\dot\gamma}^{\alpha} J_{ijkl\alpha\dot\alpha)} = \frac{1}{2}\nabla_{(\dot\gamma}^{\alpha} J_{ijkl\alpha\dot\alpha)} \ ,$$

using (41). We therefore obtain the wave equation

$$\Box g_{ijkl\dot\alpha\dot\beta} = -\frac{1}{2}\{\lambda_{[i\alpha}, \nabla_{(\dot\alpha}^{\alpha}\chi_{jkl]\dot\beta)}\} + \frac{1}{2}[\nabla_{(\dot\alpha}^{\alpha}W_{[ij}, \nabla_{\alpha\dot\beta)}W_{kl]}]. \tag{42}$$

Remarkably, the equations of motion obtained hitherto for the partial supermultiplet $\{A_{\alpha\dot{\beta}}, \lambda_{i\alpha}, W_{ij}, \chi_{ijk\dot{\alpha}}, g_{ijkl\dot{\alpha}\dot{\beta}}\}$ are Euler-Lagrange equations for the simultaneous variation of the $\binom{N}{4}$ superfield functionals

$$\mathcal{L}_{ijkl} = \mathrm{Tr}\ \left(f^{\dot{\alpha}\dot{\beta}} g_{ijkl\dot{\alpha}\dot{\beta}} + \chi^{\dot{\alpha}}_{[ijk} \nabla_{\alpha\dot{\alpha}} \lambda^{\alpha}_{l]} + W_{[ij} \Box W_{kl]} - W_{[ij}\{\lambda^{\alpha}_{k}, \lambda_{l]\alpha}\}\right), \tag{43}$$

whose leading (ϑ-independent) terms yield, for $N = 4$, the action functional $S = \int dx^4 \epsilon^{ijkl} \mathcal{L}_{ijkl}$, i.e. (19).

Consider now,

$$\begin{aligned} \nabla_{i\dot{\alpha}} g_{jklm\dot{\beta}\dot{\gamma}} &\equiv \tfrac{1}{3}\left(\nabla_{i(\dot{\alpha}} g_{jklm\dot{\beta}\dot{\gamma})} + \epsilon_{\dot{\alpha}(\dot{\beta}} \nabla^{\dot{\delta}}_{i} g_{jklm\dot{\gamma})\dot{\delta}}\right) \\ &= \psi_{ijklm\dot{\alpha}\dot{\beta}\dot{\gamma}} + \epsilon_{\dot{\alpha}(\dot{\beta}} \Xi_{ijklm\dot{\gamma})} , \end{aligned} \tag{44}$$

which defines a spin $\frac{3}{2}$ superfield $\psi_{ijklm\dot{\alpha}\dot{\beta}\dot{\gamma}}$. Now from (37) and using (32), (38),

$$\begin{aligned} 2\nabla^{\dot{\gamma}}_{i} g_{jklm\dot{\beta}\dot{\gamma}} &= \nabla^{\dot{\gamma}}_{i}(\nabla_{j(\dot{\beta}} \chi_{klm\dot{\gamma})}) \\ &= [[\nabla^{\dot{\gamma}}_{i}, \nabla_{j(\dot{\beta}}], \chi_{klm\dot{\gamma})}] - \nabla_{j\dot{\beta}}(\nabla^{\dot{\gamma}}_{i} \chi_{klm\dot{\gamma}}) - \nabla_{j\dot{\gamma}}(\nabla^{\dot{\gamma}}_{i} \chi_{klm\dot{\beta}}) \\ &= 6[W_{ij}, \chi_{klm\dot{\beta}}] - 2\nabla_{j\dot{\beta}}([W_{i[k}, W_{lm]}]) + \nabla^{\dot{\gamma}}_{j}(g_{iklm\dot{\gamma}\dot{\beta}} + \epsilon_{\dot{\gamma}\dot{\beta}}[W_{i[k}, W_{lm]}]) \\ &= 6[W_{ij}, \chi_{klm\dot{\beta}}] - 3[\chi_{ji[k\dot{\beta}}, W_{lm]}] - 3[W_{i[k}, \chi_{lm]j\dot{\beta}}] + \nabla^{\dot{\gamma}}_{j} g_{iklm\dot{\beta}\dot{\gamma}} . \end{aligned}$$

So on adding parts symmetric and skew-symmetric in i, j we obtain

$$\nabla^{\dot{\gamma}}_{i} g_{jklm\dot{\beta}\dot{\gamma}} = 2[W_{i[j}, \chi_{klm]\dot{\beta}}] - [W_{[jk}, \chi_{lm]i\dot{\beta}}] , \tag{45}$$

yielding, from (44), the relation between $\psi_{ijklm\dot{\alpha}\dot{\beta}\dot{\gamma}}$ and lower-spin fields,

$$\nabla_{i\dot{\alpha}} g_{jklm\dot{\beta}\dot{\gamma}} = \psi_{ijklm\dot{\alpha}\dot{\beta}\dot{\gamma}} + \epsilon_{\dot{\alpha}(\dot{\beta}} \left(\frac{2}{3}[W_{i[j}, \chi_{klm]\dot{\gamma})}] - \frac{1}{3}[W_{[jk}, \chi_{lm]i\dot{\gamma})}]\right). \tag{46}$$

The spin $\frac{3}{2}$ equation of motion follows in virtue of lower-spin equations on application of $\nabla^{\dot{\gamma}}_{\alpha}$ to both sides, namely,

$$\nabla^{\dot{\gamma}}_{\alpha} \psi_{ijklm\dot{\alpha}\dot{\beta}\dot{\gamma}} = J_{ijklm\alpha\dot{\alpha}\dot{\beta}} , \tag{47}$$

with

$$J_{ijklm\alpha\dot{\alpha}\dot{\beta}} = [\lambda_{[i\alpha}, g_{jklm]\dot{\alpha}\dot{\beta}}] + \frac{2}{3}[\nabla_{\alpha(\dot{\alpha}} W_{[ij}, \chi_{klm]\dot{\beta})}] - \frac{1}{3}[W_{[ij}, \nabla_{\alpha(\dot{\alpha}} \chi_{klm]\dot{\beta})}], \tag{48}$$

a covariantly conserved current satisfying

$$\nabla^{\alpha\dot{\alpha}} J_{ijklm\alpha\dot{\alpha}\dot{\beta}} = 0 \tag{49}$$

in virtue of lower spin equations of motion. Similarly,

$$\nabla_{i\dot{\alpha}} \psi_{jklmn\dot{\beta}\dot{\gamma}\dot{\delta}} = C_{ijklmn\dot{\alpha}\dot{\beta}\dot{\gamma}\dot{\delta}} + \epsilon_{\dot{\alpha}(\dot{\beta}} \Xi_{ijklmn\dot{\gamma}\dot{\delta})} \tag{50}$$

where the spin 2 superfield is defined by

$$C_{ijklmn\dot{\alpha}\dot{\beta}\dot{\gamma}\dot{\delta}} = \frac{1}{4}\nabla_{i(\dot{\alpha}}\psi_{jklmn\dot{\beta}\dot{\gamma}\dot{\delta})} \tag{51}$$

and

$$\begin{aligned} \Xi_{ijklmn\dot{\beta}\dot{\gamma}} &\equiv \tfrac{1}{4}\nabla_i^{\dot{\alpha}}\psi_{jklmn\dot{\alpha}\dot{\beta}\dot{\gamma}} = \tfrac{1}{12}\nabla_i^{\dot{\alpha}}\left(\nabla_{j(\dot{\alpha}}g_{klmn\dot{\beta}\dot{\gamma})}\right) \quad \text{from (44)}, \\ &= \tfrac{1}{6}[\chi_{i[jk\dot{\beta}}, \chi_{lmn]\dot{\gamma}}] + \tfrac{1}{2}[W_{i[j}, g_{klmn]\dot{\beta}\dot{\gamma}}] + \tfrac{1}{6}[W_{[jk}, g_{lmn]i\dot{\beta}\dot{\gamma}}] \end{aligned} \tag{52}$$

using (45). Covariant differentiation of both sides of (50) yields the spin 2 dynamical equation

$$\nabla_\alpha^{\dot{\delta}} C_{ijklmn\dot{\alpha}\dot{\beta}\dot{\gamma}\dot{\delta}} = J_{ijklmn\dot{\alpha}\dot{\beta}\dot{\gamma}} \, , \tag{53}$$

with the covariantly conserved current

$$\begin{aligned} J_{ijklmn\alpha\dot{\beta}\dot{\gamma}\dot{\delta}} &= \{\lambda_{[i\alpha}, \psi_{jklmn]\dot{\beta}\dot{\gamma}\dot{\delta}}\} + \tfrac{1}{6}\{\chi_{[ijk(\dot{\beta}}, \nabla_{\alpha\dot{\gamma}}\chi_{lmn]\dot{\delta})}\} \\ &\quad + \tfrac{1}{2}[\nabla_{\alpha(\dot{\beta}}W_{[ij}, g_{klmn]\dot{\gamma}\dot{\delta})}] - \tfrac{1}{6}[W_{[ij}, \nabla_{\alpha(\dot{\beta}}g_{klmn]\dot{\gamma}\dot{\delta})}] \, . \end{aligned} \tag{54}$$

By iteration of this procedure, superfields $\phi_{i_n \ldots i_1 \dot{\alpha}_{n-2}\dot{\alpha}_1 \ldots \dot{\alpha}_{n-1}}$ of $sl(N)$ rank n of spin $s = \frac{(n-2)}{2}$ may be produced for higher spins. The super-Jacobi identities and the constraints (31) admit such superfields for every $n \leq N$ and for arbitrarily large N. We therefore have the following superspace recursion relations between $sl(N)$ rank n and rank $(n+1)$ superfields:

$$\nabla_{i\dot{\alpha}} f_{\alpha\beta} = \frac{1}{2}\nabla_{(\alpha\dot{\alpha}}\lambda_{i\beta)} \tag{55a}$$

$$\nabla_{i\dot{\alpha}}\lambda_{j\alpha} = 2\nabla_{\alpha\dot{\alpha}}W_{ij} \tag{55b}$$

$$\nabla_{i\dot{\alpha}}W_{jk} = \chi_{ijk\dot{\alpha}} \tag{55c}$$

$$\nabla_{i_{n+2}\dot{\alpha}_1}\phi_{i_{n+1}\ldots i_1\dot{\alpha}_2\ldots\dot{\alpha}_n} = \phi_{i_{n+2}\ldots i_1\dot{\alpha}_1\ldots\dot{\alpha}_n} + \epsilon_{\dot{\alpha}_1(\dot{\alpha}_2}\Xi_{i_{n+2}\ldots i_1\dot{\alpha}_3\ldots\dot{\alpha}_n)} \, , \quad n \geq 2, \tag{55d}$$

where the spin $\frac{n}{2}$ superfield is defined recursively by

$$\phi_{i_{n+2}\ldots i_1\dot{\alpha}_1\ldots\dot{\alpha}_n} = \frac{1}{n}\nabla_{i_{n+2}(\dot{\alpha}_1}\phi_{i_{n+1}\ldots i_1\dot{\alpha}_2\ldots\dot{\alpha}_n)} \tag{56}$$

and

$$\Xi_{i_{n+2}\ldots i_1\dot{\alpha}_1\ldots\dot{\alpha}_{n-2}} = \frac{1}{n}\nabla_{i_{n+2}}^{\dot{\alpha}_{n-1}}\phi_{i_{n+1}\ldots i_1\dot{\alpha}_1\ldots\dot{\alpha}_{n-1}} \tag{57}$$

are completely determined in terms of lower spin superfields. The first two equations in this series are given by (38) and (46) with

$$\begin{aligned} \Xi_{ijkl} &= [W_{i[j}, W_{kl]}] \\ \Xi_{ijklm\dot{\gamma}} &= \tfrac{2}{3}[W_{i[j}, \chi_{klm]\dot{\gamma}}] - \tfrac{1}{3}[W_{[jk}, \chi_{lm]i\dot{\gamma}}]. \end{aligned}$$

In the general case, the action of $\nabla_{i_{n+3}}^{\dot{\alpha}_n}$ on (56) yields

$$\begin{aligned} n\nabla_{i_{n+3}}^{\dot{\alpha}_n}\phi_{i_{n+2}\ldots i_1\dot{\alpha}_1\ldots\dot{\alpha}_n} &= 2(n+1)\ [W_{i_{n+3}i_{n+2}}, \phi_{i_{n+1}\ldots i_1\dot{\alpha}_1\ldots\dot{\alpha}_{n-1}}] \\ &\quad + \nabla_{i_{n+2}}^{\dot{\alpha}_n}\phi_{i_{n+3}i_{n+1}\ldots i_1\dot{\alpha}_1\ldots\dot{\alpha}_n} - (n+1)\ \nabla_{i_{n+2}(\dot{\alpha}_1}\Xi_{i_{n+3}i_{n+1}\ldots i_1\dot{\alpha}_2\ldots\dot{\alpha}_{n-1})} \, . \end{aligned} \tag{58}$$

On adding the parts symmetric and antisymmetric in $i_{n+2}i_{n+3}$ we obtain

$$\begin{aligned}(n+1)\,\Xi_{i_{n+3}\dots i_1\dot\alpha_1\dots\dot\alpha_{n-1}} &= \nabla^{\dot\alpha_n}_{i_{n+3}}\phi_{i_{n+2}\dots i_1\dot\alpha_1\dots\dot\alpha_n}\\ = 2\,[[W_{i_{n+3}i_{n+2}},\phi_{i_{n+1}\dots i_1\dot\alpha_1\dots\dot\alpha_{n-1}}] &- \tfrac{1}{2}\nabla_{[i_{n+2}(\dot\alpha_1}\Xi_{i_{n+3}]i_{n+1}\dots i_1\dot\alpha_2\dots\dot\alpha_{n-1})}\\ -\tfrac{n+1}{2n-2}\,\nabla_{(i_{n+3}(\dot\alpha_1}\Xi_{i_{n+2})\dots i_1\dot\alpha_2\dots\dot\alpha_{n-1})}\ .\end{aligned} \tag{59}$$

The relations (55) contain both dynamical as well as kinematical information. As we have explicitly seen for the lower spin fields, the dynamical content may be extracted by covariant differentiation of (55) and use of the constraints (31). In general, the higher-spin superfields have equations of motion

$$\nabla^{\dot\alpha_n}_{\alpha}\phi_{i_{n+2}\dots i_1\dot\alpha_1\dots\dot\alpha_n} = J_{i_{n+2}\dots i_1\alpha\dot\alpha_1\dots\dot\alpha_{n-1}}\ , \tag{60}$$

where

$$\begin{aligned} J_{i_{n+2}\dots i_1\alpha\dot\alpha_1\dots\dot\alpha_{n-1}} &= \nabla_{i_{n+2}\dot\alpha_1}(\nabla^{\dot\alpha_n}_{\alpha}\phi_{i_{n+1}\dots i_1\dot\alpha_2\dots\dot\alpha_n})\\ &\quad +[[\nabla^{\dot\alpha_n}_{\alpha},\nabla_{i_{n+2}\dot\alpha_1}],\phi_{i_{n+1}\dots i_1\dot\alpha_2\dots\dot\alpha_n}] - \nabla_{\alpha(\dot\alpha_1}\Xi_{i_{n+2}\dots i_1\dot\alpha_2\dots\dot\alpha_{n-1})}\\ &= \nabla_{i_{n+2}\dot\alpha_1}J_{i_{n+1}\dots i_1\alpha\dot\alpha_2\dots\dot\alpha_{n-1}} + [\lambda_{i_{n+2}\alpha},\phi_{i_{n+1}\dots i_1\dot\alpha_1\dots\dot\alpha_{n-1}}]\\ &\quad -\nabla_{\alpha(\dot\alpha_1}\Xi_{i_{n+2}\dots i_1\dot\alpha_2\dots\dot\alpha_{n-1})}\ ;\end{aligned} \tag{61}$$

a recursion relation for the higher spin source currents. The latter therefore allow explicit construction in an iterative fashion, starting from the known ones above and using the relations (55) in order to determine $\nabla_{i_{n+2}\dot\alpha_1}J_{i_{n+1}\dots i_1\alpha\dot\alpha_2\dots\dot\alpha_{n-1}}$. We therefore see that the constraints (23), or equivalently (31), in virtue of the super-Jacobi identities, recursively reveal an unending chain of dynamical equations for superfields of increasing spin beginning with the spin $\frac{1}{2}$ equation for $\chi_{ijk\dot\alpha}$. These equations moreover have the form of the interacting Dirac–Fierz equations (11) with source currents, $J_{i_{n+2}\dots i_1\gamma\dot\alpha_1\dots\dot\alpha_{n-1}}$, which are functionals of all fields of spin $\leq \frac{n}{2}$. Consistency of the linear equations of motion (60) requires covariant constancy of these currents,

$$\nabla^{\gamma\dot\alpha_1}J_{i_{n+2}\dots i_1\gamma\dot\alpha_1\dots\dot\alpha_{n-1}} = 0\ ; \tag{62}$$

conditions satisfied non-trivially in virtue of lower-spin equations of motion. The equations of motion (60) have the general form

$$\begin{aligned}\partial^{\dot\alpha_n}_{\alpha}\phi_{i_{n+2}\dots i_1\dot\alpha_1\dots\dot\alpha_n} &= j_{i_{n+2}\dots i_1\gamma\dot\alpha_1\dots\dot\alpha_{n-1}}\\ &= J_{i_{n+2}\dots i_1\gamma\dot\alpha_1\dots\dot\alpha_{n-1}} - [A^{\dot\alpha_n}_{\alpha},\phi_{i_{n+2}\dots i_1\dot\alpha_1\dots\dot\alpha_n}]\ ,\end{aligned} \tag{63}$$

where the currents on the right (which are symmetric in their dotted indices) are divergence-free,

$$\partial^{\alpha\dot\alpha_1}j_{i_{n+2}\dots i_1\alpha\dot\alpha_1\dots\dot\alpha_{n-1}} = 0. \tag{64}$$

The spin 1 current in this chain, the source $j_{ijkl\alpha\dot\beta}$ for the spin 1 field $g_{ijkl\dot\alpha\dot\beta}$ is precisely the Noether current corresponding to global gauge invariance of the funtional $\mathcal{L}_{ijkl}$ (43).

We have seen that the constraints imply not only the existence of higher-spin superfields but also their equations of motion. In fact, the superfield equations of motion are not only implied by the supercurvature constraints (31), but are actually *equivalent* to them. Whereas above we have assumed the constraints in order to derive the superfield equations of motion, the former may instead be seen to arise as consequences of the latter. This converse implication follows from the linear equations for ϕ:

$$\begin{aligned} \nabla^{\dot\alpha_n}_{i_{n+3}} \phi_{i_{n+2}\dots i_1 \dot\alpha_1 \dots \dot\alpha_n} &= (n+1)\, \Xi_{i_{n+3}\dots i_1 \dot\alpha_1 \dots \dot\alpha_{n-1}} \\ \nabla^{\dot\alpha_n}_{\alpha} \phi_{i_{n+2}\dots i_1 \dot\alpha_1 \dots \dot\alpha_n} &= J_{i_{n+2}\dots i_1 \alpha \dot\alpha_1 \dots \dot\alpha_{n-1}} \ . \end{aligned} \tag{65}$$

The consistency conditions for these equations are precisely the constraint equations (23), or equivalently (31). It is in fact sufficient to consider the first-order spin-one superfield equations (45) and (39), for which eqs. (23) are the compatibility conditions. Explicitly, covariantly diffentiating (39) with respect to $\nabla^{\dot\alpha}_{\beta}$ yields

$$[\nabla^{\dot\alpha}_{\beta}, \nabla^{\dot\beta}_{\alpha}] g_{ijkl\dot\alpha\dot\beta} = \nabla^{\dot\alpha}_{\beta} J_{ijkl\alpha\dot\alpha} - \nabla^{\dot\beta}_{\alpha} J_{ijkl\beta\dot\beta} = \epsilon_{\beta\alpha} \nabla^{\gamma\dot\gamma} J_{ijkl\gamma\dot\gamma} = 0$$

in virtue of (41). Therefore, since $g_{ijkl\dot\alpha\dot\beta}$ is symmetric in $\dot\alpha, \dot\beta$, (23c) follows as a compatibility condition for (39). Similarly, acting with the spinorial derivative on (45) yields

$$\begin{aligned} \nabla^{\dot\alpha}_{p} \left(\nabla^{\dot\beta}_{n} g_{ijkl\dot\alpha\dot\beta} \right) &= 2\{\chi^{\dot\alpha}_{pn[i}, \chi_{jkl]\dot\alpha}\} + \{\chi^{\dot\alpha}_{n[ij}, \chi_{kl]p\dot\alpha}\} \\ &\quad +2[W_{n[i}, [W_{|p|j}, W_{kl]}]] - [W_{[ij}, [W_{|p|k}, W_{l]n}]] + [W_{[ij}, [W_{kl]}, W_{pn}]] \ , \end{aligned}$$

which implies

$$\{\nabla^{\dot\alpha}_{p}, \nabla^{\dot\beta}_{n}\} g_{ijkl\dot\alpha\dot\beta} = 0,$$

a relation equivalent to the curvature constraint (23a) since $g_{ijkl\dot\alpha\dot\beta}$ is symmetric in its spinor indices. Finally,

$$\begin{aligned} &\nabla^{\dot\alpha}_{\alpha}(\nabla^{\dot\beta}_{n} g_{ijkl\dot\alpha\dot\beta}) - \nabla^{\dot\beta}_{n}(\nabla^{\dot\alpha}_{\alpha} g_{ijkl\dot\alpha\dot\beta}) = \nabla^{\dot\alpha}_{\alpha}(2[W_{n[i}, \chi_{jkl]\dot\alpha}] - [W_{[ij}, \chi_{kl]n\dot\alpha}]) - \nabla^{\dot\beta}_{n} J_{ijkl\alpha\dot\beta} \\ &= 2[W_{n[i}, [\lambda_{j\alpha}, W_{kl]}]] - 2[W_{[ij}, [\lambda_{k\alpha}, W_{l]n}]] - 2[\lambda_{\alpha[i}, [W_{jk}, W_{l]n}]] \ \equiv \ 0 \end{aligned}$$

in virtue of the Jacobi identities, so the curvature constraint (23b) follows.

3 The $\mathcal{D}$-gauge and component expansions of superfields

Evaluating the superfield equations of motion of the previous section at $\bar\vartheta = 0$ yields equations of motion of the identical form for the leading component fields, with all fields including the connection in $\nabla_{\alpha\dot\beta}$ being ordinary fields on $\mathcal{M}^4$ and transforming according to $\bar\vartheta$-independent gauge-transformations (16). Explicitly, we have

$$
\begin{aligned}
\overset{\circ}{f}_{\dot\alpha\dot\beta} &= 0 \\
\overset{\circ}{\nabla}{}^{\alpha\dot\beta}\overset{\circ}{\lambda}_{i\alpha} &= 0 \\
\overset{\circ}{\Box}\overset{\circ}{W}_{ij} &= \tfrac{1}{2}\{\overset{\circ}{\lambda}{}^{\alpha}_{i}, \overset{\circ}{\lambda}_{j\alpha}\} \\
\overset{\circ}{\nabla}_{\gamma}{}^{\dot\alpha}\overset{\circ}{\chi}_{ijk\dot\alpha} &= [\overset{\circ}{\lambda}_{[i\gamma}, \overset{\circ}{W}_{jk]}] \\
\overset{\circ}{\nabla}_{\gamma}{}^{\dot\alpha_n}\overset{\circ}{\phi}_{i_{n+2}\ldots i_1\dot\alpha_1\ldots\dot\alpha_n} &= \overset{\circ}{J}_{i_{n+2}\ldots i_1\gamma\dot\alpha_1\ldots\dot\alpha_{n-1}} \quad ;\ 4 \le (n+2) \le N.
\end{aligned}
\tag{66}
$$

In the previous section we demonstrated the equivalence of the supercurvature constraints (23) to the superfield relations (55) and the set of superfield equations of motion (24c), (27), (30), (33) and (60). In fact we shall prove that:

The following three sets of data are pairwise equivalent, up to gauge transformations.

(i) The superconnection $\{A_{i\dot\alpha}, A_{\alpha\dot\alpha}\}$ on $\mathcal{M}^{4|2N}$ subject to the supercurvature constraints (23) (or equivalently (31)).

(ii) Superfields $\{A_{\alpha\dot\beta}, \lambda_{i\alpha}, W_{ij}, \chi_{ijk\dot\alpha}, \phi_{i_{n+2}\ldots i_1\dot\alpha_1\ldots\dot\alpha_n}; 2 \le n \le (N-2)\}$ on $\mathcal{M}^{4|2N}$ satisfying the superfield relations (55), which imply the superfield equations of motion.

(iii) The set of component fields $\{\overset{\circ}{A}_{\alpha\dot\beta}, \overset{\circ}{\lambda}_{i\alpha}, \overset{\circ}{\phi}_{i_{n+2}\ldots i_1\dot\alpha_1\ldots\dot\alpha_n}; 0 \le n \le (N-2)\}$ on $\mathcal{M}^4$ satisfying the component super self-duality equations (66).

We have already proven that from a superconnection satisfying (23), we may recursively construct superfields $\{f_{\alpha\beta}, \lambda_{i\alpha}, W_{ij}, \chi_{ijk\dot\alpha}, \phi_{i_{n+2}\ldots i_1\dot\alpha_1\ldots\dot\alpha_n}; 2 \le n \le (N-2)\}$ which automatically satisfy the superfield equations of motion (24c), (27), (30), (33) and (60). To prove the remaining equivalences, we shal! closely follow the technique for such equivalence proofs developed by Harnad et al in [13], where the equivalence between the conventional superspace constraints and the $N = 3$ field equations was given. A similar proof for the d=10 super Yang-Mills theory was given in [20]. To obtain (*i*) or (*ii*) from (*iii*), we need to be able to reconstruct the superfield data on $\mathcal{M}^{4|2N}$ from the leading component fields on $\mathcal{M}^4$; and the proof of the inverse implications requires $\bar\vartheta$- expansions of the superfield data. Both reconstruction of superfields from leading components, as well as the $\bar\vartheta$- expansion of the latter, clearly require some gauge–fixing, for gauge transformations of superfields have parameters depending on the coordinates $\{x, \bar\vartheta\}$ of $\mathcal{M}^{4|2N}$, whereas the component fields have gauge transformations depending only on the x-coordinates of $\mathcal{M}^4$. In order to perform $\bar\vartheta$- expansions or reconstructions of the superfields we clearly need to choose a gauge for them which eliminates all

$\bar{\vartheta}$-dependence of their transformation parameters. Following [13, 20] we use a 'transverse' gauge condition on the odd components of the superconnection which effectively eliminates the local gauge freedom associated with the $\bar{\vartheta}$-coordinates, viz.

$$\bar{\vartheta}^{i\dot{\alpha}} A_{i\dot{\alpha}} = 0 \, . \tag{67}$$

This is tantamount to the requirement that the Euler operator measuring the degree of homogeneity in the $\bar{\vartheta}$ variables is equal to its gauge covariantisation, i.e.

$$\mathcal{D} \equiv \bar{\vartheta}^{i\dot{\alpha}} \frac{\partial}{\partial \bar{\vartheta}^{i\dot{\alpha}}} = \bar{\vartheta}^{i\dot{\alpha}} \nabla_{i\dot{\alpha}} \, . \tag{68}$$

Contracting (22a) with $\bar{\vartheta}^{i\dot{\alpha}}$ we see that in this gauge the parameter of gauge transformations satisfies $\mathcal{D}\tau(x, \bar{\vartheta}) = 0$, and is therefore homogeneous of degree zero in the odd variables; i.e. it is $\bar{\vartheta}$-independent. This gauge is therefore a suitable one in which to perform $\bar{\vartheta}$- expansions. The condition (67) implies no restriction on the x-dependence of the parameters and the $\mathcal{M}^4$ gauge transformations of the component fields therefore remain intact.

In this gauge, the implication $(ii) \Rightarrow (iii)$ trivially follows on evaluating all the superfields at $\bar{\vartheta} = 0$.

Now, in virtue of the fact that the Euler operator $\mathcal{D}$ is gauge–covariant in this gauge, we can immediately write down its action on all the superfields by contracting the superspace relations (55) by $\bar{\vartheta}^{i\dot{\alpha}}$:

$$\begin{aligned}
\mathcal{D} f_{\alpha\beta} &= \tfrac{1}{2}\, \bar{\vartheta}^{i\dot{\alpha}} \nabla_{(\alpha\dot{\alpha}} \lambda_{i\beta)} \\
\mathcal{D} \lambda_{j\alpha} &= 2\, \bar{\vartheta}^{i\dot{\alpha}} \nabla_{\alpha\dot{\alpha}} W_{ij} \\
\mathcal{D} W_{jk} &= \bar{\vartheta}^{i\dot{\alpha}} \chi_{ijk\dot{\alpha}} \\
\mathcal{D} \chi_{jkl\dot{\beta}} &= \bar{\vartheta}^{i\dot{\alpha}} (g_{ijkl\dot{\alpha}\dot{\beta}} + \epsilon_{\dot{\alpha}\dot{\beta}} [W_{i[j}, W_{kl]}]) \\
\mathcal{D} g_{jklm\dot{\beta}\dot{\gamma}} &= \bar{\vartheta}^{i\dot{\alpha}} \psi_{ijklm\dot{\alpha}\dot{\beta}\dot{\gamma}} - \bar{\vartheta}^{i}_{(\dot{\beta}} \left(\tfrac{2}{3} [W_{i[j}, \chi_{klm]\dot{\gamma})}] - \tfrac{1}{3} [W_{[jk}, \chi_{lm]i\dot{\gamma})}] \right) \\
\mathcal{D} h_{jklmn\dot{\beta}\dot{\gamma}\dot{\delta}} &= \bar{\vartheta}^{i\dot{\alpha}} C_{ijklmn\dot{\alpha}\dot{\beta}\dot{\gamma}\dot{\delta}} - \bar{\vartheta}^{i}_{(\dot{\beta}} \Xi_{ijklmn\dot{\gamma}\dot{\delta})} \\
\mathcal{D} \phi_{i_{n+1} \dots i_1 \dot{\alpha}_1 \dots \dot{\alpha}_{n-1}} &= \bar{\vartheta}^{i_{n+2}\dot{\alpha}_n} \phi_{i_{n+2} \dots i_1 \dot{\alpha}_1 \dots \dot{\alpha}_n} - \bar{\vartheta}^{i_{n+2}}_{(\dot{\alpha}_1} \Xi_{i_{n+2} \dots i_1 \dot{\alpha}_2 \dots \dot{\alpha}_{n-1})} .
\end{aligned} \tag{69}$$

Further, contracting the constraints (31) by $\bar{\vartheta}^{i\dot{\alpha}}$, we obtain

$$\begin{aligned}
(1 + \mathcal{D}) A_{i\dot{\alpha}} &= 2\, \bar{\vartheta}^{j}_{\dot{\alpha}} W_{ij} \\
\mathcal{D} A_{\alpha\dot{\alpha}} &= -\, \bar{\vartheta}^{i}_{\dot{\alpha}} \lambda_{i\alpha} \, ,
\end{aligned} \tag{70}$$

or equivalently

$$\begin{aligned}
[\mathcal{D}, \nabla_{i\dot{\alpha}}] + \nabla_{i\dot{\alpha}} &= 2\, \bar{\vartheta}^{j}_{\dot{\alpha}} W_{ij} \\
[\mathcal{D}, \nabla_{\alpha\dot{\alpha}}] &= -\, \bar{\vartheta}^{i}_{\dot{\alpha}} \lambda_{i\alpha} \, .
\end{aligned} \tag{71}$$

Now, since the action of the Euler operator $\mathcal{D}$ on any polynomial in $\bar{\vartheta}$ yields the same polynomial with each term multiplied by its degree of $\bar{\vartheta}$-homogeneity, these relations actually determine the superfields from their leading components uniquely. In a $\bar{\vartheta}$-expansion, the k-th order terms on the left-hand sides of (69) are given by the $(k-1)$-th order terms of the superfield expressions multiplying $\bar{\vartheta}$ on the right-hand sides. These relations therefore recursively define the superfields from their leading components; and they do so in a unique manner. In fact the $\mathcal{D}$-recursions (69) encode the non-dynamical content of the relations (55); and similarly (70) contain the non-dynamical part of the constraints (31). By repeated application of (69) and (70), the leading components may be seen to determine the entire $\bar{\vartheta}$-expansion of the superfields. The leading terms are:

$$\begin{aligned}
f_{\alpha\beta} &= \overset{\circ}{f}_{\alpha\beta} + \tfrac{1}{2}\bar{\vartheta}^{i\dot{\alpha}}\overset{\circ}{\nabla}_{(\alpha\dot{\alpha}}\overset{\circ}{\lambda}_{i\beta)} + \quad \cdots \\
\lambda_{j\alpha} &= \overset{\circ}{\lambda}_{j\alpha} + 2\bar{\vartheta}^{i\dot{\alpha}}\overset{\circ}{\nabla}_{\alpha\dot{\alpha}}\overset{\circ}{W}_{ij} + \bar{\vartheta}^{i\dot{\alpha}}\bar{\vartheta}^{l\dot{\beta}}\left(\overset{\circ}{\nabla}_{\alpha\dot{\alpha}}\overset{\circ}{\chi}_{lij\dot{\gamma}} - \epsilon_{\dot{\alpha}\dot{\gamma}}[\overset{\circ}{\lambda}_{l}, \overset{\circ}{W}_{ij}]\right) + \quad \cdots \\
W_{jk} &= \overset{\circ}{W}_{jk} + \bar{\vartheta}^{i\dot{\alpha}}\overset{\circ}{\chi}_{ijk\dot{\alpha}} + \tfrac{1}{2}\bar{\vartheta}^{i\dot{\alpha}}\bar{\vartheta}^{l\dot{\beta}}\left(\overset{\circ}{g}_{lijk\dot{\alpha}\dot{\beta}} + \epsilon_{\dot{\alpha}\dot{\beta}}[\overset{\circ}{W}_{l[i}, \overset{\circ}{W}_{jk]}]\right) + \quad \cdots \\
\chi_{jkl\dot{\beta}} &= \overset{\circ}{\chi}_{jkl\dot{\beta}} + \bar{\vartheta}^{i\dot{\alpha}}\left(\overset{\circ}{g}_{ijkl\dot{\alpha}\dot{\beta}} + \epsilon_{\dot{\alpha}\dot{\beta}}[\overset{\circ}{W}_{i[j}, \overset{\circ}{W}_{kl]}]\right) \\
&\quad + \tfrac{1}{2}\bar{\vartheta}^{i\dot{\alpha}}\bar{\vartheta}^{m\dot{\gamma}}\overset{\circ}{\psi}_{mijkl\dot{\alpha}\dot{\beta}\dot{\gamma}} + \bar{\vartheta}^{i\dot{\alpha}}\bar{\vartheta}^{m}_{\dot{\beta}}\left([\overset{\circ}{W}_{[jk}, \overset{\circ}{\chi}_{l]mi]\dot{\alpha}}] - \tfrac{1}{3}[\overset{\circ}{W}_{[mi}, \overset{\circ}{\chi}_{jkl]\dot{\alpha}}]\right) \\
&\quad + \bar{\vartheta}^{i\dot{\alpha}}\bar{\vartheta}^{m}_{\dot{\alpha}}[\overset{\circ}{W}_{i[j}, \overset{\circ}{\chi}_{kl]m\dot{\beta}}] + \quad \cdots \\
g_{jklm\dot{\beta}\dot{\gamma}} &= \overset{\circ}{g}_{jklm\dot{\beta}\dot{\gamma}} + \bar{\vartheta}^{i\dot{\alpha}}\overset{\circ}{\psi}_{ijklm\dot{\alpha}\dot{\beta}\dot{\gamma}} - \bar{\vartheta}^{i}_{(\dot{\beta}}\left(\tfrac{2}{3}[\overset{\circ}{W}_{i[j}, \overset{\circ}{\chi}_{klm]\dot{\gamma})}] - \tfrac{1}{3}[\overset{\circ}{W}_{[jk}, \overset{\circ}{\chi}_{lm]i\dot{\gamma})}]\right) \\
&\quad + \tfrac{1}{2}\bar{\vartheta}^{i\dot{\alpha}}\bar{\vartheta}^{n\dot{\delta}}\overset{\circ}{C}_{nijklm\dot{\alpha}\dot{\beta}\dot{\gamma}\dot{\delta}} + \quad \cdots,
\end{aligned} \tag{72}$$

and in general,

$$\begin{aligned}
\phi_{i_{n+2}\ldots i_1\dot{\alpha}_1\ldots\dot{\alpha}_n} &= \overset{\circ}{\phi}_{i_{n+2}\ldots i_1\dot{\alpha}_1\ldots\dot{\alpha}_n} + \bar{\vartheta}^{i_{n+1}\dot{\alpha}_{n+1}}\overset{\circ}{\phi}_{i_{n+3}\ldots i_1\dot{\alpha}_1\ldots\dot{\alpha}_{n+1}} \\
&\quad - \bar{\vartheta}^{i_{n+2}}_{(\dot{\alpha}_1}\overset{\circ}{\Xi}_{i_{n+2}\ldots i_1\dot{\alpha}_2\ldots\dot{\alpha}_{n-1})} + \quad \cdots
\end{aligned}$$

and for the superconnection components

$$\begin{aligned}
A_{j\dot{\alpha}} &= \bar{\vartheta}^{k\dot{\alpha}}\overset{\circ}{W}_{jk} + \tfrac{2}{3}\bar{\vartheta}^{k\dot{\alpha}}\bar{\vartheta}^{i\dot{\alpha}}\overset{\circ}{\chi}_{ijk\dot{\alpha}} \\
&\quad + \tfrac{1}{4}\bar{\vartheta}^{k\dot{\alpha}}\bar{\vartheta}^{i\dot{\alpha}}\bar{\vartheta}^{l\dot{\beta}}\left(\overset{\circ}{g}_{lijk\dot{\alpha}\dot{\beta}} + \epsilon_{\dot{\alpha}\dot{\beta}}[\overset{\circ}{W}_{l[i}, \overset{\circ}{W}_{jk]}]\right) + \quad \cdots \\
A_{\alpha\dot{\alpha}} &= \overset{\circ}{A}_{\alpha\dot{\alpha}} - \bar{\vartheta}^{j}_{\dot{\alpha}}\overset{\circ}{\lambda}_{j\alpha} - \bar{\vartheta}^{j}_{\dot{\alpha}}\bar{\vartheta}^{i\dot{\beta}}\overset{\circ}{\nabla}_{\alpha\dot{\beta}}\overset{\circ}{W}_{ij} \\
&\quad - \tfrac{1}{3}\bar{\vartheta}^{j}_{\dot{\alpha}}\bar{\vartheta}^{i\dot{\beta}}\bar{\vartheta}^{l\dot{\gamma}}\left(\overset{\circ}{\nabla}_{\alpha\dot{\beta}}\overset{\circ}{\chi}_{lij\dot{\gamma}} - \epsilon_{\dot{\beta}\dot{\gamma}}[\overset{\circ}{\lambda}_{l}, \overset{\circ}{W}_{ij}]\right) \\
&\quad - \tfrac{1}{12}\bar{\vartheta}^{j}_{\dot{\alpha}}\bar{\vartheta}^{i\dot{\beta}}\bar{\vartheta}^{l\dot{\gamma}}\bar{\vartheta}^{m\dot{\delta}}\overset{\circ}{\nabla}_{\alpha\dot{\beta}}\overset{\circ}{g}_{mlij\dot{\gamma}\dot{\delta}} + \quad \cdots
\end{aligned} \tag{73}$$

It is easy to check that expansions (72), (73) satisfy the $\mathcal{D}$-recursion relations (69)-(71); and it is clear how higher terms may be obtained from the latter. The supersymmetry transformations of the component fields (17) may now be obtained immediately from the action of $\delta = \bar{\eta}^{i\dot{\alpha}}\frac{\partial}{\partial\bar{\vartheta}^{i\dot{\alpha}}} + \eta^{\alpha}_{i}\frac{\partial}{\partial\vartheta^{\alpha}_{i}}$ on the non-chiral superfield $\widehat{f}_{ij}$ in (21) with its chiral superfield components $f_{ij} = 2W_{ij}, f_{i\alpha} = \lambda_{i\alpha}, f_{\alpha\beta}$ having the above the expansions.

The implication $(iii) \Rightarrow (i)$, that the superconnection satisfies the constraints (23) provided that the equations of motion (66) hold for the components may now be directly verified order by order in a $\bar{\vartheta}$- expansion by inserting (73) in the constraints (23b) and (23c). An inductive proof of this implication is given in the next section.

4 The equivalence between superfield equations and component equations

In the previous sections we have obtained the proof of the implications $(i) \Rightarrow (ii) \Rightarrow (iii)$ between our three sets of data. The chain of the inverse implications, $(iii) \Rightarrow (ii) \Rightarrow (i)$, and therefore the full equivalence between the three sets, may be proven by induction on the degree of $\bar{\vartheta}$-homogeneity from the zero-order (component) equations, following the method of Harnad et al [13, 20].

The relations (69)-(71) imply the following further $\mathcal{D}$-recursions:

$$
\begin{aligned}
\mathcal{D}f_{\dot\alpha\dot\beta} = \mathcal{D}[\nabla^{\alpha}_{\dot\alpha}, \nabla_{\alpha\dot\beta}] &= -\bar{\vartheta}^{i}_{(\dot\beta}\nabla^{\alpha}_{\dot\alpha)}\lambda_{i\alpha} \\
\mathcal{D}(\nabla^{\alpha}_{\dot\alpha}\lambda_{i\alpha}) &= \bar{\vartheta}^{j}_{\dot\alpha}\left(\{\lambda^{\alpha}_{j}, \lambda_{i\alpha}\} + 2\Box W_{ij}\right) \\
\mathcal{D}&\left(\nabla^{\alpha\dot\beta}\nabla_{\alpha\dot\beta}W_{ij} - \{\lambda^{\alpha}_{i}, \lambda_{j\alpha}\}\right) \\
&= \bar{\vartheta}^{k\dot\beta}\left(2[\lambda^{\alpha}_{k}, \nabla_{\alpha\dot\beta}W_{ij}] + \nabla^{\alpha\dot\beta}\nabla_{\alpha\dot\beta}\chi_{ijk\dot\beta} + 2[\nabla^{\alpha}_{\dot\beta}W_{k[i}, \lambda_{j]\alpha}]\right) \\
&= \bar{\vartheta}^{k\dot\beta}\left(\nabla^{\alpha\dot\beta}\nabla_{\alpha\dot\beta}\chi_{ijk\dot\beta} - 2[\lambda^{\alpha}_{[i}, \nabla_{\alpha\dot\alpha}W_{jk]}]\right) \\
\mathcal{D}&\left(\nabla^{\alpha\dot\alpha}\chi_{ijk\dot\alpha} - [\lambda^{\alpha}_{[i}, W_{jk]}]\right) \\
&= \bar{\vartheta}^{k\dot\beta}\ \left(\{\lambda^{\alpha}_{l}, \chi_{ijk\dot\alpha}\} + \nabla^{\alpha\dot\alpha}g_{lijk\dot\alpha\dot\beta} + \nabla^{\alpha}_{\dot\beta}[W_{l[i}, W_{jk]}] - 2[\nabla^{\alpha}_{\dot\beta}W_{l[i}, W_{jk]}]\right. \\
&\qquad \left. -\{\lambda^{\alpha}_{[i}, \chi_{jk]l\dot\beta}\}\right) \\
&= \bar{\vartheta}^{k\dot\beta}\left(\nabla^{\alpha\dot\alpha}g_{lijk\dot\alpha\dot\beta} - J^{\alpha}_{lijk\dot\alpha}\right)
\end{aligned}
\tag{74}
$$

and so on. In the general case we have

$$
\begin{aligned}
&\mathcal{D}\left(\nabla^{\dot\alpha_n}_{\alpha}\phi_{i_{n+2}\dots i_1\dot\alpha_1\dots\dot\alpha_n} - J_{i_{n+2}\dots i_1\alpha\dot\alpha_1\dots\dot\alpha_{n-1}}\right) \\
&\quad = \bar{\vartheta}^{i_{n+3}\dot\alpha_n}\left(\nabla^{\dot\alpha_{n+1}}_{\alpha}\phi_{i_{n+3}\dots i_1\dot\alpha_1\dots\dot\alpha_{n+1}} - J_{i_{n+3}\dots i_1\alpha\dot\alpha_1\dots\dot\alpha_n}\right)
\end{aligned}
\tag{75}
$$

Since these relations are linear in $\bar{\vartheta}$ on the right, the $(n+1)$-st order terms on the left are determined by the n- th order terms of the coefficients of $\bar{\vartheta}$ on the right. So if we assume that the superfield equations hold to order n in $\bar{\vartheta}$, the right-hand-sides of (74), (75) vanish up to order $(n+1)$. Therefore, since the homogeneity operator is positive on the expressions above, it follows by induction that:

The superfield equations hold if the (zeroth-order) component equations (66) are satisfied, i.e. $(iii) \Rightarrow (ii)$.

These relations also demonstrate that the entire tower of higher–spin component equations are contained in the superfield equation (33) for $\chi_{ijk\dot\alpha}$, or equivalently in the superfield equation (39) for $g_{ijkl\dot\alpha\dot\beta}$.

We may similarly prove that the $\mathcal{D}$-recursions (69)-(71) imply the relations (55). The first step of the induction follows since at zero ($\bar\vartheta$-independent) order, (55) are manifestly implied by (69)-(71). We now proceed to show that given the $\mathcal{D}$-recursions (69)-(71), the superfield relations (55) hold to order $(n+1)$ in $\bar\vartheta$ provided they are valid to order n.

Applying $\mathcal{D}$ to (55a) and using the $\mathcal{D}$-recursions (69)-(71), we obtain

$$\begin{aligned}&\mathcal{D}(\nabla_{i\dot\alpha} f_{\alpha\beta} - \tfrac{1}{2}\nabla_{(\alpha\dot\alpha}\lambda_{i\beta)})\\ &\quad= -\nabla_{i\dot\alpha} f_{\alpha\beta} + 2\bar\vartheta^j_{\dot\alpha}[W_{ij}, f_{\alpha\beta}] + \tfrac{1}{2}\nabla_{i\dot\alpha}(\bar\vartheta^{j\dot\beta}\nabla_{(\alpha\dot\beta}\lambda_{j\beta)})\\ &\qquad+\tfrac{1}{2}\bar\vartheta^j_{\dot\alpha}\{\lambda_{j(\alpha}, \lambda_{i\beta)}\} - \nabla_{(\alpha\dot\alpha}\bar\vartheta^{j\dot\beta}\nabla_{\beta)\dot\beta}W_{ji}\end{aligned}$$

Therefore

$$\begin{aligned}&(1+\mathcal{D})(\nabla_{i\dot\alpha} f_{\alpha\beta} - \tfrac{1}{2}\nabla_{(\alpha\dot\alpha}\lambda_{i\beta)}) \quad \text{(to order } (n+1)\text{)}\\ &= 2\bar\vartheta^{j\dot\beta}[W_{ij}, \epsilon_{\dot\alpha\dot\beta} f_{\alpha\beta} - [\nabla_{(\alpha\dot\beta}, \nabla_{\beta)\dot\alpha}]] \quad \text{by the inductive hypothesis}\\ &= 0\,,\end{aligned}$$

in virtue of (24c), which holds to n-th order as a consequence of the inductive hypothesis. The relation (55a) therefore follows, since $(1+\mathcal{D})$ is a positive-definite operator.

Similarly, applying $\mathcal{D}$ to (55b) and using the $\mathcal{D}$-recursions (69)-(71) yields

$$\begin{aligned}&\mathcal{D}(\nabla_{i\dot\alpha}\lambda_{j\alpha} - 2\nabla_{\alpha\dot\alpha}W_{ij})\\ &= [[\mathcal{D}, \nabla_{i\dot\alpha}], \lambda_{j\alpha}] + 2\nabla_{i\dot\alpha}(\bar\vartheta^{k\dot\gamma}\nabla_{\alpha\dot\gamma}W_{kj})\\ &= \quad +2\bar\vartheta^k_{\dot\alpha}[W_{ij}, \lambda_{k\alpha}] - 2\bar\vartheta^{k\dot\gamma}\nabla_{\alpha\dot\alpha}\chi_{kij\dot\gamma}\end{aligned}$$

and using the inductive hypothesis, we obtain

$$\begin{aligned}&(1+\mathcal{D})(\nabla_{i\dot\alpha}\lambda_{j\alpha} - 2\nabla_{\alpha\dot\alpha}W_{ij}) \quad \text{(to order } (n+1)\text{)}\\ &= -2\,\bar\vartheta^k_{\dot\alpha}\left(\nabla^{\dot\gamma}_{\alpha}\chi_{ijk\dot\gamma} - [\lambda_{[i\alpha}, W_{jk]}]\right)\\ &= 0\,,\end{aligned}$$

in virtue of (33), which, to n-th order, is a consequence of the inductive hypothesis.

Now, applying $\mathcal{D}$ to (55c) and using the $\mathcal{D}$-recursions (69)-(71) yields

$$\mathcal{D}(\nabla_{i\dot\alpha}W_{jk} - \chi_{ijk\dot\alpha}) \quad = \quad -\nabla_{i\dot\alpha}W_{jk} + 2\bar\vartheta^l_{\dot\alpha}[W_{il}, W_{jk}] + \nabla_{i\dot\alpha}(\bar\vartheta^{l\dot\beta}\chi_{ljk\dot\beta}).$$

The inductive hypothesis therefore implies that

$$\begin{aligned}&(1+\mathcal{D})(\nabla_{i\dot\alpha}W_{jk} - \chi_{ijk\dot\alpha}) \quad \text{(to order } (n+1)\text{)}\\ &= \bar\vartheta^l_{\dot\alpha}(2\,[W_{il}, W_{jk}] - [W_{i[l}, W_{jk]}] + [W_{l[i}, W_{jk]}])\\ &\equiv 0,\end{aligned}$$

yielding (55c). Note that this and the proofs of all further relations in (55) follow from just the inductive hypothesis, whereas the proofs of (55a,b) above require the satisfaction of the superfield equations of motion (24c) and (33) to n-th order in $\bar{\vartheta}$.

Similarly, for the next relation, the $\mathcal{D}$-recursions (69)-(71) imply that

$$
\begin{aligned}
&(1+\mathcal{D})(\nabla_{i\dot{\alpha}}\chi_{jkl\dot{\beta}} - g_{ijkl\dot{\alpha}\dot{\beta}} - \epsilon_{\dot{\alpha}\dot{\beta}}[W_{i[j}, W_{kl]}]) \quad \text{(to order } (n+1)\text{)} \\
&= 2\bar{\vartheta}^m_{\dot{\alpha}}[W_{im}, \chi_{jkl\dot{\beta}}] - \bar{\vartheta}^{m\dot{\gamma}}\nabla_{i\dot{\alpha}}\left(g_{ijkl\dot{\alpha}\dot{\beta}} + \epsilon_{\dot{\alpha}\dot{\beta}}[W_{i[j}, W_{kl]}]\right) - \bar{\vartheta}^{m\dot{\gamma}}\psi_{mijkl\dot{\gamma}\dot{\alpha}\dot{\beta}} \\
&\quad + \bar{\vartheta}^m_{(\dot{\alpha}}\left(\tfrac{2}{3}[W_{m[i}, \chi_{jkl]\dot{\beta})}] - \tfrac{1}{3}[W_{[ij}, \chi_{kl]m\dot{\beta})}]\right) - \bar{\vartheta}^m_{[\dot{\alpha}}\left([\chi_{mi[j\dot{\beta}]}, W_{kl]}] - [W_{i[j}, \chi_{kl]m\dot{\beta}]}]\right) \\
&= 0
\end{aligned}
$$

if (46) is assumed to hold to order n. All remaining relations in (55) follow similarly from the positive-definiteness of $(1+\mathcal{D})$ and the fact that

$$
\begin{aligned}
&(1+\mathcal{D})\left(\nabla_{i_{n+2}\dot{\alpha}_1}\phi_{i_{n+1}\ldots i_1\dot{\alpha}_2\ldots\dot{\alpha}_n} - \phi_{i_{n+2}\ldots i_1\dot{\alpha}_1\ldots\dot{\alpha}_n}\right. \\
&\qquad\qquad \left. - \epsilon_{\dot{\alpha}_1(\dot{\alpha}_2}\Xi_{i_{n+2}\ldots i_1\dot{\alpha}_3\ldots\dot{\alpha}_n)}\right) \qquad \text{(to order } (n+1)\text{)} \\
&= 0 \quad \text{by the inductive hypothesis.}
\end{aligned}
$$

We may now proceed to prove by induction that the $\mathcal{D}$-recursions (69)-(71), together with (55) to order n, imply the constraint equations (31) to order $(n+1)$. The zero-th order relations are manifest in virtue of the recursion relations (70) and the definition of the curvature $f_{\alpha\beta}$. The rest follows by action of $\mathcal{D}$ on the constraints (31). Thus

$$
\begin{aligned}
&\mathcal{D}\left(\{\nabla_{i\dot{\alpha}}, \nabla_{j\dot{\beta}}\} - 2\epsilon_{\dot{\alpha}\dot{\beta}}W_{ij}\right) \\
&= -2\{\nabla_{i\dot{\alpha}}, \nabla_{j\dot{\beta}}\} + 2\nabla_{i\dot{\alpha}}(\bar{\vartheta}^k_{\dot{\beta}}W_{jk}) + 2\nabla_{j\dot{\beta}}(\bar{\vartheta}^k_{\dot{\alpha}}W_{ik}) - 2\epsilon_{\dot{\alpha}\dot{\beta}}\bar{\vartheta}^{k\dot{\gamma}}\chi_{ijk\dot{\gamma}} \,.
\end{aligned}
$$

Assuming (55c) to order n, we therefore have that

$$
(2+\mathcal{D})\left(\{\nabla_{i\dot{\alpha}}, \nabla_{j\dot{\beta}}\} - 2\epsilon_{\dot{\alpha}\dot{\beta}}W_{ij}\right) \quad \text{(to order } (n+1)\text{)} \quad = 0 \,,
$$

from which the first equation in (31) follows since $(2+\mathcal{D})$ is a positive operator. Similarly, (69)-(71) imply that

$$
\mathcal{D}\left([\nabla_{i\dot{\alpha}}, \nabla_{\beta\dot{\beta}}] - \epsilon_{\dot{\alpha}\dot{\beta}}\lambda_{i\beta}\right) = -[\nabla_{i\dot{\alpha}}, \nabla_{\beta\dot{\beta}}] - 2\bar{\vartheta}^k_{\dot{\alpha}}\nabla_{\beta\dot{\beta}}W_{ik} - \nabla_{i\dot{\alpha}}(\bar{\vartheta}^k_{\dot{\beta}}\lambda_{k\beta}) - 2\bar{\vartheta}^k_{[\dot{\alpha}}\nabla_{\beta\dot{\beta}]}W_{ki}
$$

and assuming the validity of (55b) to order n, we have the relation

$$
(1+\mathcal{D})\left([\nabla_{i\dot{\alpha}}, \nabla_{\beta\dot{\beta}}] - \epsilon_{\dot{\alpha}\dot{\beta}}\lambda_{i\beta}\right) \quad \text{(to order } (n+1)\text{)} \quad = 0 \,,
$$

which in turn implies the second equation in (31). Finally,

$$
\begin{aligned}
&\mathcal{D}\left([\nabla_{\alpha\dot{\alpha}}, \nabla_{\beta\dot{\beta}}] - \epsilon_{\dot{\alpha}\dot{\beta}}f_{\alpha\beta}\right) \quad \text{(to order } (n+1)\text{)} \\
&= \bar{\vartheta}^i_{\dot{\alpha}}\nabla_{\beta\dot{\beta}}\lambda_{i\alpha} - \bar{\vartheta}^i_{\dot{\beta}}\nabla_{\alpha\dot{\alpha}}\lambda_{i\beta} - \tfrac{1}{2}\epsilon_{\dot{\alpha}\dot{\beta}}\bar{\vartheta}^{i\dot{\gamma}}\nabla_{(\alpha\dot{\gamma}}\lambda_{i\beta)} \\
&= 0
\end{aligned}
$$

if the equation of motion for $\lambda_{i\alpha}$ are assumed to hold to order n, since $\nabla_{\beta\dot\beta}\lambda_{i\alpha} = \frac{1}{2}\nabla_{(\beta\dot\beta}\lambda_{i\alpha)}$ in virtue of (27).

This completes the proof of the chain of implications $(iii) \Rightarrow (ii) \Rightarrow (i)$. We have therefore demonstrated the full equivalence between our three sets of data.

5 The conserved tensor supercurrents

Gauge–invariance of the $\binom{N}{4}$ superfield functionals (43) yields this number of second rank traceless (i.e. satisfying $\epsilon^{\alpha\beta}\epsilon^{\dot\alpha\dot\beta}T_{ijkl\alpha\dot\alpha,\beta\dot\beta} = 0$) conserved superfield tensors having the form

$$\begin{aligned} T_{ijkl\alpha\dot\alpha,\beta\dot\beta} &= \mathrm{Tr}\ \Big(g_{ijkl\dot\alpha\dot\beta}f_{\alpha\beta} + \nabla_{\alpha\dot\beta}\lambda_{[i\beta}\chi_{jkl]\dot\alpha} - \lambda_{[i\alpha}\nabla_{\beta\dot\alpha}\chi_{jkl]\dot\beta} \\ &\quad +\tfrac{1}{2}\lambda_{[i\beta}\nabla_{\alpha\dot\alpha}\chi_{jkl]\dot\beta} - \tfrac{1}{2}\nabla_{\alpha\dot\alpha}\lambda_{[i\beta}\chi_{jkl]\dot\beta} + \tfrac{2}{3}\epsilon_{\dot\beta\dot\alpha}\epsilon_{\beta\alpha}\{\lambda^{\gamma}_{[i},\lambda_{j\gamma}\}W_{kl]} \\ &\quad -\tfrac{1}{3}\nabla_{(\alpha\dot\alpha}W_{[ij}\nabla_{\beta)\dot\beta}W_{kl]} + \tfrac{1}{3}W_{[ij}\nabla_{\alpha\dot\alpha}\nabla_{\beta\dot\beta}W_{kl]}\ \Big). \end{aligned} \tag{76}$$

The Tr in these expressions denotes the gauge algebra trace. The expression (76) is in fact the unique traceless linear combination of the three existing second rank conserved tensors,

$$\begin{aligned} T^{(1)}_{ijkl\alpha\dot\alpha,\beta\dot\beta} &= \mathrm{Tr}\ \Big(g_{ijkl\dot\alpha\dot\beta}f_{\alpha\beta} - \lambda_{[i\alpha}\nabla_{\beta\dot\alpha}\chi_{jkl]\dot\beta} - \nabla_{\beta\dot\alpha}W_{[ij}\nabla_{\alpha\dot\beta}W_{kl]}\Big) \\ T^{(2)}_{ijkl\alpha\dot\alpha,\beta\dot\beta} &= \mathrm{Tr}\ \Big(\tfrac{1}{2}\lambda_{[i\beta}\nabla_{\alpha\dot\alpha}\chi_{jkl]\dot\beta} - \tfrac{1}{2}\nabla_{\alpha\dot\alpha}\lambda_{[i\beta}\chi_{jkl]\dot\beta} + \nabla_{\alpha\dot\beta}\lambda_{[i\beta}\chi_{jkl]\dot\alpha} \\ &\quad +\epsilon_{\beta\alpha}\epsilon_{\dot\beta\dot\alpha}\{\lambda^{\gamma}_{[i},\lambda_{j\gamma}\}W_{kl]}\Big) \\ T^{(3)}_{ijkl\alpha\dot\alpha,\beta\dot\beta} &= \mathrm{Tr}\ \tfrac{1}{3}\Big(W_{[ij}\nabla_{\alpha\dot\alpha}\nabla_{\beta\dot\beta}W_{kl]} - \nabla_{\alpha\dot\alpha}W_{[ij}\nabla_{\beta\dot\beta}W_{kl]} + 2\nabla_{\beta\dot\alpha}W_{[ij}\nabla_{\alpha\dot\beta}W_{kl]} \\ &\quad -\epsilon_{\beta\alpha}\epsilon_{\dot\beta\dot\alpha}\{\lambda^{\gamma}_{[i},\lambda_{j\gamma}\}W_{kl]}\Big)\ . \end{aligned} \tag{77}$$

The conservation of these tensors is a non-trivial consequence of the superfield equations of motion. Thus, using the equation of motion (39), together with the operator identity

$$\nabla^{\alpha\dot\alpha}\nabla_{\beta\dot\alpha} = \delta^{\alpha}_{\beta}\Box + f^{\alpha}_{\beta} \tag{78}$$

and the cyclic property of the trace, we obtain

$$\begin{aligned} \partial^{\alpha\dot\alpha}T^{(1)}_{ijkl\alpha\dot\alpha,\beta\dot\beta} &= \mathrm{Tr}\ \nabla^{\alpha\dot\alpha}\Big(g_{ijkl\dot\alpha\dot\beta}f_{\alpha\beta} - \lambda_{[i\alpha}\nabla_{\beta\dot\alpha}\chi_{jkl]\dot\beta} - \nabla_{\beta\dot\alpha}W_{[ij}\nabla_{\alpha\dot\beta}W_{kl]}\Big) \\ &= \mathrm{Tr}\ \Big(-\lambda_{[i\beta}\Box\chi_{jkl]\dot\beta} - 2\Box W_{[ij}\nabla_{\beta\dot\beta}W_{kl]}\Big) \\ &= 0 \quad \text{in virtue of (33) and (34).} \end{aligned}$$

Similarly,

$$\begin{aligned} \partial^{\alpha\dot\alpha}T^{(2)}_{ijkl\alpha\dot\alpha,\beta\dot\beta} &= \mathrm{Tr}\ \Big(\lambda_{[i\beta}\Box\chi_{jkl]\dot\beta} + \nabla_{\alpha\dot\beta}\lambda_{[i\beta}\nabla^{\alpha\dot\alpha}\chi_{jkl]\dot\alpha} + \nabla_{\beta\dot\beta}(\{\lambda^{\gamma}_{[i},\lambda_{j\gamma}\}W_{kl]})\Big) \\ &= 0 \end{aligned}$$

also in virtue of (33) and (34). Finally (78) and the further operator identity

$$\nabla^{\alpha\dot\alpha}\nabla_{\alpha\dot\alpha}\nabla_{\beta\dot\beta} = 2\nabla_{\beta\dot\beta}\Box + 2f_\beta^\alpha\nabla_{\alpha\dot\beta} \tag{79}$$

implies that

$$\begin{aligned}\partial^{\alpha\dot\alpha}T^{(3)}_{ijkl\alpha\dot\alpha,\beta\dot\beta} &= \tfrac{2}{3}\mathrm{Tr}\ \left(2W_{[ij}\nabla_{\beta\dot\beta}\Box W_{kl]} + 2\Box W_{[ij}\nabla_{\beta\dot\beta}W_{kl]} - \nabla_{\beta\dot\beta}(\{\lambda^\gamma_{[i},\lambda_{j\gamma}\}W_{kl]})\right)\\ &= 0 \quad \text{in virtue of (30).}\end{aligned}$$

The gauge–invariant tensors (77) have conserved superpartners. The lower rank conserved spin–tensors are

$$\begin{aligned}
T_{ijk\alpha\dot\alpha,\beta} &= \mathrm{Tr}\ (2f_{\alpha\beta}\chi_{ijk\dot\alpha} - \nabla_{\alpha\dot\alpha}\lambda_{[i\beta}W_{jk]} + \lambda_{[i\beta}\nabla_{\alpha\dot\alpha}W_{jk]} - 2\lambda_{[i\alpha}\nabla_{\beta\dot\alpha}W_{jk]})\\
T^{(1)}_{ijklm\alpha\dot\alpha,\dot\beta} &= \mathrm{Tr}\ (4\lambda_{i\alpha}g_{jklm\dot\alpha\dot\beta} - \lambda_{[j}g_{klm]i\dot\alpha\dot\beta} - 4\chi_{i[jk\dot\alpha}\nabla_{\alpha\dot\beta}W_{lm]}\\
&\qquad +6\nabla_{\alpha\dot\beta}W_{i[j}\chi_{klm]\dot\alpha} - 5\epsilon_{\dot\alpha\dot\beta}W_{i[j}[\lambda_{k\alpha},W_{lm]}])\\
T^{(2)}_{ijklm\alpha\dot\alpha,\dot\beta} &= \mathrm{Tr}\ (\nabla_{\alpha\dot\alpha}W_{i[j}\chi_{klm]\dot\beta} - W_{i[j}\nabla_{\alpha\dot\alpha}\chi_{klm]\dot\beta} - 2\nabla_{\alpha\dot\beta}W_{i[j}\chi_{klm]\dot\alpha}\\
&\qquad +2\epsilon_{\dot\alpha\dot\beta}W_{i[j}[\lambda_{k\alpha},W_{lm]}])\\
T^{(3)}_{ijklm\alpha\dot\alpha,\dot\beta} &= \mathrm{Tr}\ (\nabla_{\alpha\dot\alpha}\chi_{i[jk\dot\beta}W_{lm]} - \chi_{i[jk\dot\beta}\nabla_{\alpha\dot\alpha}W_{lm]} + 2\chi_{i[jk\dot\alpha}\nabla_{\alpha\dot\beta}W_{lm]}\\
&\qquad +2\epsilon_{\dot\alpha\dot\beta}W_{i[j}[\lambda_{k\alpha},W_{lm]}])\\
T_{ijkl\alpha\dot\alpha} &= \mathrm{Tr}\ (3\lambda_{i\alpha}\chi_{jkl\dot\alpha} + \lambda_{[j\alpha}\chi_{kl]i\dot\alpha} + 2\nabla_{\alpha\dot\alpha}W_{i[j}W_{kl]} - 2W_{i[j}\nabla_{\alpha\dot\alpha}W_{kl]})\ .
\end{aligned} \tag{80}$$

All these tensors satisfy the conservation law

$$\partial^{\alpha\dot\alpha}T_{i\ldots m\alpha\dot\alpha,\ldots} = 0$$

in virtue of the equations of motion, for instance,

$$\begin{aligned}
\partial^{\alpha\dot\alpha}T_{ijkl\alpha\dot\alpha} &= \mathrm{Tr}\ \nabla^{\alpha\dot\alpha}\left(3\lambda_{i\alpha}\chi_{jkl\dot\alpha} + \lambda_{[j\alpha}\chi_{kl]i\dot\alpha} + 2\nabla_{\alpha\dot\alpha}W_{i[j}W_{kl]} - 2W_{i[j}\nabla_{\alpha\dot\alpha}W_{kl]}\right)\\
&= \mathrm{Tr}\ \Big(3\lambda_{i\alpha}[\lambda^\alpha_{[j},W_{kl]}] + 2\lambda_{[j\alpha}[\lambda^\alpha_{[k},W_{l]i}] + \lambda^\alpha_i[\lambda_{[j\alpha},W_{kl]}]\\
&\qquad +2\{\lambda^{i\alpha},\lambda_{[j\alpha}\}W_{kl]} - 2W_{i[j}\{\lambda^\alpha_k,\lambda_{l]\alpha}\}\Big)\\
&= 0\ .
\end{aligned}$$

These may be used to couple $N \geq 4$ self-dual gauge theories to gravity and supergravity, whereas there are no appropriate conservation laws for such couplings of $N \leq 3$ self-dual theories.

6 Concluding remarks

We have demonstrated that the self-dual Yang-Mills equations afford supersymmetrisation beyond the conventionally 'maximal' $N = 4$ extension and thus yield non-trivial four-dimensional Lorentz covariant systems of equations invariant under N-extended rigid Poincaré supersymmetry for arbitrary values of N.

The self-duality constraints for the supercurvature have been shown to imply the existence of superfields of arbitrarily high spin and we have also demonstrated the complete equivalence of these constraints to the component equations of motion, which for $N \geq 4$ provide possibly the unique consistently coupled realisations of the zero rest-mass Dirac–Fierz equations for arbitrary spin fields. The consistency of our systems is actually a consequence of the *matreoshka phenomenon*: the N-extended system nestles within the $(N+1)$-extended system completely intact. The extra fields of the latter have interactions governed by a source current which depends only on lower spin fields and which does not require modification on further supersymmetrisation.

We have further demonstrated that the $N \geq 4$ systems of equations imply the conservation of a stress tensor, and supersymmetric generalisations. They may therefore be coupled nontrivially to the Einstein equations and to supergravity. This is unlike the $N \leq 3$ self-dual Yang-Mills theories, which have no appropriate stress-like tensors.

Our $N = 5$ theory, moreover, is probably the unique supersymmetric theory in which a spin $\frac{3}{2}$ field is coupled to a vector field, without requiring a spin 2 coupling as well for consistency [21]. However, our systems also allow *locally* supersymmetric generalisations, i.e. arbitrary–N self-dual supergravities. These have spin 1, spin $\frac{3}{2}$ and spin 2 gauge–invariances, as well as both Yang-Mills and gravitational coupling constants. In chiral superspace these arbitrarily extended self-dual supergravity equations take the form (23) as well, with the covariant derivatives being *generally covariant* ones in chiral superspace [22]. The systematic unravelling of these constraints requires a generalisation of the procedure presented in this paper. The appropriate generalisation of the $\mathcal{D}$-gauge, which eliminates all $\bar{\vartheta}$- dependence of diffeomorphism as well as gauge–transformation parameters has been discussed by us in [23].

Our super self-dual systems provide an infinitely large enhancement of an already rich class of conformally invariant exactly soluble systems in four dimensions (the $N = 0$ self-duality equations), a supersymmetric version of the twistor transform providing a method of constructing explicit solutions [12, 19]. In view of recent discussions about the centrality of the self-dual Yang-Mills equations and their twistor transform in the theory of integrable systems (e.g. [24]), the significance of these extensions seems obvious. Reductions are likely to yield all possible integrable couplings of the lower dimensional systems hitherto found to descend from the self-dual Yang-Mills equations. Our systems indeed provide a non-trivial *self-dual Yang-Mills hierarchy* of integrable systems. It seems likely that the recently discussed large–N extensions of the KdV equations [25] are reductions of our systems.

In memoriam V. I. Ogievetsky (1928 – 1996)

Victor Isaakovich Ogievetsky died on 23rd March, 1996. The work reported in this paper was largely completed in the autumn of 1995 and Victor Isaakovich worked courageously and with unfailing enthusiasm on this text during the difficult last months of his life. The memory of an imaginative physicist, an inspiring teacher and a generous soul will always endure.

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