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SELF-CONSISTENT EQUILIBRIA IN A CYLINDRICAL REVERSED-FIELD PINCH

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SUMMARY

The object of this work is to study the Self-consistent Magnetofluidstatic Equilibria of a 2-region (plasma + gas) Reversed-Field Pinch (RFP) in cylindrical approximation (namely, with vanishing inverse aspect ratio). Differently from what happens in a Tokamak, in a RFP a significant part of the plasma current is driven by a Dynamo Electric Field (DEF), in its turn mainly due to plasma turbulence. So, we have worked out a reasonable mathematical model of the above Self-consistent Equilibria under the following main points:
a) to the lowest order, and according to a standard ansatz, the turbulent DEF, say $\varepsilon^{(t)}$, *is expressed as a homogeneous transform of the magnetic field B of degree 1,* $\varepsilon^{(t)} = \alpha \cdot B$, *with* $\alpha \equiv$ *a given 2-nd rank tensor, homogeneous of degree 0 in B and generally depending on the plasma state.*
b) $\varepsilon^{(t)}$ *does not explicitly appear in the plasma energy balance, as it were produced by a Maxwell demon able ot extract the corresponding Joule power from the plasma.*
In particular, we show that, if both α *and the resistivity tensor* η *are isotropic and constant, the magnetic field is force-free with abnormality equal to* $\alpha\mu_0/\eta$, *in the limit of vanishing* β*; that is, we recover the well-known J.B. Taylor's result, in these particular conditions, starting from ideas quite different from the usual ones (minimization of total magnetic energy under constrained total elicity). Finally, the general problem is solved numerically under circular (besides cylindrical) symmetry, for simplicity neglecting the existence of the gas region (i.e., assuming the plasma in direct contact with the external wall).*

RIASSUNTO

Oggetto del presente lavoro è lo studio degli Equilibri Magnetofluidostatici Autoconsistenti di un Pinch "a Campo Invertito" (Reversed-Field Pinch, RFP) a due regioni (plasma + gas), in approssimazione cilindrica (cioè, con rapporto d'aspetto inverso uguale a zero).
A differenza di quanto avviene in un Tokamak, in un RFP una parte significativa della corrente di plasma è "tirata" da un Campo Elettrico "Dinamo" (Dynamo Electric Field, DEF), a sua volta principalmente sostenuto dalla turbolenza del plasma. Si è quindi elaborato un ragionevole modello matematico dei detti Equilibri Autoconsistenti essenzialmente caratterizzato dai seguenti punti:
a) il DEF turbolento, diciamo $\varepsilon^{(t)}$, è espresso (al più basso ordine significativo, e secondo un ansatz standard) come trasformata omogenea di grado 1 del campo magnetico B, $\varepsilon^{(t)} = \alpha \cdot B$ con $\alpha \equiv$ a tensore di rango 2, funzione di stato data, omogenea di grado 0 in B.
b) $\varepsilon^{(t)}$ non compare esplicitamente nel bilancio energetico del plasma, come se fosse prodotto da un demone di Maxwell che estrae dal plasma la corrispondente potenza Joule.
Si dimostra in particolare che, se α e il tensore di resistività η sono entrambi isotropi e costanti, il campo magnetico di plasma è force-free con anormalità pari a $\alpha\mu_0/\eta$ nel limite di $\beta \to 0$; vale a dire, il ben noto risultato di J.B. Taylor viene riprodotto, in queste particolari condizioni, su una base concettuale essenzialmente diversa da quella usuale (minimizzazione dell'energia magnetica totale a elicità totale constante). Il problema generale è infine risolto numericamente sotto simmetria circolare (oltre che cilindrica), trascurando per semplicità l'esistenza della regione di gas (cioè supponendo il plasma in diretto contatto con la parete).

Keywords: RFP (Reversed-Field Pinch), Equilibria, Selfconsistent, Dynamo.

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SUMMARY

In a recent paper [1], one of us (C.L.S.) used a MagnetoFluidDynamic (MFD), dissipative model to consistently compute the $(\psi, f, p, T) =$ (poloidal magnetic flux per unit axial length, axial magnetic field, pressure, (plasma) temperature) *stationary* profiles in a cylindrical, two-region (plasma + surrounding gas) Tokamak. Our present aim is that of extending the same analysis to a (cylindrical, two-region) Reversed-Field Pinch (RFP), i.e. to a physical system where a significant Dynamo Electric Field (DEF) drives current in addition to the usual effective electric field $E + v \times B$. Even though this DEF will substantially be thought of as due to *MFD turbulence*, we shall keep on the field laminarity, treating the DEF as being produced by some unspecified mechanism (in principle, a Maxwell demon). Apart from the modifications coming from the presence of the DEF in the Ohm's law, this will allow us to work with exactly the same equations as in [1]. Of course, the DEF will eventually be expressed in terms of the laminar fields in order to close the model, and we shall make reference to the *turbulent* DEF to this end.

In the simplest case where the resistivity η is isotropic, and both η and α (the usual (scalar) factor in front of B giving the turbulent DEF under turbulence's isotropy, to the lowest order) are constant, for vanishing small β we find that the corresponding (force-free) magnetic configuration has (constant) abnormality $\alpha\mu_0/\eta$.

We also consider the general situation where all the transport coefficients and α are generic functions of the plasma (laminar) state (and, possibly, of B and/or its derivatives). Limiting ourselves to *circular* (besides cylindrical) symmetry, and to a plasma column ideally in contact with the shell (one-region problem), we work out an extended

numerical campaign, covering a variety of models for the (tensor) resistivity $(\eta_{\|}, \eta_{\perp})$, the perpendicular thermal conductivity $\lambda_{\perp}$ and α. Also, a special case where αB is replaced by $\alpha \cdot B$, with α being a symmetric hyperbolic tensor, has been considered.

In conclusion, our model provides a conceptually different, and much more flexible, route to the interpretation of RFP's equilibria, as compared with the classical one [2].

1. Introduction

The following cylindrical configuration was considered in [1]. An open cylinder C with axis z and arbitrarily given, but simply-connected normal section Ω (an open domain of the $\mathrm{x} \equiv (x, y)$-plane) contains a cylinder C_p (open) with unknown, but similarly simply-connected normal section Ω_p, filled by plasma. The (by assumption, not empty) difference-cylinder $C_g = C \setminus \overline{C}_p$ (here the overbar means closure, as usual) with (topologically annular) normal section $\Omega_g = \Omega \setminus \overline{\Omega}_p$ is filled by gas. The various fields of interest (in particular, the poloidal Magnetic Flux (MFx) per unit axial length ψ, the Magnetic Field (MF) $B = f\hat{z} + \hat{z} \times \nabla\psi$, $\hat{z} \equiv$ unit-vector along z, the pressure p, the temperature T) are defined and sufficiently smooth in $\Omega_p \cup \Omega_g$ and extended to $\overline{\Omega}$ by continuity *from inside*. The MF component normal to $\hat{z}$ is assumed to vanish in exactly one point m (the magnetic axis) in $\overline{\Omega}$ (actually, m turns out to belong to Ω_p), with $\nabla\nabla\psi|_m : \xi\xi > 0$ (say), $\forall \xi \in \mathbb{R}^2$ and $\neq 0$, and the axial MF f not to vanish at m. (Under continuity of ψ and of its gradient through $\partial\Omega_p$, the first condition allows to use ψ as a radius-like coordinate in $\overline{\Omega}$, with pole at m).

A one-fluid, dissipative model with infinite parallel thermal conductivity (implying constant temperature along magnetic lines) was used for the plasma, including mass/internal-energy conservation equations (with no source), standard MagnetoFluidStatic (MFS) momentum balance, generalized (i.e. with generally non-diagonal tensor conductivities) Ohm's/Fourier's laws, and pre-Maxwell (stationary) system, with mass/internal-energy densities, and transport coefficients, given as functions of the plasma state (p, T)[1] and, respectively, of the plasma state and $|B|$. As for the gas, apart from pre-Maxwell's system, we have (trivially) constant p (no force density) and constant f, harmonic ψ (no current density). Furthermore some "boundary" condition is assumed to hold on $\partial\Omega_p$ for T, so that the temperature problem in the gas decouples, and can be ignored.

Neglecting *convection* current density in the (plasma) energy conservation equation[2], all of the above made it possible to decouple four equations for (ψ, f, p, T) in Ω_p, and one equation for ψ in Ω_g, from the full set of stationary equations. Under convenient interface (= on $\partial\Omega_p$) and boundary (= on $\partial\Omega$), or special accessory, conditions, this system is likely to have solutions (possibly, unique "in the small"); but no solution procedure seems available (reasonably, from both the theoretical and computational points of view) other than recursive. Everything becomes mathematically trivial, however, on assuming circular (besides cylindrical) symmetry; then we are led to an Ordinary Differential (OD), two-point problem (with the radius r as independent variable), which can be solved by standard numerical techniques. If in addition we assume a diagonal Ohm's law with isotropic, constant resistivity, the solution is found an-

[1] In practice, we do not use this information.

[2] Strictly speaking, this should justified a posteriori, since we *can* evaluate both charge density and velocity, though incompletely, see Sect 6e.

alytically by quite elementary means, giving a parabolic profile for ψ (with $\psi|_{\partial\Omega_p} = 0$), $p - p^0$ proportional to ψ ($p^0 \doteq p|_{\partial\Omega_p} > 0$) and constant $f = f^0 \doteq f|_{\partial\Omega_p}$, in the plasma; and logarithmic ψ, and constant $p = p^0$ and $f = f^0$ (of course), in the gas.

The present paper purposes to study the same physical configuration in the presence of a significant Dynamo Electric Field (DEF) $\mathcal{E}$ in the plasma, in practice to be thought of as due to MFD turbulence. Hopefully, this will provide a reasonable theoretical interpretation of the (almost) stationary part of a RFP's discharge. In contrast to a Tokamak, a RFP is featured by a high-shear MF, with its axial (toroidal) component being typically reversed in the vicinity of the plasma boundary w.r.t. its orientation on axis. This is expected to enable the confinement of larger-β plasmas, and to make less dangerous certain MFD instabilities. Since a typical RFP discharge lasts significantly longer than the resistive-diffusion time $a^2\mu_0/\eta$ ($a \equiv$ a typical plasma dimension), one usually argues that the magnetic configuration be "regenerated" by some dynamo mechanism. The best candidate in this sense seems to be the (statistically stationary) turbulent dynamo due to random ($\sim$) fluctuations of v (velocity) and B, giving rise to the average EF $\mathcal{E}^{(t)} \doteq \overline{\tilde{v} \times \tilde{B}}$ (with the overbar here meaning stochastic average over the space/time turbulent scales). As we shall see in detail, an appropriate introduction of this DEF into our model leads to an interpretation of RFP's "equilibria" which is alternative to, and much more flexible than, J.B. Taylor's conjecture, and provides the same force-free solution as a basic, special case[3].

[3]As well known, Taylor's conjecture leads to the variational problem of minimizing the magnetic energy, for fixed (gauge invariant) helicity, in a class of MF's conveniently conditioned on the boundary. This results into a force-free (or Beltrami's) MF with constant abnormality ω equal to 2 times the related Lagrange multiplier. Force-free fields with *constant* abnormality are sometimes called Trkal's fields, after a paper by V. Trkal

Our model has many limitations (some of which can hopefully be removed at the cost of more refinements), but is logically consistent. However, it suffers from the typical trouble of every fluid theory: to compute, we need to know effective transport coefficients and α as functions of the (mean) magnetoplasma state. Our present poor knowledge in this sense gives our results a somewhat aleatory flavour; but, from a positive point of view, we can also exploit the situation the reversed way, getting some information about the above objects by comparing our numerical results with the available measurements (of the mean fields, or of convenient functionals of them).

2. Turbulent Dynamo and Stationary-Turbulent Equations

Contrary to a similar problem arising in viscous hydrodynamics, where no satisfactory theory has been established to express the stochastic dyadics $\overline{vv}$ in terms of the mean fields $(\overline{v}, \overline{p})$, some theoretical information about the bi-linear average $\mathcal{E}^{(t)}$ is available since a long time[4]. In particular, a simple homogeneity argument proves that, if $\mathcal{E}^{(t)}$ is homogeneous of degree 1 w.r.t. $\overline{\eta}$, then it must be equally homogeneous of degree 1 w.r.t. $\overline{B}$. Assuming this property to hold, and requiring a sufficient "localization" of the associated phenomena, one expects $\mathcal{E}^{(t)}$ to be expressible as a (rapidly converging) series of type

$$\mathcal{E}_i^{(t)} = \alpha_{ij}\overline{B}^j + \beta_{ijk}\overline{B}^{k/j} + \gamma_{ijkh}\overline{B}^{h/jk} + \dots \tag{1}$$

where α_{ij}, β_{ijk}, γ_{ijkh}, ... are tensors of rank 2, 3, 4 ..., homogeneous of degree 0 w.r.t. $\overline{B}$ and of degree 1 w.r.t. $\overline{\eta}$, and a priori depending on

[3] published in 1919. Some general information about Trkal's fields can be found in [4]. For a short discussion of Trkal's problem in a toroid and its solution, see [5].

[4] Seemingly, this was due to the linearity of the induction equation, with the Ohm's law replaced in it, w.r.t. B.

the other mean fields (see [6], Chpt. 7, for a general introduction). If in addition we assume isotropy, α_{ij}, β_{ijk}, ... will be isotropic as well, or

$$\alpha_{ij} = \alpha_0 g_{ij} \ , \quad \beta_{ijk} = \beta_0 \varepsilon_{ijk} \ , \quad \gamma_{ijkh} = \gamma_0 g_{ih} g_{jk} + \gamma_1 g_{ik} g_{jh} + \gamma_2 g_{ij} g_{kh} \ , \ldots$$

where $g_{ij} \equiv$ fundamental tensor, $\varepsilon_{ijk} \equiv$ Ricci's (antisymmetric) fundamental tensor, and $\alpha_0, \beta_0, \gamma_0, \gamma_1, \gamma_2 \ldots$ are scalars. Then one finds:

$$(1') \qquad \mathcal{E}^{(t)} = \alpha_0 \overline{B} + \mu_0 \beta_0 \overline{J} - \mu_0 \gamma_0 \nabla \times \overline{J} + \ldots$$

with $\mu_0 \overline{J} = \nabla \times \overline{B}$. When this expression is replaced in the averaged Ohm's law (neglecting convection current, and with E being the usual "laboratory" EF)

$$(2) \qquad \overline{\eta} \cdot \overline{J} + \overline{\tilde{\eta} \cdot \tilde{J}} = \overline{E} + \overline{v} \times \overline{B} + \mathcal{E}^{(t)} \ ,$$

one sees that the β_0-term just modifies the isotropic part of $\overline{\eta}$ by the addition of the isotropic tensor $-\mu_0 \beta_0 g_{ij}$. By including this term in an "effective" mean resistivity tensor, still to be denoted by $\overline{\eta}$, and neglecting terms from those in $\nabla \nabla \overline{B}$ on, one ends up with

$$(1'') \qquad \mathcal{E}^{(t)} = \alpha_0 \overline{B} \ .$$

We stress that eq. (1″) is a *parallel*, but generally *non-linear*, (although homogeneous of degree 1) transform of $\overline{B}$.

Coming back to the general expansion (1), the more or less accurate and complete determination of α_{ij}, β_{ijk}, ... depends on the (possibly non-linear) instability theory we shall make use of in order to evaluate the fluctuation field $(\tilde{v}, \tilde{B})$. For instance, an expression of $\mathcal{E}^{(t)}$, up to normalizing factors, has been derived in [7] from a model of resistive-interchange modes in a cylindrical-circular plasma column. Then both

$\overline{B}$ and $\mathcal{E}^{(t)}$ (strictly speaking, that DEF should not be classified as "turbulent") have no r-components, and, with reference to standard cylindrical coordinates (r, θ, z), with the usual orientation, it is found that:

$$(1_1''') \qquad \mathcal{E}_\theta^{(t)} = \gamma \overline{B}_\theta + \nu \overline{B}_z$$

$$(1_2''') \qquad \mathcal{E}_z^{(t)} = \gamma \overline{B}_z + \nu \overline{B}_\theta$$

where (γ, ν) are scalars, homogeneous of degree 0 w.r.t. $\overline{B}$ and proportional to η (assumed isotropic and constant), defined up to numerical factors, and with ν positive. The latter condition gives $(1''')$ an intrinsically chiral character.

Whatever the expression of $\mathcal{E}^{(t)}$, anyhow, we must guess at the other equations we have to use, following the general lines of [1]. Agreeing to the turbulent picture of DEF, in principle we have to start with the fully non-stationary MFD system. Granted, for the sake of argument, that the same basic approximations made for the (stationary) laminar equations of [1] keep valid for the general ones, the stationary-turbulent system will then be derived from the stationary-laminar one by formally replacing the generic (laminar) field X by $\overline{X} + \tilde{X}$ and averaging. Indeed, time derivatives appear linearly, with constant coefficients ($= 1$), in the mass, momentum, and energy conservation (non-stationary) equations, respectively as $\partial_t \rho$, $\partial_t(\rho v)$, $\partial_t \mathcal{H}$ (with $(\rho, \mathcal{H}) \equiv$ (mass, energy) density), so that their contributions vanish after time averaging over the turbulent time-scale (also assuming stationary averages on that scale). The same simple recipe holds valid for the equations with no time-derivatives (of course), like the Ohm's law eq. (2).

The stationary-turbulent equations derived this way are rigorous (to within the basic approximations we have started with), but *of no use* unless the several n-linear $(2 \leq n \leq 3)$ stochastic averages coming into play are known in terms of mean fields, or possibly neglected on order-of-magnitude evaluation. Unfortunately, we have very poor (or no) information about these averages (apart, partly, for $\mathcal{E}^{(t)}$), and, on the other hand, the possibility of neglecting all of them but $\mathcal{E}^{(t)}$ appears quite unattainable, even in principle. Differently said, in practice the above stationary-turbulent equations are mere "rephrasings" of the original (non-stationary) equations.

A way out of such an impasse can be indicated as follows. We keep on the laminar-stationary picture as in [1], but take into account the existence of an additional EF $\mathcal{E}$ (eventually to be thought of as DEF) impressed by some unspecified, artificial mechanism (a Maxwell demon), and uniquely determined as "quasi-local" functions of the laminar fields (i.e., functions of the related values of them and of their lowest-order gradients), according to whatever law. In spite of its somewhat expedient nature, this idea can be made consistent with the conservation laws.

Based on the above, we shall (obviously) leave unchanged the mass/momentum conservation equations, as well as the (by assumption, diagonal) Fourier's law and the mass/internal-energy densities expressions (indeed, they do not contain the effective EF). As for the Ohm's law, its version

$$\eta \cdot J = E^{*} \doteq E + v \times B + \mathcal{E} \tag{2bis}$$

is not against first principles, and can be used. What requires some more careful consideration, instead, is the internal-energy balance. Of course in presence of an impressed EF $\mathcal{E}$ and of a current density J, a Joule power

$\mathcal{E} \cdot J$ must be released in the plasma; but the addition of this power in the RHS of the standard internal-energy conservation equation (see for inst. [8], Chpt. 1, p. 24, neglecting independent sources, viscous losses and convection current density, in agreement with our basic assumptions), i.e. in

$$\nabla \cdot (v\mathcal{H} + q) = E + v \times B - p\nabla \cdot v \ , \tag{3}$$

with $q \equiv$ (conductional) heat flux, would be deceptive. The reason is that the Maxwell demon, who "keeps $\mathcal{E}$ alive", must also provide the related power $\mathcal{E} \cdot J$ (if ≥ 0), by extracting it from the plasma itself (at least, in the average over a "magnetic layer" bounded by the magnetic surfaces ψ and $\psi + \delta\psi$). This can easily be proved in a direct way.

Indeed, let us consider the (cylindrical) magnetic layer $(\psi, \psi + \delta\psi)$, per unit axial length. By assumption, our physical system is merged into the curl-less EF $E = -U\hat{z} - \nabla\varphi$, where $(Uz, \varphi) \equiv$ (axial, meridional) part of the electrostatic potential, $U =$ const and $\varphi = \varphi(\mathrm{x})$ (in force of the assumed cylindrical symmetry). The uniform EF $E_z = -U$ penetrates into the system through the external boundary ∂C (by assumption, insulating along z)[5], and must be thought of as due to the variation in time, at a constant rate, of an external poloidal MFx per unit axial length ϕ, taken positive by the cork-screw rule, and of course independent of z, according to $U = \mathrm{d}_t\phi$.

Due to the mass conservation equation (with positive mass density), it can easily be shown that $\langle v \cdot \nabla\psi \rangle = 0$, where $\langle\cdot\rangle$ is the familiar

[5]This simulates the existence of one or more meridional cuts along the shell, in the observance of the cylindrical symmetry.

(cylindrical) "magnetic average" operator defined by:

$$\langle\cdot\rangle(\psi) \doteq (I(\psi))^{-1} \oint_{\psi(\mathrm{x})=\psi} \mathrm{d}l/|\nabla\psi|(\cdot) \quad , \quad I(\psi) \doteq \oint_{\psi(\mathrm{x})=\psi} \mathrm{d}l/|\nabla\psi| \ .$$

Since the internal energy $\mathcal{H}$ is constant along $\psi(\mathrm{x}) = \mathrm{const}$ (indeed, it depends on the plasma state (p, T) only, and $\nabla\psi\times\nabla p = 0$, $\nabla\psi\times\nabla T = 0$), no internal energy can be *convected* through the layer's "side" boundary Σ:

$$\int_\Sigma \mathrm{d}\,\Sigma\cdot v\mathcal{H} = \delta\psi\langle I\mathcal{H}v\cdot\nabla\psi\rangle_\psi \equiv 0$$

(where subscript ψ means derivative w.r.t. ψ). The same holds valid for all functions which are constant along $\psi(\mathrm{x}) = \mathrm{const}$ (and only for those); hence, *not* for the kinetic energy density $\rho v^2/2$. But the latter density must be neglected for consistency with the MFS momentum balance, where ρ was actually put equal to zero.

The thermal losses through Σ are immediately evaluated on the same basis as

$$\delta Q \doteq \int_\Sigma \mathrm{d}\,\Sigma\cdot q = \delta\psi\langle Iq\cdot\nabla\psi\rangle_\psi$$

The corresponding (internal energy, heat) fluxes through the "extremity" boundary of the layer vanish by symmetry, and so we must equal δQ to the total power (mechanical + electric) made from the outside on the layer. The mechanical power through the whole boundary is zero, because $\int_\Sigma \mathrm{d}\,\Sigma\cdot vp = 0$ and because the same integral over the extremity boundary again vanishes by symmetry. As for the electric power, there is no charge flux through Σ ($J\cdot\mathrm{d}\,\Sigma = 0$); but there is a charge flux $I\delta\psi\langle J_z\rangle$ entering the layer at the section $z = 0$ with electric potential $U = 0$ (say), and leaving it from the section $z = 1$, with potential U. In conclusion, also taking into account that $\langle J_z\rangle \equiv J_z$ (due to the radial equilibrium equation) the desired power balance reduces to:

$$I^{-1}Q \equiv I^{-1}\langle Iq \cdot \nabla\psi\rangle_\psi = -UJ_z \quad , \quad Q \doteq \frac{\delta Q}{\delta \psi} \ . \tag{4}$$

Note that

$$-U\int_{\Omega_p} d_2 x J_z = -U \oint_{\partial\Omega_p} \mathrm{d}\, l \cdot \nabla\psi = \int_{\Omega_p} d_2 x I^{-1} Q \geq 0 \ ,$$

otherwise we would have a thermoelectric generator with unit efficiency. Since $\oint_{\partial\Omega_p} \mathrm{d}\, l \cdot \nabla\psi > 0$ (see next Section), we conclude that $U \leq 0$.

Let us now compare eq. (4) with the magnetic average of eq. (3). The LHS of eq. (3) reduces to $I^{-1}Q$ after averaging, due to $\langle v \cdot \nabla\psi\rangle = 0$. As for the RHS, we have $\langle E \cdot J\rangle = -UJ_z$, due to the assumed simple-connectedness of Ω_p, in turn implying single-valuedness of φ, or $\langle\nabla\varphi \cdot J\rangle =$ $= -\mu_0^{-1}\langle\nabla\varphi \cdot \hat{z} \times \nabla\psi\rangle f_\psi \equiv 0$; and finally, $v \times B \cdot J - p\nabla \cdot v = \nabla \cdot (pv)$, which gives zero on taking the magnetic average. In conclusion, we end up again with eq. (4). This means that, if we insist at including the Joule power $\mathcal{E} \cdot J$ in the RHS of eq. (3), we must also add a sink $-S$ (in the same RHS) such that $\langle S\rangle = \langle\mathcal{E} \cdot J\rangle$. If $\langle\mathcal{E} \cdot J\rangle \geq 0$, this implies that the thermal power released in the plasma be "rechanneled" into some form of (well-organized) energy able to keep $\mathcal{E}$ alive, *against* the 2nd principle of thermodynamics. But *by definition* a Maxwell demon must keep to conservational laws only, and not necessarily to the 2nd principle.

The situation changes, of course, if eq. (3), with $\mathcal{E} \cdot J - S$ added in the RHS (say, eq. (3bis)) is reconsidered in the turbulent framework (also identifying the old laminar fields with the stochastically averaged ones). Then $-S$ does exactly amount to the contribution from all the stochastic averages which should appear in it (put in the RHS), apart from $\mathcal{E}^{(t)} \cdot \overline{J}$.

But in this case we could not maintain that $\langle \mathcal{E}^{(t)} \cdot \overline{J} \rangle = \langle S \rangle$, unless all other equations are kept unvaried w.r.t. their laminar versions (and with the averaged fields in place of the laminar ones). This is quite an artificial requirement, and so we are driven back to the "laminar-Maxwell demon" model as a possible compromise.

We note that, although $\nabla^2\psi|_m > 0$ and $\nabla\psi \neq 0$ in $\Omega_p \setminus \{m\}$, J_z is not necessarily ≥ 0 in the whole of $\overline{\Omega}_p$; and the same, of course, is true for $\langle \mathcal{E} \cdot J \rangle = -UJ_z$ (for $U < 0$). Instead, since $\eta \cdot J\,J > 0$ for $J \neq 0$ ($\eta \cdot J\,J$ is a definite positive quadratic form by assumption (dissipative plasma)), $J \neq 0 \Rightarrow \langle (E + \mathcal{E}) \cdot J \rangle > 0$. It is interesting to examine the sign of $\langle \mathcal{E} \cdot J \rangle$ for $\mathcal{E} = \alpha_0 B$ (eq. (1″)). We shall do that for the simple case where $\eta_{\parallel} = \eta_{\perp} = \eta_*$, and η_* is independent of $|B|$. It then turns out that (see next Section) $p_\psi = U/\eta_*$ and (writing α_0 as α from now on) $f_\psi = -\alpha\mu_0/\eta_*$, giving $\mathcal{E} \cdot J = \eta_*^{-1}(\alpha^2|B|^2 - \alpha U f)$. If $U = 0$, $\mathcal{E} \cdot J \geq 0$; if $U < 0$, from $0 \leq |J|^2 = \eta_*^{-2}(U^2 - 2\alpha f + \alpha^2|B|^2)$ it follows that $-\alpha U f \geq -\frac{1}{2}(U^2 + \alpha^2|B|^2)$, hence that $(\alpha^2|B|^2 \geq U^2) \Rightarrow \mathcal{E} \cdot J \geq 0$ in any case.

3. The model equations and the interface/boundary conditions

3.1. Equations.

First of all we recapitulate the full system of equations we shall (and did) make use of:

a) in the plasma

(1) $\langle v \cdot \nabla \psi \rangle = 0$ (from mass conservation without source and $\rho > 0$);

(2_1) $\nabla\psi \times \nabla f = 0$,

(2_2) $\nabla\psi \times \nabla p = 0$,

(2_3) $\nabla^2\psi + f\,f_\psi + \mu_0 p_\psi = 0$, (from MFS momentum balance $J \times B = \nabla p$

and $\nabla \cdot B = 0$);

(3) $\mu_0 J \equiv \nabla \times B = \hat{z}\nabla^2\psi - f_\psi \hat{z} \times \nabla\psi$ (Ampère's law);

(4) $\eta \cdot J = E + v \times B + \mathcal{E} \equiv E^* + \mathcal{E}$ (Ohm's law with total current density J identified with conductional current density); here

$$\eta \doteq \eta_{\|}\hat{B}\hat{B} + \eta_{\perp}(\hat{\hat{\delta}} - \hat{B}\hat{B}) - \eta_{\wedge}\hat{B} \times \hat{\hat{\delta}} \quad , \quad \eta_{\|} > 0 \ , \ \eta_{\perp} > 0 \ ,$$

$\hat{\hat{\delta}} \equiv$ unit dyadics, $\hat{X} \doteq X/|X|$ (actually, Hall resistivity $\eta_\wedge$ will play no role in what follows);

(5) $E = -U\hat{z} - \nabla\varphi$ (Faraday's law under cylindrical symmetry);

(6) $-q_\perp = \lambda_\perp T_\psi \nabla\psi$ (Fourier's law with $\lambda_{\|} \to \infty$), $\lambda_\perp > 0$;

(7) $I^{-1}\langle Iq \cdot \nabla\psi\rangle_\psi = -UJ_z$ (energy balance of the magnetic layer, see Sect 2)

The above eqs. (1÷7) must be completed by giving $\mathcal{E}$ in terms of the other fields: we shall mainly use ansatz (2.1″), α (scalar) given as a (quasi-local) field function, homogeneous of degree 0 w.r.t. B. Similarly given as functions of the magnetoplasma state are η and $\lambda_\perp$.

By combining eqs. (1÷5) and (2.1″), we get:

(8) $$f\langle\eta_{\|}\rangle p_\psi + \langle\eta_{\|}|B|^2\rangle\mu_0^{-1}f_\psi = -\langle\alpha|B|\rangle + fU \equiv \mathcal{A} \ ,$$

(9) $$\langle\mathcal{N}\rangle p_\psi + f\langle\eta_{\|}\rangle\mu_0^{-1}f_\psi = -\langle\alpha\rangle f + U = \mathcal{B} \ ,$$

where $\mathcal{N} \doteq |B|^{-2}(\eta_{\|}f^2 + \eta_\perp h^2)$, $h \doteq |\nabla\psi|$, $|B|^2 = f^2 + h^2$.[6]

Eqs. (8, 9) provide unique (p_ψ, f_ψ) if the determinant $\mathcal{D} \doteq f^2\langle\eta_{\|}\rangle^2 - \langle\eta_{\|}|B|^2\rangle\langle\mathcal{N}\rangle$ is not zero, according to:

(10) $$p_\psi = \mathcal{D}^{-1}(\mathcal{A}f\langle\eta_{\|}\rangle - \mathcal{B}\langle\eta_{\|}|B|^2\rangle) \ ,$$

[6]From eqs. (2_3, 8, 9) it is immediate to see that, if (ψ, f, p) is a solution to them corresponding to α, $(\psi, -f, p)$ is a solution corresponding to $-\alpha$.

$$f_\psi = \mathcal{D}^{-1}(\mathcal{B}f\langle\eta_\|\rangle - \mathcal{A}\langle\mathcal{N}\rangle) \ . \tag{11}$$

Knowledge of (p, f) on the basis of eqs. (10, 11) is what we mean by "self-consistence" of the solution (in particular, of eq. (2_3)).

If $(\eta_\|, \alpha)$ do not vary along $\psi = \text{const}$, eq. (10) simplifies to

$$p_\psi = -U\langle h^2\rangle(\eta_\| f^2 - \langle|B|^2\rangle\langle\mathcal{N}\rangle) \ ; \tag{10bis}$$

and so, the plasma turns out force-free, in this case, when $U = 0$. If in addition we assume isotropic η, $\eta_\| = \eta_\perp = \eta_*$, we get:

$$p_\psi = U/\eta_* \ , \tag{12}$$

$$f_\psi = -\alpha\mu_0/\eta_* \ . \tag{13}$$

(This result has been used in Sect 2). Under the further restriction that $\alpha\mu_0/\eta_* \equiv \omega = \text{const}$, eq. ($2_3$) gives

$$\nabla^2\psi + \omega^2\psi = -U\mu_0/\eta_* + \omega f^0 \quad , \quad f^0 \doteq f(\psi = 0) \ . \tag{14}$$

If $U = 0$, this is the same as $(\nabla \times B)_z = \omega B_z$; and since $\hat{z} \times (\nabla \times B) = = \hat{z} \times \omega B$ follows from eq. (13), we conclude that $\nabla \times B = \omega B$: if $U = 0$, B is force-free with (constant) abnormality ω. Taylor's main result is thus recovered, in this special case, along a different conceptual route. Also, note that if both f^0 and α are reversed, ψ does not change (eq. (14)), whereas f is reversed. This fact remains true in general conditions, as it can easily be checked from eqs. (2_3, 8, 9), see again([6]).

Eqs. (2_3, 6, 7) then give:

$$-I^{-1}\langle I\lambda_\perp T_\psi h^2\rangle_\psi = U(\mu_0^{-1} f f_\psi + p_\psi) \ . \tag{15}$$

In conclusion, if $\mathcal{D} \neq 0$, and if $(\eta, \lambda_\perp, \alpha)$ can be expressed as functions of $(\psi, \nabla\psi, f, p, T)$, we end up with the closed system (2_3, 10, 11, 15) for

(ψ, f, p, T). The nature of this system is quite exotic: for given $(f,p)(\psi)$, eq. (2_3) is a quasi-linear, elliptic PDE, and for given $\psi(\mathrm{x})$, eqs. (10, 11, 15) are quasi-linear ODE's, of the 1st order w.r.t. (f,p) and of the 2nd order w.r.t. T, with ψ as independent variable. The "symultaneous" eqs. (2_3, 10, 11, 15) give rise to a problem which could be classified as (singular)-functional-differential one; we believe that its analysis (or rather, of more fundamental versions of it) could raise interesting theoretical questions.

Syst. (2_3, 10, 11, 15) definitely simplifies under circular symmetry: then $\langle\ \rangle$ reduces to identity, and one ends up with the 1st-order ODS in normal form ($r \equiv$ radius):

(16_1) $\quad \mathrm{d}_r\psi = h$,

(16_2) $\quad \mathrm{d}_r h = -(h/r + fg + U\mu_0/\eta_\perp)$,

(16_3) $\quad \mathrm{d}_r f = hg$,

(17_1) $\quad \mathrm{d}_r T = \delta$,

(17_2) $\quad \mathrm{d}_r\delta = -\lambda_\perp^{-1}[\delta(\lambda_\perp/r + \mathrm{d}_r\lambda_\perp) + U(fg/\mu_0 + U/\eta_\perp)]$,

(17_3) $\quad \mathrm{d}_r p = U\lambda_\perp/\eta_\perp$.

Here $g \equiv f_\psi$ is shorthand for $\mu_0\left[-\frac{\alpha}{\eta_\parallel} + \frac{Uf}{|B|^2}\left(\frac{1}{\eta_\parallel} - \frac{1}{\eta_\perp}\right)\right]$, and $\mathrm{d}_r\lambda_\perp$ should be understood as a (known) linear combination of $\mathrm{d}_r(p, T, |B|)$. If in particular $(\eta_\parallel, \eta_\perp, \alpha)$ are independent of (p, T), the triple of eqs. ($16_{1\div3}$) decouples from the triple ($17_{1\div3}$), and can be solved by itself. Finally, if $(\eta_\parallel, \eta_\perp, \alpha)$ are constant and $\eta_\parallel = \eta_\perp$, eqs. ($16_{1\div3}$) are solved by elementary analytical means (see Sect 4). This case will be referred to as "standard" in the following.

b) in the gas

As we already know, here ψ is harmonic, f and p are constant and T is decoupled.

c) other unknowns

There are five more plasma unknowns, and one gas unknown, decoupled, namely v, φ, $q_{\|} \equiv$ parallel heat flux (in the plasma), and φ (in the gas), see Sect 6, e).

3.2. Interface/Boundary Conditions.

We shall assume a non-singular B, and similarly exclude both singular current and singular force densities on $\partial\Omega_p$. This implies continuity of ψ, and respectively of B and of p through $\partial\Omega_p$. On the other hand, it is quite reasonable to define $\partial\Omega_p$ as a (closed) line where (f, p) are not analytic. Then it is easy to show that $\lim_{\mathrm{x}\to\partial\Omega_p} \psi(\mathrm{x})$ must be a constant. With no loss of generality, we shall agree to put this constant equal to zero.

By similar reasons, we shall require continuity of φ through $\partial\Omega_p$ (not of its gradient). As for T, we shall prescribe

$$\lim_{\mathrm{x}\to\partial\Omega_p^-} T(\mathrm{x}) = \text{const } = T^0 > 0$$

(alternatively, we could require that no conductional heat flows to outside, of course providing an appropriate sink in the energy equation, etc).

On $\partial\Omega$, we shall give ψ as a (positive) constant (typically), and φ as a constant (typically). The gas values $(f^0, p^0 > 0)$ will be thought of as given.

3.3. Hints for a Discussion of the Free-Boundary Problem.

Let us first consider the (degenerate) one-region problem, with $\Omega \equiv \equiv \Omega_p$. For $\mathrm{x} \to \partial\Omega^-$, we have $\psi \to 0^-$, $p \to p^0 > 0$, $f \to f^0$, $T \to T^0 > 0$. If $\mathcal{D} \neq 0$, syst. (2_3, 10, 11, 15) is likely to provide a unique (in the small) solution under these conditions. A hint is as follows. Assuming $(f,p)(\psi)$ to be given for $\psi < 0$, eq. (2_3) is known to provide a unique (in the small) solution vanishing for $\mathrm{x} \to \partial\Omega^-$ under precisely stated restrictions ("quasi-linear Dirichlet Problem"). Eq. (15) is a 2nd-order quasi-linear ODE for T (for the given $(f,p)(\psi)$ and the just computed $\psi(\mathrm{x})$), having $\langle\lambda_\perp h^2\rangle$ as principal coefficient, > 0 for $\psi_{\min} < \psi \leq 0$, and $\to 0^+$ for $\psi \to \psi_{\min^+}$. This singularity at $\psi_{\min}$ ensures that the equation has a *unique* solution under the *single* end-point condition at $\psi \to 0^-$, if we require regularity at the singular end-point $\psi = \psi_{\min}$. Finally, eqs. (10, 11) are quasi-linear, 1st-order ODE's for the computed (ψ, T), and provide a unique solution for (f,p) (the "next approximation") under the end-point conditions at $\psi \to 0^-$. This is just an example of recursive procedure, the real problem consisting in proving its possible convergence (under convenient restrictions). Of course the solution (ψ, f, p, T), if it exists, should fulfil all the accessory requirements (for instance $p > 0$ everywhere, or $f(\psi_{\min}) \neq 0$) in order to be accepted. For given, $(f^0, p^0 > 0, T^0 > 0)$, and given transport coefficients and α, the only (operational) parameter of this problem, occurring in the model equations, is U. If a unique solution exists in an interval about $\overline{U}$, we can compute the total axial current in that interval:

$$(18) \qquad \mathcal{I} \doteq \mu_0^{-1} \oint_{\psi=0} \mathrm{d}\,l \cdot \nabla\psi = \mu_0^{-1} I^{-1} \langle h^2 \rangle|_{\psi=0} = \mathcal{I}(U) \ ;$$

assuming this function to be invertible at $U = \overline{U}$, we can use $\mathcal{I}$ as operational parameter in place of U about $\overline{\mathcal{I}} = \mathcal{I}(\overline{U})$.

The following important remark is appropriate. Suppose a (non-trivial) solution of eq. (14) (hence, under the conditions leading to eqs. (12, 13), and for $\omega =$ const) to exist for $-U\mu_0|_{\eta_*} + \omega f^0 = 0$; since $\psi \to 0^-$ on $\partial\Omega$, we conclude that ψ is eigensolution to the Helmoltz equation $\nabla^2\psi + \omega^2\psi = 0$, and that ω^2 is the corresponding eigenvalue. Now, it is well-known that $(\nabla^2 + k)u = 0$, $u|_{\partial\Omega} = 0$, admits of an infinite set of positive (non-degenerate) eigenvalues, and that only the eigensolution corresponding to the smallest of them, say k_1, can meet our requirements on ψ. So, if (19) $\omega^2 = k_1 \equiv k_{1,\Omega}$, the original inhomogeneous problem will lead to a "Fredholm-like" (so to speak) alternative. This suggests that our general problem (2_3, 10, 11, 15), as well as its circular version (16, 17), may share the same property, having no, or (infinitely) many, solutions under a criticality condition of type (19). A special case of that is met in the standard problem (Sect 4), where it reads: $|\omega|a = j_0 \equiv$ the 1st zero of the Bessel function J_0.

No special difficulties are encountered in the actual free-boundary (two-region) problem. When the value of $\psi|_{\partial\Omega} \doteq \psi(\mathrm{x} \to \partial\Omega^-)$ is given, say $\psi|_{\partial\Omega} = \Psi =$ const > 0, we have again a quasi-linear Dirichlet problem for ψ for given $(f,p)(\psi < 0)$. Existence/uniqueness theorems are still available in this case (indeed, the lack of analyticity of (f,p) at $\psi = 0$ plays no significant role), and we can start a recursive procedure as before. If a solution is found, the curve $\psi = 0$ describes the requested free boundary. Again, the total axial current $\mathcal{I}$ can be used as alternative parameter; most suitably, in this case, in place of Ψ. Then the iterational problem for ψ has to be solved under $\psi|_{\partial\Omega} =$ const (unspecified) and $\mathcal{I}$ given (see for instance [9]). Conditions ensuring existence/uniqueness are known in this case too, and we can similarly start our recursive procedure. Of the two alternative versions of this free-boundary problem (given Ψ vs. given $\mathcal{I}$), the second one is definitely more interesting from the physical point of view.

Everything becomes almost trivial under circular symmetry: we have merely to solve the ODS (16, 17) in the plasma, under the the additional, self-evident, end-point conditions $h(0) = 0$, $\delta(0) = 0$, matching the (logarithmic) ψ in the gas region at $r = a$ (plasma radius), up to the 1st derivative included.

4. The "Standard" Case

We already know that the problem is linear in this case. In fact we obtain the Bessel equation

$$\frac{1}{r}\,\mathrm{d}_r(r\,\mathrm{d}_r\psi) + \omega^2\psi = -H\omega^2 \quad , \quad H\omega^2 = U\mu_0/\eta - f^0\omega$$

(in this Section, we shall write η for η_*). Assuming $\omega \neq 0$, the solution is

$$\psi(r) = CJ_0(\omega r) - H \ , \tag{1}$$

where the constant C is found from

$$0 = \psi(a) = CJ_0(\omega a) - H \tag{2}$$

if $J_0(\omega a) \neq 0$. On discarding the trivial solution $\psi = 0$, we have that $J_0(\omega a) = 0 \Leftrightarrow H = 0$. This is the critical condition in the standard case. Then we compute:

$$f(r) = -\omega\psi(r) + f^0 = -\omega CJ_0(\omega r) + U/\alpha \ , \tag{3}$$

(4) $$h(r) = -C\omega J_1(\omega r) \ ,$$

hence

(5) $$\mathcal{I} = -2\pi a \mu_0^{-1} C\omega J_1(\omega a) \ .$$

Eqs. (2,5) give an explicit relation between H and $\mathcal{I}$, namely

(6) $$J_0(\omega a)\mathcal{I} + 2\pi\omega a \mu_0^{-1} J_1(\omega a) H = 0 \ ,$$

if both $J_0(\omega a)$ and $J_1(\omega a)$ are $\neq 0$. However, from eqs. (1,2) we see that $\psi(r)$ can be < 0 in $0 \leq r < a$ if and only if $|\omega| a < j_1 \equiv$ the 1st zero of J_1; hence $J_1(\omega a) \neq 0$ automatically, and $h(r) > 0$ in $0 < r \leq a$, as it must be. In conclusion, we shall require $|\omega| a < j_1$ and $|\omega| a \neq j_0$. If the second condition is not met, H must vanish to obtain a non-trivial solution; then ψ is determined by giving $\mathcal{I}$, according to

$$\psi(r) = -\frac{\mathcal{I} J_0(\omega r)\mu_0}{2\pi j_0 J_1(j_0)} \qquad (J_1(j_0) \sim 0.52) \ .$$

If $j_0 < |\omega| a < j_1$, the reversal of f may occur for U sufficiently small, since

$$f(0) = \frac{f^0}{J_0(\omega a)} + \left(1 - \frac{1}{J_0(\omega a)}\right) \frac{U\mu_0}{\eta\omega} \ ,$$

and $J_0(\omega a) < 0$.

Some important quantities can be immediately computed, e.g.:

i) the RFP "functional" parameters $F \doteq \frac{f^0}{((f))}$ and $\Theta \doteq \frac{h(a)}{((f))}$, with $((\cdot)) \equiv$ linear average over the plasma section. In fact,

$$((f)) = \frac{\mathcal{I}\mu_0}{\pi a}\left[\frac{1}{\omega a} - \frac{J_0(\omega a)}{2J_1(\omega a)}\right] \quad \text{and} \quad 2\pi a h(a) = \mathcal{I}\mu_0 \ ;$$

ii) the total magnetic energy per unit axial length

$$W = \frac{1}{2\mu_0} 2\pi \int_0^a \mathrm{d}\, r\, r |B|^2 = \frac{C^2\pi}{\mu_0}\int_0^{\omega a} \mathrm{d}\, x\, x [J_0^2(x) + J_1^2(x)] =$$
$$= \frac{C^2\pi}{\mu_0}\left\{x^2\left[J_0^2(x) + J_1^2(x) - \frac{J_0(x)J_1(x)}{x}\right]\right\}_{x=\omega a} \ ;$$

iii) the MF abnormality, or

$$\mu(r) \doteq \mu_0 \frac{J\cdot B}{|B|^2}(r) = \omega - \mu_0 \left(\frac{f p_\psi}{|B|^2}\right)(r) \ ,$$

which is $\simeq \omega$ for sufficiently small U;

iv) finally, a little algebra proves that $U^2 \leq \alpha^2|B|^2$ becomes in this case

$$2UJ_0(\omega r) \leq \alpha\omega C[J_0^2(\omega r) + J_1^2(\omega r)] \ ;$$

as we know, $\alpha(J\cdot B)(r) \geq 0$ is ensured under this condition.

We stress that the linear problem we have just described has three free parameters, e.g. ω, H and f^0, with the latter playing the role of a normalization. If desired, ω and H can be expressed as functions of (Θ, F) (for given f^0, and if $U \neq 0$), through

$$\frac{1}{\Theta} = \frac{2}{\omega a} - \frac{J_0}{J_1}(\omega a) \quad , \quad \frac{\Theta f^0}{F} = -H\omega\frac{J_1}{J_0}(\omega a) \ .$$

This is no longer possible if $U = 0$: then $H\omega = -f^0$, and the second equation reduces to an identity in force of eqs. (3,4) at $r = a$. Indeed, a Taylor's state ($U = 0$) has only two parameters free, abnormality and normalization, and the latter cannot be specified through (Θ, F).

No difficulties are encountered to solve the two-region problem. For given radius of the shell b (a is unknown, but $< b$), we get $\psi(r)$ from eq. (1) for $r \leq a$, and $\psi(r) = \frac{\mathcal{I}}{2\pi}\ln\frac{r}{a}$ for $r \geq 0$. It is immediate to check that both ψ and $\mathrm{d}_r\psi$ are continuous at $r = a$. If $\mathcal{I}$ is given, a is found from eq. (6). If instead $\Psi \equiv \psi(b)$ is given, a is found by combining eq. (6) with $\Psi = \frac{\mathcal{I}}{2\pi}\ln\frac{b}{a}$.

In both the one- and two-region problems, $p - p^0 = \frac{U}{\eta}\psi$ in the plasma. As for the plasma temperature, assuming constant $\lambda_\perp$ one finds:

$$T(r) - T^0 = \frac{U}{\lambda_\perp \mu_0}\psi(r) \ . \qquad (7)$$

5. Loop-Voltage Anomaly and "Antidynamo" Effect

Till now, we have worked with $\mathcal{E} = \alpha B$, or ansatz (2.1″), $\alpha =$ a scalar, homogeneous of degree 0 w.r.t. B and of degree 1 w.r.t. η. As we shall show in Sect. 7, however, to get reasonable numerical profiles of our unknowns (in the circular, one-region problem we are concerned with), we have to choose such $|\alpha|$ and $|U|$ that $|U|$ be a relatively small fraction of a typical $|\alpha B_z|$, and this $|U|$ might turn out too small if compared with the experimental one, unless the resistivity is sufficiently increased above the Spitzer's value[7]. This fact shows the same trend as the well-known (and so-called) "Loop-Voltage Anomaly" (experimental $|U|$ significantly larger than the roughest estimate $|U| \sim \eta\mathcal{I}/(\pi a^2)$, $\eta \equiv \eta_{\text{spitzer}}$) and suggests that the introduction of $\mathcal{E}$ through ansatz (2.1″) is still not adequate.

With $\mathcal{E}$ expressed by ansatz (2.1″), independently of the sign of α, αf and $-U$ play the same role, in the z-component of the Ohm's law and in the bulk of the plasma, of driving a (positive) J_z; and αf dominates. On the other hand, both 3D-numerical simulations [10] and experiments [11] suggest that an "antidynamo" EF tends to *decrease* J_z near the axis. Also, it is commonly accepted, and it has been proved under a number of severe restrictions/conjectures[8] [7], that in the circular, one-region problem

$$\int_{\Omega} d_2 x \mathcal{E} \cdot \overline{B} \simeq O(\varepsilon) \tag{1}$$

[7]Due to its amount, hardly the last effect can be attributed to the fact that the resistivity tensor we are working with be the sum of the real resistivity tensor and $-\mu_0\beta_0 g_{ij}$, see Sect 2; not to mention, anyhow, that no trustworthy information is available about β_0, and not even about its sign.

[8]Among the restrictions, here we mention the requirements that η be scalar and constant, that ρ (mass density) be similarly constant, and that the plasma be in contact with a fully (i.e. in every direction) ideally conductive shell.

$$(2) \qquad \int_\Omega d_2x\left(\mathcal{E}\cdot\overline{J}+\eta\overline{\overline{J^2}}\right)\simeq O(\varepsilon)$$

where $\varepsilon \doteq l/a$, $l \equiv$ a typical fluctuation wave-length, as $\varepsilon \to 0$.

Giving the above relations (1,2) the trust they deserve, if we put $\alpha\overline{B}$ in place of $\mathcal{E}$ in them, and make $\varepsilon \to 0$, we immediately see that:

i) α cannot have constant sign along the radius because of (1);

ii) $\int_\Omega d_2x\alpha\overline{B}\cdot\overline{J}$ should be negative because of (2), whereas we know that (writing $\overline{\overline{\mu}}$ for $\mu_0\overline{J}\cdot\overline{B}/|\overline{B}|^2$) $\alpha\overline{B}\cdot\overline{J}=\alpha\overline{\overline{\mu}}|\overline{B}|^2/\mu_0 \simeq \frac{\alpha^2}{\eta}|\overline{B}|^2>0$ for small $\overline{p}_\psi$

All of the above prompts us to look for, and to make use of, a more refined dynamo model than that expressed by ansatz (2.1″). As we already mentioned in Sect 2, in the same reference [7] an almost completely explicit expression (2.1‴) of $\mathcal{E}$ has been worked out. Apart from unspecified numerical factors, the γ and the ν of the above ansatz are given as functions of $\overline{B}$ and its r-derivatives *up to the 3rd order*[9], homogeneous of degree 0 w.r.t. $\overline{B}$ and proportional to η (assumed isotropic and constant). When α is a tensor, our basic system (3.(2_3, 1, 8, 9)) is simply generalized by replacing $\langle\alpha : BB\rangle$ for $\langle\alpha|B|^2\rangle$ in eq. (3.8) and, respectively, $\langle\alpha : BB/|B|^2\rangle$ for $\langle\alpha\rangle$ in eq. (3.9). This will result in a more general ODS than syst. (3.16, 3.17), say syst. (3.16′, 3.17′), under circular symmetry. By introducing the B-dependence (as described above) in the γ and ν occurring in this system through ansatz (2.1‴), we end up with a *1st-order* ODS with four more unknowns than before. In conclusion, apart from critical situations, we need four more end-point conditions to get a unique solution; and this raises new difficulties, especially in our model, where the shell is assumed insulating along z. Anyhow, starting with ansatz (2.1‴), and dropping overbars from now on for brevity, we

[9] It can be directly recognized that this is a *substantial* dependence.

find $\mathcal{E} \cdot B = \gamma |B|^2$, which implies that γ cannot have constant sign along the radius (again due to (1)). Furthermore, this implies that

$$\mathcal{E} \cdot J = \gamma J \cdot B + \nu (J \times B)_r = \gamma \mu |B|^2 + \nu \, \mathrm{d}_r p \ ,$$

which *can* be negative in the section-average.

Having renounced the upgrading of our OD problem in the sense illustrated above, (at least, in the present study) we propose an alternative, much simpler ansatz for the $\mathcal{E}$ of the same cylindrical-circular problem, given by:

$$(3_1) \qquad \mathcal{E}_\theta = \alpha_\theta B_\theta \qquad\qquad (3_2) \qquad -\mathcal{E}_z = \alpha_z B_z$$

with $(\alpha_\theta, \alpha_z)$ being prescribed, under $\alpha_\theta, \alpha_z > 0$ as the α of $(2.1'')$; namely, in the simplest case, as constants. We stress that, differently from ansatz $(2.1'')$ - which is essentially based on the turbulence isotropy - and from ansatz $(2.1''')$ - which follows from a relatively sophisticated physical model - ansatz (3) is a purely ad-hoc expedient whose purpose is that of (hopefully) remedying the difficulties we have just illustrated. We also stress that the α's corresponding to the three ansätze of above are definitely different object from the tensor-algebra standpoint. Indeed, ansatz $(2.1'')$ corresponds to a spherical tensor α; ansatz $(2.1''')$ with $\nu > 0$ corresponds to a tensor equal to the sum of a spherical tensor and of a (non vanishing) antisymmetric tensor with prescribed sign; and finally, our ansatz $(3_{1,2})$ corresponds to a symmetric (generally non-rectangular) hyperbolic tensor α. It goes without saying that a symmetric-hyperbolic α is definitely in contrast with isotropy.

Anyway, by use of ansatz (3) we get

i) $\mathcal{E} \cdot B = \alpha_\theta B_\theta^2 - \alpha_z B_z^2$, and this *can* change sign along r (because $B_\theta \to 0$ for $r \to 0$, and B_z becomes small for $r \to a$);

ii) $\mathcal{E}\cdot J = \alpha_\theta B_\theta J_\theta - \alpha_z B_z J_z$, and this *can* be negative in the average, because $B_\theta > 0$ (with the usual orientation of (θ, z)), $\mu_0 J_\theta = -\mu_0 \, \mathrm{d}_r f > 0$, B_z is prevailingly > 0 if $\alpha_\theta > 0$, and J_z is prevailingly > 0.

It is worthwhile to point out that the standard case can easily be generalized when the (tensor) α changes from spherical to (generally non-rectangular) hyperbolic. The main difference is that *modified* Bessel functions replace the standard ones in the final formulae, e.g. to give (compare eq. (4.1)):

$$\psi(r) = C I_0(\omega r) - K \ , \quad \omega \doteq \frac{\mu_0}{\eta}\sqrt{\alpha_\theta \alpha_z} \ , \quad -K\omega^2 \doteq \frac{\mu_0}{\eta}(U + \alpha_z f^0) \ .$$

Similarly, we have:

$$\mathcal{I} = 2\pi a C \mu_0^{-1} I_1(\omega a)$$

(compare eq. (4.5)),

$$f(r) = -\alpha_\theta \mu_0 \psi(r)/\eta + f^0 = -C I_0(\omega r)\alpha_\theta \mu_0/\eta - U/\alpha_z$$

(compare eq. (4.3)), and so on.

We shall show some numerical results obtained by working with ansatz $(3_{1,2})$ (and conveniently chosen $(\alpha_\theta, \alpha_z)$) in Sect 7.

6. Additional Remarks

a) $\alpha = 0$ is not a singular limit-value for the (linear) standard-case problem. This is quite evident from the nature of the differential equations, and can directly be checked (at the cost of a little annoying work with the Bessel function expansions for small argument), by taking the limit for $\alpha \to 0$ of the solution. The parabolic profile of ψ is thus recovered.

b) In this study, we have left aside every question concerning the stability of the computed stationary states. However, one expects they can be stable against perturbations which leave unvaried the helicity if $|\alpha B| \gg$ $\gg |E + v \times B|$.

c) An important limitation of the present model is that it does not take into account any (given) plasma source/sink, which would lead to a mass conservation equation of type $\nabla \cdot (\rho v) = \mathcal{U}$ ($\equiv$ plasma source if positive). Should the momentum balance keep the standard MFS structure in spite of $\mathcal{U} \neq 0$, one could hope to conveniently extend the present treatment to cover this case.

d) In our model, ∂C has been assumed insulating along z. As it was already mentioned, this simulates the presence of one or more meridional cuts of the real (toroidal) machine. If toroidal (in our cylindrical approximation, axial) cuts exist, it can be shown that eq. (2.4) must be modified according to (for a *single* cut):

$$I^{-1} Q = -U J_z - \frac{f_\psi}{\mu_0} I^{-1} V \ , \tag{1}$$

where $V \doteq$ potential jump through the cut, positive if V increases counter-clockwise. Note that $f_\psi < 0$, and so, if $V > 0$, the second term cooperates with the first one, leading to a larger $\lambda_\perp$(to keep the temperature peacking more or less unvaried).

From a topological point of view, the existence of an axial cut is equivalent to "open a hole" in Ω, about the magnetic axis. To our knowledge, it is still ignored how the helicity definition must be modified (w.r.t. its usual "upgraded" definition for a "solid" torus [12]) for this topology of the domain of interest, in order to keep gauge-invariance. As it is clear from eq. (1), no difficulty is met in our model.

e) Once our problem has been solved for (ψ, f, p, T) under the convenient interface and boundary conditions, the remaining unknowns $(\varphi, v, q_{\|})$ are determined only incompletely. Precisely, one finds

$$\begin{aligned}
\varphi(\psi, l) &= \varphi_0(\psi, l) + k^{(\varphi)}(\psi) \\
v_{\perp}(\psi, l) &= v_{\perp_0}(\psi, l) + C_1(\psi, l) k_{\psi}^{(\varphi)}(\psi) \\
v_{\|}(\psi, l) &= v_{\|_0}(\psi, l) + B\left[k^{(v_{\|})}(\psi) + C_2(\psi, l) k_{\psi}^{(\varphi)}(\psi)\right] \\
q_{\|}(\psi, l) &= q_{\|_0}(\psi, l) + B k^{(q_{\|})}(\psi)
\end{aligned}$$

where $(\varphi_0, v_{\perp_0}, v_{\|_0}, q_{\|_0}, C_1, C_2)$ are known and periodic w.r.t. l, and $k^{(\varphi)}$, $k^{(v_{\|})}$, $k^{(q_{\|})}$, are arbitrary. In particular, one has:

$$\begin{aligned}
C_1 &\doteq |B|^{-2}(f\hat{z} \times \nabla\psi - \hat{z}|\nabla\psi|^2) \quad (C_1 \cdot B = 0 \ , \ \langle \nabla \cdot C_1 \rangle = 0) \\
C_2 &= -\int_0 \frac{\mathrm{d}\,l}{|\nabla\psi|} \nabla \cdot C_1 \ ,
\end{aligned}$$

l oriented counter-clockwise. The above equations have been derived in [13].

7. Numerical Investigations

7.1. Tentative Completion of the Model.

As it has been pointed out, we need to guess at $(\eta_{\|}, \eta_{\perp}, \lambda_{\perp})$ and α (spherical, ansatz (2.1″)) or $(\alpha_\theta, \alpha_z)$ (hyperbolic, ansatz $(5.3_{1,2})$) as functions of $(p, T, |B|)$, and, respectively, as quasi-local functions of (p, T, B), in order to actually use our ODS (3.16, 3.17) or (3.16′, 3.17′). This is a most crucial aspect of the present research, at least until the interest is restricted to the (somewhat naïve) object of achieving "quantitatively precise" results.

To start with, considering the typical operational conditions of an RFP, it seems reasonable to assume

(1)

$\eta_\perp/\eta_\parallel \simeq \text{const} = c$ a few units ([14], since $|B|$ is roughly constant) ,

and

(2) $$\eta_\parallel \simeq \eta_0 T^{s''} \quad , \quad \eta_0 \equiv \text{const}$$

(a typical RFP-plasma resistivity), with $s'' = -3/2$, according to Spitzer's formula. This is suggested by several measurements [15,16], even though they are partly hidden by the loop-voltage anomaly (in its turn, supposedly due to either MFD- or kinetic-dynamo processes). Also, the independence of p, in $\eta_\parallel$, fits with the typically low-β character of the RFP equilibria.

As for $\lambda_\perp$, there are only indirect indications, based on estimations of either the energy confinement time [17] or the energy flux at the plasma edge [18]. They suggest that $\lambda_\perp$ be substantially determined by stochastic-transport mechanisms. According to that, $\lambda_\perp$ should roughly be proportional to $L_{\text{corr}} v_{\text{th}} \overline{(\tilde{B}/\overline{B})^2}$, where $L_{\text{corr}} \equiv$ a correlation length (of the order of a in an RFP), and $v_{\text{th}} \equiv$ thermal velocity in the plasma. Again neglecting the dependence of $\lambda_\perp$ on p and $|B|$ (for similar reasons as before), one expects that a scaling

$$\lambda_\perp \simeq \lambda_0 T^s \quad ,$$

$\lambda_0 \doteq \text{const}$ (a typical RFP (perpendicular) thermal conductivity), should have s roughly in the range $(-1, 1/2)$, where the upper limit would correspond to $\overline{(\tilde{B}/\overline{B})^2}$ independent of T (for example according to 3D numerical simulations [19]) and the lower one would presuppose the most favourable scaling $\overline{(\tilde{B}/\overline{B})^2} \propto T^{-3/2}$ [20].

Even less is known about α (isotropic case): in practice, neither direct or indirect measurements exist to our knowledge. Still in force of the low-β nature of the RFP equilibria, we can neglect the dependence on p; and so, limiting our attention to the dependence on T, the scaling

$$(4) \qquad \alpha \simeq \alpha_0 T^{s'} \quad , \quad \alpha_0 = \text{const} \quad ,$$

should have $s' \gtrsim -3/2$. In fact, we know from the standard case that, for an almost force-free discharge, the abnormality μ is $\simeq \alpha\mu_0/\eta$; but the experimental μ is a slowly decreasing function of r, and the experimental T is also decreasing (more or less parabolically) with r; hence μ should be a (slowly) increasing function of T, or $s' \gtrsim -3/2$. As for the typical λ_0 and α_0, they can be evaluated as $\lambda_0 \simeq \frac{\eta_0|B_0|^2}{\mu_0^2 T_0^2}$, and respectively as $|\alpha_0| \simeq \frac{\eta_0}{a\mu_0}$, where (B_0, T_0) is a typical (MF, temperature), say on axis. As we know, choosing a positive, rather than negative, α is not against generality. Finally, in the case of the hyperbolic tensor $(\alpha_\theta, \alpha_z)$ (again, the choice $\alpha_\theta > 0, \alpha_z > 0$ is not against generality), we can assume them to depend on T like α, and to begin with, to be constant.

7.2. Some Introductory Remarks.

Accepting the (monomial) scaling scheme as described above, a specific solution will be completely identified by a relatively small number of parameters. With no loss of generality, we can always eliminate a from that set by taking it as unit length. The remaining parameters can be grouped in two classes:

i) $(U < 0, f^0, p^0 > 0, T^0 > 0) \equiv$ "operational" parameters. With our assumptions, however, p^0 can be taken $= 0$. Furthermore, in the light of eq. (3.(8,9)), we see that, choosing a positive α, $f(0)$ is positive too; hence, at least for computational purposes, it can be more convenient (even though less significant from the physical standpoint) to take this $f(0)$ as operational parameter in place of f^0 (this will save us a shooting run). Again to save shootings, a similar choice can be (and sometimes has been) made with reference to T.

ii) $(\eta_0, c, s'', \lambda_0, s, \alpha_0, s')$ (for instance with reference to the isotropic-case) $\equiv$ "physical" parameters.

Plainly, (f^0, T^0) fix end-point conditions for (f, T), whereas U, and the full set of the physical parameters, occur *substantially* in our ODS, i.e. they, and all of them independently (in general), do specify the differential laws the unknowns must fulfil.

If we consider the situation from Taylor's theory point of view ($U = 0, \alpha = \text{const}\,, \eta_{\|} = \eta_{\perp} = \eta_* = \text{const}$), we know that in this case the MF (f, h) is solution to a linear, homogeneous ODS (three 1st-order equations) which is completely specified by the *single* parameter ω; hence, on excluding criticality, (f, h) is completely determined by this ω and by a normalization, say, by (ω, f^0). For given f^0, there is a one-to-one relation between ω and the (gauge invariant) helicity K for unit axial length. In fact, the direct $(\omega \mapsto K)$ functional nature of this relation is self-evident, whereas the indirect one $(K \mapsto \omega)$ can easily be proved on the basis that the extremal MF (with that K, and for which $\delta W = 0$), corresponds to the *absolute minimum* of W[10].

According to our model, instead, and always excluding criticality, for given (f^0, T^0) the prescription of K specifies an m-dimensional variety

[10] Alternatively, one could explicitly compute $K(\omega)$ for the given f^0, showing that $d_\omega K$ never vanishes.

V_K in the $(m+1)$-dimensional space of the m physical parameters and U (in our present case $m=7$); hence, a corresponding "variety" of (f,h,T) stationary states (out of the class of all the (f,h,T) fields with the same (f^0,T^0)). In particular, putting $U=0$, $c=1$, $s=s'=s''=0$, $\alpha_0\mu_0/\eta_0 = \omega(K)$ in this V_K, we obtain $(f,h)\equiv$ Taylor's state $(f^0,\omega(K))$ for any λ_0 (the temperature problem decouples in this case), in correspondence to the least W compatible with f^0 and K. We stress that all points of V_K correspond to a genuine (f,h,T) stationary state according to our model, even though only those of them with (f,h) equal to Taylor's state $(f^0,\omega(K))$ do reach the minimum W. Of course one expects that, the smaller the W of a (f,h,T) stationary state in V_K (for the given (f^0,T^0)), the closer that state will be to Taylor's state $(f^0,\omega(K))$. An example of a 1-dimensional sub-variety V_K is shown in fig. (19), where the $(m+1)$-dimensional parameter space has been reduced to the plane (α_0,λ_0).

One more remark is in order. We know that, in the standard case with $U\neq 0$, two of the three configurational parameters, for instance ω and H, can uniquely be expressed in terms of the functional parameters (Θ,F). We can do the same in our more general model, but, of course, the plane (Θ,F) will be one-to-one (at least, locally) with a 2-dimensional variety in the $(m+1)$-dimensional parameter space. We shall show some examples of this possibility, where we have released the parameter pair (λ_0,α_0). This can be of help, because the information about λ_0, and even more that about α_0, is poor, whereas the experimental (Θ,F) are rather well known (due to the easiness of their measurement). This is just an instance of what has been suggested in the end of Sect 1.

7.3. Numerical Results.

In order to numerically integrate our ODS (3.16, 3.17) (or (3.16′, 3.17)) in the one-region configuration, first of all we shall introduce dimensionless (cap) quantities based on the following set of "reference" physical quantities:

$a \equiv$ plasma radius

$\eta_0 \equiv$ a reference resistivity (as a rule, the on-axis resistivity)

$B_0 \equiv$ a reference MF (nearly always, the on-axis MF)

$T_0 \equiv$ a reference temperature (either the wall or on-axis temperature).

For Spitzer resistivity, $\eta_0 = RZ_{\text{eff}}T_{\text{axis}}^{-3/2}$, where R is a constant with dimensions (length)^6 (mass)$^{5/2}$ (electric charge)$^{-2}$ (time)$^{-4}$, and equal to $\simeq 3.10^{-2}\sqrt{m_e}e^2/\varepsilon_0^2$ (all symbols having their usual meaning, and with T in energy units).

Dimensionless variables are defined as follows:

$$\hat{r} \doteq r/a \ , \quad (\hat{f},\hat{h}) \doteq (f,h)/|B_0| \ , \quad \hat{\eta} \doteq \eta/\eta_0 \ , \quad \hat{T} \doteq T/T_0 \ ,$$

$$\hat{\psi} \doteq \psi/(a|B_0|) \ , \quad (\hat{U},\hat{\mathcal{E}}) \doteq (U,\mathcal{E})a\mu_0/(\eta_0|B_0|) \ , \quad \hat{J} \doteq \mu_0 Ja/|B_0| \ ,$$

$$\hat{p} = p\mu_0/|B_0|^2, \ \ \hat{\lambda}_\perp \doteq \lambda_\perp \mu_0^2 T_0/(\eta_0|B_0|^2), \ \ \hat{\alpha} \doteq \alpha\mu_0 a/\eta_0, \ \ (\hat{\mu},\hat{\omega}) \doteq (\mu,\omega)a \ .$$

Rewritten in terms of these dimensionless variables, the ODS is integrated by means of a standard 4th-order Runge-Kutta (and shooting) routine, under the six end-points conditions:

$$\hat{h}(0) = 0 \ , \quad \hat{\delta}(0) = 0 \ , \quad \hat{\psi}(1) = 0 \ , \quad \hat{p}(1) = 0 \ ,$$

$\hat{f}(1) \equiv$ given (or $\hat{f}(0) \equiv$ given > 0), $\hat{T}(1) \equiv$ given > 0 (or $\hat{T}(0) \equiv$ given > 0).

We shall show some of our numerical findings in the following of this Section. (For the sake of typographical convenience we shall neglet caps to denote dimensionless quantities).

To begin with, we have reproduced the analytical solution of a standard case. This is shown in fig. (1) for $-U = 0.1$, $\omega = 2.6$, $f(0) = 1$. Triangles denote the analytical solution.

As a matter of fact, we have used a Spitzer-like resistivity ($s'' = -\frac{3}{2}$) and $\eta_0 = 1$, $c = 2$ in all of the numerical computations whose results are reported in this study. These results have been grouped in four sets, namely:

i) families of profiles (of $B, J, T, \mu \dots$ etc) for isotropic α;

ii) families of profiles for non-isotropic (hyperbolic) α;

iii) curves $\lambda_0 =$ const, with α_0 (or α_{θ_0}) varying along them, in the (Θ, F)-plane;

iv) special curves.

The various cases are specified and commented upon as follows.

i) (isotropic α)

fig.	λ_0	s	α_0	s'	$-U$	$f(0)$
2	0.38	0	5.6,6.0	0	0.8	1
3	0.12	0	2.4,2.6	0	0.25	1
4	0.32	0	2.2,2.6	0	0.5	1
5	0.34	0	4.2	0	0.8	0.95,1,1.05
6	0.30, 0.34, 0.38	0	4.6	0	0.8	1
7	0.36	-1	2.4,2.6	-1	0.8	1
8	0.15	-1	3.6,4.2	-1	0.8	1

In all cases of above $T(1) = 0.1$. In particular, figs. (5,6) show the profile sensitivity w.r.t. variations of $f(0)$ and, respectively, of λ_0. Arrows point out direction of the *increasing* parameter.

ii) (hyperbolic α)

fig.	λ_0	s	α_{θ_0}	α_{z_0}	s'
9	3	0	4.5	3.0	0
10	2.5	0	5.2	comp.	0
11	2.5	-1	3.5	comp.	-1
12	2.6	0	3.0	2	-1

In all cases of above, $-U = 5.0$, $f(0) = 1$, $T(0) = 1$. In case (9) we find a non-monotonic decrease of the pressure; case (12) avoids this occurrence by means of a different parameter choice. In cases (10,11), instead, the same phenomenon (which is clearly due to our ignorance about $(\alpha_{\theta_0}, \alpha_{z_0})$) has been eschewed in a more radical fashion, namely by determining α_{z_0}, for the given α_{θ_0}, so that the resulting pressure profile best-fits a reasonable $1 - r^6$. As one can see, both U and λ_0 increase by a typical factor $5 \div 10$ w.r.t. the values obtained in the previous group i).

iii)

fig.	s	s'	$-U$
13	0	0	0.8
14	-1	-1	0.8
15	0	0	5
16	-1	-1	5

Here, $f(0) = 1$, $T(0) = 1$.

Cases (13,14) have isotropic α, with α_0 varying along the curves parametrized in λ_0; cases (15,16) have hyperbolic α, with α_{θ_0} varying along them (again parametrized in λ_0), and α_{z_0} determined consistently with the above requirement on the pressure profile. In all cases, most experimental (Θ, F)-points (for various machines and operational conditions) fall in the shaded area.

iv)

Fig. (17): this shows three curves in the (Θ, F)-plane for isotropic α, (with α_0 varying along the curves), parametrized in $s' = (-1, -0.5, 0)$, and for fixed $\lambda_0 = 0.38$, $-U = 0.8$, $f(0) = 1$, $T(1) = 0.1$.

Fig. (18): this shows two curves in the (Θ, F)-plane for isotropic α, (with α_0 varying along the curves), with $\lambda_0 = 0.34$, $s = -0.25$ (dotted) and $\lambda_0 = 0.15$, $s = -1$ (continuous); the other parameters are $s' = -1$, $-U = 0.8$, $f(0) = 1$, $T(1) = 0.1$ for both curves.

Fig. (19): this is a family of $\lambda_0 =$ const-curves, α_0 varying along them, in the (W, K)-plane, for $-U = 0.8$, $f(0) = 1$, $T(0) = 1$, $s = s' = -1$, as mentioned before in this Section. Note that λ_0 decreases with W along $K =$ const.

Fig. (20): the same as fig. (19), but with the role of λ_0 and α_0 interchanged. Note that α_0 increases as W decreases along $K =$ const.

Apart from the smallness of $|U|$ and λ_0, the profiles of group (i) look reasonable enough from the experimental point of view, both for $s = s' = 0$ and $s = s' = -1$. The just mentioned difficulties have been remedied in the profiles of group (ii); but example (9), which exhibits a "hollow" pressure, proves that the parameter choice is critical. This problem has been eliminated in the subsequent examples (12) (which is featured by $s = 0$, $s' = -1$) and (10,11) (constrained pressure profile). Anyway, the fact that ansatz (5.3) has no *independent* justification makes the full set (ii) a little baffling.

8. Conclusions

We have set up a steady-state model of a cylindrical RFP. On the basis of this model, we have recovered the familiar Bessel-function solution (under cylindrical-circular symmetry) for both η and α isotropic and constant and $U = 0$, the MF abnormality being equal to $\alpha\mu_0/\eta$. Our steady states, however, do exist unique (apart from criticality's occurrence) in much more general situations, in practice for any dependence of (η (=tensor), $\lambda_\perp$) on the magnetoplasma state $(p, T, |B|)$ and, respectively, of α (generally, a tensor) on $(p, T, B, \nabla B)$, homogeneous of degree 0 w.r.t. B (we did exclude the dependence of α on higher-order derivatives of B to keep unchanged the differential order of the problem). For scalar α, these states reproduce reasonably well the main experimental findings, apart from a loop-voltage anomaly and a (related) smallness of the perpendicular thermal conductivity. A hyperbolic tensor α, *if carefully selected*, can remove these troubles, but, seemingly, it lacks an independent justification.

As a basic comment, we stress that, if on one hand we have developed a precise macroscopic theory (open to several important, reasonably feasible, refinements, see below), on the other this theory must be "fed" by a number of microscopic parameters whose present knowledge is too poor (especially with reference to the (tensor) α) to allow, or to expect, a really accurate reproduction of specific experimental results.

We leave for future work the consideration of a mass-source term in the mass conservation equation, of possible higher-order (not necessarily "turbulent" in the strict sense) dynamo mechanisms, of the influence of the toroidal cuts of the shell, and, eventually, of the toroidal effects.

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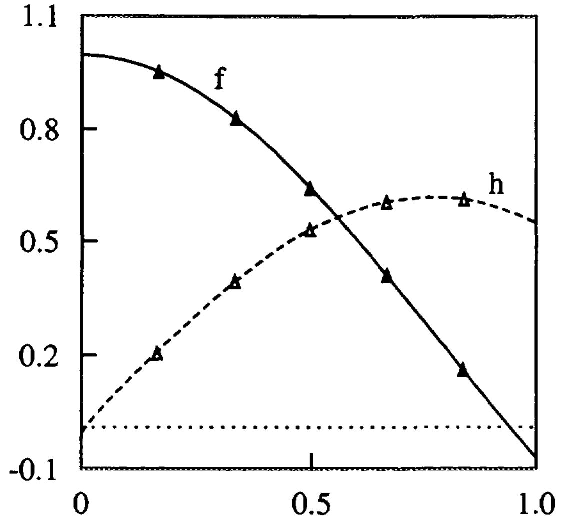
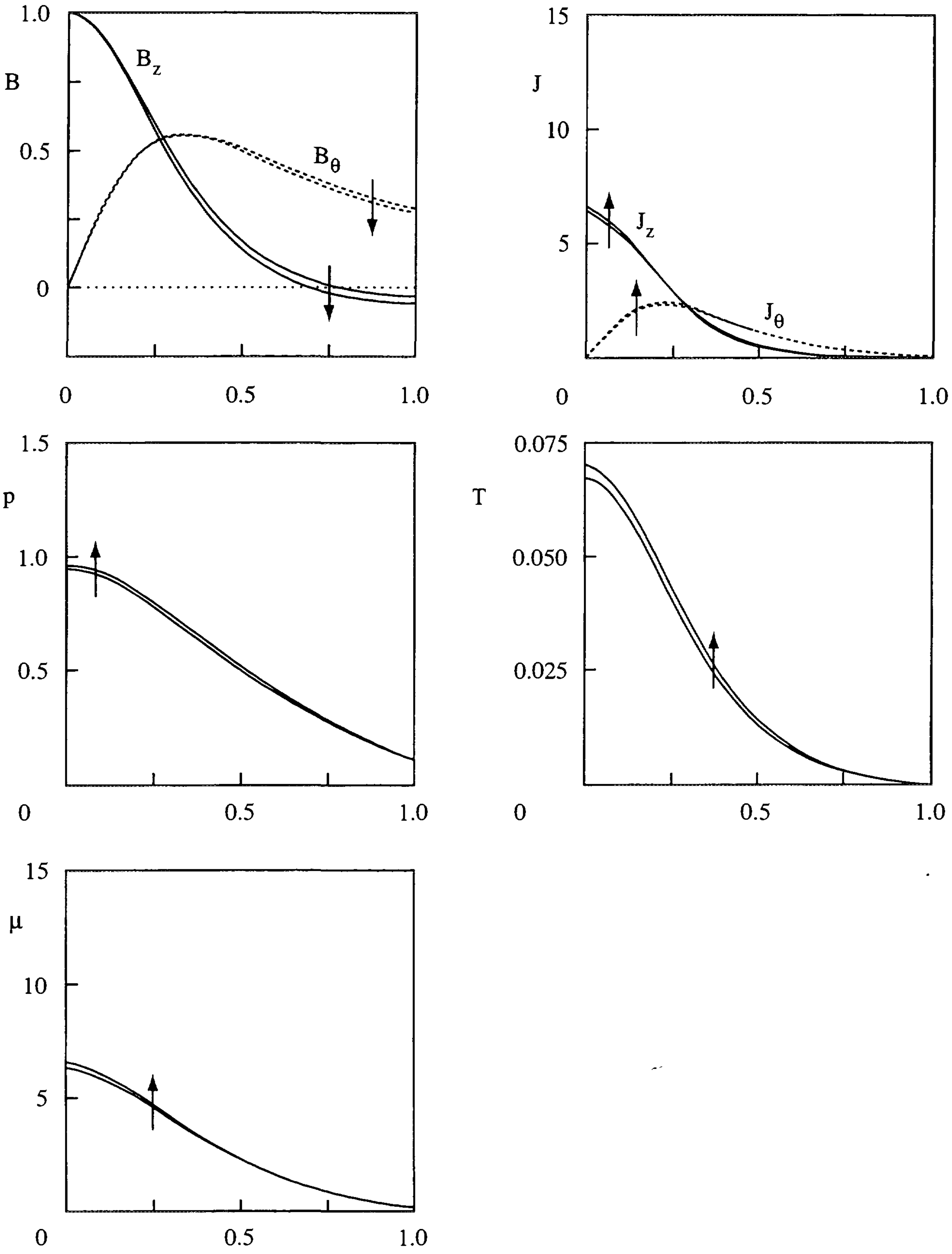


Fig.1



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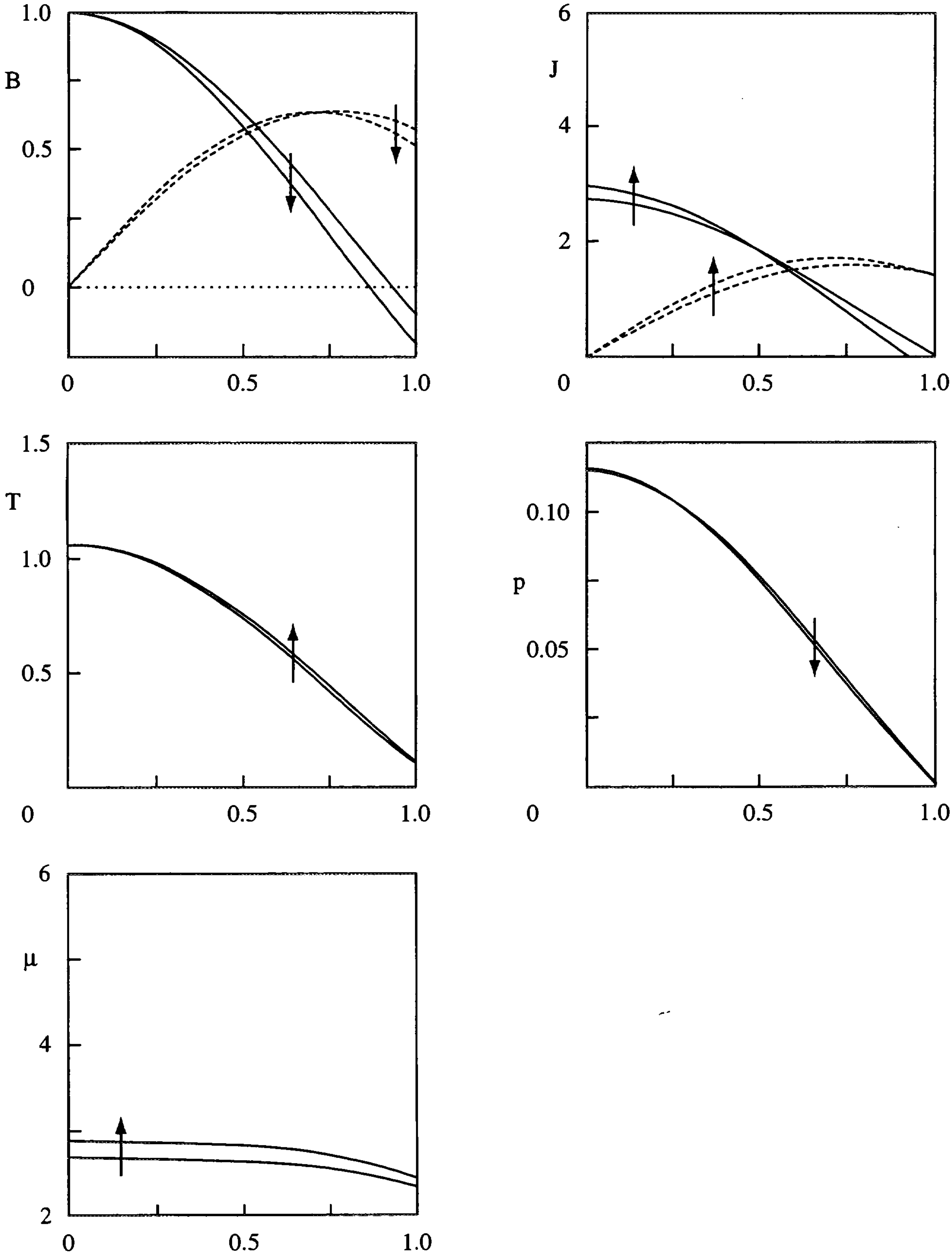

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Fig.3

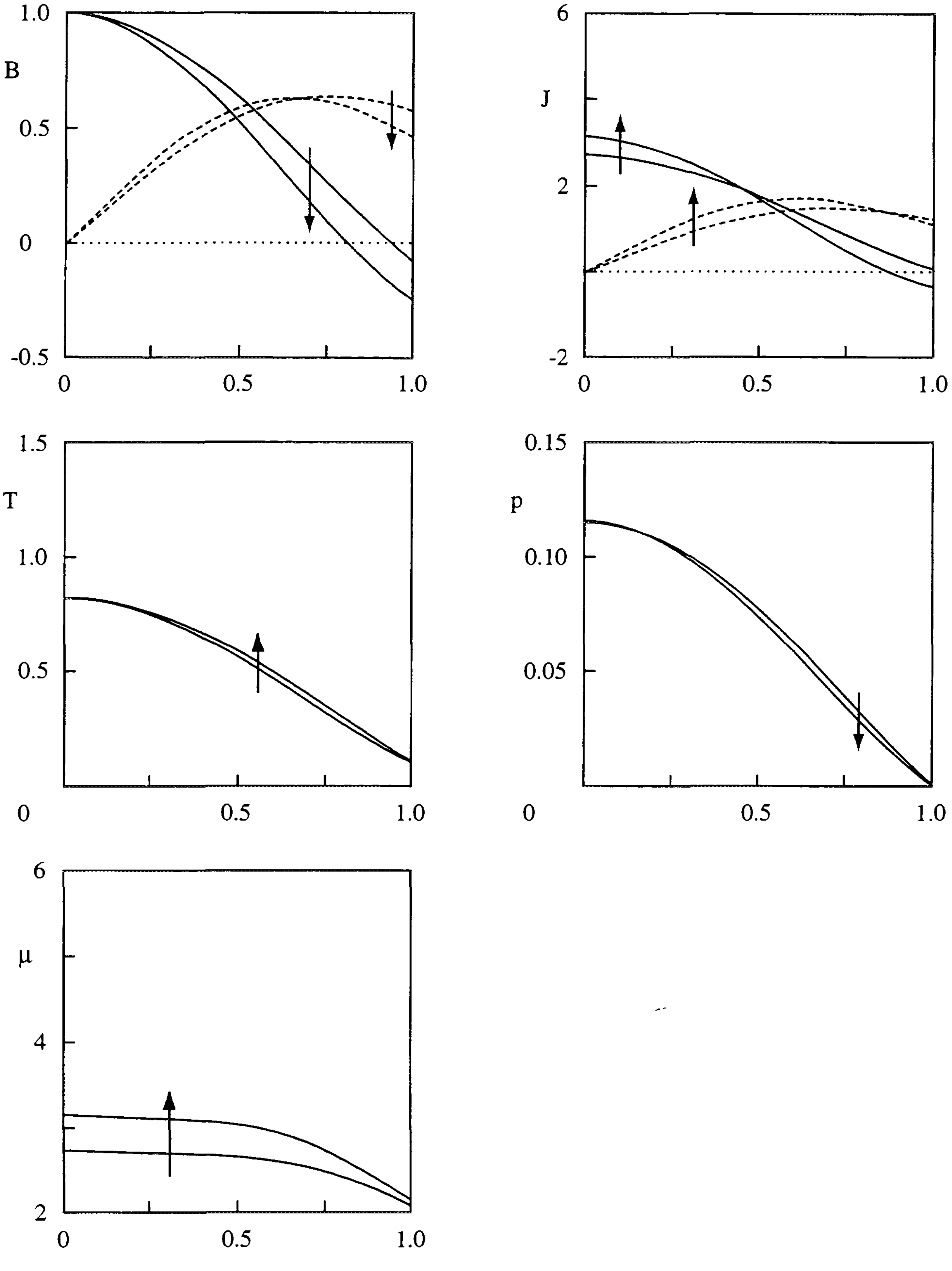
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Fig.4

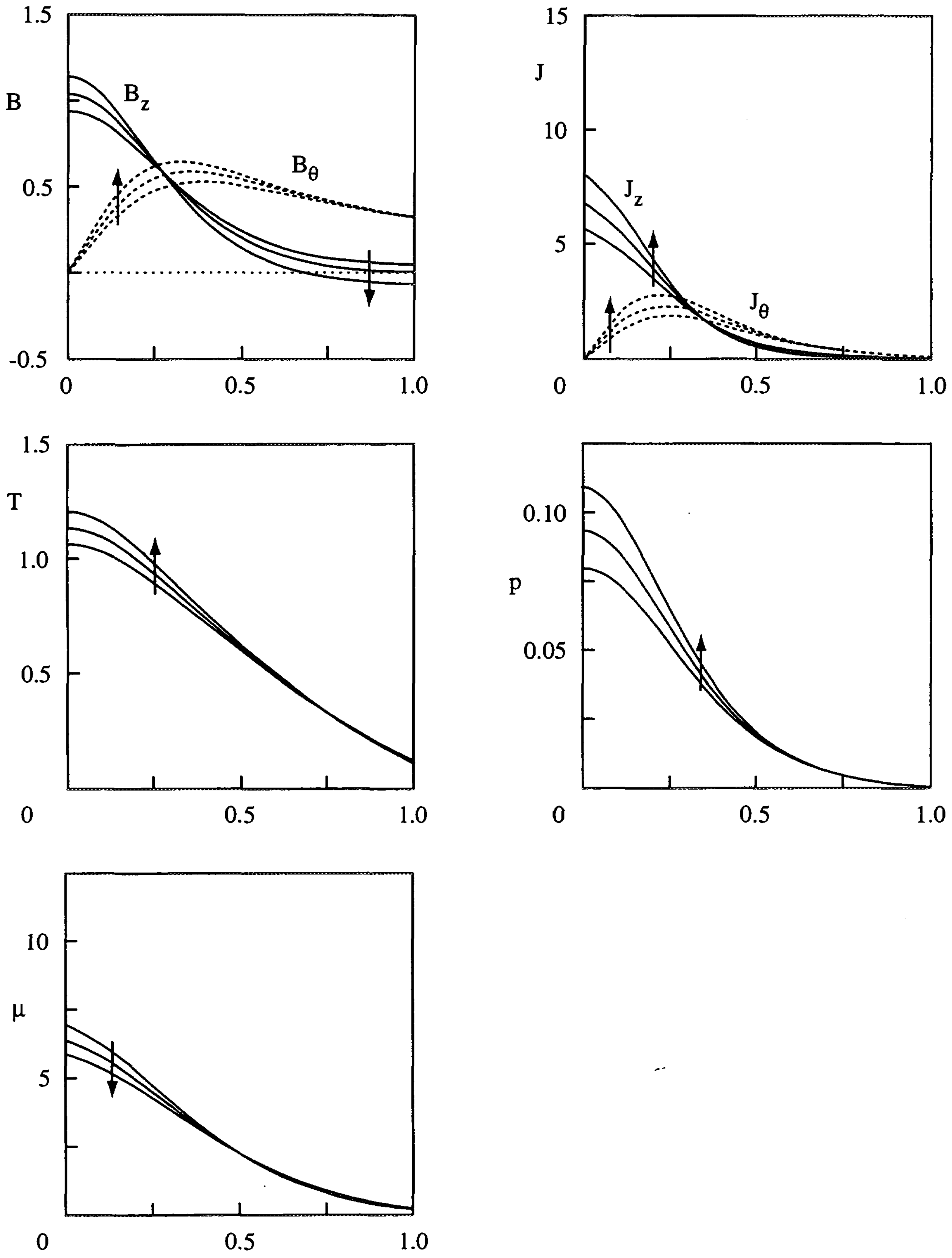

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Fig.5

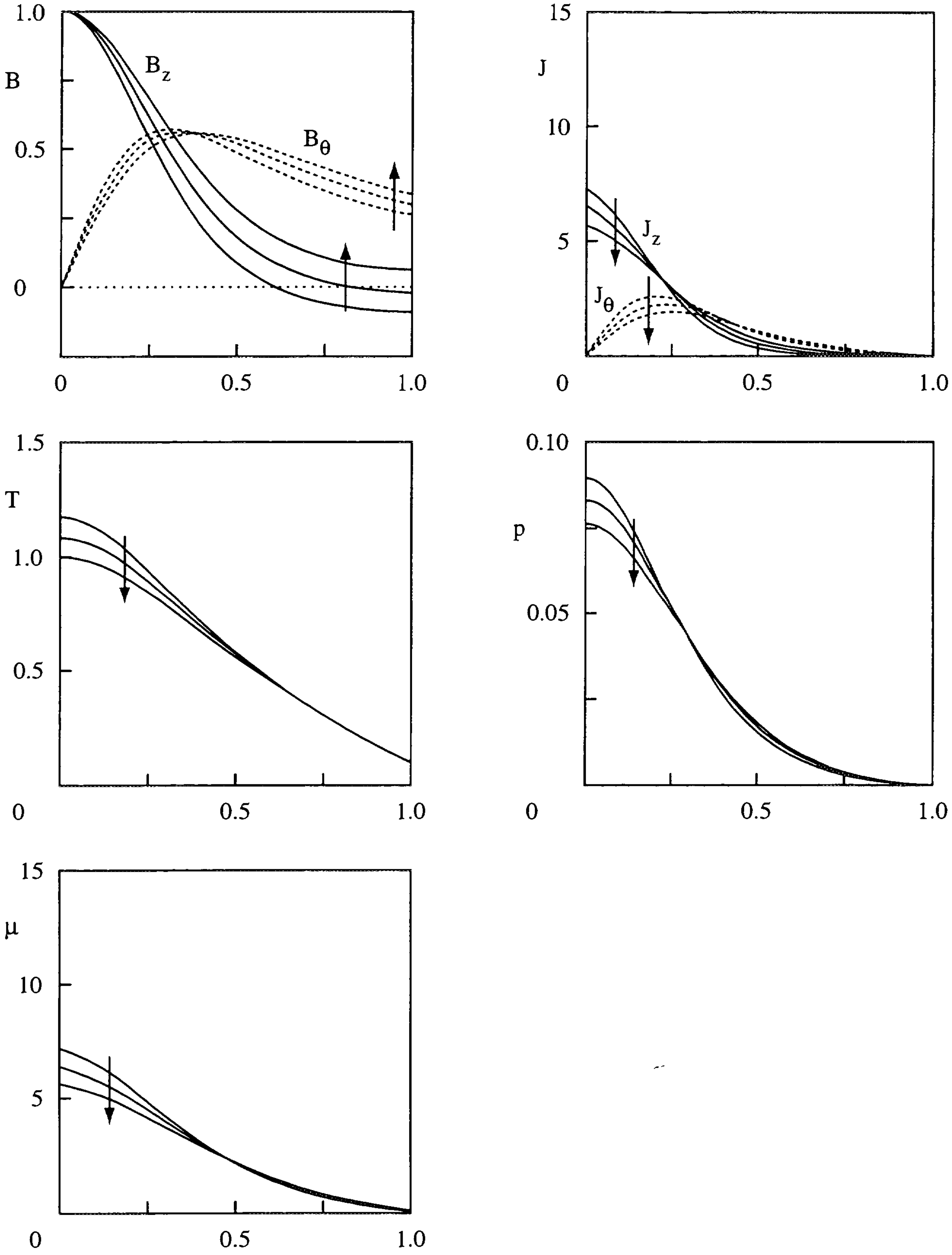

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Fig.6

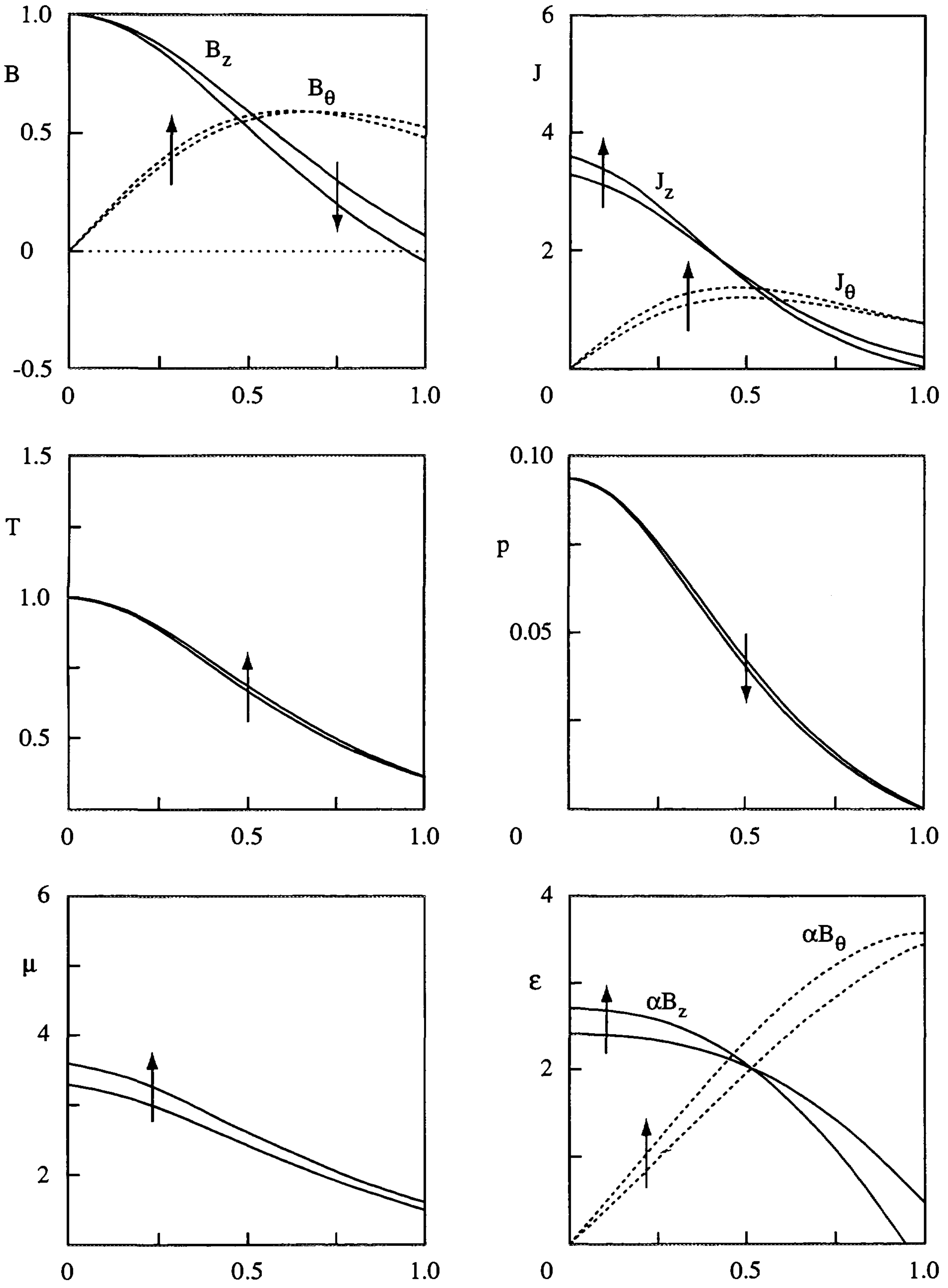

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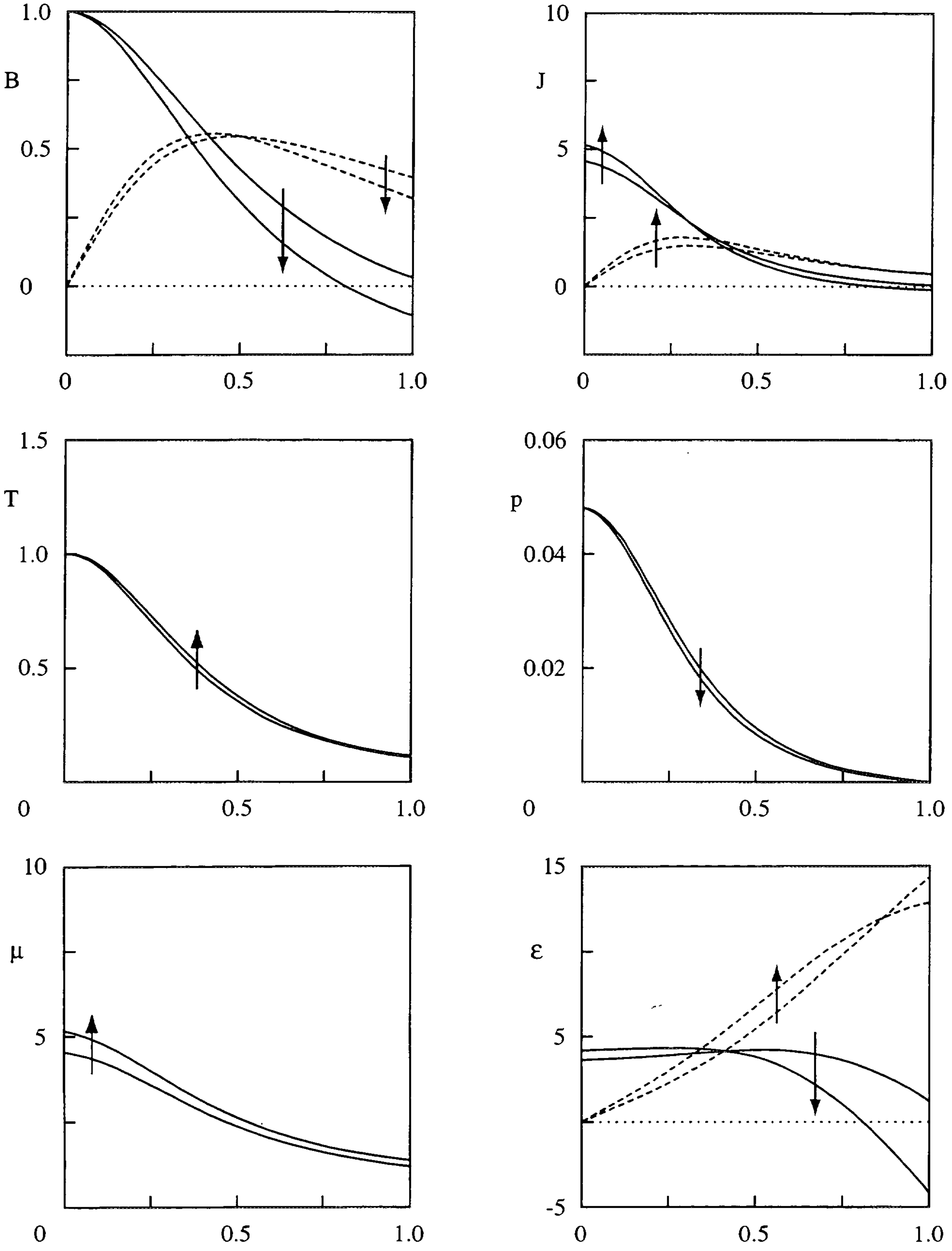
Fig.7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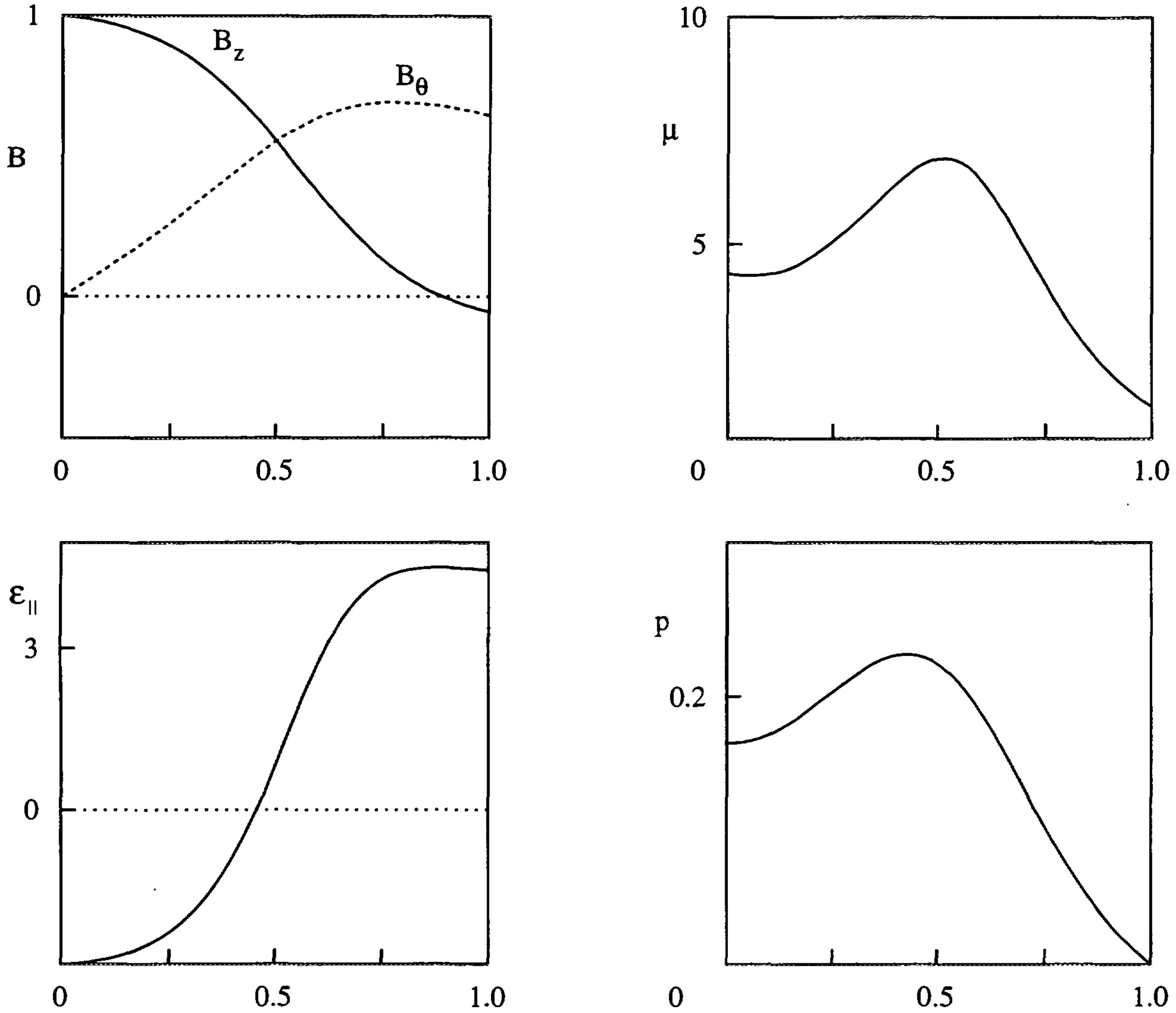
1.0

Fig.8



1
B
0
B_z
B_θ
0
0.5
1.0
10
μ
5
0
0.5
1.0
$\varepsilon_{||}$
3
0
0
0.5
1.0
p
0.2
0
0.5
1.0

Fig.9

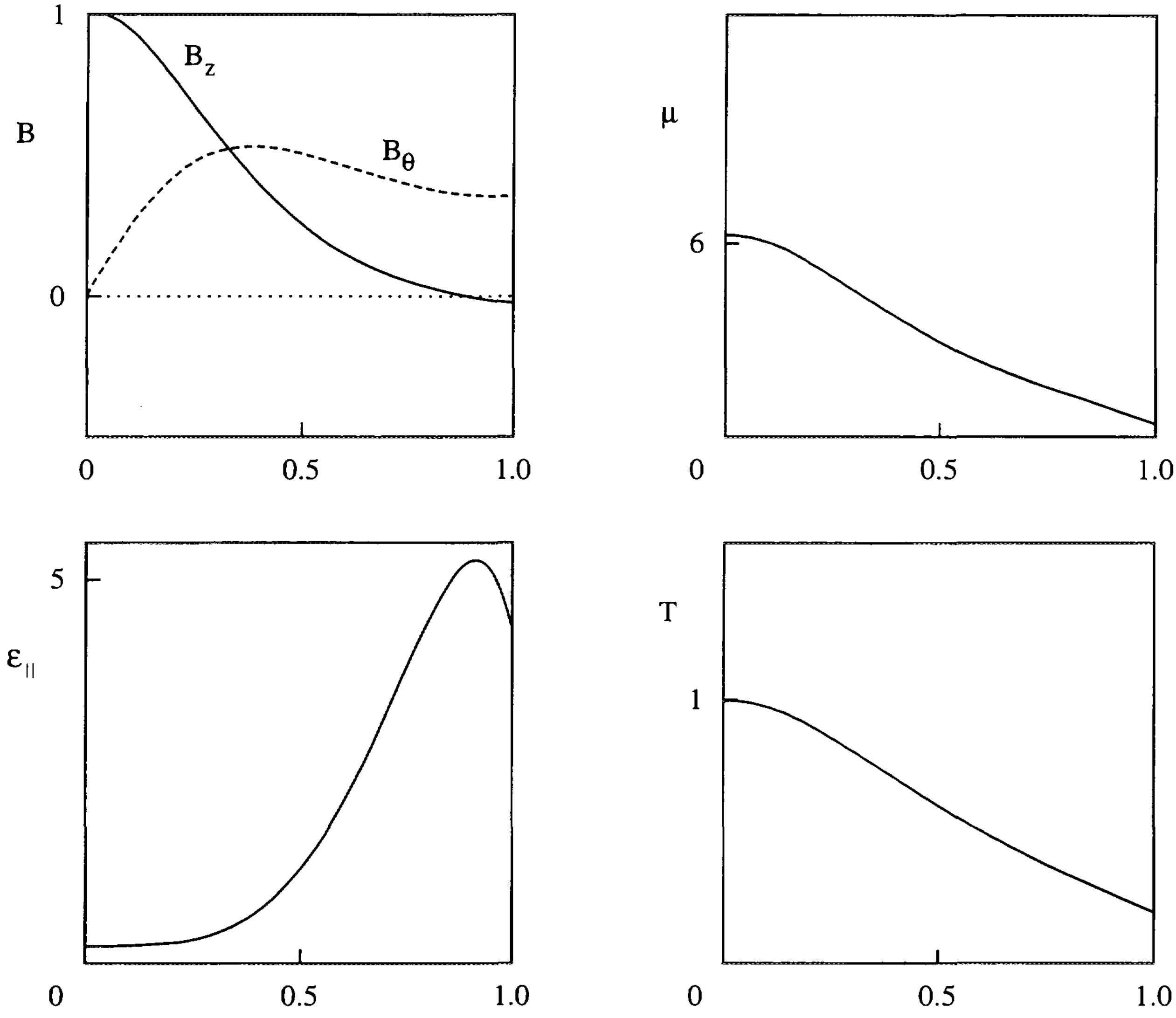
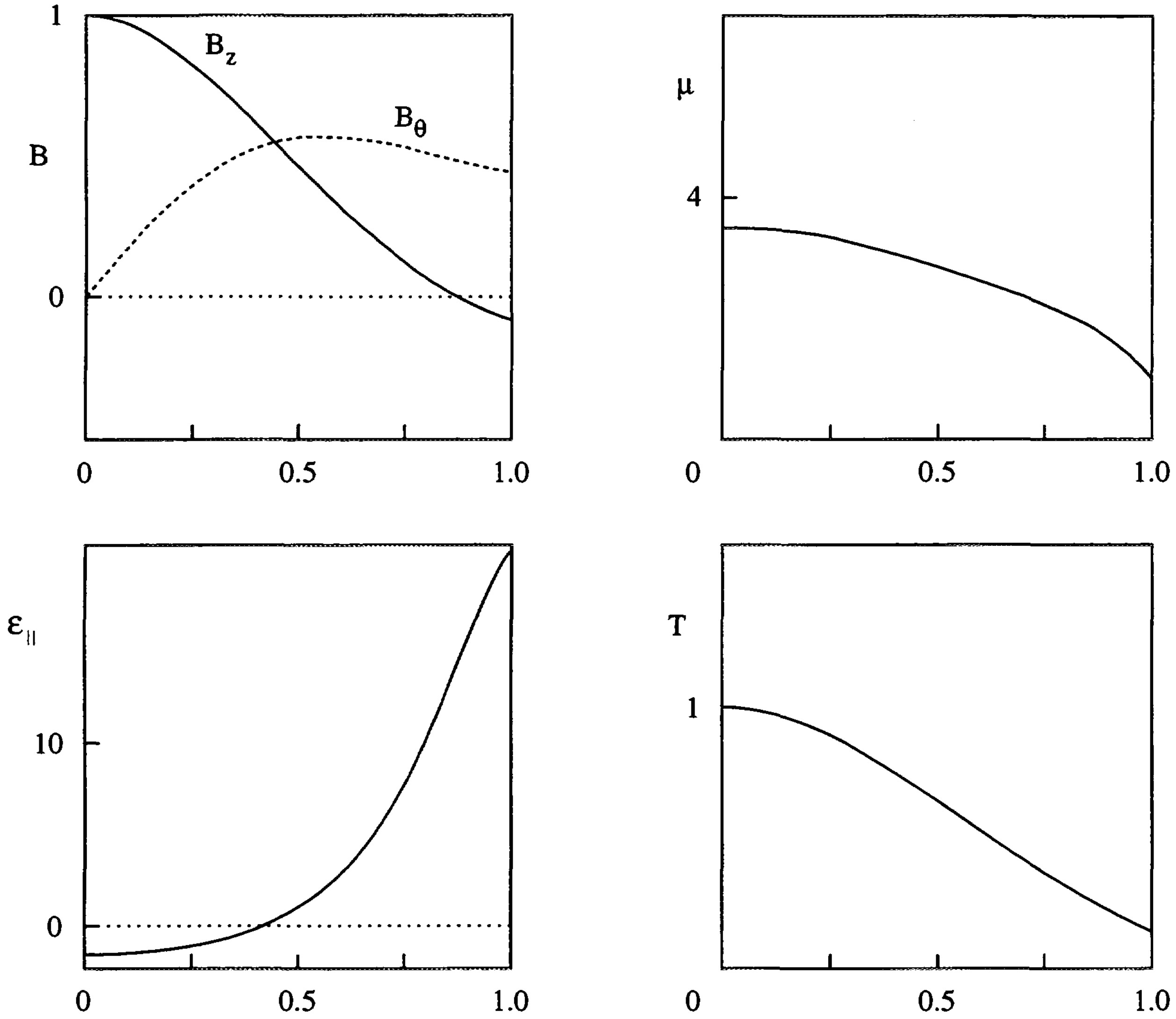


Fig.10



1
B
B_z
B_θ
0
0
0.5
1.0
μ
4
0
0.5
1.0
$\varepsilon_{\parallel}$
10
0
0
0.5
1.0
T
1
0
0.5
1.0

Fig.11

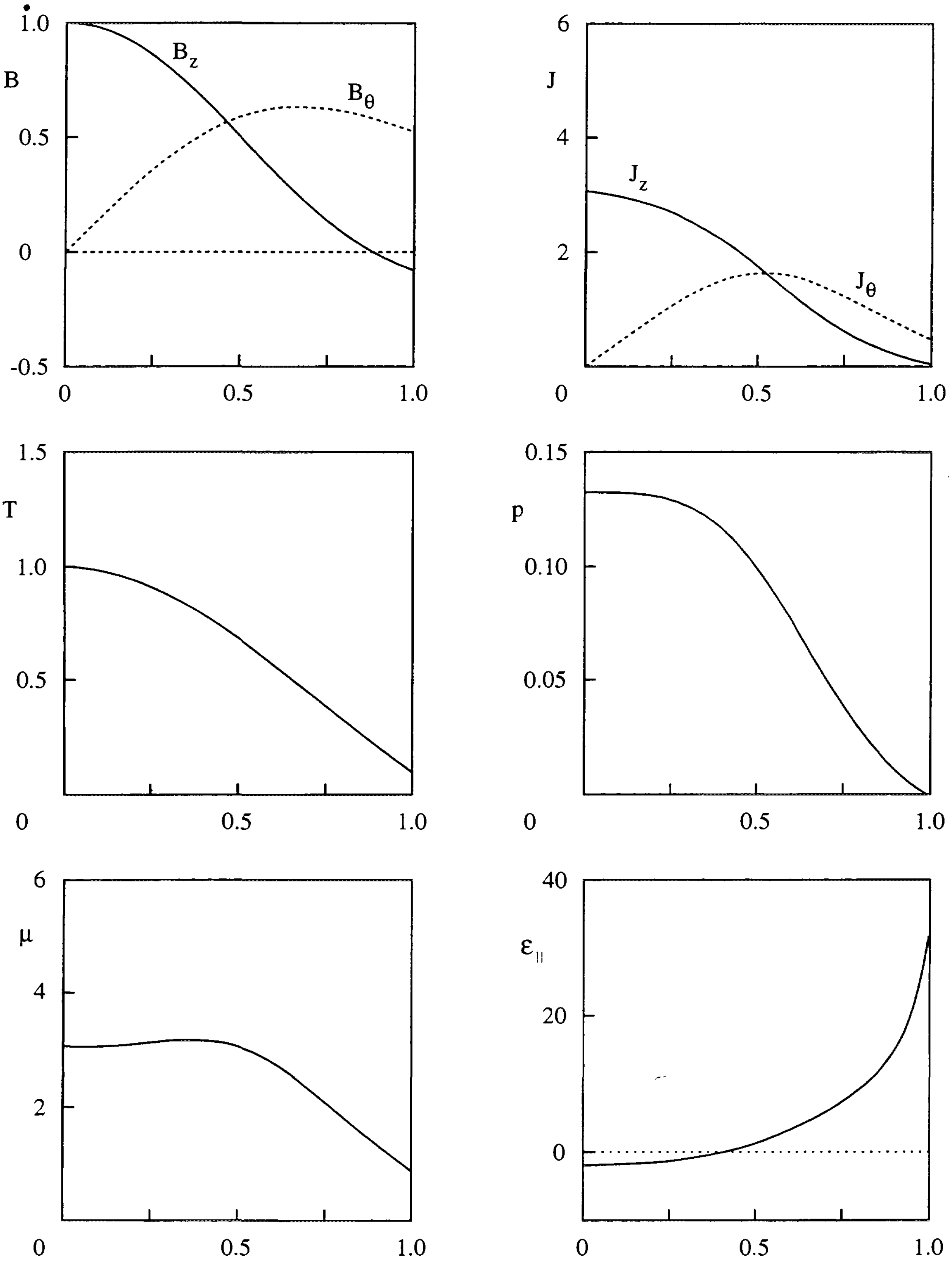

B
1.0
0.5
0
-0.5
B_z
B_θ
0
0.5
1.0
J
6
4
2
J_z
J_θ
0
0.5
1.0
T
1.5
1.0
0.5
0
0.5
1.0
p
0.15
0.10
0.05
0
0.5
1.0
μ
6
4
2
0
0.5
1.0
$\varepsilon_{\parallel}$
40
20
0
0
0.5
1.0

Fig.12

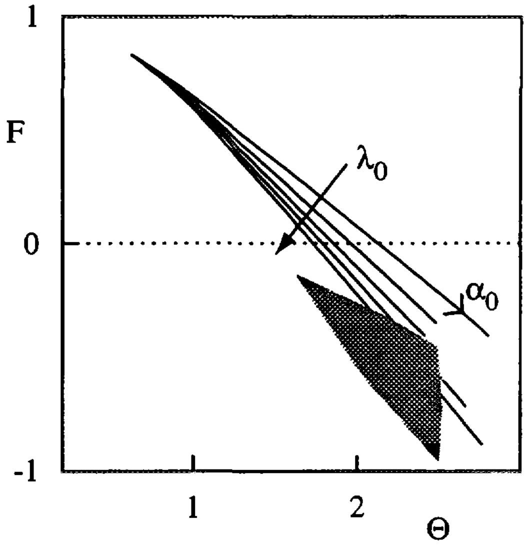


Fig.13

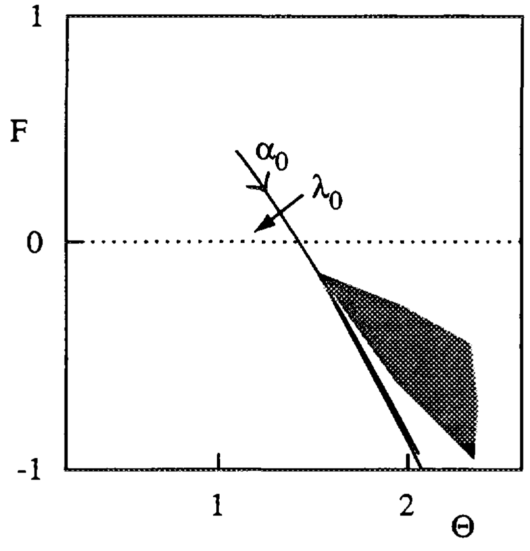


Fig.14

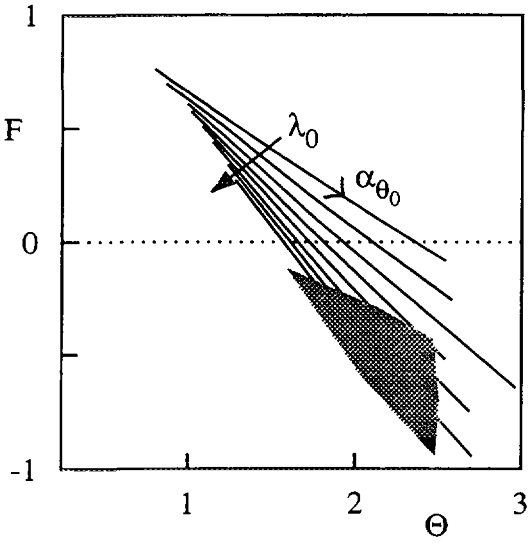

1
F
0
-1
1
2
3
Θ
λ_0
α_{θ_0}

Fig.15

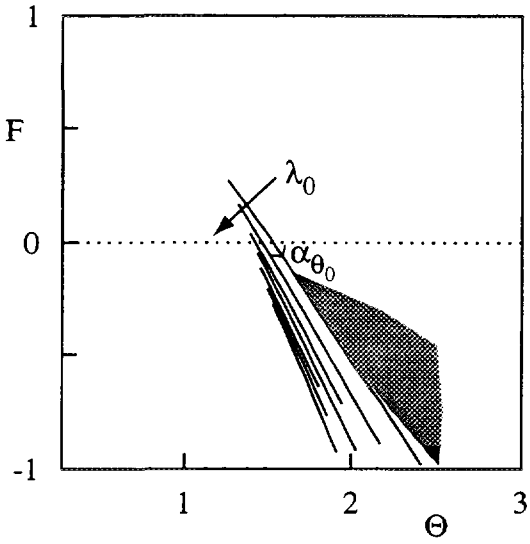

1
F
0
-1
1
2
3
Θ
λ_0
α_{θ_0}

Fig.16

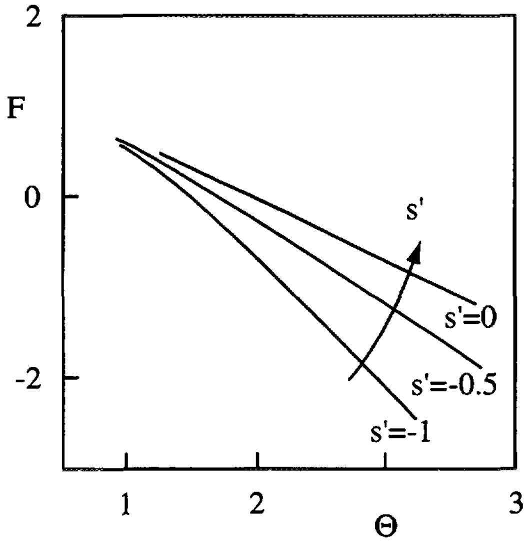


Fig.17

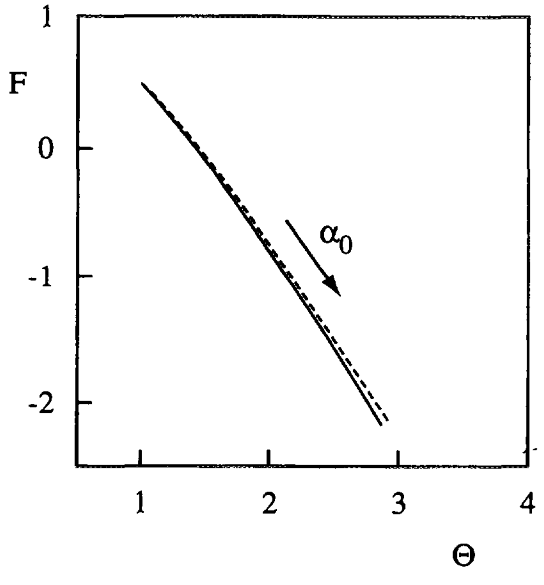


Fig.18

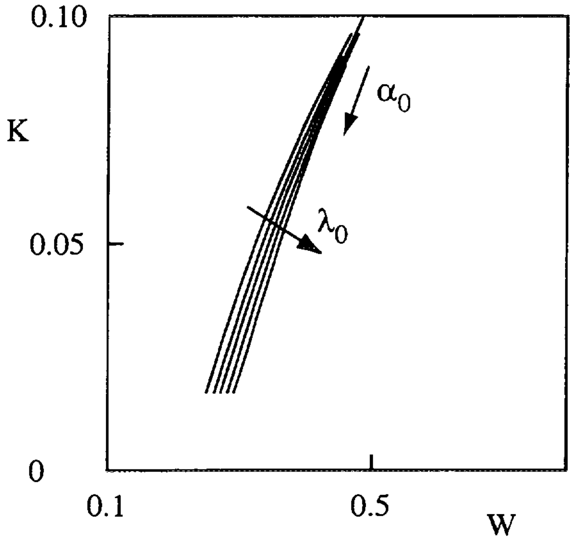


Fig.19

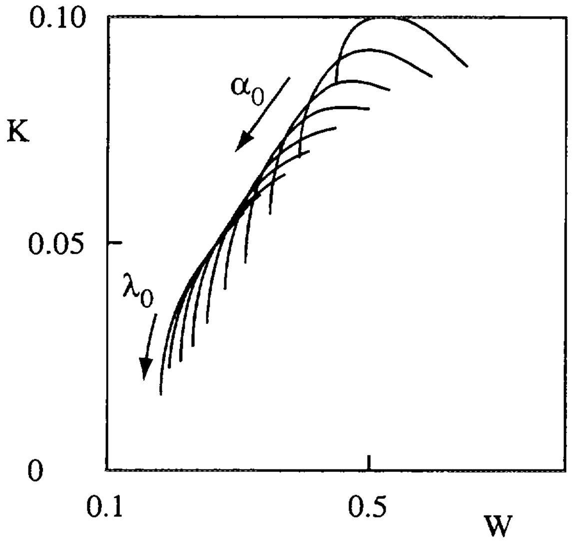


Fig.20

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