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SPIN AND MADELÜNG FLUID*

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ABSTRACT – Starting from the Pauli current we obtain (in the ordinary tensorial language) the decomposition of the non-relativistic local velocity in two parts: one parallel and the other orthogonal to the impulse. The former is recognized to be the "classical" part, that is, the center-of-mass (CM) velocity, and the latter the "quantum" one, that is, the velocity of the motion in the CM frame (namely, the internal "spin motion" or *zitterbewegung*). Inserting this complete, composite expression of the velocity into the kinetic energy term of the classical non-relativistic (i.e. newtonian) lagrangian, we straightforwardly get the appearance of the socalled quantum potential associated as it is known, to the Madelüng fluid. In such a way, *the quantum mechanical behaviour of particles appears to be strictly correlated to the existence of spin and zitterbewegung.*

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1 Variational approaches to the Madelüng fluid

As well-known, the lagrangian for a non-relativistic scalar particle may be assumed to be:

$$\mathcal{L} = \frac{i\hbar}{2}(\psi^* \partial_t \psi - (\partial_t \psi^*)\psi) - \frac{\hbar^2}{2m}\nabla\psi^*\nabla\psi - U\psi^*\psi \tag{1}$$

where U is the potential due to the external forces, the other symbols meaning as usual. Taking the variations with respect to ψ, ψ^*, (i.e. working out the Euler-Lagrange equations), we get the Schrödinger equations for ψ^* and ψ, respectively.

The most general scalar wavefunction $\psi \in \mathbb{C}$ may be factorized as follows:

$$\psi = \sqrt{\rho}e^{i\frac{\varphi}{\hbar}}, \tag{2}$$

where $\rho, \varphi \in \mathbb{R}$. By this position, eq.(1) becomes:

$$\mathcal{L} = -\left[\partial_t\varphi + \frac{1}{2m}(\nabla\varphi)^2 + \frac{\hbar^2}{8m}\left(\frac{\nabla\rho}{\rho}\right)^2 + U\right]\rho. \tag{3}$$

Taking the variations with respect to ρ and φ we obtain[1,2] the two well-known equations for the socalled Madelüng[3] fluid, which, taken together, are equivalent to the Schrödinger equation, i.e.:

$$\partial_t\varphi + \frac{1}{2m}(\nabla\varphi)^2 + \frac{\hbar^2}{4m}\left[\frac{1}{2}\left(\frac{\nabla\rho}{\rho}\right)^2 - \frac{\Delta\rho}{\rho}\right] + U = 0, \tag{4}$$

where

$$\frac{\hbar^2}{4m}\left[\frac{1}{2}\left(\frac{\nabla\rho}{\rho}\right)^2 - \frac{\Delta\rho}{\rho}\right] \equiv -\frac{\hbar^2}{2m}\frac{\Delta|\psi|}{|\psi|} \tag{5}$$

is often called the "quantum potential"; and

$$\partial_t\rho + \nabla\cdot(\rho\nabla\varphi/m) = 0. \tag{6}$$

Eqs.(4),(6) are the *Hamilton–Jacobi* and the *continuity* equations for this probabilistic fluid respectively, and constitute the "hydrodynamic" formulation of the Schrödinger theory. Usually, they are not obtained by the above variational approach, but by inserting position (2) directly into the Schrödinger equation and subsequently separating away the real and imaginary part. This second way of proceeding does obey merely to mathematical requirements, and does no longer gives any physical insight of the Madelüng fluid. On the contrary our variational approach, as we are going to see, can provide us with a physical interpretation of the non-classical terms appearing in eqs.(3-4).

The early physical interpretation of the quantum potential was forwarded by de Broglie's pioneering theory of the *pilot wave*(4); in the fifties, Bohm(5) revisited and completed de Broglie's approach in a systematical way (sometimes, Bohm's theoretical formalism is referred to as the "Bohm formulation of Quantum Mechanics", alternative and complementary to the Heisenberg (matrices and Hilbert spaces), Schrödinger (wavefunctions), and Feynman (path integrals) ones. From Bohm's up to the present days, several conjectures about the origin of that mysterious quantum potential have been put forth, by postulating "subquantal" forces, the presence of ether, and so on. Particularly important are the derivations of the Madelūng fluid within the *stochastic mechanics* framework. In such theories, the origin of the non-classical term (5) appears as substancially *kinematical*. In fact to the classical *drift* (or *convective, translational*) velocity p/m, it is added therein a non-classical, stochastic *diffusion* component $v_{\rm diff}$ (either of *markovian*(6) or *not markovian*(2) type). By adopting markovian–brownian assumptions, the Hamilton–Jacobi eq.(4) is obtained in the form of a "generalized"#1 Newton equation $F = ma$; the continuity equation comes out instead from the simple sum of the "forward" and "backward" Fokker–Planck equations.

In the present paper we shall correlate, at variance with the above theories, the quantum potential with the spinning nature of the elementary particles constituting matter. The starting point is the existence of the socalled *zitterbewegung* (zbw)(7–11) expected to enter any spinning particle theories. A spinning particle endowed with zbw appears as an extended-type object so that the non-classical component of the global velocity is actually related to the "internal" motion [i.e. to the motion observed in the center-of-mass frame (CMF), which is the one where $\boldsymbol{p} = 0$ by definition]. The existence of an "internal" motion is denounced, besides by the mere presence of spin, by the remarkable fact that, according to the standard Dirac theory, the particle impulse $\boldsymbol{p}$ is *not* parallel to the velocity: $\boldsymbol{v} \neq \boldsymbol{p}/m$#2; moreover, in the *free* case, while $[\widehat{\boldsymbol{p}}, \widehat{H}] = 0$ so that $\boldsymbol{p}$ is a conserved

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#1 "Generalized" in that the acceleration a is defined by means of "forward" and "backward" time derivatives.

#2 The non-relativistic limit of Dirac theory does not entail necessarily $\boldsymbol{v} \longrightarrow 0$ but actually only $\boldsymbol{p} \longrightarrow 0$, so that the zbw, in general, does not disappear even in the framework of Schrödinger theory.

quantity, quantity $\boldsymbol{v}$ is *not* a constant of the motion: $[\hat{\boldsymbol{v}}, \widehat{H}] \neq 0$ ($\hat{\boldsymbol{v}} \equiv \vec{\alpha}$ being the usual vector matrix of Dirac theory).

Let us explicitly notice that assuming the zbw is equivalent[8,11] to splitting the motion variables as follows (the dot meaning derivation with respect to time)

$$\boldsymbol{x} \equiv \boldsymbol{\xi} + \boldsymbol{X} \; ; \quad \dot{\boldsymbol{x}} \equiv \boldsymbol{v} = \boldsymbol{w} + \boldsymbol{V} \; , \tag{7}$$

where $\boldsymbol{\xi}$ and $\boldsymbol{w} \equiv \dot{\boldsymbol{\xi}}$ describe the motion of the CM in the chosen reference frame, whilst $\boldsymbol{X}$ and $\boldsymbol{V} \equiv \dot{\boldsymbol{X}}$ describe the internal motion referred to the CMF. From a dynamical point of view the conserved electric current is associated to the helical trajectories of the electrical charge (i.e. to $\boldsymbol{x}$), whilst the center of the particle coulombian field is associated to the geometrical centers of such trajectories (i.e. to $\boldsymbol{w}$). Going back to lagrangian (3), it is now possible, starting by the above assumptions, to attempt an interpretation of the non-classical term $\frac{\hbar^2}{8m}\left(\frac{\nabla\rho}{\rho}\right)^2$ appearing therein. So, the first term in the r.h.s. of eq.(3) represents, apart from the sign, the total energy

$$\partial_t \varphi = -E \; ; \tag{8}$$

whereas the second term is recognized to be the kinetic energy $\boldsymbol{p}^2/2m$ *of* the CM, if one assumes that

$$\boldsymbol{p} = -\nabla\varphi. \tag{9}$$

The third term, that gives origin to the quantum potential, is instead interpreted as the kinetic energy *in* the CMF, that is, the internal energy due to the zbw motion, provided that we re-write it in the following form:

$$\frac{\hbar^2}{8m}\left(\frac{\nabla\rho}{\rho}\right)^2 \longrightarrow \frac{1}{2}m\boldsymbol{V}^2, \tag{10}$$

$$\mathcal{L} \longrightarrow -\left[\partial_t\varphi + \frac{1}{2m}(\nabla\varphi)^2 + \frac{1}{2}m\boldsymbol{V}^2 + U\right]\rho. \tag{11}$$

Eq.(11) holds if and only if

$$|\boldsymbol{V}| = \frac{\hbar}{2}\frac{|\nabla\rho|}{m\rho}. \tag{12}$$

At this point it is easily realized that in lagrangian (11) the sum of the two kinetic energy terms, $\boldsymbol{p}^2/2m$ and $\frac{1}{2}m\boldsymbol{V}^2$, is nothing but *a mere application of the well-known König theorem*.

In this note we shall show that assumption (12) can be easily got from the nonrelativistic (NR) analog of the so-called Gordon decomposition[12] that is the *Pauli current*[13], together with a constraint derived from the hydrodynamics of the Dirac equation in the NR limit.

2 Spin and quantum potential

During the last thirty years Hestenes[14] did sistematically employ the Clifford algebras language in the description of the geometrical, kinematical and hydrodynamical (i.e., *field*) properties of spinning particles, both in relativistic and NR frameworks. He applied the Clifford formalism to Dirac and to Schrödinger–Pauli theory. In the small-velocity limit of the Dirac equation, or directly from Pauli's, Hestenes got the following decomposition of the particle velocity:

$$\boldsymbol{v} = \frac{\boldsymbol{p} - e\boldsymbol{A}}{m} + \frac{\nabla \times (\rho \boldsymbol{s})}{m\rho} \tag{13}$$

where the light speed c is assumed equal to 1, quantity e is the electric charge, $\boldsymbol{A}$ the external electromagnetic vector potential, $\boldsymbol{s}$ is the *local spin vector*#3 $\boldsymbol{s} \equiv \rho^{-1}\psi^{\dagger}\widehat{\boldsymbol{s}}\psi$, and $\widehat{\boldsymbol{s}}$ is the spin operator usually represented by the Pauli matrices as:

$$\widehat{\boldsymbol{s}} \equiv \frac{\hbar}{2}(\sigma_x;\ \sigma_y;\ \sigma_z). \tag{14}$$

As a consequence, the internal zbw velocity reads:

$$\boldsymbol{V} \equiv \frac{\nabla \times (\rho \boldsymbol{s})}{m\rho}. \tag{15}$$

As a particular case, the *Schrödinger* one arises; namely when no external magnetic field is present ($\boldsymbol{A} = 0$) and the local spin vector has no precession, then $\boldsymbol{s}$ is constant in time and uniform in space. In this case, we may explicitate the previous equation as follows

$$\boldsymbol{V} = \frac{\nabla\rho \times \boldsymbol{s}}{m\rho}. \tag{16}$$

Notice that, even in the Schrödinger theoretical framework, the zbw *does not vanish*, except for plane waves, i.e., for the $\boldsymbol{p}$-eigenfunctions, for which not only $\boldsymbol{s}$, but also ρ

#3 Hereafter, every quantity is a *local* or *field* quantity: $\boldsymbol{v} \equiv \boldsymbol{v}(\boldsymbol{x};t)$; $\boldsymbol{p} \equiv \boldsymbol{p}(\boldsymbol{x};t)$; $\boldsymbol{s} \equiv \boldsymbol{s}(\boldsymbol{x};t)$; and so on.

is constant and uniform, so that $\nabla\rho = 0$. Notice also that the continuity equation (6) $\partial_t\rho + \nabla\cdot(\rho\boldsymbol{p}/m) = 0$ can be still re-written in the usual form, namely $\partial_t\rho + \nabla\cdot(\rho\boldsymbol{v}) = 0$. In fact $\nabla\cdot\boldsymbol{V} \equiv \nabla\cdot[\nabla\times(\rho\boldsymbol{s})]$, being the divergence of a rotor, is identically equal to zero, so that $\nabla\cdot(\boldsymbol{p}/m) = \nabla\cdot\boldsymbol{v}$.

By the ordinary tensor language, *without employing Clifford algebras*, we will now show that the decomposition (13) is straightforwardly obtainable from the familiar expression of the socalled Pauli current[13] (that is, from the Gordon decomposition of the Dirac current in the NR limit):

$$\boldsymbol{j} = \frac{i\hbar}{2m}[(\nabla\psi^\dagger)\psi - \psi^\dagger\nabla\psi] - \frac{e\boldsymbol{A}}{m}\psi^\dagger\psi + \frac{1}{m}\nabla\times(\psi^\dagger\hat{\boldsymbol{s}}\psi). \tag{17}$$

A spinning NR particle can be described through a Pauli 2-component spinor Φ:

$$\psi \equiv \sqrt{\rho}\,\Phi \tag{18}$$

where Φ, if we want to have $|\psi| = \rho$, has to obey the normalization constraint

$$\Phi^\dagger\Phi = 1. \tag{19}$$

Since we have by definition $\rho\boldsymbol{s} \equiv \psi^\dagger\hat{\boldsymbol{s}}\psi \equiv \rho\,\Phi^\dagger\hat{\boldsymbol{s}}\Phi$, the insertion of the factorization (18) for ψ into the above expression (17) for the Pauli current gives, together with eq.(9), just the equation:

$$\boldsymbol{j} \equiv \rho\boldsymbol{v} = \rho\frac{\boldsymbol{p} - e\boldsymbol{A}}{m} + \frac{\nabla\times(\rho\boldsymbol{s})}{m} \tag{20}$$

which is nothing but equation (13).

The Schrödinger subcase (i.e., as above said, the case of local spin vector constant and uniform) corresponds to *spin eigenstates*, and then we have to require a wave-function factorizable into the product of a "non-spin" part $\sqrt{\rho}e^{i\varphi}$ (scalar) and of a "spin" part χ (Pauli spinor):

$$\psi \equiv \sqrt{\rho}\,e^{i\frac{\varphi}{\hbar}}\chi, \tag{21}$$

χ being *constant with regard to time and space*. Now we have $\boldsymbol{s} \equiv \chi^\dagger\hat{\boldsymbol{s}}\chi =$ constant, and, as above seen, $\boldsymbol{V}$ will be given by eq.(16).

Because of the following, known mathematical property of the square of the vector product between two generic vectors $\boldsymbol{a}$ and $\boldsymbol{b}$,

$$(\boldsymbol{a}\times\boldsymbol{b})^2 = \boldsymbol{a}^2\boldsymbol{b}^2 - (\boldsymbol{a}\cdot\boldsymbol{b})^2 \tag{22}$$

we have

$$V^2 = \left(\frac{\nabla\rho \times \boldsymbol{s}}{m\rho}\right)^2 = \frac{(\nabla\rho)^2\boldsymbol{s}^2 - (\nabla\rho\cdot\boldsymbol{s})^2}{(m\rho)^2}. \tag{23}$$

Let us now insert into equation (23) the NR limit of a constraint found by Hestenes in his hydrodynamic formulation of the Dirac theory. Being β the Takabayasi angle[15], Hestenes derived from the Dirac equation, by means of Clifford algebras, the following relation:

$$\nabla\cdot(\rho\boldsymbol{s}) = -m\rho\sin\beta. \tag{24}$$

In the NR limit, entailing $\beta \cong 0$ (so that only the two positive-energy components do not vanish in the standard Dirac bispinor), equation (24) reduces to

$$\nabla\cdot(\rho\boldsymbol{s}) = 0. \tag{25}$$

In the Schrödinger case $\boldsymbol{s} = \text{const.}$, so that $\nabla\cdot\boldsymbol{s} = 0$; then, we can write

$$\nabla\rho\cdot\boldsymbol{s} = 0. \tag{26}$$

Putting this result into eq.(23), we easily get

$$V^2 = \boldsymbol{s}^2\left(\frac{\nabla\rho}{m\rho}\right)^2; \tag{27}$$

then, since $|\boldsymbol{s}| = \hbar/2$, we are finally able to deduce just eq.(12)

$$|V| = \frac{\hbar}{2}\frac{\nabla\rho}{m\rho}.$$

Let us remark that, after having inserted eq.(27), respectively in lagrangian (11), the Hamilton–Jacobi and the Schrödinger equations can be re-written as follows:

$$\partial_t\varphi + \frac{1}{2m}(\nabla\varphi)^2 + \frac{\boldsymbol{s}^2}{m}\left[\frac{1}{2}\left(\frac{\nabla\rho}{\rho}\right)^2 - \frac{\Delta\rho}{\rho}\right] + U = 0, \tag{28}$$

$$-\frac{2\boldsymbol{s}^2}{m}\Delta\psi = E\psi. \tag{29}$$

Note that the quantity $2|\boldsymbol{s}|$ replaces $\hbar$, the latter quantity appearing no longer; in a way we might say that it is more appropriate to write $\hbar = 2|\boldsymbol{s}|$, rather than $|\boldsymbol{s}| = \hbar/2$ ...!

In conclusion, we first achieved the Gordon-like non-relativistic decomposition of the local velocity by the ordinary tensorial language. Secondly, we have derived the quantum potential, no longer within the traditional stochastic framework, but (without the *ad hoc* postulates and the *a priori* assumptions characterizing stochastic quantum mechanics) by relating in a natural way, the non-classical energy term to zbw and spin. All that carries further evidence that the quantum behaviour of microsystems may be a *direct consequence* of the existence of spin. In fact, when $s = 0$ we consequently have a vanishing quantum potential in the Hamilton–Jacobi equation, which becomes then totally *classical* and newtonian. In this way we are induced to conjecture that no really elementary *quantum* scalar particles exist, but that such particles are always constituted by spinning objects endowed with zbw[#4]; and up to present no contrary experimental evidence has been found.

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[#4] As, e.g., pion composed by quarks.

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