On the distribution of eigenvalues of certain matrix ensembles

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Abstract

Invariant random matrix ensembles with weak confinement potentials of the eigenvalues, corresponding to indeterminate moment problems, are investigated. These ensembles are characterized by the fact that the mean density of eigenvalues tends to a continuous function with increasing matrix dimension contrary to the usual cases where it grows indefinitely. It is demonstrated that the standard asymptotic formulae are not applicable in these cases and that the asymptotic distribution of eigenvalues can deviate from the classical ones.

The model with $V(x) = \log^2(|x|)/\beta$ is considered in detail. It is shown that when $\beta \to \infty$ the unfolded eigenvalue distribution tends to a limit which is different from any standard random matrix ensembles but which is the same for all three symmetry classes: unitary, orthogonal and symplectic.

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1. Random matrices ensembles are widely used for the description of statistical properties of energy levels of complex quantum systems. Although initially they were supposed to be applied only to many-body systems with complicated interactions like heavy nuclei (see e.g.[1]), it was later [2, 3] conjectured that they can be used even for low-dimensional quantum models with the requirement that the classical motion of such systems should have strong chaotic properties. The important feature of this conjecture consist in suppose that, *after a proper rescaling of the eigenvalues, the statistical properties of the spectrum of a generic chaotic quantum system should be close to one of the three classical random matrix ensembles*: the unitary(GUE), the orthogonal(GOE) or the symplectic(GSE) depending only on the symmetry of the model [1]-[5][1].

The strong argument in favor of this conjecture is the fact that many different random matrix ensembles, at the scale of the mean level separation, give the same level spacing distribution [4].

Considering only the case of ensembles invariant under all possible rotations of the eigenvectors (compatible with the imposed symmetry), the joint probability distribution $P(M)$ of the matrix elements of a $N \times N$ matrix M is defined [4] by choosing in

$$P(M) = C \exp[-Tr(V(M))] \tag{1}$$

the function $V(x)$. Integrating (1) over the parameters related to the eigenvectors one can obtain the well known [4] joint probability distribution of the eigenvalues

$$P(x_1, x_2, ..., x_N) = C_N \exp[-\sum_{i=1}^{N} V(x_i)] \prod_{i<j} |x_i - x_j|^\gamma, \tag{2}$$

where C_N is a normalization constant and γ is a symmetry parameter equal to 1,2 or 4 for GOE, GUE and GSE respectively.

In the simplest case of the unitary ensemble ($\gamma = 2$) all n-point correlation functions can be written in terms of a single function [4]

$$R_n(x_1, \dots, x_n) = \det\left[K_N(x_i, x_j)\right]_{i,j=1,2,\dots,n}, \tag{3}$$

[1]For generic classically integrable models one expects that energy levels are independent and their spacing distribution is close to the Poisson distribution [6].

where

$$K_N(x,y) = \exp\left[-\frac{1}{2}V(x) - \frac{1}{2}V(y)\right] \sum_{n=0}^{N-1} P_n(x)P_n(y), \qquad (4)$$

and $P_n(x)$, $n = 1,2,\ldots$, are polynomials orthogonal with respect to the measure $\exp[-V(x)]$, i.e.,

$$\int \exp[-V(x)]\, P_n(x)P_m(x)dx = \delta_{mn}. \qquad (5)$$

The correlation functions for the orthogonal and the symplectic ensembles can be expressed in terms of the so-called skew-orthogonal polynomials [4, 7].

By the Christoffel-Darboux formula [8] the kernel (4) can be rewritten as

$$K_N(x,y) = \exp\left[-\frac{1}{2}V(x) - \frac{1}{2}V(y)\right] \frac{a_{N-1}}{a_N} \left[\frac{P_N(x)P_{N-1}(y) - P_N(y)P_{N-1}(x)}{x-y}\right], \qquad (6)$$

where a_n is the coefficient of the term x^n in $P_n(x)$ ($P_n(x) = a_n x^n + \ldots$). Further progress in the explicit computation of the n-point correlation function, in the natural limit $N \to \infty$, depends on the knowledge of the asymptotic behaviour of the polynomials $P_n(x)$ when $n \to \infty$. In principle, $K_N(x,y)$ and all the other correlation functions, in particular, the average level density $\rho_N(x)$ of eigenvalues

$$\rho_N(x) \equiv K_N(x,x) = \exp[-V(x)] \frac{a_{N-1}}{a_N} \left[P'_N(x)P_{N-1}(x) - P_N(x)P'_{N-1}(x)\right] \qquad (7)$$

depend on the potential and on N. The universal behaviour is only expected in the limit $N \to \infty$ and after unfolding [1]-[5], i.e., after rescaling the eigenvalues by choosing a new variable $\xi = \xi(x)$ from the relation

$$\frac{d\xi}{dx} = \rho_N(x). \qquad (8)$$

In terms of these variables the n-point correlation functions are still given by Eq. (3) but with $K_N(x,y)$ replaced by the kernel

$$\overline{K}_N(\xi_1,\xi_2) = \frac{K_N[x(\xi_1), x(\xi_2)]}{\sqrt{\rho_N[x(\xi_1)]\rho_N[x(\xi_2)]}}. \qquad (9)$$

It is evident that these new ξ-variables will have by construction an average level density equal to one, from which follows the name of this procedure - unfolding the spectrum.

The hypothesis of universal behaviour of the matrix ensembles assumes that in these unfolded variables and in the limit $N \to \infty$ the above kernel tends to universal function

$$\overline{K}_0(\xi_1, \xi_2) = \frac{\sin\left[\pi(\xi_1 - \xi_2)\right]}{\pi(\xi_1 - \xi_2)} \tag{10}$$

independent of the $V(x)$. (The corresponding limiting functions for GOE and GSE are given [4, 5]).

2. For many potentials this conjecture has been already verified (see e.g. [4, 9]). In fact, there exists a simple WKB-type ansatz for the asymptotics of orthogonal polynomials of an "arbitrary" $V(x)$ [10] that, as it is shown in Ref. [14], leads to the scaling limit (10). However it can be proved only for special classes of potentials.

The main ingredient of this asymptotics is the calculation of the mean eigenvalue density $\overline{\rho}_N(x)$ as the function which gives an extremum of the total measure (2) of the ensemble of NxN matrices. Consider (2) written as

$$P(x_1, x_2, ..., x_N) = C \exp\left[-\mathcal{F}(x_1, x_2, ..., x_N)\right], \tag{11}$$

where

$$\mathcal{F}(x_1, x_2, ..., x_N) = \sum_{i=1}^{N} V(x_i) - \frac{\gamma}{2} \sum_{i \neq j} \ln|x_i - x_j|. \tag{12}$$

Assuming $\overline{\rho}_N(x)$ to be a smooth function of x only non-zero in the interval $a < x < b$ one has

$$\mathcal{F}\left[\overline{\rho}_N(x)\right] = \int_a^b \overline{\rho}(x) V(x) dx - \frac{\gamma}{2} \int_a^b \int_a^b \overline{\rho}_N(x) \ln|x - x'| \overline{\rho}_N(x') dx dx', \tag{13}$$

with the normalization condition

$$\int_a^b \overline{\rho}(x) dx = N. \tag{14}$$

The extremal function $\overline{\rho}(x)$ in (13) is defined by condition $\delta\mathcal{F}/\delta\rho = 0$ that yields

$$V(x) = \gamma \int_a^b \overline{\rho}_N(t) \ln|x - t| dt + C, \tag{15}$$

or, by differentiation,

$$\mathcal{P}\int_a^b \frac{\overline{\rho}_N(t)}{x-t}dt = \frac{1}{\gamma}V'(x), \tag{16}$$

where the symbol $\mathcal{P}$ denotes the Cauchy principal value. The general solution of this singular integral equation (often called in this context the Dyson equation) is well known (see e.g. [12, 13]) and can be written in many different forms. For simplicity we assume that $V(x)$ is an even convex function and $a = -R$, $b = R$. In this case,

$$\overline{\rho}_N(x) = \frac{N}{\pi\sqrt{R^2-x^2}} + \frac{1}{\gamma\pi^2}\mathcal{P}\int_{-R}^{R} \frac{dt}{t-x}\sqrt{\frac{R^2-t^2}{R^2-x^2}}V'(t), \tag{17}$$

where the value of R has to be determined from the equation

$$\frac{1}{\gamma\pi}\int_{-R}^{R} dt\sqrt{\frac{R+t}{R-t}}V'(t) = N. \tag{18}$$

Integrating over x one obtains the useful relation

$$\int_0^x \overline{\rho}_N(x') \quad dx' = \frac{N}{\pi}\arcsin(\frac{x}{R}) + \frac{1}{\gamma\pi^2}\int_{-R}^{R} dtV'(t)\ln\left|\frac{R^2-tx+\sqrt{(R^2-t^2)(R^2-x^2)}}{t-x}\right|. \tag{19}$$

If for example $V(x) = k\gamma|x|^\alpha$, where k is constant, then [9, 11],

$$\overline{\rho}(x) = \frac{N}{R_N}f(\frac{x}{R_N}), \tag{20}$$

with

$$f(x) = \frac{\alpha}{\pi}\int_{|x|}^1 \frac{\tau^{\alpha-1}}{\sqrt{\tau^2-x^2}}d\tau \tag{21}$$

and

$$R_N = \left(\frac{N\sqrt{\pi}}{2k}\frac{\Gamma(\alpha/2)}{\Gamma(\alpha+1/2)}\right)^{1/\alpha}. \tag{22}$$

For the gaussian potential $V(x) = \gamma x^2/2$

$$\overline{\rho}(x) = \frac{1}{\pi}\sqrt{2N - x^2}, \tag{23}$$

which is the famous Wigner semicircle law [4].

Knowing $\overline{\rho}(x)$ the asymptotics of the N^{th} orthogonal polynomial when $N \to \infty$ can be written as follows (see e.g. [10, 11] and [14])

$$P_N(x) \mapsto \sqrt{\frac{2}{\pi}}\frac{\exp[\frac{1}{2}V(x)]}{\sqrt{R_N^2 - x^2}} \cos\left[\Phi_N(x)\right], \tag{24}$$

where

$$\Phi_N(x) = \pi \int_x^{R_N} \rho_N(x')dx' + \frac{1}{2}\arccos(\frac{x}{R_N}) - \frac{\pi}{4}. \tag{25}$$

For certain purposes it is more convenient a slightly different asymptotics when the M^{th} polynomial is expressed in terms of quantities connected with the N^{th} molynomial. If M and N are large and $M - N \ll M$

$$P_M(x) \mapsto \sqrt{\frac{2}{\pi}}\frac{\exp[\frac{1}{2}V(x)]}{\sqrt{R_N^2 - x^2}} \cos\left[\Phi_N(x) + (M - N)\arccos(\frac{x}{R_N})\right]. \tag{26}$$

A simple physical explanation of this ansatz can be found in [14]. Assuming its validity and taking into account that $a_{N-1}/a_N \to R_N/2$ as $N \to \infty$ it is easy to compute the kernel (6) (see [14])

$$K_N(x,y) = \frac{\cos\left[\Phi_N(x)\right]\cos\left[\Phi_N(y) - \phi(y)\right] - \cos\left[\Phi_N(y)\right]\cos\left[\Phi_N(x) - \phi(x)\right]}{\pi(x-y)\sqrt{|\sin[\phi(x)]\sin[\phi(y)]|}}, \tag{27}$$

where ϕ is defined from the relation $x = R_N \cos\phi(x)$, $y = R_N \cos\phi(y)$. Now, assuming $x/R_N \sim 1$ and $y/R_N \sim 1$ but their difference $|x - y| \ll R_N$, one has

$$K_N(x,y) \simeq \frac{\sin\left[\xi(x) - \xi(y)\right]}{\pi(x-y)}, \tag{28}$$

where $d\xi/dx = \rho_N(x)$ has the meaning (up to the shift) of the mean staircase function. After unfolding one obtains

$$K_N(\xi,\eta) \simeq \frac{\sin(\xi - \eta)}{\pi\left[x(\xi) - x(\eta)\right]\sqrt{\rho[x(\xi)]\rho[x(\eta)]}}, \tag{29}$$

and $x(\xi)$ as above is the inverse function of $\xi(x)$. If we suppose that the mean density does not change much on the scale $\Delta\xi \sim 1$, we can conclude that this expression coincides with (10)[2].

3. All this leaves the impression that the potential $V(x)$ plays a secondary role and that in the limit of large N, after unfolding, the statistical properties of any ensemble will follow universal functions. That this is not the whole story has been shown in Ref. [15]. In this paper, it was considered the case of a unitary ensemble, $\gamma = 2$, with a potential

$$V(x) = \sum_{n=0}^{\infty} \ln\left[1 + 2q^{n+1}\cosh(2\chi) + q^{2n+2}\right], \tag{30}$$

where $x = \sinh(\chi)$ and the parameter $q = \exp(-\beta)$ with $\beta > 0$. The main reason to choose this particular form was the fact that the asymptotics of the corresponding orthogonal polynomials, the so-called q-Hermite polynomials, can be calculated explicitly [16]. In Ref. [15] it was then obtained that in limit $N \to \infty$ the kernel (6) tends to

$$\overline{K}(\xi,\eta) = C(\beta)\Omega(\beta\xi,\beta\eta)\Theta_4(\xi,\eta;p)\frac{\theta_1\left[\pi(\xi-\eta);p\right]}{\sinh\left[\beta(\xi-\eta)/2\right]}, \tag{31}$$

$$\Theta_4(\xi,\eta;p) = \frac{\theta_4\left[\pi(\xi+\eta);p\right]}{\sqrt{\theta_4(2\pi\xi;p)\theta_4(2\pi\eta;p)}},$$

$$\Omega(u,v) = \frac{\sqrt{\cosh(u)\cosh(v)}}{\cosh\left(\frac{u+v}{2}\right)},$$

$$C(\beta) = \frac{\beta}{2\pi\theta_1'(0;p)},$$

where $\theta_1(x,p)$ and $\theta_4(x,p)$ are Jacobi's theta functions defined as in [20, p. 921], $\theta_i'(x,p) = \partial\theta_i(x,p)/\partial x$, and $p = \exp(-2\pi^2/\beta)$.

Certainly, (31) is far from the standard expectation (10). In particular when $p < 1$ i.e. $\beta < 2\pi^2$ and $\xi \approx \eta$ this kernel can be approximated by the simple expression

$$\overline{K}_\beta(\xi,\eta) = \frac{\beta}{2\pi}\frac{\sin\left[\pi(\xi-\eta)\right]}{\sinh\left[\beta(\xi-\eta)/2\right]}, \tag{32}$$

[2]These considerations are valid far from singular points of $\rho(x)$. On interesting phenomena near such points see [18].

from which follows that it tends to the GUE limit (10) only when $\beta \to 0$. This approximate expression was used in [15] to compute the nearest-neighbour spacing distribution and it was concluded that with increasing β it tends to the Poisson distribution typical of an uncorrelated sequence of eigenvalues.

The purpose of this note is two-fold. First, we clarify why the ensemble of Ref.[15] gives a result different from the standard one, Eq.(10). We will also show that the potential (30) belongs to a large class of potentials for which the usual asymptotic estimates are, strictly speaking, incorrect. Second, we shall compute directly the level spacing distribution of the eigenvalues in the limit $\beta \to \infty$ for this and similar ensembles showing that after unfolding they tend to a limiting distribution independent of β which is neither the Poisson nor the GUE distribution.

4. Before proceeding, we need a few facts from the theory of orthogonal polynomials [17] which are well known but apparently never used in the present context. Let us define the moments μ_n of the distribution $\exp[-V(x)]$

$$\mu_n = \int \exp[-V(x)]\, x^n dx, \quad n = 1, 2, \ldots, \tag{33}$$

where for simplicity we have omitted the limits of integration. Given the function $V(x)$ all the μ_n are uniquely defined. The important question, is to know if the inverse is also true, i.e., if given all the μ_n it is possible to find the unique function $V(x)$. If the answer to this question is positive we say that the "moment problem" is determined, otherwise we call it indeterminate.

It is a well known result that finite limits of integration lead always to a determined moment problem (assuming that $V(x)$ has no singularities). For the infinite interval there exists a simple condition that states that in order for the moment problem

$$\mu_n = \int_{-\infty}^{\infty} \exp[-V(x)]\, x^n dx, \quad n = 1, 2, \ldots \tag{34}$$

to be determined, it is sufficient that

$$\sum_{n=1}^{\infty} \mu_{2n}^{-\frac{1}{2n}} = \infty. \tag{35}$$

On the other hand for the moment problem on the semi-infinite interval

$$\mu_n = \int_{0}^{\infty} \exp[-V(x)]\, x^n dx, \quad n = 1, 2, \ldots \tag{36}$$

to be determined, it is sufficient that

$$\sum_{n=1}^{\infty} \mu_n^{-\frac{1}{2n}} = \infty. \tag{37}$$

If the moment problem is indeterminate then there is function $f(x)$ orthogonal to all x^n such that

$$\int \exp\left[-V(x)\right] f(x) x^n dx = 0, \text{ for all n.} \tag{38}$$

Roughly speaking one can say that slowly growing potentials lead to indeterminate problems. Thus, for example, the potential $V(x) = k|x|^\alpha$ gives a determined moment problem only when $\alpha \geq 1$ for the interval $(-\infty, +\infty)$ and only when $\alpha \geq \frac{1}{2}$ for an interval $(0, +\infty)$. That otherwise we have an indeterminate problem follows from the two easily proved identities

$$\int_{-\infty}^{\infty} \exp\left(-k|x|^\alpha\right) \cos\left(k|x|^\alpha \tan\frac{\pi\alpha}{2}\right) x^n dx = 0 \text{ for all n, if } \alpha < 1, \tag{39}$$

and

$$\int_{0}^{\infty} \exp\left(-k|x|^\alpha\right) \sin\left(k|x|^\alpha \tan\pi\alpha\right) x^n dx = 0 \text{ for all n, if } \alpha < \frac{1}{2}. \tag{40}$$

In the same way, the identity

$$\int_{0}^{\infty} \exp\left(-\frac{1}{\beta}\ln^2 x\right) \sin\left(\frac{2\pi}{\beta}\ln x\right) x^n dx = 0 \text{ for all n} \tag{41}$$

shows that the moment problems of the potential $V(x) = \frac{1}{\beta}\ln^2 x$ is always indeterminate.

The importance of the above-introduced notions of determined and indeterminate moment problems lies in the fact that these two types of models differ by the behaviour of their mean density $\rho_N(x)$ of eigenvalues of corresponding random matrix ensemble in the limit $N \to \infty$ [17, p. 50]. In fact, a necessary and sufficient condition for the moment problem to be determined is that as $N \to \infty$

$$\rho_N(x) \to \infty. \tag{42}$$

If the moment problem is indeterminate, then

$$\rho_N(x) < \infty \tag{43}$$

as $N \to \infty$ and the density tends to a continuous function of x.

It is evident that for indeterminate problems the asymptotic behaviour of the corresponding orthogonal polynomials cannot be described by the previously discussed method simply because formulae from [10, 11] define the level density and other quantities as unique function of $V(x)$. But for indeterminate problems there are infinite many different measures giving exactly the same orthogonal polynomials and there is no way to choose the "correct" one.

The difference between ensembles whose potentials give raise to a determined or indeterminate moment problem can be understood from their limits as $N \to \infty$. After unfolding, the universal behaviour is expected in the scale of ξ of order 1, but if $\rho_N \to \infty$ as $N \to \infty$ the corresponding values of the old variables x tend to zero. Therefore one is forced to consider very small values of x and the existence of universal asymptotic formulae seems natural. On the contrary, for indeterminate problems even after unfolding, the corresponding values of x are of order 1 and there is no reason why the limit should be universal.

One can conjecture that the asymptotic formula of the behaviour of the orthogonal polynomials given in [10, 14] can be applied only for determinate problems. Asymptotic properties of indeterminate problems can be completely different from standard expectations.

The asymptotic behaviour of the potential (30) is the following

$$V(x) \mapsto \frac{2}{\beta} \ln^2 x. \tag{44}$$

Therefore, the problem considered in Ref. [15] corresponds to an indeterminate moment problem and the difference between Eq. (32) and the expected value Eq. (10) is not so surprising.

5. Nevertheless, the smooth quantities even for indeterminate problems can be described by the usual formulae. For example, let us define the smooth mean level density by

$$\overline{\rho}(x_0) = \frac{1}{\Delta x} \int_{x_0 - \frac{1}{2}\Delta x}^{x_0 + \frac{1}{2}\Delta x} \rho(x) dx. \tag{45}$$

If the interval Δx includes many eigenvalues but $\Delta x \ll R_N$ it is still possible in some cases to prove Eq. (16) but the local mean density of states will be different from this value. Below we shall present the explicit calculation for a certain model that clearly illustrates this point. In some sense, it is possible to say that for indeterminate problems there is no separation between macro and micro scales.

We stress that for determined problems $\rho_N \to \infty$ when $N \to \infty$ assuming that the potential $V(x)$ does not depend on N. Sometimes one considers the N-dependent potential in such a way that the mean density of states tends to N-independent limit. By analogy with the Gaussian case one often defines the measure as $\exp(-NV(x))$ (see e.g. [14]). If it is not just rescaling of variables it can drastically change the asymptotic behaviour of all quantities as it corresponds to a particular limit when certain coupling constants tend to infinity with increasing N.

A rough description of indeterminate problems can be obtained using the above-mentioned asymptotic formulae though locally they cannot be applied. The main feature of indeterminate systems is that their mean density tends when$N \to \infty$ to a certain function independent of N.

Let us consider the example of the potential

$$V(x) = 2k|x|^{\alpha}. \tag{46}$$

From Eq. (39) it follows that is an indeterminate problem when $0 < \alpha < 1$. In fact, to find the behaviour of $\overline{\rho}(x)$ at a fixed x it is necessary to compute the function (21) as $x \to 0$. When $\alpha > 1$, $f(0)$ is infinite but if $0 < \alpha < 1$, we find[3] that when $N \to \infty$

$$\overline{\rho}(x) \mapsto k\frac{\alpha}{\pi}|x|^{\alpha-1}\tanh\frac{\pi\alpha}{2}. \tag{47}$$

Another important example is a potential of the form

$$V(x) = \frac{1}{\beta}\ln^2|x| \tag{48}$$

that when $N \to \infty$ gives

$$R_N \mapsto 2\exp(\frac{N\beta}{2}) \tag{49}$$

[3]Note that $\pi\int_0^x \rho_N(y)dy$ equals the argument of the nul function (39) which is orthogonal to all powers of x.

and

$$\bar{\rho}(x) \mapsto \frac{2}{\beta|x|}. \tag{50}$$

We stress that Eqs. (47) and (50) are approximating formulae giving only the smooth part of the mean level density. The exact mean level density has oscillations which are washed out by the above method (see below).

Assume for the moment that the expression (29) is valid for indeterminate systems and $\rho(x)$ is the limiting mean density. In most cases one is interested in the investigation of the statistical properties of a large number of eigenvalues. As $\rho(x)$ is an integrable function, in order to obtain many eigenvalues one is forced to consider large values of (x, y) and (ξ, η). Now, the important region in the kernels is the following

$$x, y \gg 1, \; \Delta\xi = \xi - \eta \sim 1 \tag{51}$$

and

$$K(\xi, \eta) = \frac{\sin(\Delta\xi)}{\pi\left[x(\xi) - x(\xi - \Delta\xi)\right]\sqrt{\rho\left[x(\xi)\right]\rho\left[x(\xi - \Delta\xi)\right]}}. \tag{52}$$

If $x(\xi) - x(\xi - \Delta\xi) - \Delta\xi dx/d\xi = o(\Delta\xi) \ll 1$ one can neglect the higher-order terms giving as result the limiting form (10). This condition is equivalent to

$$\frac{d^2x}{d\xi^2} \ll \frac{dx}{d\xi} \text{ when } \xi \to \infty. \tag{53}$$

For example, for the model (46), $\rho(x) \sim x^{\alpha-1}$ hence $x(\xi) \sim \xi^{\frac{1}{\alpha}}$ and the above condition is fulfilled if $\xi \ll 1$. This means that for this model it is possible to observe a noticeable deviation from the standard result (10) only at small values of ξ. But as there is only a small fraction of eigenvalues in this region, the asymptotics in the bulk of the spectra tends to the usual one[4].

From the above condition we can infer that in order to have a non-standard behaviour it is necessary that the second derivative of $x(\xi)$ is of the same order as the first one, which is true if, for example,

$$x \sim \exp(\beta\xi), \tag{54}$$

or, in other words, if $\rho(x) \sim 1/\beta x$. But this is exactly the case, as we have already seen, of the square logarithmic potential and, consequently, of the

[4]There is an interesting limit when $\alpha \to 0$ as $N \to \infty$ but we shall not consider it here.

potential which has been discussed in Ref. [15]. Indeed, if we substitute (54) into (29) we obtain the approximate expression (32).

These simple considerations clearly show why models with the potential growing as $\log^2(x)$ are different from the other ones. It is for these models that the statistical properties of eigenvalues deviate from the standard ones, not only near special points, but in the bulk of the spectra.

6. It is natural to ask whether Eq.(32) is valid for all values of β. Note that it was obtained only when $\beta < 2\pi^2$. In [15] it was noticed that with increasing β the distribution of eigenvalues tends to the Poisson distribution but the analysis was based on the approximate kernel (32).

We shall show that for this type of models, when $\beta \to \infty$, the kernel (6) tends (after unfolding) to a limiting function which is different from any standard ensemble but which is the same for all three symmetry types: unitary, orthogonal and symplectic.

We start considering the potential[5]

$$V(x) = \frac{\gamma}{\beta} \log^2 |x|, \tag{55}$$

for which the probability distribution (2) is

$$P(x_1, x_2, ..., x_N) = C_N \exp\left(-\frac{\gamma}{\beta}\sum_{i=1}^{N} \ln^2 |x_i|\right) \prod_{i>j} |x_i - x_j|^\gamma. \tag{56}$$

The expression of the smoothed level density suggests the convenience of introducing the new variables ξ_i connected to the x_i by

$$x_i = 2\sinh(\beta\xi_i), \tag{57}$$

whose probability distribution can be written as follows

$$\begin{aligned} P(\xi_1, ..., \xi_N) &= C'_N \exp\left(-\frac{\gamma}{\beta}\sum_{i=1}^{N} \ln^2 |2\sinh\beta\xi_i|\right) \\ &\quad \prod_{i>j} |2\sinh\beta\xi_i - 2\sinh\beta\xi_j|^\gamma \prod_{i=1}^{N} 2\cosh\beta\xi_i, \end{aligned} \tag{58}$$

[5]For convenience we have introduced the symmetry parameter γ in the potential (see (2)).

where the last term comes from the product $dx_1 \dots dx_N$. When $\beta \to \infty$ with ξ_i fixed, $x_i \to \infty$ and in the difference $|x_i - x_j|$ the term with the largest modulus $|\xi_i|$ will dominate. Let

$$|\xi_1| < |\xi_2| < \ldots < |\xi_N|. \tag{59}$$

Then in the limit of large β the probability distribution tends to the simple function

$$P(\xi_1, ..., \xi_N) = C_N \exp\left[-\gamma\beta \sum_{n=1}^{N} \xi_n^2 + \beta \sum_{n=1}^{N} |\xi_n| \left((n-1)\gamma + 1\right)\right], \tag{60}$$

or

$$P(\xi_1, ..., \xi_N) = \prod_{n=1}^{N} \frac{1}{2\sigma\sqrt{2\pi}} \exp\left[-\frac{1}{2\sigma^2}(|\xi_n| - \frac{n-1}{2} - \frac{1}{2\gamma})^2\right], \tag{61}$$

where $\sigma = 1/\sqrt{2\gamma\beta}$. As $\beta \to \infty$ each $|\xi_n|$ is distributed as the Gaussian random variable centered at $(n - 1 + 1/\gamma)/2$ with a half-width that goes to zero when $\beta \to \infty$. It means that condition (59) is fulfilled and the calculations become simple. In particular, the mean level density, equaled to the integral over all variable but one, can be written as

$$\rho(\xi) = \frac{1}{2} \sum_{n=0}^{N-1} \frac{1}{\sigma\sqrt{2\pi}} \exp\left[-\frac{1}{2\sigma^2}(|\xi| - \xi^{(n)})^2\right], \tag{62}$$

where $\xi^{(n)} = (n + 1/\gamma)/2$.

In the limit $\beta \to \infty$ this density tends to a sum of δ functions

$$\rho(\xi) = \frac{1}{2} \sum_{n=0}^{N-1} \delta(|\xi| - \xi^{(n)}). \tag{63}$$

Therefore, in this limit, the eigenvalues are located on a crystal lattice structure whose sites are separated by a distance of one half.

The difference between the exact mean density and its usual approximation obtained by the solution of the saddle point equation (16) is clearly seen. Eq. (16) gives only the smoothed part of $\rho(\xi)$, but is unable to reproduce the prominent oscillations of (62) and (63).

The next logical step is unfolding the spectrum with the correct density of states given by Eq. (62). Thus, we introduce the new variable η

$$\frac{d\eta}{d\xi} = \rho(\xi), \tag{64}$$

where, as $\beta \to \infty$, $\rho(\xi)$ can be represented as the piecewise continuous function

$$\rho(\xi) = \frac{1}{2}\frac{1}{\sigma\sqrt{2\pi}} \exp\left[-\frac{1}{2\sigma^2}(|\xi| - \frac{1}{2}(n-1+\frac{1}{\gamma}))^2\right], \text{ if } |\xi| \in I_n \tag{65}$$

and the boundaries of the intervals I_n $(n = 1, 2, \ldots, N)$ are chosen in between two peaks

$$\frac{1}{2}(n-1+\frac{1}{\gamma}) - \frac{1}{4} < I_n < \frac{1}{2}(n-1+\frac{1}{\gamma}) + \frac{1}{4}. \tag{66}$$

(The first and the last intervals being slightly different: $0 < I_1 < \frac{1}{4} + \frac{1}{2\gamma}$ and $\frac{1}{2}(N-1+\frac{1}{\gamma}) - \frac{1}{4} < I_N < \infty$.) Choosing $\eta(0) = 0$, one concludes that in the limit $\beta \to \infty$ when $|\xi| \in I_n$, $|\eta| \in J_n$ where new intervals J_n are

$$\frac{n-1}{2} < J_n < \frac{n}{2}. \tag{67}$$

To find the probability distribution in the coordinates η_i it is necessary to change variables which amounts to multiply (61) by $\prod_1^N \rho^{-1}(\xi_i)$

$$P(\eta_1, ..., \eta_N) = P\left[\xi_1(\eta_1), ..., \xi_N(\eta_N)\right] \prod_1^N \rho^{-1}\left[\xi_i(\eta_i)\right]. \tag{68}$$

As $\eta(\xi)$ is a monotonic function, the sequence of inequalities (59) transform to

$$|\eta_1| < |\eta_2| < \ldots < |\eta_N|. \tag{69}$$

Therefore, with this ordering of variables the probability distribution is given by Eq. (68). But the ordering (69) does not give any information about the distribution of the η_i inside the intervals J_n. The only restriction are the above inequalities.

Besides these possibilities there is a special configuration when all η_i belong to intervals J_i with the same number i:

$$\eta_1 \in J_1, \eta_2 \in J_2, \ldots, \eta_N \in J_N. \tag{70}$$

In this case in Eq. (68) all terms cancel and one obtains

$$P(\eta_1, \dots, \eta_N) = 1. \tag{71}$$

But there are many other possibilities when at least one η_n belongs to the interval J_m and $n \neq m$. The probability of this event is proportional to

$$\exp\left[-\frac{1}{2\sigma^2}(|\xi_n| - \frac{n-1}{2} - \frac{1}{2\gamma})^2 + \frac{1}{2\sigma^2}(|\xi_n| - \frac{m-1}{2} - \frac{1}{2\gamma})^2\right] =$$
$$\exp\left[-\frac{1}{2\sigma^2}\left(2|\xi_n| - (m+n-2+\frac{2}{\gamma})(m-n)\right)\right]. \tag{72}$$

By assumption $\eta_n \in J_m$, from which it follows that $\xi_n \in I_m$, i.e.,

$$x_{min} < |\xi_n| < x_{max}, \tag{73}$$

where $x_{min} = (m - \frac{3}{2} + \frac{1}{\gamma})/2$ and $x_{max} = (m - \frac{1}{2} + \frac{1}{\gamma})/2$.

But in this interval the function in the exponent of (72) is always positive. Indeed if $x_{min} \leq x \leq x_{max}$ then

$$\phi_{n,m}(x) = \frac{1}{2}(2x - m - n + 2 - \frac{2}{\gamma})(m-n) \geq \tag{74}$$

$$\begin{cases} \phi_{n,m}(x_{min}) = \frac{1}{2}(m-n)(m-n-1) & \text{if } m > n \\ \phi_{n,m}(x_{max}) = \frac{1}{2}(n-m)(n-m-1) & \text{if } m < n \end{cases}. \tag{75}$$

As m, n are integers $m > n$ ($m < n$) implies that $m \geq n+1$ ($m \leq n-1$) and $\phi_{mn}(x) > 0$ for $x \in I_m$ and $n \neq m$. (m=1 and m=N do not give additional difficulties.) Therefore the probability that the variable η_n will be in the interval J_m with $m \neq n$ has the factor $\exp(-\beta\phi_{n,m})$ and tends to zero as $\beta \to \infty$.

These considerations show that the model in the limit $\beta \to \infty$ after the unfolding (65) tends to the following very simple model. Let us have N points. The probability distribution for the n^{th} point is uniform in the interval J_n and has the shape shown in Fig. 1. All points are uncorrelated and the joint probability is the product of individual ones

$$P_N(y_1, \dots, y_N) = \prod_{n=1}^{N} p_n(y_n). \tag{76}$$

To rewrite the n-point correlation functions in the usual form (3) note that for all possible configurations of y_n

$$P_N(y_1,\dots,y_N) = \left(\det\left[\phi_n(y_m)\right]_{n,m}\right)^2. \tag{77}$$

where $\phi_n(x)$ are functions obeying $\phi_n^2(x) = p_n(x)$ which, evidently, forms an orthonormal system of functions

$$\int_{-\frac{N}{2}}^{\frac{N}{2}} \phi_n(x)\phi_m(x)dx = \delta_{nm}. \tag{78}$$

According to [4] all n-point correlation functions can be written in the usual form (3) with the kernel

$$k_N(x,y) = \sum_{n=1}^{N} \phi_n(x)\phi_n(y). \tag{79}$$

The space structure of this kernel is presented at Fig. 2. Note that this kernel and all correlation functions do not depend only on the difference of the coordinates and, consequently they are not translational invariant as in the classical cases [4]. Special care must therefore be payed when defining standard quantities, like spacing distribution, etc.

Let us consider the nearest-neighbour spacing distribution of eigenvalues. If $E(t_1,t_2)$ is the probability that there is no eigenvalue in the interval (t_1,t_2) then the spacing distribution $p(t_1,t_2)$, defined as the probability that there is one eigenvalue at t_1, t_1+dt_1 and a second one at t_2, t_2+dt_2 but none in between, can be computed as [4]

$$p(t_1,t_2) = -\frac{\partial^2 E(t_1,t_2)}{\partial t_1 \partial t_2}. \tag{80}$$

In our case from (80) it follows that[6]

$$E(t_1,t_2) = \prod_{n=1}^{N}\left[1 - \int_{t_1}^{t_2} p_n(y)dy\right]. \tag{81}$$

[6] It can be also rewritten as $\det(1-K_{t_1,t_2})$ where the operator K_{t_1,t_2} is defined in the usual way[4]

$$(K_{t_1,t_2}f)(x) = \int_{t_1}^{t_2} k_N(x,y)f(y)dy.$$

Let $t_1 = t$, $t_2 = t + s$ and the integer part of $2t$ equals n

$$t = \frac{n}{2} + \tau \text{ and } 0 < \tau < \tfrac{1}{2}.$$

When $n \geq 0$, $p(t, t+s)$ does not depend on n and has the following form (see Fig. 3)

$$p(t,t+s) = \begin{cases} 0, & 0 < s < 1/2 - \tau \\ 2^{1-m}, & m/2 - \tau < s < (m+1)/2 - \tau \end{cases} . \tag{82}$$

For $n = -1$ it has a similar form but instead of having the first jump at $\frac{1}{2} - \tau$ it jumps at $s = 1 - \tau$. For $n \leq -2$ it takes the form $(t = -|n|/2 + \tau)$

$$p(t,t+s) = \begin{cases} 0, & 0 < s < 1/2 - \tau \\ 2^{1-m}, & m/2 - \tau < s < (m+1)/2 - \tau \quad m = 1, \ldots, |n| - 1 \\ 2^{2-|n|}, & |n|/2 - \tau < s < (|n|+1)/2 - \tau \\ 0, & (|n|+1)/2 - \tau < s \end{cases} . \tag{83}$$

As now the nearest neighbour spacing distribution (and other statistical quantities) depends on two variables, a care is needed to compare them with the standard definitions. The most natural definition of a smoothed nearest neighbour spacing distribution is just to compute it over all possible points of the spectrum by fixing only the distance between two levels:

$$\bar{p}(s) = \frac{\sum_{n=-N/2}^{N/2-1} \int_0^{1/2} p(n/2 + \tau, n/2 + \tau + s) d\tau}{\sum_{n=-N/2}^{N/2-1} \int_0^{1/2} d\tau}. \tag{84}$$

When $N \to \infty$, $p(t, t+s)$ for almost all n has the form depicted on Fig. 3. Therefore as $N \to \infty$

$$\bar{p}(s) = 2 \int_0^{\frac{1}{2}} p(\tau, \tau + s) d\tau. \tag{85}$$

It gives

$$\bar{p}(s) = \begin{cases} 2s, & 0 < s < 1/2 \\ 2^{-n}(n + 2 - s), & n/2 < s < (n+1)/2 \quad n = 1, 2, \ldots . \end{cases} . \tag{86}$$

On Fig. 4 we display this function together with the next-to-nearest neighbour spacing distributions $\bar{p}_k(s)$ which give the probability that in the interval s, $s + ds$ there are exactly k eigenvalues.

Important property of this result is that statistical distribution of eigenvalues in the limit $\beta \to \infty$ after unfolding is the same for all three classes of symmetry: unitary, orthogonal and symplectic . In this respect it resembles the distribution of energy levels of the 3-dimensional Anderson model near the metal-insulator transition [19].

7. Up to now we have discussed the model with the logarithmic squared potential (55). To relate it with the model (30) considered in [15] it is necessary to investigate the behaviour of the kernel (31) when $\beta \to \infty$. For this purpose it is convenient to transform the theta-functions in it by usual formulae:

$$\theta_1(x;p) = i(-i\tau)^{-1/2}\exp(-\frac{ix^2}{\pi\tau})\theta_1(\frac{x}{\tau};p'),$$

$$\theta_4(x;p) = (-i\tau)^{-1/2}\exp(-\frac{ix^2}{\pi\tau})\theta_2(\frac{x}{\tau};p'),$$

where $p = \exp(i\pi\tau)$ and $p' = \exp(-i\pi/\tau)$.
For the functions in (31) $p = \exp(-2\pi^2/\beta)$ and $p' = \exp(-\beta/2)$.

After this transformation the kernel (31) can be rewritten in the form

$$k(\xi,\eta) = C(\beta)\Omega(\beta\xi,\beta\eta)\frac{f_2(\xi+\eta)f_1(\xi-\eta)}{\sqrt{f_2(2\xi)f_2(2\eta)}\sinh(\beta(\xi-\eta)/2)}, \tag{87}$$

with

$$f_2(x) = \sum_{n=-\infty}^{+\infty}\exp(-\frac{\beta}{2}(n+\frac{1}{2}-x)^2),$$

$$f_1(x) = \sum_{n=-\infty}^{+\infty}(-1)^n\exp(-\frac{\beta}{2}(n+\frac{1}{2}-x)^2),$$

$$C(\beta) = \frac{\beta}{2f_1'(0)}.$$

As $\beta \to \infty$

$$C(\beta) \to \exp(\beta/4)/2,$$

$$f_2(x) \to \exp(-\beta(\bar{n}+1/2-x)^2/2),$$

$$f_1(x) \to (-1)^{\bar{n}}\exp(-\beta(\bar{n}+1/2-x)^2/2),$$

where $\bar{n} = [|x|]$ is a value of an integer n for which the expression $(n+1/2-x)^2$ has a minimal value

$$\Delta(x) = (1/2 - \{|x|\})^2 \leq 1/4$$

($[x]$ and $\{x\}$ are integer and fractional parts of x).

Finally, one has

$$K(\xi,\eta) \to \exp(-\frac{\beta}{2}\left[||\xi| - |\eta|| + \Delta(\xi+\eta) + \Delta(\xi-\eta) - \frac{1}{2}\Delta(2\xi) - \frac{1}{2}\Delta(2\eta)\right]). \tag{88}$$

As $0 \leq \Delta \leq 1/4$, the dominant contribution comes from the region

$$|\xi| = \frac{m}{2} + \delta\xi, \quad |\eta| = \frac{m}{2} + \delta\eta, \tag{89}$$

where $0 \leq \delta\xi, \delta\eta \leq 1/2$. (Note that it means that $[|2\xi|]=[|2\eta|]$.)

Simple calculation shows that in these squares

$$K(\xi,\eta) = \begin{cases} 1, & \text{if } \xi\eta > 0 \\ (-1)^{[2\xi]}, & \text{if } \xi\eta < 0 \end{cases} . \tag{90}$$

For all other values of ξ and η $K(\xi,\eta) = 0$.

Therefore, when $\beta \to \infty$ the exact kernel (31) of the model (30) tends to approximate expression (79) of the model (55). Important point is that the latter was obtained only after non-trivial unfolding of the spectrum (62) contrary to the former one for which the mean density of states is automatically constant. But it is easy to check that for the model (30) the measure itself has prominent oscillations of the same type as oscillations in the density of states (62) for the model (55) and both models are equivalent.

8. To check the accuracy of the above predictions we have performed the Monte-Carlo simulations of the joint distribution of eigenvalues (2) for the unitary ensemble[7] taking as the potential the function

$$V(x) = \frac{1}{2\beta}\log^2(1+x^2), \tag{91}$$

which reproduces the asymptotic behaviour of the measure (30) and is non-singular for small values of x.

[7] We have checked that for other ensembles one obtains the same results.

Eigenvalues in the domain $|x| \leq 1$ will feel a quartic potential $V(x) = x^4/2\beta$, those outside this region , i.e. when $|x| \geq 1$, will be under the influence of a weak logarithmic confining potential as was discussed above. In the later domain eigenvalues are spread from $-R_N$ till R_N where for large N

$$R_N \simeq 2\exp(\frac{N\beta}{2})$$

which we have verified works well for $N\beta > 5$.

This exponential dependence of R_N with the product $N\beta$ implies a rapid spreading of the eigenvalues into the domain $|x| \geq 1$ even for relatively small values of N and β.

The usual saddle-point calculation as in (50) gives that in the limit $N \to \infty$ the mean number of levels between x_2 and x_1 tends to

$$\bar{N}(x_2) - \bar{N}(x_1) = \frac{1}{\beta}|\epsilon_2 \log|x_2| - \epsilon_1 \log|x_1||, \tag{92}$$

where $\epsilon_i = \text{sign}(x_i)$.

The asymptotic independence of a smooth staircase function and density of states of N is a typical manifestation of the indeterminate character of this problem. Actually we have observed that for $\beta > 1$, $N = 20$ is already close to the asymptotic value.

We stress that this equation can be applied only when $x_2 \gg x_1$. The local density of states will have oscillations and will deviate from the standard one (50) obtained just by differentiation of the above expression.

In order to gain confidence in the Monte Carlo simulation it is instructive to start by considering the simple case $N = 2$ where the level spacing is given by

$$P(s) = K^{-1}Af(As), \tag{93}$$

where

$$f(t) = t^2 \int_0^\infty \exp[-2V(u+t) - 2V(u-t)]du,$$

and

$$K = \int_0^\infty f(t)dt, \quad A = K^{-1}\int_0^\infty tf(t)dt.$$

In Fig. 5 it is shown that Monte Carlo simulations reproduce the above expression quite well. In particular we observe that the Monte Carlo simulation is able to reproduce, in the case $\beta = 30$, the extremely sharp peak near the origin. Although the figure may suggest a linear dependence of $P(s)$ for small values of s, careful analysis shows that, when $s \to 0$, $P(s) \to k(\beta)s^2$, and when $s \to \infty$ $R(s) \to \exp(\log s/\beta)$ for $\beta \to \infty$.

We come now to the case of large N and Fig. 6 shows the result of calculation of the mean (over many realizations) eigenvalue density as a function of the variable

$$\xi = \frac{1}{\beta}\log x \tag{94}$$

in which, according to the standard arguments, the mean eigenvalue density has to be equal to 1. We observe that the mean density of states has prominent oscillations and only its smoothed value equals 1. The solid line in this figure is the theoretical curve (62) and the agreement is quite good even for $\beta = 20$.

The existence of such oscillations modify all correlation functions. In Fig. 7 we present the nearest-neighbour distribution for $N = 20$ and $\beta = 80$ taking into account only the "first" unfolding (94). The appearance of a crystaline structure is clearly seen. But it will disappear after the unfolding with the correct density of states (62). In Fig. 8 shows this phenomenon for $N = 40$ and the same value of β. The solid line is our piece-wise formula (86). As above, the agreement is very good.

We have also considered the model with the same potential as in (91) but with x_i distributed not from $-\infty$ to $+\infty$ but from 0 to $+\infty$. It corresponds not to q-Hermite but to q-Laguerre polynomials [21]. Repeating all considerations, one concludes that $\bar{p}(s)$ should have the form:

$$\bar{p}(s) = \begin{cases} s, & 0 < s < 1 \\ 2 - s, & 1 < s < 2 \\ 0, & \text{otherwise} \end{cases} . \tag{95}$$

In Fig. 9 results from numerical simulations for the nearest-neighbour spacing distribution are compared to Eq. (95). This result means that the asymptotic behaviour, when $q \to 0$, of q-Laguerre polynomials is quite different from those of q-Hermite ones.

9. In conclusion we stress few points.

- There are two types of matrix ensembles invariant with respect to all rotations corresponding to determinated and indeterminate moment problems.

- One can conjecture (but not prove in full generality) that for the first class of ensembles the asymptotics of orthogonal polynomials are given by formulae (17), (24) and, consequently, after unfolding the eigenvalue distribution will agree with the standard results. (Strictly speaking, it was argued only for unitary ensembles. Most probably, it is also true for orthogonal and symplectic ensembles but here one has to consider the asymptotics of skew-orthogonal polynomials which is a more complicated problem.)

- For the second type of ensembles corresponding to indeterminate moment problems the general local asymptotics of orthogonal polynomials cannot exist as in this case the mean eigenvalues density tends when the matrix dimension increases to a (non-universal) function which, in general, has a structure even on the scale of a mean distance between two eigenvalues. But the quantities smoothed over a larger interval can be computed by usual formulae.

- Nevertheless, the eigenvalue distribution can be close to the standard ones even for indeterminate problems as the deviation of the exact and smoothed mean densities can be small and only a small number of levels will feel the difference.

- Models with a weak logarithmic potential like in Eq. (55) are one of the best examples of large deviations from the standard situation. In such cases the mean density has large fluctuations and tends to a series of δ-functions when the strength of the potential decreases. All limiting correlation functions can be computed analytically and after unfolding the limiting distribution is the same for all three symmetry classes: unitary, orthogonal and symplectic.

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FIGURE CAPTIONS

Figure 1. The individual probability distribution $p_n(x)$ for the model (76) in the limit $\beta \to \infty$.

Figure 2. The support of the kernel (79). The modulus of the kernel equals 1 inside the indicated squares. Outside these squares the kernel is zero.

Figure 3. The local nearest-neighbour spacing distribution $p(t, t+s)$ for $0 < t < 1/2$.

Figure 4. The smoothed nearest-neighbour spacing distribution and the smoothed next-to-nearest-neighbour distributions $p_k(s)$.

Figure 5. The nearest neighbour spacing distribution for the case $N = 2$ and for $\beta = 8$ and $\beta = 30$. The histograms are monte Carlo simulations. Also shown the Poisson and the Wigner distributions, for the sake of comparison.

Figure 6. The density of states for $\beta = 40$ and $N = 20$. The Monte Carlo simulations(the histogram) compared with the function $\theta_4(2\xi, p)$.

Figure 7. The nearest neighbour spacing distribution for $\beta = 80$ and $N = 20$ before the unfolding with the exact mean level density.

Figure 8. The nearest neighbour spacing distribution for $\beta = 80$ and $N = 20$ after the unfolding with the exact mean level density compared with the theoretical spacing distribution. Also shown, the Poisson distribution and the Wigner surmise, for the sake of comparison.

Figure 9. The nearest neighbour spacing distribution for $\beta = 80$ and $N = 20$ compared with the theoretical spacing distribution for the case of only positive eigenvalues.

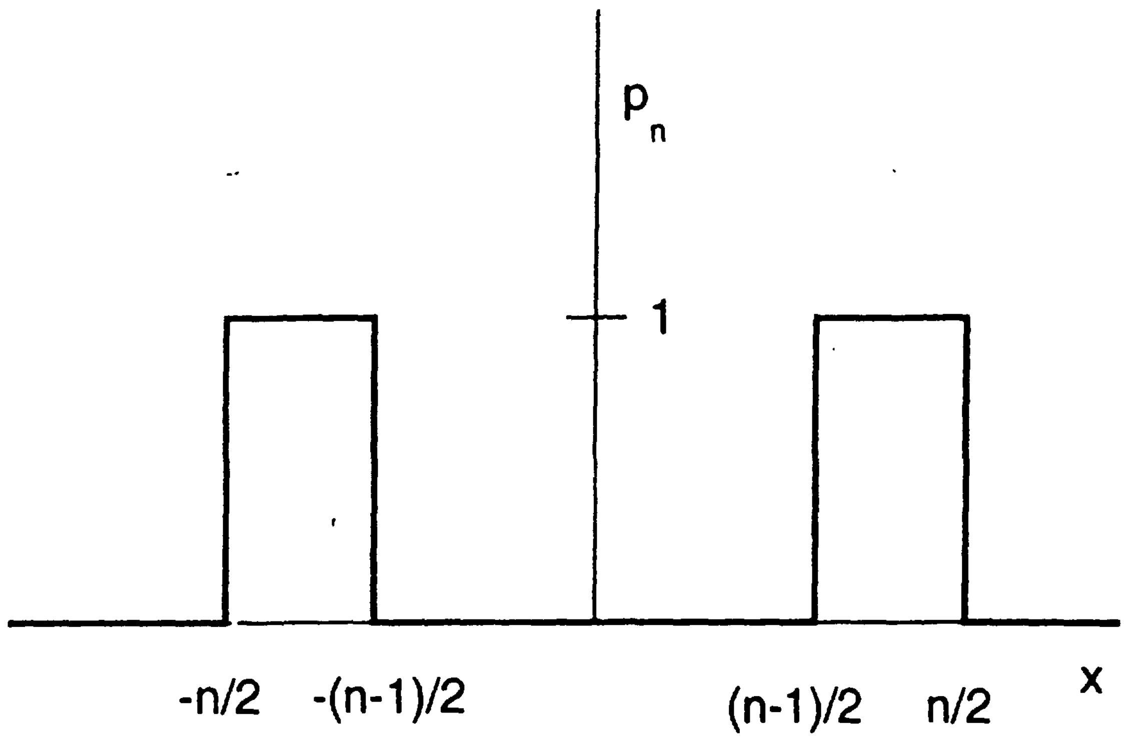

Figure 1

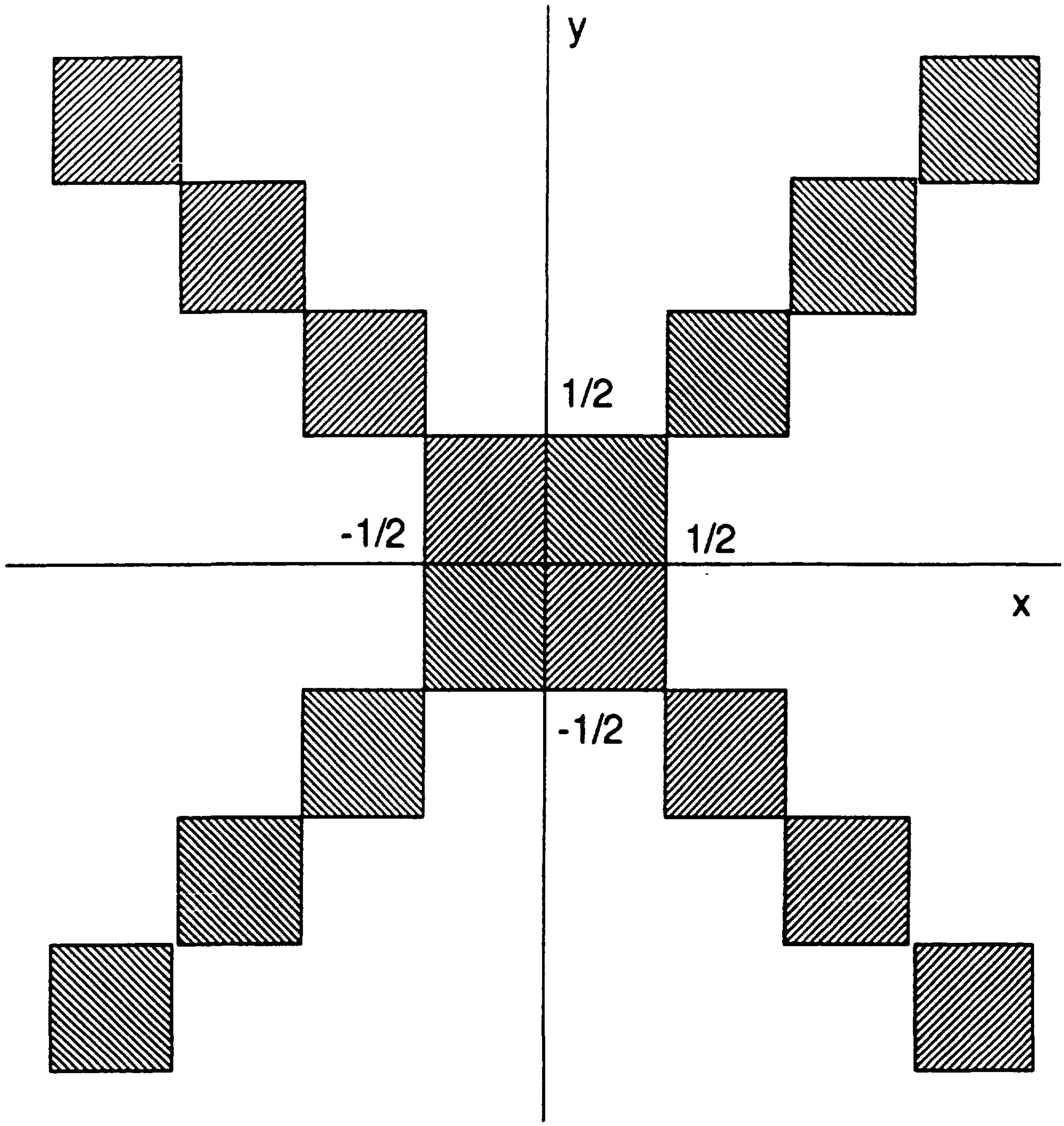

Figure 2

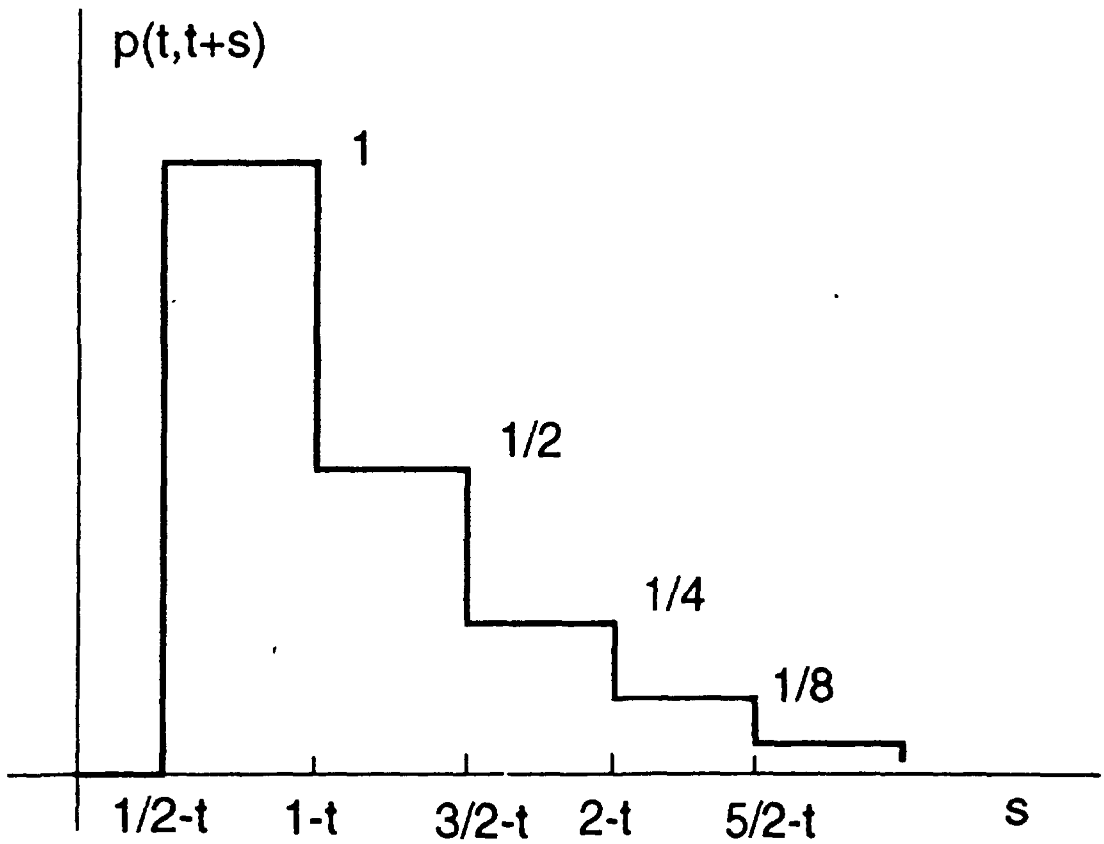

Figure 3

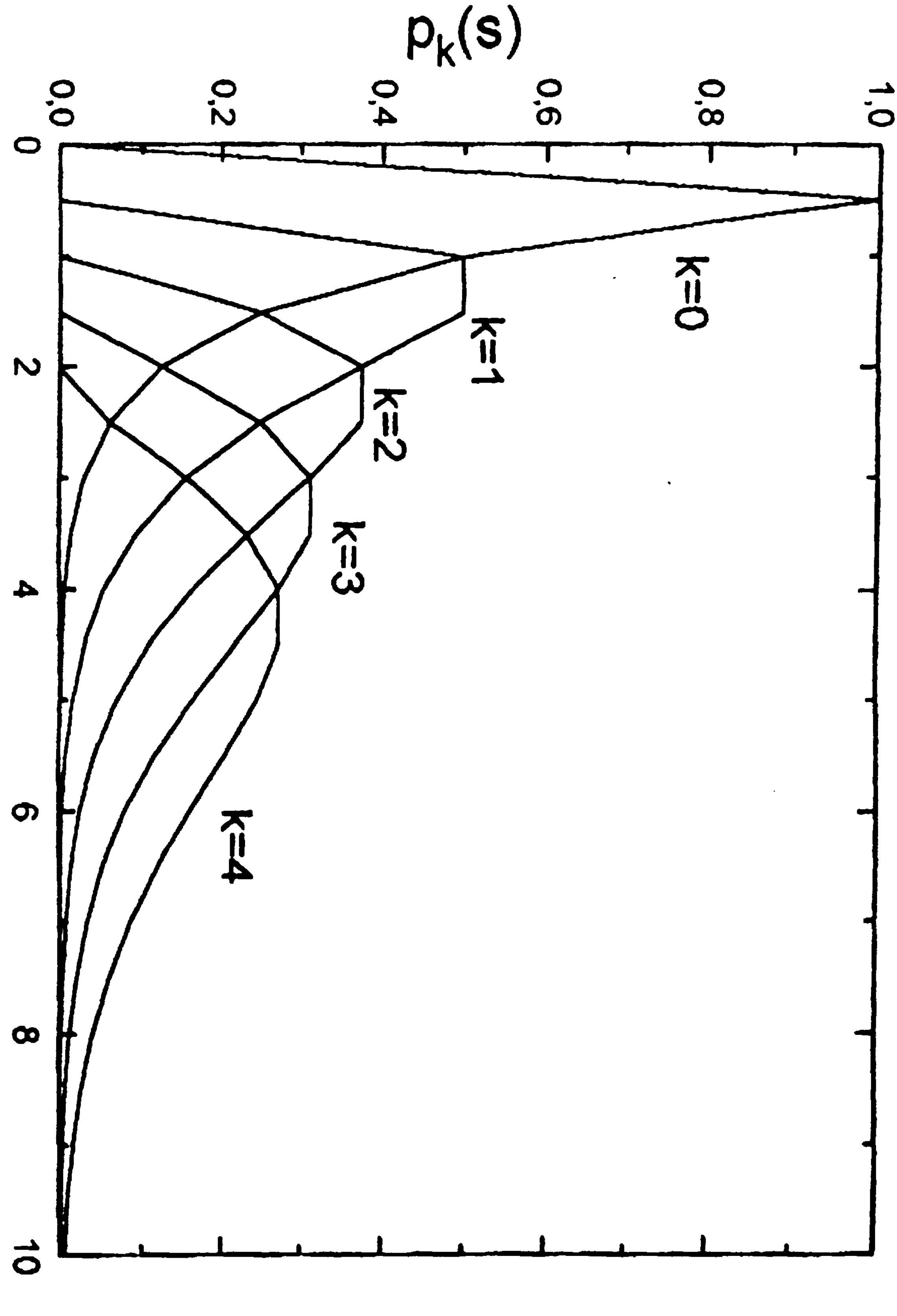

Figure 4

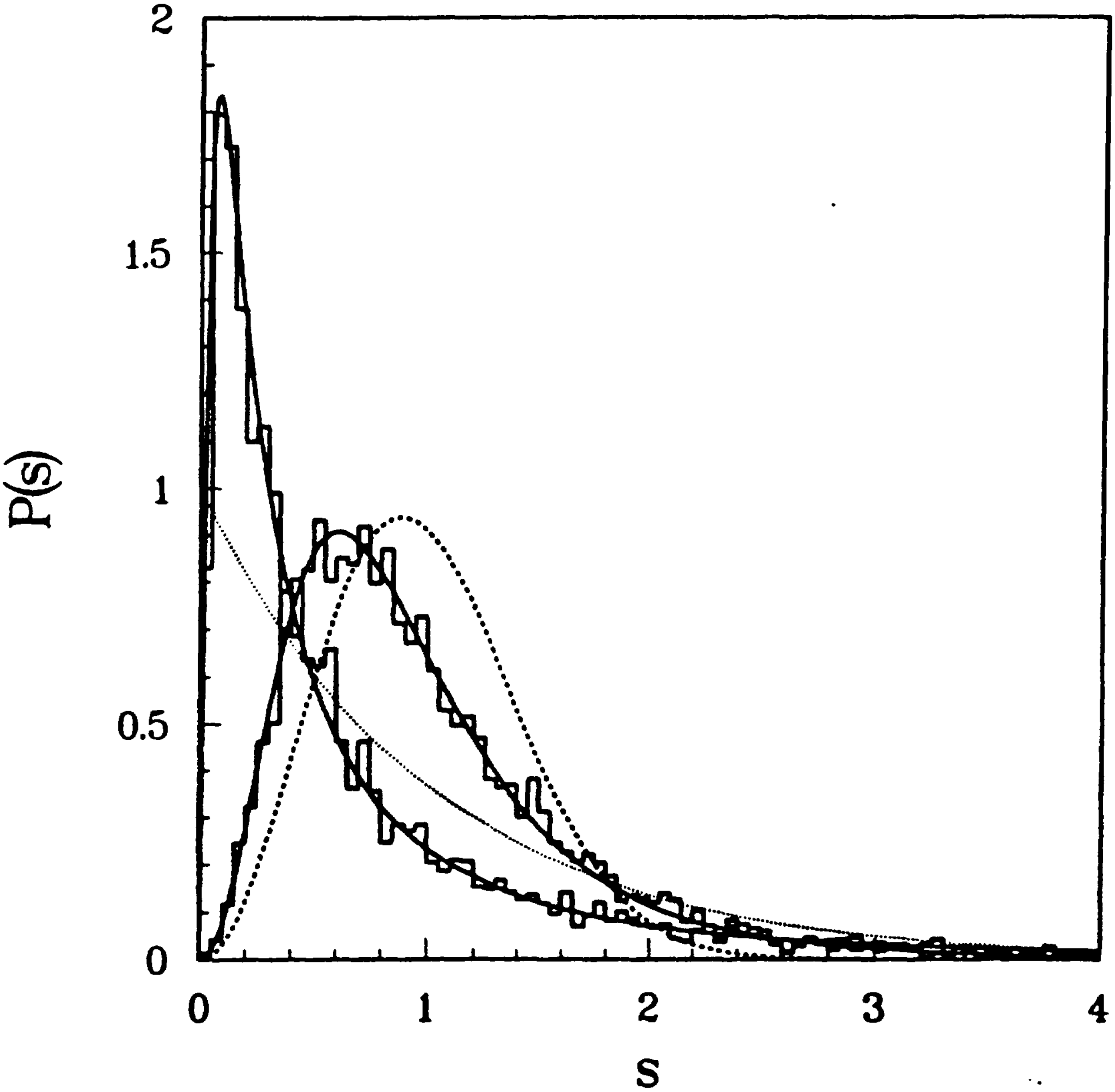

Figure 5

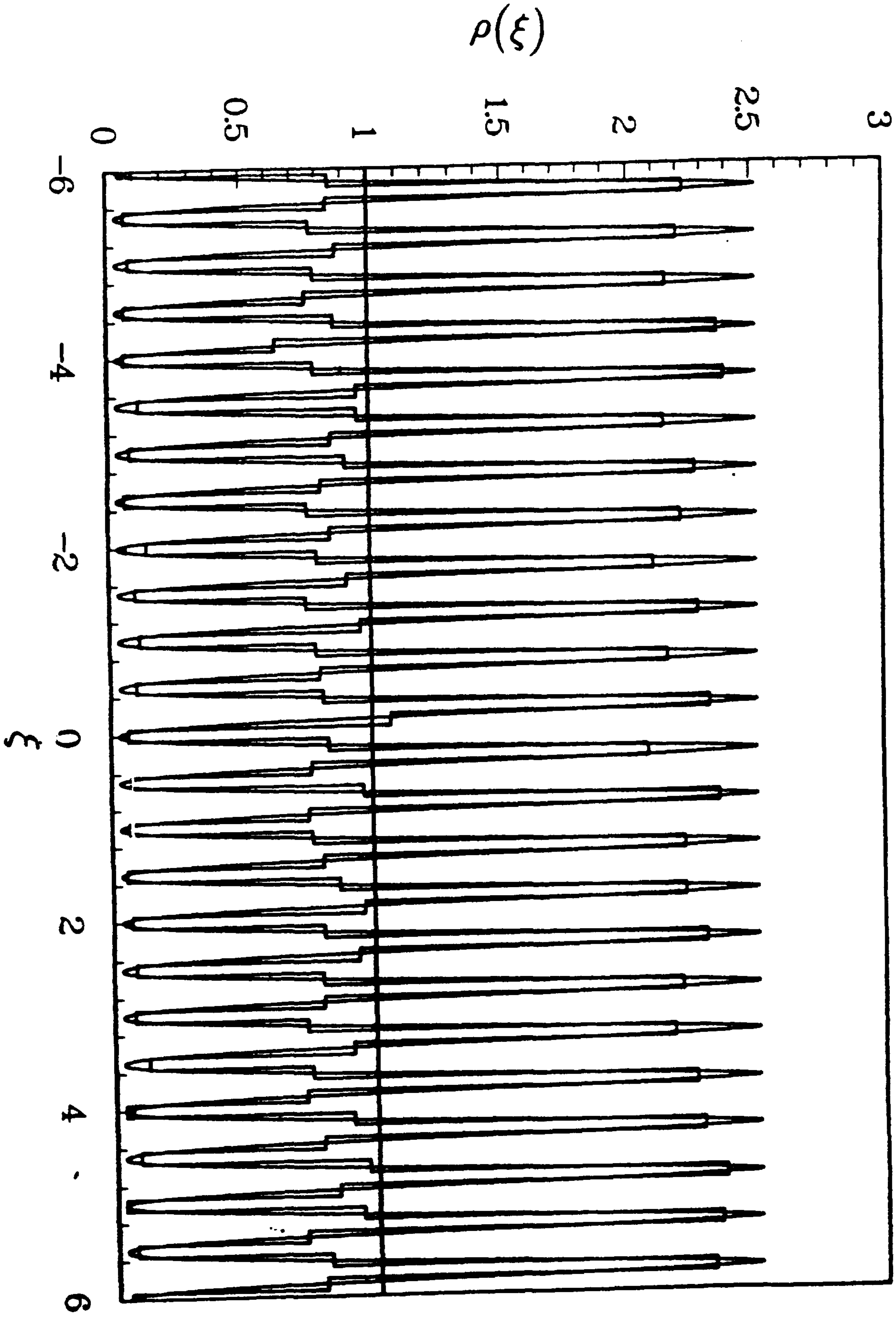

Figure 6

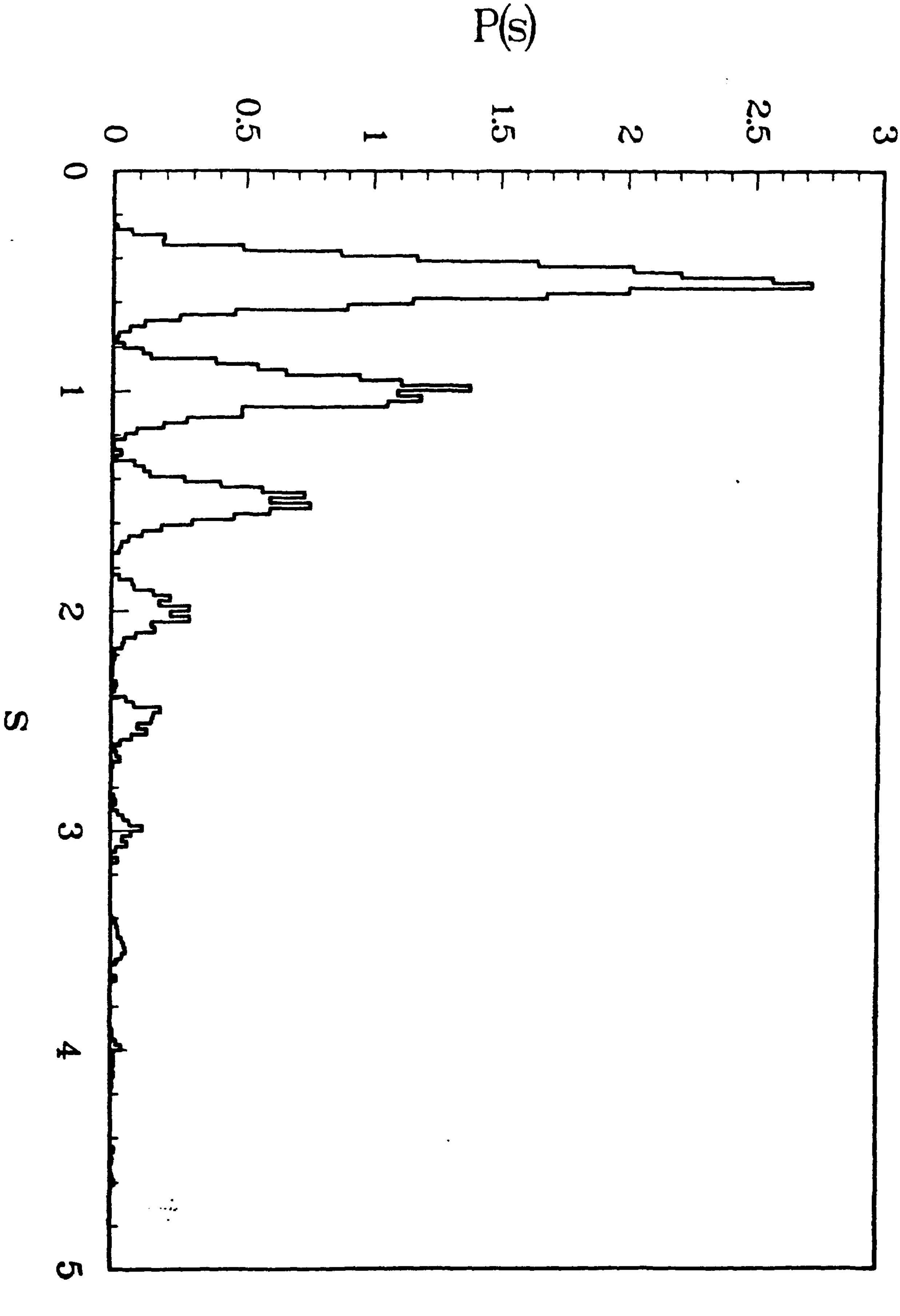

Figure 7

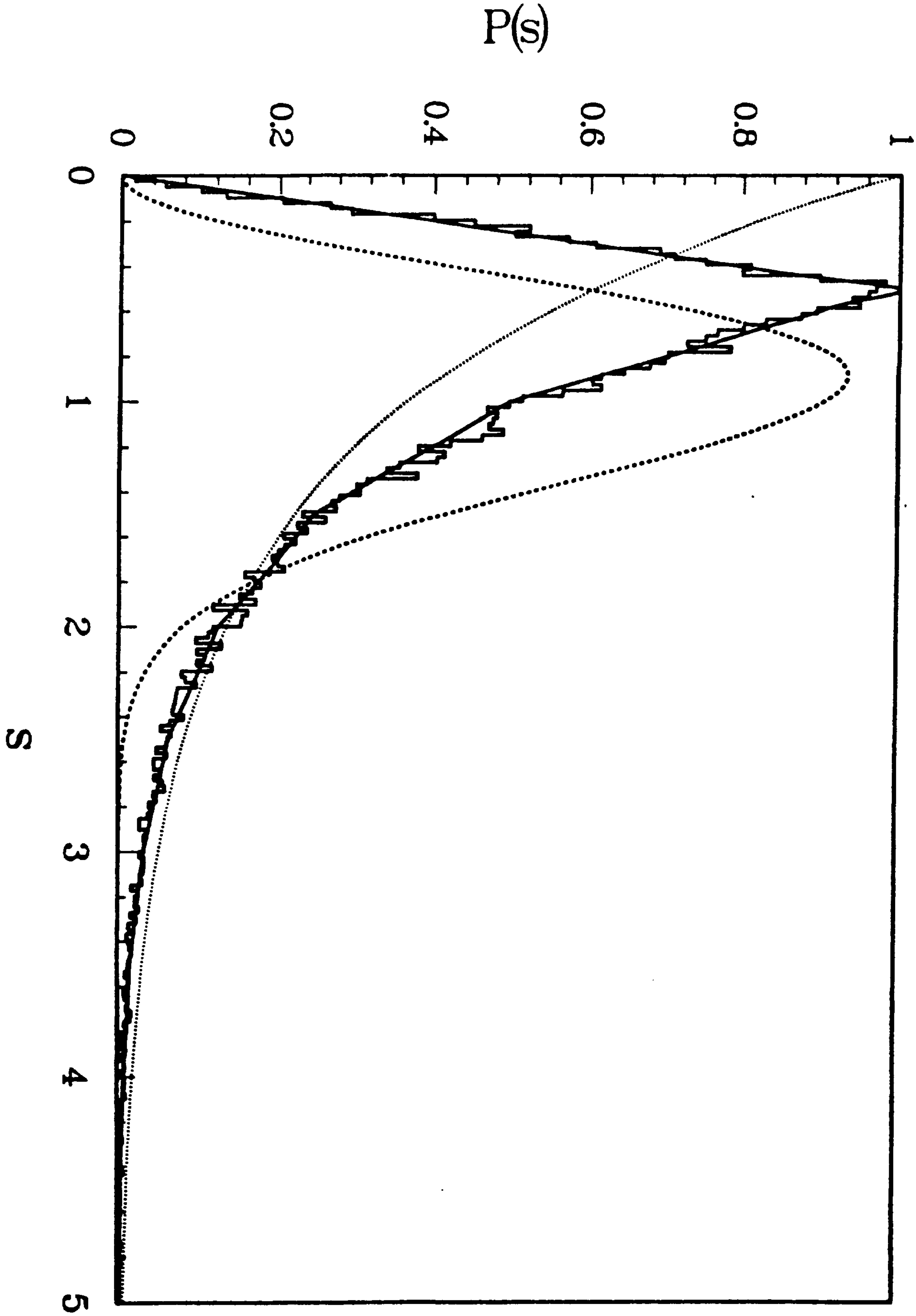

Figure 8

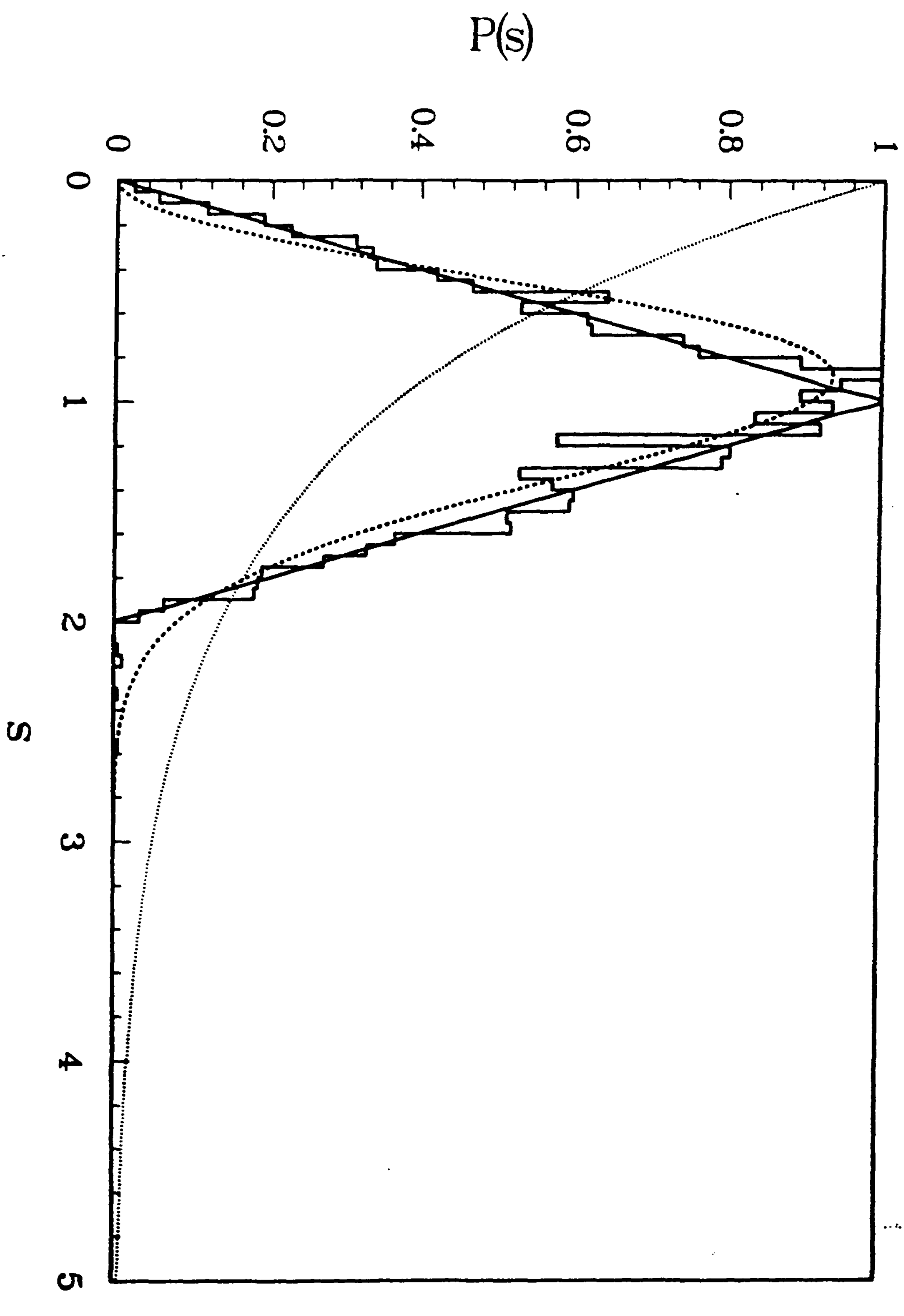

Figure 9