Quaternionic Formulation of the Exact Parity Model

S. P. Brumby, R. Foot and R. R. Volkas
Research Centre for High Energy Physics,
School of Physics, University of Melbourne,
Parkville, Victoria 3052, Australia
(February 28, 1996)

Abstract

The exact parity model (EPM) is a simple extension of the Standard Model which reinstates parity invariance as an unbroken symmetry of nature. The mirror matter sector of the model can interact with ordinary matter through gauge boson mixing, Higgs boson mixing and, if neutrinos are massive, through neutrino mixing. The last effect has experimental support through the observed solar and atmospheric neutrino anomalies. In this paper we show that the exact parity model can be formulated in a quaternionic framework. This suggests that the idea of mirror matter and exact parity may have profound implications for the mathematical formulation of quantum theory.

Typeset using REVTeX

I. INTRODUCTION

The exact parity model (EPM) is a simple extension of the Standard Model which reinstates parity invariance as an exact symmetry of nature [1,2]. The mirror matter sector of the model can interact with ordinary matter through gauge boson mixing, Higgs boson mixing and, if neutrinos are massive, through neutrino mixing. The last effect has experimental support through the observed solar and atmospheric neutrino anomalies [3]. In this paper we show that the exact parity model can be formulated in a quaternionic framework. This suggests that the idea of mirror matter and exact parity may have profound implications for the mathematical formulation of quantum theory.

The Standard Model (SM), and any extension thereof, can be augmented with a mirror matter sector in order to reinstate parity invariance as an exact symmetry (neither explicitly nor spontaneously broken). Take the SM as a concrete example. The gauge group $G_{SM} = SU(3)_c \otimes SU(2)_L \otimes U(1)_Y$ of the SM is first extended by postulating that the gauge theory of the world is $G_{SM} \times G'_{SM}$. The ordinary fermion and Higgs fields of the SM are placed into the usual representations of G_{SM} and are taken to be singlets under G'_{SM}. One then introduces *mirror matter* as fields that are singlets under G_{SM} but have the standard assignments under G'_{SM}. In order to make the theory parity symmetric, the mirror fermion fields are taken to have the opposite chirality to the ordinary fermion fields, and an exact parity symmetry which interchanges ordinary and mirror fields is introduced. It is simple to show that this non-standard parity symmetry is not spontaneously broken by the Higgs potential for a large range of parameters [2].

Ordinary and mirror matter share common gravitational interactions, which makes mirror matter interesting for cosmology [4]. Perhaps of even more interest is that ordinary and mirror matter will generically also interact through non-gravitational interactions. In the case of the exact parity extension of the SM these non-gravitational interactions are induced by photon–mirror-photon, Z–mirror-Z and Higgs mixing. Photon mixing causes mirror particles to have ordinary electric charge. Since mirror matter has yet to be observed electromagnetically, an upper bound on the photon mixing parameter can be derived [5]. This phenomenological constraint forces the ordinary electric charges of mirror particles to be very small, or "minicharged" using terminology in the literature. A very interesting and characteristic prediction of the EPM is thus that there will exist mirror leptons and quarks which are mass degenerate with ordinary leptons and quarks, and which have exactly the same electric charge *ratios* as the ordinary fermions. The Z boson mixing phenomenon is constrained to be very small because it is controlled by the same parameter that induces photon mixing. This effect is so small that it is phenomenologically uninteresting. Higgs boson mixing also yields a characteristic prediction: because mass eigenstates must also be parity eigenstates, the physical neutral Higgs boson will be maximally mixed with its mirror partner. Thus each mass eigenstate Higgs boson will decay half of the time into ordinary matter and the other half into mirror matter [2,6]. This prediction is testable provided that the Higgs boson mixing parameter is large enough.[1]

---

[1]Actually, there will be quite strong constraints on the Higgs boson mixing parameter if one

If neutrinos are massive, then ordinary and mirror neutrinos will generically mix. The exact parity symmetry of the theory forces the mixing angle between and ordinary neutrino and its mirror partner to be maximal just as for Higgs boson mixing. In the absence of large inter-generational neutrino mixing (as suggested by the observed form of the Kobayashi-Maskawa matrix), this leads for a large range of parameters to the observationally supported prediction that the flux of both solar and atmospheric neutrinos should be half of the standard expectation. This scenario will be tested further by the forthcoming SNO, SuperKamiokande and Borexino experiments, as well as by the continuation of the GALLEX and SAGE experiments.

The EPM is thus of great interest for both theoretical and phenomenological reasons. In this paper, we wish to explore a possible connection between the EPM and quaternions. Our starting point is the following observation: In the EPM, every particle has a distinct mirror partner. In other words, the number of degrees of freedom is exactly doubled. Our idea is that this doubling of degrees of freedom may be related to the doubling of degrees of freedom concomitant with the extension of complex quantum mechanics to quaternionic quantum mechanics. Therefore, we will introduce quaternionic fields and construct our theory so that the extra fields correspond to the particles in the mirror sector of the EPM.

While the fields in our theory will be quaternionic, we will consider the underlying quantal structure to be that of ordinary complex quantum mechanics. That is, we will ascribe special significance to a particular pure imaginary unit quaternion and identify it with the i of normal (complex) quantum mechanics.

Although there are some interesting ideas in the literature about how one may write down a genuinely quaternionic quantum theory [7-10], there is no empirical evidence which would allow the selection of the best candidate. Furthermore, it is possible to conceive of mechanisms by which quaternionic processes operating in the high energy (or short distance) regime become negligible in a low energy (or spatially asymptotic) limit [9] retrieving standard complex dynamics but with possible extra (remnant, physical) degrees of freedom. We will therefore take it as premature to try to develop the EPM on the basis of a full quaternionic quantal structure, although we do not preclude this as a future development. Indeed we hope that the present work may act as a spur for such developments. For the moment we will take the first step: to rewrite the usual Lagrangian of the EPM using quaternionic fields. The quaternionic parts of each individual term in the Lagrangian will however exactly cancel with the quaternionic parts of another term in the Lagrangian. In this sense, we will not be doing anything radical. However, we will set the stage for a fully quaternionic treatment. We hope that you will agree with us that even our limited exercise leads to interesting mathematics, and that interesting physics may develop from it in the future. It is certainly non-trivial that the EPM can be rewritten in this particular manner.

Before we begin, we need to state our notational conventions. We choose the metric tensor to be given by $g^{\mu\nu} = diag(-1, 1, 1, 1)$, with the Dirac matrices obeying the usual relations

---

demands that the mirror sector does not come into equilibrium with the ordinary sector in the early universe (in the context of the Hot Big Bang model). It may be that these constraints imply that the physical Higgs boson mass eigenstates cannot be resolved in experiment.

$\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}$. We will work in the Majorana representation for the Dirac matrices. With the above metric, each of the γ^μ is a real matrix in the Majorana representation. Further, $\gamma^{0T} = -\gamma^0$ while $\gamma^{iT} = \gamma^i$. The matrix $\gamma_5 \equiv i\gamma^0\gamma^1\gamma^2\gamma^3$ is pure imaginary. The fact that the γ^μ's are real while γ_5 is imaginary will prove extremely convenient when we need to commute quaternion units with Dirac matrices.

II. LAGRANGIAN FOR A FREE FERMION

Consider a fermionic field Ψ which is quaternion valued. We can write Ψ as

$$\Psi = \psi_0 + i\psi_1 + j\psi_2 + k\psi_3, \tag{1}$$

where $\psi_0, ..., \psi_3$ are real-valued and i, j and k are the quaternion units. They obey the quaternionic multiplication rules,

$$i^2 = j^2 = k^2 = -1 \quad \text{and} \quad ijk = -1. \tag{2}$$

Note that the quaternions are not commutative.

As discussed in the Introduction, we will not be pursuing quaternionic quantum mechanics as such. Instead we choose i to play the usual role of $\sqrt{-1}$ in quantum mechanics. That is, i commutes with all the operators corresponding to generalised co-ordinates and their conjugate momenta, and appears in all the fundamental commutators. We anticipate that complex analytical tools (such as the Fourier transform) can be extended to our quaternionic Hilbert space, with bells and whistles attached, but our present work will be limited to recasting the standard Lagrangian of the (complex) EPM into an exactly equivalent form written in terms of quaternions, and we shall avoid attempting to formulate a full theory of quantised, quaternionic fields[2].

Having singled out i to play a special role, we take advantage of the symplectic representation of Ψ, whereby

$$\Psi = \psi + j\psi'. \tag{3}$$

The fields $\psi \equiv \psi_0 + i\psi_1$ and $\psi' \equiv \psi_2 - i\psi_3$ are "complex"-valued fields (i.e., complex with respect to the sub-algebra generated by 1 and i).

We will first rewrite the Lagrangian for a free Dirac fermion in terms of Ψ. It is given by $\mathcal{L}_K$ where

$$\mathcal{L}_K = \frac{1}{2}(i\overline{\Psi}\partial\!\!\!/L\Psi + \overline{\Psi}\partial\!\!\!/L\Psi i), \tag{4}$$

where $\partial\!\!\!/ \equiv \gamma^\mu\partial_\mu$, and L is the usual left-handed chiral projection operator $(1 - \gamma_5)/2$. Note that the second term above is the Hermitian conjugate of the first term, and it follows from

---

[2]There is no simple quaternionic generalisation of Fourier transform technology [11]. Adler [9] has proposed using (real) Fourier-sine and -cosine transforms.

the algebraic relations amongst the quaternionic units, Eqs (2), that we have completely removed the j and k dependence of the Lagrangian.

We can equally well write this Lagrangian as

$$\mathcal{L}_K = \frac{1}{2}(i\overline{\Psi}_L \not{\partial}\Psi_L + \overline{\Psi}_L \not{\partial}\Psi_L i), \tag{5}$$

where $\Psi_L \equiv L\Psi$ is a chiral quaternionic fermion field.

The relevance of quaternions to the notion of exact parity can be simply illustrated by examining Ψ_L more closely. We know that

$$\Psi_L \equiv L\Psi = L\psi + Lj\psi' = L\psi + jR\psi', \tag{6}$$

where $R \equiv (1 + \gamma_5)/2$ is the usual right-handed chiral projection operator. Since γ_5 in the adopted notational scheme is pure imaginary, it anticommutes with the quaternion unit j. We have therefore established that the left-handed chiral quaternionic spinor Ψ_L is a symplectic linear combination of an ordinary left-handed Weyl spinor $\psi_L \equiv L\psi$ and an ordinary right-handed Weyl spinor $\psi'_R \equiv R\psi'$.

Under the parity transformation $x^\mu \rightarrow x_\mu$, $\psi_L \rightarrow -\gamma^0 \psi'_R$ and $\psi'_R \rightarrow \gamma^0 \psi_L$ (the minus sign in this transformation ensures that it is a Z_2 transformation). There are two useful ways to establish this. First one can simply substitute for Ψ in terms of the complex-valued fields ψ and ψ' in $\mathcal{L}_K$. The result is

$$\mathcal{L}_K = i\overline{\psi}\not{\partial}L\psi + i\overline{\psi}'\not{\partial}R\psi' = i\overline{\psi}_L\not{\partial}\psi_L + i\overline{\psi}'_R\not{\partial}\psi'_R \tag{7}$$

which is manifestly parity invariant in the usual way. There is also an elegant quaternionic derivation. The parity transformation written in terms of the quaternionic field Ψ is

$$x^\mu \rightarrow -x_\mu \quad \text{and} \quad \Psi_L \rightarrow j\gamma^0\Psi_L. \tag{8}$$

It is easy to show that $\mathcal{L}_K$ is invariant under this transformation. If $\Psi_L \rightarrow j\gamma^0\Psi_L$ is written in terms of the symplectic components one obtains $\psi_L \rightarrow -\gamma^0\psi'_R$ and $\psi'_R \rightarrow \gamma^0\psi_L$ which is just the usual parity transformation.

The simple calculation above establishes an interesting connection between parity symmetry and quaternions. However, as is well known there are many ways to extend the SM Lagrangian into a parity symmetric form. One way is through the use of the idea of "left-right symmetry" [12] and another is the mirror matter formulation we are interested in here. The two are distinguished for three reasons. First, for phenomenological reasons left-right symmetry can only be a symmetry of the Lagrangian, not of the vacuum. By contrast, parity invariance as implemented through mirror matter can be a symmetry of both the Lagrangian and the vacuum. Second, the left-right symmetric model sees parity-partner fields sharing common $SU(3)_c \otimes U(1)_{B-L}$ gauge interactions, while mirror matter is constructed to be neutral under all G_{SM} gauge forces (modulo the neutral gauge boson mixing effects mentioned earlier). Third, in left-right symmetric models the chiral partners forming a Dirac fermion are also parity partners, whereas the parity partners in the EPM are *not* also mass partners. Does the quaternionic formulation of parity distinguish between these two scenarios?

5

III. FERMION MASS

We will begin to answer this question by examining the third of the distinguishing features listed above. Let us examine the varieties of fermion mass possible in the quaternionic framework.

One may construct meaningful mass terms purely from the chiral quaternionic field Ψ_L. These are in a sense the simplest mass terms one can have, because they do not involve the introduction of additional degrees of freedom through a right-handed quaternionic partner field Ψ_R. Consider the term

$$B_1 \equiv \overline{\Psi}_L j \Psi_L. \tag{9}$$

This term is Lorentz invariant and nonzero, because the right chiral projector R that multiplies $\overline{\Psi}$ from the right turns into L after commutation through j: $B_1 = \overline{\Psi} j L \Psi$. Furthermore, it is Hermitian as can be seen from the following simple computation:

$$\left(\Psi_L^\dagger \gamma^0 j \Psi_L\right)^\dagger = \Psi_L^\dagger(-j)\gamma^{0\dagger}\Psi_L = \Psi_L^\dagger \gamma^0 j \Psi_L, \tag{10}$$

where in the last step we have used the fact that $\gamma^{0\dagger} = -\gamma^0$ in our notation and that j commutes with the real matrix γ^0. Finally, B_1 is parity symmetric under Eq. (8). In terms of symplectic components,

$$B_1 = \overline{\psi}'_R \psi_L - \overline{\psi}_L \psi'_R + \overline{\psi}_L j \psi_L + \overline{\psi}'_R j \psi'_R, \tag{11}$$

where the minus sign between the first two terms combines with the phase factor of -1 obtained from Eq. (8) to render the first two terms a parity symmetric pair. The last two terms are also obviously parity symmetric. However, the last two terms are j-dependent and our programme requires that we expunge them. In a true quaternionic quantum mechanics it may be interesting to ascribe physical significance to such peculiar j-dependent mass terms, but for our present purposes we must render them unphysical. This is achieved by a brute force cancellation. We simply require that the physically allowed mass term modelled on B_1 be given by $\mathcal{B}_1$ [3] where

$$\mathcal{B}_1 \equiv \frac{1}{2}(B_1 - iB_1 i). \tag{12}$$

The relation $-iji = -j$ then ensures that all j-dependent pieces of B_1 are systematically cancelled by the second term. In symplectic components we then simply have that

$$\mathcal{B}_1 = \overline{\psi}'_R \psi_L - \overline{\psi}_L \psi'_R. \tag{13}$$

Note that this term identifies parity partners as also mass partners. It is therefore suitable for use in a possible quaternionic formulation of the usual left-right symmetric model, but

---

[3]From a mathematical point of view, we are projecting the mass term into the i-complex subalgebra of the quaternions. The properties of quaternionic mappings of the type $Q \rightarrow (Q - qQq)/2$ are investigated by de Leo, et al [13].

not for the EPM. It will ultimately be gauge invariance that distinguishes between left-right symmetry and the EPM. The above mass term will be chosen to be gauge invariant in the left-right symmetric case (after electroweak symmetry breaking), but it will not be gauge invariant in the EPM.

There are several other suitable fermion bilinears. We will discuss them all below, but we will not provide the (simple) derivations of their stated properties as we did in the warm up example above.

Our second bilinear is

$$\mathcal{B}_2 \equiv \frac{1}{2}(B_2 - iB_2i), \tag{14}$$

where

$$B_2 \equiv \overline{\Psi}_L k \Psi_L. \tag{15}$$

This is simply the parity-odd version of $\mathcal{B}_1$. A linear combination of $\mathcal{B}_1$ and $\mathcal{B}_2$ with arbitrary real coefficients will then yield the most general hermitian Dirac mass term connecting the symplectic components.

The next class of bilinears provide Majorana masses. In the Majorana representation of the Dirac matrices, the charge conjugation matrix $\mathcal{C}$ is simply given by

$$\mathcal{C} = -i\gamma^0. \tag{16}$$

Using this matrix we can write down the charge-conjugate of Ψ_L, given by

$$(\Psi_L)^c \equiv \mathcal{C}\left(\overline{\Psi}_L\right)^T = (\psi_L)^c + j(\psi_R')^c. \tag{17}$$

We therefore see that the $(\Psi_L)^c$ has opposite chirality to Ψ_L as usual.

The parity even or symmetric Majorana bilinear is given by

$$\mathcal{B}_3 \equiv \frac{1}{2}(B_3 - iB_3i) + H.c. \tag{18}$$

where

$$B_3 \equiv \overline{\Psi}_L(\Psi_L)^c. \tag{19}$$

The parity odd term is

$$\mathcal{B}_4 \equiv \frac{1}{2}(B_4 - iB_4i) + H.c. \tag{20}$$

where

$$B_4 \equiv i\overline{\Psi}_L i(\Psi_L)^c. \tag{21}$$

A general Majorana mass is then an arbitrary linear combination of the two terms above, giving different Majorana masses to ψ_L and ψ_R'.

7

This concludes the catalogue of all acceptable mass terms that can be formed using Ψ_L only. We have obviously covered all of the possibilities offered by the ordinary symplectic components ψ_L and ψ'_R: all varieties of acceptable mass involving these fields are included above. We can therefore be sure that we have left no independent possibilities at the quaternionic level unaccounted for.

We now introduce a right-handed quaternionic spinor $\Psi_R = \psi_R + j\psi'_L$ in addition to Ψ_L. The complete list of Lorentz-invariant, Hermitian and nonzero terms that do not involve charge conjugation is:

$$B_5 \equiv i\overline{\Psi}_L\Psi_R + \overline{\Psi}_R\Psi_L i, \tag{22}$$

$$B_6 \equiv \overline{\Psi}_L i\Psi_R + \overline{\Psi}_R i\Psi_L, \tag{23}$$

$$B_7 \equiv j\overline{\Psi}_L\Psi_R + \overline{\Psi}_R\Psi_L j, \tag{24}$$

$$B_8 \equiv \overline{\Psi}_L\Psi_R j + j\overline{\Psi}_R\Psi_L, \tag{25}$$

$$B_9 \equiv k\overline{\Psi}_L\Psi_R + \overline{\Psi}_R\Psi_L k, \tag{26}$$

$$B_{10} \equiv \overline{\Psi}_L\Psi_R k + k\overline{\Psi}_R\Psi_L. \tag{27}$$

Other possibilities such as $\overline{\Psi}_L j\Psi_R + \overline{\Psi}_R j\Psi_L$ are identically equal to zero. These terms are then made j-independent using the trick introduced earlier. One then obtains the allowed set

$$\mathcal{B}_{5-10} \equiv \frac{1}{2}(B_{5-10} - iB_{5-10}i), \tag{28}$$

of standard mass terms rewritten in terms of quaternionic fields.

In symplectic components,

$$\mathcal{B}_5 = i(\overline{\psi}_R\psi_L + \overline{\psi}'_L\psi'_R + \overline{\psi}_L\psi_R + \overline{\psi}'_R\psi'_L). \tag{29}$$

These terms produce a degenerate pair of Dirac fermions given by $\psi \equiv \psi_L + \psi_R$ and $\psi' \equiv \psi'_L + \psi'_R$. (The unusual overall factor of i is simply due to our chosen notational scheme.) Furthermore, using Eq. (8) one sees that $\mathcal{B}_5$ is parity invariant. The interesting observation is that the parity partner of, say, ψ_L is ψ'_R rather than its mass partner ψ_R. This type of mass term is precisely the one we will need to use to construct the quaternionic reformulation of the EPM.[4]

The term $\mathcal{B}_6$ is just the parity-odd version of $\mathcal{B}_5$. In terms of symplectic components it is given by

---

[4]One can define another version of the parity transformation as $\Psi_{L,R} \to i\gamma^0\Psi_{R,L}$ under which $\mathcal{B}_5$ is also invariant. This is actually another way of defining standard parity, because the parity partners $\psi_{L,R}$ and $\psi'_{L,R}$ are again also mass partners. However this definition of standard parity is only possible if the number of degrees of freedom is doubled (one has ψ' as well as ψ). Clearly $\mathcal{B}_5$ is invariant under both standard parity and the mirror matter version of parity. With it one could construct a left-right symmetric model augmented by a mirror matter sector, a perfectly consistent and reasonable model which however we will not pursue any further. Who needs a broken mirror in addition to a good mirror when just one good mirror may be enough to reflect reality?

$$\mathcal{B}_6 = i(\overline{\psi}_R \psi_L - \overline{\psi}'_L \psi'_R + \overline{\psi}_L \psi_R - \overline{\psi}'_R \psi'_L). \tag{30}$$

If parity symmetry was not imposed on the Lagrangian, then the most general mass Lagrangian would be $m\mathcal{B}_5 + m'\mathcal{B}_6$. The ordinary complex Dirac fermions ψ and ψ' would then no longer be mass degenerate.

The term $\mathcal{B}_7$ is an interesting case. Focus first of all on just the $j\overline{\Psi}_L \Psi_R$ part. A simple computation shows that

$$j\overline{\Psi}_L \Psi_R = j\overline{\psi}_L j\psi'_L - j\overline{\psi}_R j\psi'_R + j(\overline{\psi}_L \psi_R + \overline{\psi}'_R \psi'_L). \tag{31}$$

The latter two terms have a nonzero quaternionic piece which will be cancelled by other terms in $\mathcal{B}_7$, so we can ignore them. To interpret the remaining terms, it is necessary to first remove the j's by bringing them to adjacent positions and then using $j^2 = -1$. One finds that

$$j\overline{\psi}_L j\psi'_L = -\psi_L^T \gamma^0 \psi'_L = i\overline{(\psi_L)^c}\psi'_L. \tag{32}$$

The term $\mathcal{B}_7$ therefore describes off-diagonal Majorana masses linking ψ with ψ'.

The other terms $\mathcal{B}_{8,9,10}$ just describe other possibilities for these off-diagonal Majorana masses. For instance $\mathcal{B}_8$ is the parity partner of $\mathcal{B}_7$. Similarly $\mathcal{B}_9$ and $\mathcal{B}_{10}$ are parity partners. Adding $\mathcal{B}_7 - \mathcal{B}_{10}$ with arbitrary coefficients produces the most general allowed off-diagonal Majorana masses.

The ingredients $\psi_{L,R}$ and $\psi'_{L,R}$ can only produce the Dirac and Majorana masses covered by $\mathcal{B}_5 - \mathcal{B}_{10}$ above. We can therefore be sure that all meaningful independent mass terms that one can write using quaternionic fields are included in this list. Any other candidates one could think of, such as $\overline{\Psi}_L j(\Psi_R)^c$ for example, cannot be independent of the terms already considered.

To summarise this section then, we have shown above that both left-right symmetric parity and mirror matter parity can be accommodated by mass terms formed out of quaternionic spinors. Parity asymmetric masses can also be included. If $\mathcal{B}_5$ is chosen as the preferred mass, then the symplectic components of the fermion fields form the ordinary and mirror sectors. It will ultimately be gauge invariance that distinguishes between these cases. It should also be clear that the symplectic components of a quaternionic spinor can be constructed to be fermions that interchange under any Z_2 discrete symmetry. As well as parity, one can also incorporate discrete symmetries such as charge conjugation or a Z_2 horizontal symmetry. We will not explicitly consider these possibilities here, but the reader should be aware of them.

If parity symmetry (or any other Z_2 symmetry) is not imposed, then the fermions in the theory still form two sectors, but there is no symmetry between them and thus no mass degeneracy. In the context of the SM, the parity non-invariant quaternionic formulation can for instance lead to a $G_{SM} \otimes G'_{SM}$ gauge theory with a completely independent set of Yukawa coupling constants and other parameters for each of the sectors. We will henceforth have to explicitly impose parity invariance on the theory.

9

IV. GLOBAL ABELIAN TRANSFORMATIONS

Let us now consider symmetry transformations on the quaternionic fermion field Ψ_L. We begin with global Abelian transformations. Because the fermion field Ψ_L is quaternionic (and hence non-commutative), there are two possible phase invariances: multiplication on the left and multiplication on the right. If we do these simultaneously the most general transformation is

$$\Psi_L \rightarrow e^{i\alpha}\Psi_L e^{i\beta}, \tag{33}$$

where α and β are independent of each other.[5] In symplectic components this transformation is

$$\psi_L + j\psi_R' \rightarrow e^{i(\alpha+\beta)}\psi_L + je^{i(-\alpha+\beta)}\psi_R', \tag{34}$$

where $e^{i\alpha}$ has been commuted through j. The complex fields themselves therefore simply undergo two independent Abelian transformations:

$$\psi_L \rightarrow e^{i(\alpha+\beta)}\psi_L \quad \text{and} \quad \psi_R' \rightarrow e^{i(-\alpha+\beta)}\psi_R'. \tag{35}$$

This is of course exactly what we need in order to have ordinary and mirror matter transform under independent groups (the two independent U(1)$_Y$ factors of $G_{SM} \otimes G_{SM}'$ for instance).

Furthermore, it is clear from Eqs. (35) and (7) that $\mathcal{L}_K$ is invariant under these global Abelian transformations. A problem that will continue to arise in the ensuing discussion is how best to establish the invariance of a candidate Lagrangian that is written in terms of quaternionic fields. The observation we have just made about the invariance of $\mathcal{L}_K$ under global Abelian transformations relied on us decomposing the quaternionic fields into their complex components. Although it is rather helpful to do this, it would be more elegant if an explicitly quaternionic way could be found to prove invariance. We will make some progress towards this goal, although we will see that in many ways it will remain easier to prove invariance at the complex rather than at the quaternionic level. We regard the use of complex decomposition as an unfortunate device, and it is to be hoped that a transparent quaternionic method eventually gets constructed.

In order to facilitate all further discussion of this and other issues, we now introduce a simple but powerful idempotent operator technology. Consider operators $\hat{C}_+$ and $\hat{C}_-$ defined by [13]

$$\hat{C}_+\Psi_L \equiv -\frac{i}{2}\{i, \Psi_L\} = \psi_L \quad \text{and} \quad \hat{C}_-\Psi_L \equiv -\frac{i}{2}[i, \Psi_L] = j\psi_R'. \tag{36}$$

These operators obey the relations

$$[i, \hat{C}_\pm] = 0 \,,$$

10

$$\hat{C}_+^2 = \hat{C}_+, \quad \hat{C}_-^2 = \hat{C}_-, \tag{37}$$

$$\hat{C}_+\hat{C}_- = \hat{C}_-\hat{C}_+ = 0, \quad \hat{C}_+ + \hat{C}_- = 1. $$

They therefore project out the symplectic components from Ψ_L, with $\hat{C}_+$ projecting out ψ_L and $\hat{C}_-$ projecting out $j\psi_R'$. The global U(1)$\otimes$U(1) transformation can therefore be rewritten as

$$\Psi_L \to e^{i(\alpha+\beta)}\hat{C}_+\Psi_L + e^{i(\alpha-\beta)}\hat{C}_-\Psi_L. \tag{38}$$

Two special cases are of note. If $\beta = 0$, then the transformation is, trivially,

$$\Psi_L \to e^{i\alpha}(\hat{C}_+ + \hat{C}_-)\Psi_L = e^{i\alpha}\Psi_L, \tag{39}$$

as required. If $\alpha = 0$, the transformation is

$$\Psi_L \to (e^{i\beta}\hat{C}_+ + e^{-i\beta}\hat{C}_-)\Psi_L, \tag{40}$$

which can be more neatly written as

$$\Psi_L \to e^{i\beta(\hat{C}_+ - \hat{C}_-)}\Psi_L = \Psi_L e^{i\beta} \tag{41}$$

using Eqs (37). The idempotent operators can therefore be used to rewrite multiplication on the right by a phase as multiplication on the left by an operator valued phase. This will be useful in writing non-Abelian transformations.

An instructive digression is warranted at this point. The idempotent operator formalism just discussed is closely analogous to the case of Abelian chiral transformations. Consider a standard massless Dirac field $\chi = \chi_L + \chi_R$ where the chiral projection operators L and R are analogous to $\hat{C}_+$ and $\hat{C}_-$. We know that the Lagrangian of a free massless Dirac field is invariant under independent phase rotations of its left and right handed components. This can written as $\chi \to \exp[i(\alpha + \beta)]L\chi + \exp[i(-\alpha + \beta)]R\chi$ which is similar to Eq. (38). If $\beta = 0$ then $\chi \to \exp(i\alpha)(L + R)\chi = \exp(i\alpha)\chi$. If $\alpha = 0$ then

$$\chi \to (e^{i\beta}L + e^{-i\beta}R)\chi = e^{i\beta(L-R)}\chi = e^{-i\beta\gamma_5}\chi. \tag{42}$$

One useful perspective is that the operators $\hat{C}_+$ and $\hat{C}_-$ project out the parity partners that form a chiral quaternionic spinor, while L and R project out the parity partners that form an ordinary complex Dirac fermion. Another perspective is that $\hat{C}_+$ and $\hat{C}_-$ project out irreducible representations of the Lorentz group from the reducible representation under which Ψ_L transforms, while L and R project out the irreducible parts of χ.

We will now revisit the proof of the invariance of $\mathcal{L}_K$ under global Abelian transformations, and use the idempotent operators to construct a quaternionic derivation. We first note that multiplication of Ψ_L on the left by $e^{i\alpha}$ poses no problems; invariance at the quaternionic level is manifest. Our focus is thus necessarily on multiplication by $e^{i\beta}$ on the right as per Eq. (41). By Hermitian conjugation Eq. (41) implies that

$$\overline{\Psi}_L \to \overline{\Psi}_L e^{-i\beta(\hat{C}_+^\dagger - \hat{C}_-^\dagger)}, \tag{43}$$

11

where the action of $\hat{C}_\pm^\dagger$ on $\overline{\Psi}_L$ follows immediately from the definition of the operators:

$$\overline{\Psi}_L \hat{C}_+^\dagger = \overline{\psi}_L \quad \text{and} \quad \overline{\Psi}_L \hat{C}_-^\dagger = -\psi_R' j. \tag{44}$$

The technical obstacle we face is what to do with the statement that

$$\overline{\Psi}_L \partial\!\!\!/ \Psi_L \to \overline{\Psi}_L e^{-i\beta(\hat{C}_+^\dagger - \hat{C}_-^\dagger)} e^{i\beta(\hat{C}_+ - \hat{C}_-)} \partial\!\!\!/ \Psi_L, \tag{45}$$

where it is not immediately clear how to combine the two exponentials in order to establish gauge invariance. Note in particular that $\hat{C}_\pm$ are *not* Hermitian (and are thus not projection operators in the strict mathematical sense).

However, using Eq. (40) we easily see that

$$e^{-i\beta(\hat{C}_+^\dagger - \hat{C}_-^\dagger)} e^{i\beta(\hat{C}_+ - \hat{C}_-)} = \hat{C}_+^\dagger \hat{C}_+ + \hat{C}_-^\dagger \hat{C}_- + e^{-2i\beta} \hat{C}_+^\dagger \hat{C}_- + e^{2i\beta} \hat{C}_-^\dagger \hat{C}_+. \tag{46}$$

The first two terms are β-independent and thus constitute the invariant part of the product of the exponentials. The last two terms are however β-dependent and therefore *not* invariant. Notice, though, that the last two terms have products of $\hat{C}_+$ with $\hat{C}_-$ (up to Hermitian conjugation) and so they give rise to j-dependent terms when inserted between the quaternionic spinors in $\mathcal{L}_K$. Since all j-dependence cancels from $\mathcal{L}_K$, and since the non-invariant terms are necessarily j-dependent, we conclude that the invariance of the free fermion Lagrangian can be established by the use of Eq. (46) together with the cancellation of j-dependence. This is the best we can do in proving invariance at the quaternionic level. It is certainly interesting that the j-dependent terms we systematically eliminate are also non-invariant, because it suggests that a true quaternionic quantum mechanics may break symmetries that are preserved by the complex subspace.

V. LOCAL ABELIAN TRANSFORMATIONS

The possibility of independent phase rotations acting on the left and right provides a natural motivation for the doubling of the gauge symmetry. Indeed, local U(1)⊗U(1) transformations are given by

$$\Psi_L(x) \to e^{i\alpha(x)} \Psi_L(x) e^{i\beta(x)} \tag{47}$$

or, equivalently,

$$\begin{aligned} \Psi_L(x) &\to e^{i[\alpha(x)+\beta(x)]} \hat{C}_+ \Psi_L(x) + e^{i[\alpha(x)-\beta(x)]} \hat{C}_- \Psi_L(x) \\ &= e^{i[\alpha(x)+\beta(x)(\hat{C}_+ - \hat{C}_-)]} \Psi_L(x). \end{aligned} \tag{48}$$

The quaternionic covariant derivative is given by

$$\mathcal{D}_\mu \Psi_L \equiv \partial_\mu \Psi_L + i\hat{A}_\mu \Psi_L, \tag{49}$$

where $\hat{A}_\mu$ is the quaternionic gauge field with gauge transformation law

$$\hat{A}_\mu \to \hat{A}_\mu - \partial_\mu \alpha - \partial_\mu \beta(\hat{C}_+ - \hat{C}_-). \tag{50}$$

12

It is interesting to display the detailed proof that the above really leads to a covariant derivative. What we need to establish is that $\mathcal{D}_\mu \Psi_L$ transforms the same way as Ψ_L, namely that

$$\mathcal{D}_\mu \Psi_L \to e^{i[\alpha + \beta(\hat{C}_+ - \hat{C}_-)]} \mathcal{D}_\mu \Psi_L. \tag{51}$$

Under a gauge transformation we have that

$$\begin{aligned}
\mathcal{D}_\mu \Psi_L \to{} & e^{i[\alpha + \beta(\hat{C}_+ - \hat{C}_-)]} \partial_\mu \Psi_L \\
& + i[\partial_\mu \alpha + \partial_\mu \beta (\hat{C}_+ - \hat{C}_-)] e^{i[\alpha + \beta(\hat{C}_+ - \hat{C}_-)]} \Psi_L \\
& + i[\hat{A}_\mu - \partial_\mu \alpha - \partial_\mu \beta (\hat{C}_+ - \hat{C}_-)] e^{i[\alpha + \beta(\hat{C}_+ - \hat{C}_-)]} \Psi_L \\
={} & e^{i[\alpha + \beta(\hat{C}_+ - \hat{C}_-)]} \partial_\mu \Psi_L + i\hat{A}_\mu e^{i[\alpha + \beta(\hat{C}_+ - \hat{C}_-)]} \Psi_L,
\end{aligned} \tag{52}$$

so the result we seek follows if $\hat{A}_\mu$ commutes with the exponential. Since

$$[\hat{C}_\pm, \hat{C}_+ - \hat{C}_-] = 0 \tag{53}$$

it follows that $\hat{A}_\mu$ can be written as a linear combination of the identity and $\hat{C}_\pm$ with vector fields as coefficients. Since the identity is equal to $\hat{C}_+ + \hat{C}_-$, we can without loss of generality write that

$$\hat{A}_\mu = g A_\mu \hat{C}_+ - g' A'_\mu \hat{C}_-, \tag{54}$$

where the parameters g and g' are coupling constants, while A_μ and A'_μ are real gauge fields. Using Eq. (50), we see that they have the transformation laws

$$A_\mu \to A_\mu - \frac{1}{g} \partial_\mu (\alpha + \beta) \tag{55}$$

and

$$A'_\mu \to A'_\mu - \frac{1}{g'} \partial_\mu (-\alpha + \beta). \tag{56}$$

The gauge invariant kinetic energy Lagrangian is

$$\mathcal{L}_{gK} = \frac{1}{2} (i \overline{\Psi}_L \slashed{\mathcal{D}} \Psi_L + \overline{\Psi}_L \slashed{\mathcal{D}} \Psi_L i). \tag{57}$$

The proof that this is gauge invariant follows exactly the same steps as the proof that the free fermion Lagrangian is invariant under global transformations, because we have a properly defined covariant derivative.

In symplectic components the quaternionic covariant derivative is given by

$$\mathcal{D}_\mu \Psi_L = D_\mu \psi_L + j D'_\mu \psi'_R, \tag{58}$$

and the kinetic energy Lagrangian is simply

$$\mathcal{L}_{gK} = i \overline{\psi}_L \slashed{D} \psi_L + i \overline{\psi}'_R \slashed{D}' \psi'_R, \tag{59}$$

where

$$D_\mu \psi_L \equiv \partial_\mu \psi_L + i g A_\mu \psi_L \quad \text{and} \quad D'_\mu \psi'_R \equiv \partial_\mu \psi'_R + i g' A'_\mu \psi'_R. \tag{60}$$

We will discuss the constraint due to parity invariance after we write down the gauge boson kinetic energy terms in the next section.

VI. GAUGE BOSON KINETIC ENERGY TERMS

The gauge field $\hat{\mathcal{A}}_\mu$ is a natural object to use when describing the coupling of gauge bosons to fermions, but it is not so appropriate for the description of the gauge boson kinetic energy terms. This is simply because $\hat{\mathcal{A}}_\mu$ depends on $\hat{C}_\pm$ which have been defined to act on quaternionic fields. Rather than trying to extend the definition of these operators to include action on, for instance, state vectors, it is more convenient to adopt the approach explained below.

Note that

$$(gA_\mu + jg'A'_\mu)(\hat{C}_+ + j\hat{C}_-) = \hat{\mathcal{A}}_\mu + j(gA_\mu\hat{C}_- + g'A'_\mu\hat{C}_+), \tag{61}$$

which allows us to rewrite the gauge invariant fermion kinetic energy Lagrangian as

$$\mathcal{L}_{gK} = \frac{1}{2}\left[i\overline{\Psi}_L\left(\not{\partial} + \frac{1}{2}(i\not{\mathcal{A}}\,\hat{C} + \not{\mathcal{A}}\,\hat{C}i)\right)\Psi_L + \overline{\Psi}_L\left(\not{\partial} + \frac{1}{2}(i\not{\mathcal{A}}\,\hat{C} + \not{\mathcal{A}}\,\hat{C}i)\right)\Psi_L i\right] \tag{62}$$

where

$$\mathcal{A}_\mu \equiv gA_\mu + jg'A'_\mu \tag{63}$$

and

$$\hat{C} \equiv \hat{C}_+ + j\hat{C}_-. \tag{64}$$

This is true because the j piece on the right-hand side of Eq. (61) cancels between the two terms in $\mathcal{L}_{gK}$. The field $\mathcal{A}_\mu$ is clearly a natural object in a sense, because it is the gauge boson analogue of Ψ_L. (Note that the symplectic components of $\mathcal{A}_\mu$ are real fields rather than complex fields.)

The gauge boson kinetic energy terms are now easy to write down in a quaternionic fashion. The field strength tensor $\mathcal{F}_{\mu\nu}$ is formed from $\mathcal{A}_\mu$ by applying the quaternionic covariant derivative Eq.(49),

$$\mathcal{F}_{\mu\nu} \equiv \mathcal{D}_\mu\mathcal{A}_\nu - \mathcal{D}_\nu\mathcal{A}_\mu. \tag{65}$$

This is well defined, as the $\hat{C}_\pm$ act on quaternion valued fields. In terms of the symplectic components of $\mathcal{A}$,

$$\mathcal{D}_\mu\mathcal{A}_\nu = (\partial_\mu + igA_\mu\hat{C}_+ - ig'A'_\mu\hat{C}_-)(gA_\nu + jg'A'_\nu) \tag{66}$$

$$= (g\partial_\mu A_\nu + ig^2 A_\mu A_\nu) + j(g'\partial_\mu A'_\nu + ig'^2 A'_\mu A'_\nu). \tag{67}$$

Hence, for our Abelian A_μ and A'_μ fields,

$$\mathcal{F}_{\mu\nu} = g(\partial_\mu A_\nu - \partial_\nu A_\mu) + jg'(\partial_\mu A'_\nu - \partial_\nu A'_\mu), \tag{68}$$

which is manifestly gauge invariant.

We define a complimentary gauge field by

14

$$\mathcal{A}_*^\mu \equiv gA^\mu - g'A'^\mu j. \tag{69}$$

The corresponding gauge covariant derivative is

$$\mathcal{D}_*^\mu \equiv \partial^\mu + \hat{A}_*. \tag{70}$$

$$\hat{A}_* \equiv i\mathcal{A}_*\hat{C} + \mathcal{A}_*\hat{C}i = gA\hat{C}_+ - g'A'\hat{C}_-. \tag{71}$$

Hence, the corresponding field strength tensor is

$$\mathcal{F}_*^{\mu\nu} \equiv \mathcal{D}_*^\mu\mathcal{A}_*^\nu - \mathcal{D}_*^\nu\mathcal{A}_*^\mu = g(\partial^\mu A^\nu - \partial^\nu A^\mu) - g'(\partial^\mu A'^\nu - \partial^\nu A'^\mu)j. \tag{72}$$

To form the gauge boson kinetic energy Lagrangian, we observe that

$$\mathcal{F}_*^{\mu\nu}\mathcal{F}_{\mu\nu} = g^2 F^{\mu\nu}F_{\mu\nu} + g'^2 F'^{\mu\nu}F'_{\mu\nu} + j \text{ term}. \tag{73}$$

If parity symmetry is imposed, then $g = g'$. Hence, the gauge boson kinetic energy Lagrangian is

$$-\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{4}F'_{\mu\nu}F'^{\mu\nu} = \frac{-1}{8g^2}(\mathcal{F}_*^{\mu\nu}\mathcal{F}_{\mu\nu} - i\mathcal{F}_*^{\mu\nu}\mathcal{F}_{\mu\nu}i). \tag{74}$$

Our final task in this section is to write down the quaternionic version of the gauge invariant kinetic energy mixing term between the gauge fields: $F^{\mu\nu}F'_{\mu\nu}$. This term is responsible for photon–mirror-photon and Z–mirror-Z mixing in the EPM (note that it is parity invariant). It is simply given by

$$F^{\mu\nu}F'_{\mu\nu} = \frac{-1}{8gg'}\left(i\mathcal{F}_*^{\mu\nu}ji\mathcal{F}_{\mu\nu} + \mathcal{F}_*^{\mu\nu}ji\mathcal{F}_{\mu\nu}i\right). \tag{75}$$

This term is then multiplied by an arbitrary constant and added to the diagonal kinetic energy terms for A_μ and A'_μ. The magnitude of photon–mirror-photon and Z–mirror-Z mixing is controlled by this parameter [2,5].

Before proceeding to non-Abelian transformations, we want to point out that the operator $\hat{C}$ is interesting in its own right. It belongs to a class of idempotent operators given by $\hat{C}_+ + \exp(ia)j\hat{C}_-$ where a is an arbitrary phase. Its action on quaternionic fields is simply

$$\hat{C}\Psi_R = \psi_R - \psi'_L. \tag{76}$$

It therefore maps a chiral quaternionic spinor onto a Dirac-like field. The field $\psi_R - \psi'_L$ is not a Dirac field in the usual sense because ψ_R and $-\psi'_L$ are not mass partners in the EPM. However, it behaves as a Dirac field under Lorentz transformations. Just like $\hat{C}_\pm$, $\hat{C}$ projects one representation of the Lorentz group onto another. Unlike $\hat{C}_\pm$, $\hat{C}$ projects a reducible representation onto another reducible representation. The arbitrary phase a just redefines the relative phases of the two chiral components of the Dirac-like field. We will meet another of these operators when we study Yukawa interactions.

VII. NON-ABELIAN TRANSFORMATIONS

We now turn to non-Abelian gauge transformations. Suppose there are a number of chiral quaternionic fermion fields which are placed into a column matrix Ψ_L. In the Abelian case we were able in a natural way to introduce two independent U(1) symmetries by left multiplication and right multiplication. What is the analogue of this for non-Abelian transformations? A sensible definition is obtained by writing down the non-Abelian extension of Eq. (48). The operators $\hat{C}_\pm$ allow one to write group action from the right as effectively group action from the left. The non-Abelian gauge transformation of the column matrix Ψ_L is then

$$\Psi_L \to e^{i\alpha^a T^a} e^{i\beta^b T^b(\hat{C}_+ - \hat{C}_-)} \Psi_L, \tag{77}$$

where the T's are the generators of the appropriate representation of an arbitrary group G, while the α's and β's are independent group parameters. For convenience we will not combine the two exponentials into one, although one could do so by using the Campbell-Baker-Hausdorff relation (this was of course trivial in the Abelian case). To understand what this transformation does, it is best to consider two special cases. Consider first the subset of transformations defined by $\alpha^a = \beta^a$. One then has

$$\begin{aligned}
\Psi_L &\to e^{i\alpha^a T^a} e^{i\alpha^b T^b(\hat{C}_+ - \hat{C}_-)} \Psi_L \\
&= e^{i\alpha^a T^a} \left(e^{i\alpha^b T^b} \hat{C}_+ + e^{-i\alpha^b T^b} \hat{C}_- \right) \Psi_L \\
&= e^{i2\alpha^a T^a} \hat{C}_+ \Psi_L + \hat{C}_- \Psi_L,
\end{aligned} \tag{78}$$

which in symplectic components is simply

$$\psi_L \to e^{i2\alpha^a T^a} \psi_L \quad \text{and} \quad \psi'_R \to \psi'_R. \tag{79}$$

The complex left handed field transforms, but its mirror matter partner does not. Now consider the subset of transformations defined by $\alpha^a = -\beta^a$. This transformation is given by

$$\begin{aligned}
\Psi_L &\to e^{i\alpha^a T^a} e^{-i\alpha^b T^b(\hat{C}_+ - \hat{C}_-)} \Psi_L \\
&= e^{i\alpha^a T^a} \left(e^{-i\alpha^b T^b} \hat{C}_+ + e^{i\alpha^b T^b} \hat{C}_- \right) \Psi_L \\
&= \hat{C}_+ \Psi_L + e^{i2\alpha^a T^a} \hat{C}_- \Psi_L,
\end{aligned} \tag{80}$$

which in symplectic components now sees ψ'_R transform and ψ_L remain invariant:

$$\psi_L \to \psi_L \quad \text{and} \quad \psi'_R \to e^{-i2\alpha^a T^{a*}} \psi'_R. \tag{81}$$

This shows that the transformations of Eq. (77) belong to the group $G \otimes G$, just as the Abelian transformations considered earlier were members of U(1)$\otimes$U(1).

At first sight, it appears that if ψ_L transforms as the $(R, 1)$ representation of $G \otimes G$ then ψ'_R must transform as the $(1, R^*)$ representation. This is because iT^a gets turned into $-iT^{a*}$ after being commuted through j. While this is perfectly true, it is equally valid

16

to say that ψ'_R transforms under the $(1, R)$ representation. Any transformation in R can be obtained from an appropriate transformation in R^* by redefining the group parameters. Consider the fundamental representation of SU(N) for instance. The generators form two classes: those that are real and those that are pure imaginary. In order to go from R^* to R one simply reverses the signs of all group parameters that multiply the real generators. One can therefore always reinterpret $\exp(-i\alpha^a T^{a*})$ as $\exp(i\alpha'^a T^a)$ by relating the α^a with α'^a in this way. Note that this does *not* mean that R and R^* are necessarily equivalent representations in the usual sense of being related to each other through a change of basis of the representation space (similarity transformation).

The gauge covariant derivative for non-Abelian $G \otimes G$ transformations can now be written down in close analogy to the Abelian case described above. It is given by

$$\mathcal{D}_\mu \Psi_L \equiv \partial_\mu \Psi_L + i\hat{\mathcal{W}}_\mu \Psi_L. \tag{82}$$

In terms of the real components W_μ^a and $W_\mu'^b$, the quaternionic gauge field is given by

$$\hat{\mathcal{W}}_\mu \equiv (gT^a W_\mu^a \hat{C}_+ - g'T^{*a} W_\mu'^a \hat{C}_-), \tag{83}$$

where g and g' are independent gauge coupling constants.

In terms of symplectic components, the covariant derivative of Ψ_L is given by

$$\begin{aligned}\mathcal{D}_\mu \Psi_L &= (\partial_\mu + igW_\mu^a T^a)\psi_L + j(\partial_\mu + ig'W_\mu'^b T^b)\psi'_R \\ &\equiv D_\mu \psi_L + jD'_\mu \psi'_R.\end{aligned} \tag{84}$$

The gauge covariant kinetic energy term is

$$\mathcal{L}_{gK} = \frac{1}{2}(i\overline{\Psi}_L \not{\mathcal{P}} \Psi_L + \overline{\Psi}_L \not{\mathcal{P}} \Psi_L i), \tag{85}$$

which in terms of the symplectic components is simply

$$\mathcal{L}_{gK} = i\overline{\psi}_L \not{\mathcal{P}} \psi_L + i\overline{\psi}'_R \not{\mathcal{P}}' \psi'_R. \tag{86}$$

This is precisely analogous to the Abelian case.

By demanding that the previous Lagrangian be gauge invariant, the non-Abelian gauge field $\hat{\mathcal{W}}_\mu$ must undergo the gauge transformation

$$\hat{\mathcal{W}}_\mu \rightarrow U_\alpha U_\beta \hat{\mathcal{W}}_\mu U_\beta^{-1} U_\alpha^{-1} + i(\partial_\mu U_\alpha)U_\alpha^{-1} + iU_\alpha(\partial_\mu U_\beta)U_\beta^{-1}U_\alpha^{-1}. \tag{87}$$

where

$$U_\alpha \equiv e^{i\alpha^a T^a} \quad \text{and} \quad U_\beta \equiv e^{i\beta^b T^b(\hat{C}_+ - \hat{C}_-)}, \tag{88}$$

and $U_\beta^{-1} = U_{-\beta}$.

The infinitesimal version of this law is

$$\hat{\mathcal{W}}_\mu \rightarrow \hat{\mathcal{W}}_\mu + i[\alpha^a T^a, \hat{\mathcal{W}}_\mu] + i[\beta^b T^b(\hat{C}_+ - \hat{C}_-), \hat{\mathcal{W}}_\mu] - \partial_\mu \alpha^a T^a - \partial_\mu \beta^b T^b(\hat{C}_+ - \hat{C}_-). \tag{89}$$

17

The analogy with the Abelian case is sufficiently strong to enable us to form a quaternionic non-Abelian gauge field

$$\mathcal{W}^\mu \equiv (gT^a W_\mu^a + jg'T^a W_\mu'^a),$$ (90)

and to rewrite the gauge invariant fermion kinetic Lagrangian as

$$\mathcal{L}_{gK} = \frac{1}{2}\left(i\overline{\Psi}_L[\not{\partial} + \frac{1}{2}(i\mathcal{W}\hat{C} + \mathcal{W}\hat{C}i)]\Psi_L + \overline{\Psi}_L[\not{\partial} + \frac{1}{2}(i\mathcal{W}\hat{C} + \mathcal{W}\hat{C}i)]\Psi_L i\right).$$ (91)

From Eq.(89) we obtain the infinitesimal transformation laws for the components:

$$W_\mu^a \to W_\mu^a + f^{abc}(\alpha^c + \beta^c)W_\mu^b - \frac{1}{g}\partial_\mu(\alpha^a + \beta^a)$$ (92)

and

$$W_\mu'^a \to W_\mu'^a + f^{abc}(\alpha^c - \beta^c)W_\mu'^b + \frac{1}{g'}\partial_\mu(\alpha^a - \beta^a),$$ (93)

where f^{abc} are the real totally antisymmetric structure constants of G. This demonstrates that the W_μ^a are the gauge fields for one of the factors in $G \otimes G$, while the $W_\mu'^a$ are the gauge fields for the other factor.

A quaternionic derivation of the gauge invariance of $\mathcal{L}_{gK}$ can be given by close analogy with the proof in the Abelian case. Since this is straightforward, we will not write down the details.

The field strength tensor for the quaternionic non-Abelian gauge field is formed by applying the quaternionic covariant derivative Eq.(82) to $\mathcal{W}$,

$$\mathcal{F}_{\mu\nu} \equiv \mathcal{D}_\mu\mathcal{W}_\nu - \mathcal{D}_\nu\mathcal{W}_\mu.$$ (94)

In terms of the symplectic components of $\mathcal{W}$,

$$\mathcal{D}_\mu\mathcal{W}_\nu = (\partial_\mu + igT^b W_\mu^b \hat{C}_+ - ig'T^{*b}W_\mu'^b \hat{C}_-)(gT^a W_\mu^a + jg'T^a W_\mu'^a)$$
$$= (gT^a\partial_\mu W_\nu^a + ig^2 T^b T^a W_\mu^b W_\nu^a) + j(g'T^a\partial_\mu W_\nu' + ig'^2 T^b T^a W_\mu'^b W_\nu'^a).$$ (95)

Hence, for our non-Abelian W_μ^a and $W_\mu'^a$ fields,

$$\mathcal{F}_{\mu\nu}^a = gT^a(\partial_\mu W_\nu^a - \partial_\nu W_\mu^a - gf^{abc}W_\mu^b W_\nu^c)$$
$$+ jg'T^a(\partial_\mu W_\nu'^a - \partial_\nu W_\mu'^a - g'f^{abc}W_\mu'^b W_\nu'^c).$$ (96)

The symplectic components of this expression are gauge invariant in precisely the fashion we expect, indicating that $\mathcal{F}_{\mu\nu}^a$ is the natural object from which to construct the free field Lagrangian[6]. As for the Abelian case, in order to write down the gauge boson kinetic energy terms a complimentary gauge field must be introduced,

---

[6]The gauge transformation law of its companion (operator valued) field strength tensor is $\hat{\mathcal{F}}_{\mu\nu} \to U_\beta U_\alpha \hat{\mathcal{F}}_{\mu\nu} U_\alpha^{-1} U_\beta^{-1}$, as would be expected.

18

$$\mathcal{W}_*^\mu \equiv (gT^a W^{a\mu} - g'T^a W'^{a\mu}j). \tag{97}$$

Denoting the corresponding gauge covariant field strength by $\mathcal{F}_*$, the gauge boson kinetic energy Lagrangian is

$$\mathcal{L}_{K_\mu} = \frac{-1}{8g^2}\left(Tr\{\mathcal{F}_*^{\mu\nu}\mathcal{F}_{\mu\nu}\} - iTr\{\mathcal{F}_*^{\mu\nu}\mathcal{F}_{\mu\nu}\}i\right), \tag{98}$$

where parity symmetry has been imposed, $g = g'$. Note that gauge covariance prevents the appearance of a kinetic energy mixing term in the non-Abelian case. The above Lagrangian produces the standard expressions for the kinetic energy terms for the real fields W_μ^a and $W_\mu'^a$.

VIII. HIGGS FIELDS

Let us now consider spin-0 fields. We will first determine the free kinetic energy term, which we will then make gauge invariant. After that we will write down the Higgs potential. Finally, we will examine Yukawa couplings, which will complete our construction of the quaternionic redrafting of the EPM.

The quaternionic Higgs field Φ is defined by analogy with Ψ:

$$\Phi \equiv \phi + j\phi', \tag{99}$$

where ϕ and ϕ' are ordinary complex Higgs fields. Its free kinetic energy term is simply

$$\mathcal{L}_{K\Phi} = \frac{1}{2}\left(\partial^\mu \Phi^\dagger \partial_\mu \Phi - i\partial^\mu \Phi^\dagger \partial_\mu \Phi i\right), \tag{100}$$

which in symplectic components becomes

$$\mathcal{L}_{K\Phi} = \partial^\mu \phi^\dagger \partial_\mu \phi + \partial^\mu \phi'^\dagger \partial_\mu \phi'. \tag{101}$$

The second term is needed to remove j-dependence from the Lagrangian. One could also argue that it is not necessary, because the j-dependent term from the first term is a 4-divergence and hence would not contribute to the equations of motion even if it were there. We will nevertheless remove it.

Both Abelian and non-Abelian gauge transformations are defined in exactly the same way as for fermions. The gauge covariant derivatives are also identical, leading to a gauge invariant Lagrangian which simply sees ∂_μ replaced by the appropriate covariant derivative. Since this is straightforward, we will not write down the details.

We will now construct the Higgs potential for a Higgs multiplet Φ that transforms under a complex representation $(R, 1) \oplus (1, R)$ of $G \otimes G$. We will further suppose that R is chosen so that $R \otimes R \otimes R$ and $R \otimes R \otimes R \otimes R$ do not contain singlets. Therefore the only gauge singlet we need consider for a renormalisable Higgs potential is that contained in $R^* \otimes R$. These conditions simply reproduce those applicable to the Higgs multiplet of the Standard Model, where ϕ^3 and ϕ^4 terms are not gauge invariant.

In terms of the symplectic components ϕ and ϕ', the most general Higgs potential is thus

$$V = -\mu^2 \phi^\dagger \phi - \mu'^2 \phi'^\dagger \phi' + \lambda(\phi^\dagger \phi)^2 + \lambda'(\phi'^\dagger \phi')^2 + \kappa \phi^\dagger \phi \phi'^\dagger \phi'. \tag{102}$$

The relevant terms to consider for Φ are as follows:

$$v_+ \equiv \frac{1}{2} \left(\Phi^\dagger \Phi - i\Phi^\dagger \Phi i \right) = \phi^\dagger \phi + \phi'^\dagger \phi'; \tag{103}$$

$$v_- \equiv -\frac{1}{2} \left(i\Phi^\dagger i \Phi - \Phi^\dagger i \Phi i \right) = \phi^\dagger \phi - \phi'^\dagger \phi'.$$

The most general renormalisable Higgs potential is thus

$$V = -\mu_+^2 v_+ - \mu_-^2 v_- + \lambda_+ v_+^2 + \lambda_- v_-^2 + \lambda_\pm v_+ v_-, \tag{104}$$

which is clearly equal to the Higgs potential in Eq. (102) using appropriate relations between the coefficients of the various terms.

Under parity in the EPM, $\phi \leftrightarrow \phi'$. In the quaternionic framework the involution

$$\Phi \to j\Phi i, \tag{105}$$

achieves this result up to a phase:

$$\phi + j\phi' \to -i\phi' + ji\phi. \tag{106}$$

The terms v_+ and v_- are thus parity-even and parity-odd respectively, and the Higgs potential of the EPM is

$$V_{EPM} = -\mu_+^2 v_+ + \lambda_+ v_+^2 + \lambda_- v_-^2. \tag{107}$$

An equivalent, and more convenient form is

$$V_{EPM} = \kappa_+ (v_+ - 2u^2)^2 + \kappa_- v_-^2. \tag{108}$$

This is precisely the Higgs potential first written down in Ref. [2] for the EPM. (By way of reminder, in the parameter space region $\kappa_\pm > 0$, the Higgs potential is minimised by setting $\langle \phi \rangle = \langle \phi' \rangle = u$ which is the parity conserving breakdown pattern we want.)

IX. YUKAWA INTERACTIONS

Finally, let us consider Yukawa couplings. We will restrict ourselves to reproducing the Yukawa Lagrangian found in the EPM. A typical term in the Lagrangian is $\bar{q}_L \phi u_R + \bar{q}'_R \phi' u'_L$, where q_L is the left-handed quark doublet, u_R the right-handed up quark, ϕ the Higgs doublet, and the primed fields are the mirror matter partners of the standard fields. In terms of our notation above, we therefore seek a quaternionic reformulation of the Yukawa pattern $\bar{\psi}_L \phi \psi_R + \bar{\psi}'_R \phi' \psi'_L$. This will allow us to write down the entire Yukawa Lagrangian of the EPM.

The required quaternionic reformulation is

$$\mathcal{L}_{Yuk} = \frac{h}{2} \left(i\overline{\Psi}_L \frac{1}{2} (\Phi \hat{C}_* - i\Phi \hat{C}_* i)\Psi_R + \overline{\Psi}_L \frac{1}{2} (\Phi \hat{C}_* - i\Phi \hat{C}_* i)\Psi_R i \right), \tag{109}$$

20

where

$$\hat{C}_* \equiv \hat{C}_+ - j\hat{C}_- \qquad (110)$$

and h is the Yukawa coupling constant. The symmetric treatment of multiplication by i is again required to remove j-dependence from the Lagrangian. It is easy to check that this Lagrangian yields the parity symmetric combination quoted in the previous paragraph. (The parity odd combination is obtained by inserting an additional factor of i between $\overline{\Psi}_L$ and Ψ_R.)

X. EXACT PARITY MODEL

By applying all of the techniques developed above, the Lagrangian of the Exact Parity Model can be rewritten in terms of quaternionic fields. We sketch the outline of this below.

The EPM has gauge group $G_{SM} \otimes G_{SM}$ under which a generation of standard fermions has the multiplet structure

$$q_L \sim (3, 2, 1/3; 1, 1, 0), \quad u_R \sim (3, 1, 4/3; 1, 1, 0), \quad d_R \sim (3, 1, -2/3; 1, 1, 0),$$

$$f_L \sim (1, 2, -1; 1, 1, 0), \quad e_R \sim (1, 1, -2; 1, 1, 0), \qquad (111)$$

while a generation of mirror fermions is given by

$$q_R' \sim (1, 1, 0; 3, 2, 1/3), \quad u_L' \sim (1, 1, 0; 3, 1, 4/3), \quad d_L' \sim (1, 1, 0; 3, 1, -2/3),$$

$$f_R' \sim (1, 1, 0; 1, 2, 1), \quad e_L' \sim (1, 1, 0; 1, 1, -2). \qquad (112)$$

Ordinary and mirror pairs can now be written as the symplectic components of chiral quaternionic spinors:

$$Q_L = q_L + jq_R', \quad U_R = u_R + ju_L', \quad D_R = d_R + jd_L',$$
$$F_L = f_L + jf_R', \quad E_R = e_R + je_L'. \qquad (113)$$

The $G_{SM} \otimes G_{SM}$ gauge transformations are then defined as explained in Sections 5 and 7 above. Similarly, the gauge boson and mirror gauge boson fields are assembled into quaternionic fields as per Sections 6 and 7 and their kinetic energy terms are written down.

We next introduce the Higgs doublet ϕ and its mirror partner ϕ',

$$\phi \sim (1, 2, 1; 1, 1, 0) \quad \text{and} \quad \phi' \sim (1, 1, 0; 1, 2, 1) \qquad (114)$$

which we then incorporate into a quaternionic scalar field Φ:

$$\Phi = \phi + j\phi'. \qquad (115)$$

The Higgs boson gauge invariant kinetic energy terms and the Higgs potential are then written down exactly as in Section 8. Yukawa interactions are introduced as is Section 9,

with the constraints of gauge and parity invariance imposed. After spontaneous electroweak symmetry breakdown, the Yukawa interactions lead to fermion mass terms of the form given by $\mathcal{B}_5$ in Section 3.

Parity symmetry is of course imposed by demanding invariance under

$$Q_L \to j\gamma^0 Q_L, \quad U_R \to j\gamma^0 U_R, \quad etc. \tag{116}$$

for the fermion fields and

$$\Phi \to j\Phi i \tag{117}$$

for the Higgs field. The gauge boson field parity transformation is, [7]

$$\mathcal{W}^\mu \to -j\mathcal{W}_\mu i, \tag{118}$$

where $\mathcal{W}^\mu$ represents either gluons or the electroweak bosons. These parity transformations cause corresponding Yukawa and gauge coupling constants in the two sectors to be equal.

This completes the construction of the minimal EPM. Other work has shown that nonzero neutrino masses are desirable in the EPM because they can naturally explain the solar and atmospheric neutrino anomalies [3]. The easiest way to do this is to introduce a right-handed neutrino field ν_R and its mirror partner ν'_L. These constitute another chiral quaternionic spinor N_R where

$$N_R = \nu_R + j\nu'_L. \tag{119}$$

Being a gauge singlet, N_R can have bare mass in addition to the electroweak mass it gains because of spontaneous symmetry breaking. Two types of bare mass are possible. The first is of the $\mathcal{B}_1$ form and the second is of the $\mathcal{B}_3$ form, leading to the bare mass Lagrangian

$$\mathcal{L}_{mass} = \frac{m'}{2}(\overline{N}_R j N_R - i\overline{N}_R j N_R i) + \frac{M}{2}\left[\overline{N}_R(N_R)^c - i\overline{N}_R(N_R)^c i + H.c.\right]. \tag{120}$$

The mass m' mixes ordinary and mirror neutrinos via a $\overline{\nu}_R \nu'_L$ term, while M is a common Majorana mass for ν_R and ν'_L. The see-saw mechanism for explaining why the observed neutrinos are exceptionally light is invoked by requiring that M be much larger than any other type of neutrino mass.

XI. CONCLUSION

We conclude with some philosophical or interpretative remarks: What we have achieved in this paper is a quaternionic reformulation of the EPM, but one explicitly based on standard complex quantum mechanics. We have deliberately constructed our Lagrangian so that all of the j- and k-dependent terms cancel out when the Lagrangian is decomposed into its

---

[7]Again, only up to phases: $W^\mu + jW'^\mu \to iW'_\mu - jiW_\mu$.

constituent real and complex fields. A more profound goal would have been to incorporate the EPM into a quantal structure that was explicitly quaternionic. Such a program would be premature, however, because in the absence of empirical evidence there is no obviously best way to formulate quaternionic quantum mechanics. Nevertheless, the above analysis is interesting in that it demonstrates a connection between a specific algebraic structure (the quaternions) and the model-building idea that the gauge group of the world is a product of two isomorphic factors. This idea was originally motivated for quite different reasons, such as to reinstate parity as an exact symmetry [1,2] or, more recently, to explain the observed neutrino anomalies [3]. The present paper provides yet another reason to be interested in $G_{SM} \otimes G_{SM}$ gauge theory.

REFERENCES

[1] T. D. Lee and C. N. Yang, Phys. Rev. **104**, 254 (1956). For an early phenomenological study see V. Kobzarev, L. Okun, and I. Pomeranchuk, Sov. J. Nucl. Phys. **3**, 837 (1966).

[2] R. Foot, H. Lew, and R. R. Volkas, Phys. Lett. **B272**, 67 (1991).

[3] R. Foot, H. Lew, and R. R. Volkas, Mod. Phys. Lett. A **7**, 2567 (1992); R. Foot, *ibid.* **9**, 169 (1994); R. Foot and R. R. Volkas, Phys. Rev. D **52**, 6595 (1995); University of Melbourne report No. UM-P-95/94, hep-ph/9510312 (1995). See S. M. Barr, D. Chang, and G. Senjanovic, Phys. Rev. Lett. **67**, 2765 (1991); R. Foot and H. Lew, hep-ph/9411390 (1994); Z. G. Berezhiani and R. N. Mohapatra, Phys. Rev. D **52**, 6607 (1995) for a discussion of related models featuring spontaneously broken mirror-matter parity.

[4] S. I. Blinnikov and M. Yu. Khlopov, Sov. J. Nucl. Phys. **36**, 472 (1982); for discussions of the problem of reconciling mirror matter with Big Bang Nucleosynthesis see R. Foot, M. J. Thomson and R. R. Volkas, University of Melbourne report No. UM-P-95/90, hep-ph/9509327 (1995); R. Foot and R. R. Volkas, Phys. Rev. Lett. **75**, 4350 (1995); Z. G. Berezhiani, A. D. Dolgov and R. N. Mohapatra, INFN report No. INFN-FE 16-95, hep-ph/9511221; H. M. Hodges, Phys. Rev. D **47**, 456 (1993); E. W. Kolb, D. Seckel and M. S. Turner, Nature **514**, 415 (1985).

[5] S. L. Glashow, Phys. Lett. **B167**, 35 (1986); E. Carlson and S. L. Glashow, *ibid.* **B193**, 168 (1987).

[6] See Foot, Lew and Volkas in Ref. [3].

[7] D. Finkelstein, J. M. Jauch, S. Schiminovich, and D. Speiser, J. Math. Phys. **3**, 207 (1962); **4**, 788 (1963).

[8] L. P. Horwitz, J. Math. Phys. **34**, 3405 (1993); L. P. Horwitz and L. C. Biedenharn, Ann. Phys. **157**, 432 (1984).

[9] S. Adler, *Quaternionic quantum mechanics and quantised fields* (Oxford University, New York, 1995) and references therein; IASSNS Report No. HEP-94/26, 1994; Nuc. Phys. **B415**, 195 (1994); Phys. Rev. D **37**, 3654 (1988); Phys. Rev. Lett. **57**, 167 (1986). A. J. Davies and B. H. J. McKellar, Phys. Rev. A **40**, 4209 (1989); **46**, 3671 (1992); A. J. Davies, Phys. Rev. D **41**, 2628 (1990).

[10] B. E. Hanlon and G. C. Joshi, Int. J. Mod. Phys. A **8**, 3263 (1993). C. G. Nash and G. C. Joshi, Int. J. Theor. Phys. **31**, 965 (1992); **27**, 409 (1988).

[11] A. Sudbery, Math. Proc. Camb. Phil. Soc. **85**, 199 (1979).

[12] J. C. Pati and A. Salam, Phys. Rev. D **10**, 275 (1974); R. N. Mohapatra and J. C. Pati, *ibid.* **11**, 566 (1975); *ibid.* **11**, 2558 (1975); G. Senjanovic and R. N. Mohapatra, *ibid.* **12**, 1502 (1975).

[13] S. De Leo and P. Rotelli, IASSNS report No. HEP-95/45, 1995; Int. J. Mod. Phys. A **10**, 4359 (1995); Prog. Theor. Phys. **92**, 917 (1994); Phys. Rev. D **45**, 575 (1992).