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Magnetic Charge in an Octonionic Field Theory

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Abstract

The violation of the Jacobi identity by the presence of magnetic charge is accomodated by using an explicitly nonassociative theory of octonionic fields. Lagrangian and Hamiltonian formalisms are constructed, and issues of the quantisation discussed. Finally an extension of these concepts to string theory is contemplated.

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I. INTRODUCTION

Magnetic charge and electromagnetic duality enjoy an uncomfortable enough status in 4-dimensional field theory, let alone from the difficulties in trying to extend the concept to other dimensionalities, or to string theory [1]. The source of much of the difficulty seems to be that magnetic monopoles give rise to a breakdown of associativity, in the form of an explicit 3-cocycle [2]. Instead of trying to avoid this nonassociativity, which is the usual strategy, in this paper we attempt to include it as an essential part of the theory by using a nonassociative algebra, that of the octonions, to describe the potentials.

With these nonassociative fields, it is possible to include magnetic currents in terms of fundamental fields, rather than having to realize monopoles with instantons, as is usually the case in field theory. It is interesting at this point to note that such instanton solutions have been constructed in seven and eight dimensions using octonions, satisfying a condition analogous to a generalization of self-duality [3].

In Section II of this paper we examine the classical equations of motion that we require in order to implement our theory, and then suggest a suitable Lagrangian density for the description of coupling to dyons, with both electric and magnetic charge. The appropriate constrained Hamiltonian formalism, required for quantisation, is investigated in Section III, and various difficulties are found to arise when trying to ensure time-independence of the constraints. Finally, in Section IV we discuss a possible means for overcoming these problems, by using a Jordan algebra in the context of string theory.

II. EQUATIONS OF MOTION AND LAGRANGIAN FORMALISM

The equations of motion of standard non-Abelian field theory can be written, in terms of an "electric" current j_e^ν, as

$$[D_\mu, F^{\mu\nu}] = j_e^\nu, \tag{1}$$

$$[D_\mu, \tilde{F}^{\mu\nu}] = 0, \tag{2}$$

where $\tilde{F}^{\mu\nu}$ is the dual of the field strength tensor, $\tilde{F}^{\mu\nu} \equiv \frac{1}{2}\epsilon^{\mu\nu\rho\sigma}F_{\rho\sigma}$. When the field tensor and covariant derivative are written in terms of the potential, as $F_{\mu\nu} \equiv -[D_\mu, D_\nu] = \partial_\mu A_\nu - \partial_\nu A_\mu - [A_\mu, A_\nu]$, Eq. (2) is a consequence of the Jacobi identity, which would be violated by the inclusion of magnetic sources by putting the R.H.S. equal to some $j^\nu_{\rm m}$.

It is possible to accomodate such a violation if an explicitly nonassociative algebra is used for the fields. A convenient example of such an algebra is that of the octonions [4], which has a number of useful properties. Octonions are hypercomplex numbers with seven imaginary units, and can be expressed as a sum of these units multiplied by real components:

$$x = x^0 + x^a e_a, \tag{3}$$

where the units e_a $(a = 1\ldots 7)$ obey the multiplication relations

$$e_a e_b = -\delta_{ab} + c_{abc} e_c, \tag{4}$$

with the structure constants c_{abc} being totally antisymmetric and equal to 1 for, e.g. $(abc) = (123), (145), (176), (246), (257), (347), (365)$ (different multiplication tables are possible, and a variety are used by other authors. For the purposes of the results of this paper, the choice is arbitrary). The octonions, being a normed division algebra, possess a real scalar product defined by

$$\langle x, y\rangle = \frac{1}{2}(x\bar{y} + y\bar{x}) = x^0 y^0 + x^a y^a, \tag{5}$$

which exhibits the useful properties

$$\langle x, y\rangle = \langle y, x\rangle, \tag{6}$$

$$\langle x, [y, z]\rangle = \langle [x, y], z\rangle. \tag{7}$$

Unlike the other division algebras, the real and complex numbers and the quaternions, octonions don't associate, and hence require the introduction of an associator, analogous to the commutator:

$$[x, y, z] = (xy)z - x(yz). \tag{8}$$

While not associative, they are however what is known as *alternative*, as the associator is antisymmetric under exchange of any of its elements:

$$[x, y, z] = -[y, x, z] = -[x, z, y] = -[z, y, x]. \tag{9}$$

In such an alternative algebra the Jacobi identity becomes

$$[[x, y], z] + [[y, z], x] + [[z, x], y] = 6[x, y, z]. \tag{10}$$

If the potential $A_\mu(x)$ is an octonion-valued function, Eq. (2) is no longer necessarily zero, and magnetic sources are allowed:

$$[D_\mu, \tilde{F}^{\mu\nu}] = -\epsilon^{\mu\rho\sigma\nu}[A_\mu, A_\rho, A_\sigma] = j^\nu_{\mathrm{m}}. \tag{11}$$

It is of course usual to obtain equations of motion from variation of an action, and so now it falls on us to provide a suitable formalism from which to derive Eqs. (1) and (11). Eq. (1) on its own can naturally be obtained from the Lagrangian

$$\mathcal{L}_{\mathrm{e}}(x) = -\frac{1}{4}\langle F_{\mu\nu}, F^{\mu\nu}\rangle - \langle A_\mu, j^\mu_{\mathrm{e}}\rangle. \tag{12}$$

The scalar product here ensures that the action is real, and performs the same function as the trace over generators in non-Abelian field theory.

As for the dual equation, Eq. (11), it can be derived by using a Lagrangian density

$$\mathcal{L}_{\mathrm{m}}(x) = -\frac{1}{4}\langle F_{\mu\nu}, \tilde{F}^{\mu\nu}\rangle - \langle A_\mu, j^\mu_{\mathrm{m}}\rangle. \tag{13}$$

When expanded in the potentials we find that

$$\langle F_{\mu\nu}, \tilde{F}^{\mu\nu}\rangle = -\epsilon^{\mu\nu\rho\sigma}\langle A_\mu, [A_\nu, A_\rho, A_\sigma]\rangle + \text{total divergence}. \tag{14}$$

In the standard non-Abelian theory, with associative fields, this entire term can be left out of the Lagrangian as a total divergence doesn't contribute to the action integral. With nonassociative fields, however, it persists, and allows a coupling to magnetic sources.

Now we have two Lagrangians, an electric and magnetic, but we would like only one to satisfy our dynamical requirements. Hence we incorporate both couplings into a single action, meaning that our sources are dyonic, with electric and magnetic charge. The total Lagrangian is then

$$\mathcal{L}_{\rm em}(x) = -\frac{1}{4}\langle F_{\mu\nu}, F^{\mu\nu}\rangle - \frac{1}{4}\langle F_{\mu\nu}, \tilde{F}^{\mu\nu}\rangle - \langle A_\mu, j^\mu\rangle. \tag{15}$$

The consequence of using this "electromagnetic" formalism is that now we only have one equation of motion for both kinds of coupling:

$$[D_\mu, F^{\mu\nu}] + [D_\mu, \tilde{F}^{\mu\nu}] = j^\nu. \tag{16}$$

At this point we haven't introduced any coupling constants, which would presumably be different for the electric and magnetic charges, and so would alter the relative strengths of the two terms in Eq. (16). The assignment of such constants is at present arbitrary (i.e. as yet there is no equivalent to the Dirac quantisation condition, but this is reasonable as we are still to attempt quantisation), so we will continue to leave them out of the theory for now, but will bear in mind the possibility of a relative magnitude to the terms in our Lagrangian.

III. HAMILTONIAN FORMALISM AND QUANTISATION

In this section we will follow the standard procedure for quantisation of constrained systems [5]. For simplicity, we will at first consider only the purely "electric" dynamics of Eq. (12), but without source terms. The effect of the magnetic terms will be included later. Using our potentials $A_\mu(x)$ as the configuration variables, the conjugate momenta are the electric fields, $E^\mu(x) \equiv F^{\mu 0}(x)$. With the primary constraint $E^0(x) = 0$ included with a Lagrange multiplier, the total Hamiltonian is

$$\begin{aligned} H_T &= H + \int d^3x \langle \lambda, E^0\rangle \\ &= \int d^3x \{\frac{1}{2}\langle E^i, E^i\rangle + \frac{1}{4}\langle F_{ij}, F_{ij}\rangle - \langle A_0, \partial_i E^i + [E^i, A_i]\rangle + \langle \lambda, E^0\rangle\}. \end{aligned} \tag{17}$$

Using Hamilton's Principle, from the variation of the Hamiltonian with respect to the conjugate position and momenta, we obtain the equations of motion:

$$\frac{dA_i}{dt} = E^i + \partial_i A_0 - [A_i, A_0], \tag{18}$$
$$\frac{dE^i}{dt} = -\partial_j F_{ij} + [A_j, F_{ij}] - [E^i, A_0], \tag{19}$$
$$\frac{dA_0}{dt} = \lambda, \tag{20}$$
$$\frac{dE^0}{dt} = \partial_i E^i + [E^i, A_i]. \tag{21}$$

Because of the nonassociativity of the fields, we are not using a Poisson bracket notation. If however we expand all octonions into their real components, fully associative fields are obtained which can be treated as standard position and momenta. In the process of quantisation, then, if the operators are chosen to correspond to the real components of the fields, then they will associate, and be compatible with standard quantum mechanics — a nonassociative generalisation is not required. The nonassociativity of the original fields is then reduced to a property of the structure constants which will appear in the expansion of $F_{\mu\nu}$. Any results obtained using such a notation will be equivalent to those derived here.

To keep the primary constraint satisfied at all times, an appropriate secondary constraint $\Gamma \equiv \partial_i E^i + [E^i, A_i] = 0$ is introduced. This constraint must also be time-independent, but we find its variation to be

$$\frac{d\Gamma}{dt} = 3[F_{ij}, A_i, A_j] + [A_0, \Gamma] + 6[A_0, E^i, A_i], \tag{22}$$

in the derivation of which we have used the alternativity property Eq. (10). Up to this point, the dynamics have been the same as for non-Abelian field theory [6], except that the rate of change in Γ usually appears without the associator terms of Eq. (22), in which case Γ is time-independent under the constraint conditions. However for our non-associative theory we require yet another constraint,

$$\Lambda \equiv 3[F_{ij}, A_i, A_j] + 6[A_0, E^i, A_i] = 0. \tag{23}$$

When we check the time-dependence of this constraint, we find

$$\frac{1}{6}\frac{d\Lambda}{dt} = [\partial_i E^j, A_i, A_j] - [[\partial_i A_j, A_0], A_i, A_j] - [[E^i, A_j], A_i, A_j] + [[[A_i, A_0], A_j], A_i, A_j]$$

$$-[E^i, F_{ij}, A_j] + [F_{ij}, \partial_i A_0, A_j] - [F_{ij}, [A_i, A_0], A_j] + [A_0, E^i, \partial_i A_0]$$
$$-[A_0, E^i, [A_i, A_0]] - [A_0, \partial_j F_{ij}, A_i] + [A_0, [A_j, F_{ij}], A_i] - [A_0, [E^i, A_0], A_i]$$
$$+[\lambda, E^i, A_i]. \tag{24}$$

It is possible to simplify this equation slightly through various algebraic manipulations, but not to any great advantage. It is also possible to include the sourceless form of Eq. (11), which is fair as we have used the electric equivalent, that requires that the potentials associate, i.e. form a quaternionic subalgebra. It does not, however, guarantee that the derivatives associate, and so the associators of Eq. (24) don't all vanish under this condition. The most we can say is that the sourceless theory is *locally* quaternionic. (We could require a global quaternionic structure, i.e. essentially an SU(2) field theory, and introduce nonassociativity at the quantum level, but this isn't general enough for our requirements). Thus we cannot reduce Eq. (24), and there doesn't seem to be any λ that will give us $d\Lambda/dt = 0$, and so complete this Hamiltonian formalism and allow a satisfactory quantisation of the theory.

Now we look at the effect of including the magnetic coupling term from the Lagrangian developed earlier, by adding to our Hamiltonian a term of the form

$$H' = \int d^3x \frac{1}{2}\kappa\epsilon^{ijk}\langle E^i, F_{jk}\rangle. \tag{25}$$

This comes straight from the electromagnetic Lagrangian, except for the constant κ, which allows for a difference in electric and magnetic coupling strengths.

This new term in the Hamiltonian naturally alters the equations of motion, and it also affects the constraint behaviour, leading us to define a new Λ',

$$\Lambda' \equiv \Lambda + 3\kappa\epsilon^{ijk}[E^i, A_j, A_k], \tag{26}$$

with a different time-dependence:

$$\frac{1}{6}\frac{d\Lambda'}{dt} = \frac{1}{6}\frac{d\Lambda}{dt} - \kappa^2\{[\partial_i E^j, A_i, A_j] - [[E^i, A_j], A_i, A_j] + [E^i, F_{ij}, A_j]\}$$
$$+\frac{1}{2}\kappa\epsilon^{ijk}\{[\partial_l F_{ij}, A_l, A_k] - [[F_{ij}, A_l], A_k, A_l] - [F_{ij}, F_{kl}, A_l]$$
$$+[A_0, E^i, F_{jk}] - 2[A_0, \partial_i E^j, A_k] + 2[A_0, [E^i, A_j], A_k] + 2[E^i, \partial_j A_0, A_k]$$
$$-2[E^i, [A_j, A_0], A_k] - [\partial_l F_{il}, A_j, A_k] + [[A_l, F_{il}], A_j, A_k] - [[E^i, A_0], A_j, A_k]\}. \tag{27}$$

This doesn't solve our dynamical problems, but it is worthwhile to not that some of more difficult terms may cancel if we set $\kappa = 1$, implying an equality between electric and magnetic charge, which was the preliminary assignment given in our *ad hoc* Lagrangian Eq. (15).

IV. FURTHER POSSIBILITIES

The two main problems that seem to arise in this octonionic field theory are the difficulty of constructing an appropriate action to suit the desired equations of motion, and the failure to complete a Hamiltonian formalism and hence quantise the theory. It is interesting to note that the solutions to both these problems might be the same.

The nonassociativity of the octonion algebra seems to be the chief obstacle in the construction of a successful field theory, and yet nonassociative structures have been productively applied in the use of the M_3^8 Jordan algebra, of 3×3 matrices of octonions, in heterotic string theory [7]. The ideas developed in this paper then, while not seeming to work in 4-dimensional field theory, may instead find a home in 10-dimensional superstring theory.

As for the problem with incorporating two equations of motion, Eq. (1) and Eq. (11), into a Lagrangian or Hamiltonian formalism, Nambu has conceived of a "generalised Hamiltonian dynamics", in which a canonical triplet of three dynamical variables has equations of motion derived from two Hamiltonians, using a generalisation of the Poisson bracket with three arguments. At the classical level this method can be used to describe the motion of a rotator, but difficulties arise in quantisation when trying to find the operator equivalent of Nambu's Poisson bracket. Most of the possibilities considered by Nambu either fail to match his requirements, or are equivalent to normal one-Hamiltonian dynamics. The Jordan algebra M_3^8, however, doesn't seem to suffer from those shortcomings.

Thus it seems that use of the M_3^8 Jordan algebra, most probably as applied to the 10-dimensional heterotic string, may provide an implementation of the dual equations of motion through dual Hamiltonians, and likely a working quantum field theory with fundamental magnetic charges.

REFERENCES

[1] Witten, E., *Nucl. Phys. B* **443**, 85 (1995).

[2] Jackiw, R., *Comments Nucl. Part. Phys.* **15**, 99 (1985); *Phys. Rev. Lett.* **54**, 159 (1985); Grossman, B., *Phys. Lett.* **152B**, 93 (1985); *Phys. Rev. D* **33**, 2922 (1986).

[3] Corrigan, E., Devchand, C., and Fairlie, D.B., *Nucl. Phys. B* **231**, 452 (1983); Fairlie, D.B. and Nuyts, J., *J. Phys. A* **17**, 2867 (1984); Fubini, S. and Nicolai, H., *Phys. Lett.* **155B**, 369 (1985); Günaydin, M. and Nicolai, H., *Phys. Lett.* **351B**, 169 (1995).

[4] Sorgsepp, L. and Lôhmus, J., *Hadronic J.* **2**, 1388 (1979).

[5] Dirac, P.A.M., *Lectures on Quantum Mechanics*, Belfer Graduate School of Science, Yeshiva University, New York, 1964.

[6] Itzykson, C. and Zuber, J.-B., *Quantum Field Theory*, McGraw–Hill, Singapore, 1980.

[7] Foot, R. and Joshi, G.C., Phys. Lett. **199B**, 203 (1977); *Lett. Math. Phys.* **15**, 237 (1988); Manogue, C.A. and Sudbery, A., *Phys. Rev. D* **40**, 4073 (1989).

[8] Nambu, Y., *Phys. Rev. D* **7**, 2405 (1973).