

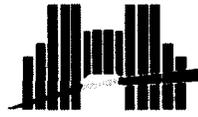


FR9700383

Groupe d'Annecy
Laboratoire
d'Annecy-le-Vieux de
Physique des Particules



ENSLAPP



Groupe de Lyon
Ecole Normale
Supérieure de Lyon

\mathcal{W} -realization of Lie algebras: Application to $so(4, 2)$ and Poincaré algebras

F. Barbarin¹, E. Ragoucy¹, and P. Sorba^{1,2}
*Laboratoire de Physique Théorique ENSLAPP **

Abstract

The property of some finite \mathcal{W} -algebras to appear as the commutant of a particular subalgebra in a simple Lie algebra \mathcal{G} is exploited for the obtention of new \mathcal{G} -realizations from a "canonical" differential one.

The method is applied to the conformal algebra $so(4, 2)$ and therefore yields also results for its Poincaré subalgebra. Unitary irreducible representations of these algebras are recognized in this approach, which is naturally compared -or associated- to the induced representation technic.

hep-th/9606014
ENSLAPP -AL-594/96
May 1996

*URA 14-36 du CNRS, associée à l'Ecole Normale Supérieure de Lyon et à l'Université de Savoie,

¹ Groupe d'Annecy: LAPP, Chemin de Bellevue BP 110, F-74941 Annecy-le-Vieux Cedex, France.

² Groupe de Lyon: ENS Lyon, 46 allée d'Italie, F-69364 Lyon Cedex 07, France.

1 Introduction

As shown in [1, 2], the construction of finite \mathcal{W} -algebras achieved in the framework of Hamiltonian reduction [3] also leads to the determination of the commutants, in the enveloping algebra¹ $\mathcal{U}(\mathcal{G})$, of particular subalgebras of a simple Lie algebra \mathcal{G} .

Such a property can be used to build, from a special realization of \mathcal{G} , a large class of \mathcal{G} -representations. Let us be more specific, and consider a graded decomposition of the simple Lie algebra \mathcal{G} of the type

$$\mathcal{G} = \mathcal{G}_{-1} \oplus \mathcal{G}_0 \oplus \mathcal{G}_{+1} \quad (1.1)$$

with the corresponding commutation relations:

$$[\mathcal{G}_0, \mathcal{G}_{\pm 1}] \subset \mathcal{G}_{\pm 1} \quad ; \quad [\mathcal{G}_{\pm 1}, \mathcal{G}_{\pm 1}] = \{0\} \quad ; \quad [\mathcal{G}_{+1}, \mathcal{G}_{-1}] \subset \mathcal{G}_0 \quad (1.2)$$

As reminded in section 2, it is possible to recognize in the commutant of a \mathcal{G} -subalgebra $\tilde{\mathcal{G}}$ of the form $\tilde{\mathcal{G}} = \mathcal{G}_{-1} \oplus \tilde{\mathcal{G}}_0$, with $\tilde{\mathcal{G}}_0 \subset \mathcal{G}_0$, a particular finite \mathcal{W} -algebra. The determination of the realization of the \mathcal{W} -algebra needed for our purpose can be obtained in a systematic way after some modification [2] of the usual Hamiltonian reduction technic [4]. Moreover, one knows how to construct a realization of the \mathcal{G} with differential operators on the space of smooth functions $\varphi(y_1, \dots, y_s)$ with $s = \dim \mathcal{G}_-$. In this picture, the abelianity of the \mathcal{G}_{-1} -part allows each \mathcal{G}_{-1} generator to act by direct multiplication:

$$\varphi(y_1, \dots, y_s) \rightarrow y_i \varphi(y_1, \dots, y_s) \quad \text{with} \quad i = 1, \dots, s \quad (1.3)$$

-cf action of the translation group- while the generators of the $\mathcal{G}_0 \oplus \mathcal{G}_+$ part will be represented by polynomials in the y_i and ∂_{y_i} (see section 3).

It is from this particular -canonical- differential realization of \mathcal{G} that new representations will be constructed with the use of the finite \mathcal{W} -algebra above mentioned. Realization of the $\tilde{\mathcal{G}}$ generators will not be affected in this approach. On the contrary, to the differential form of each generator in the $\mathcal{G} \setminus \tilde{\mathcal{G}}$ part will be added a sum of \mathcal{W} -generators, the coefficients of which are functions $f(y_i, \partial_{y_i})$. By associating a matrix differential realization to each irreducible finite dimensional \mathcal{W} -representation, one will get an action of \mathcal{G} on vector functions $\vec{\varphi} = (\varphi_1, \dots, \varphi_d)$, with $\varphi_i = \varphi_i(y_1, \dots, y_s)$, where d is the dimension of the considered \mathcal{W} -representation.

Such an approach has already been considered with some success [1] for the study of the Heisenberg quantization for a system of two particles in 1 and 2 dimensions [5]. The corresponding algebras are respectively $sp(2)$ and $sp(4)$, and in each case, it has been possible to relate the anyonic parameter to the eigenvalues of a \mathcal{W} -generator.

It has seemed to us of some interest to apply our technic on a rather well-known algebra, the $so(4, 2)$ one, also called the 4-dimensional conformal algebra. The decomposition of interest is the one in which the \mathcal{G}_{-1} part is constituted by the four translations, \mathcal{G}_0 by the Lorentz algebra $so(3, 1)$ plus the dilatation, and \mathcal{G}_+ by the four special conformal generators. The representation space is then the set of smooth functions $\mathcal{F}(M^4, \mathbb{C})$ where M^4 is the usual Minkowski space, and the finite \mathcal{W} -algebra of interest appears as a non-linear deformation of the \mathcal{G}_0 subalgebra. This algebra is the commutant in $\mathcal{U}(\mathcal{G})$ of the part $\mathcal{G}_- \oplus \tilde{\mathcal{G}}_0$, where $\tilde{\mathcal{G}}_0$ is a four dimensional subalgebra of \mathcal{G}_0 . Note also that the commutant of \mathcal{G}_- can be seen as the direct sum of this \mathcal{W} -algebra and of \mathcal{G}_- , i.e. $\mathcal{W} \oplus \mathcal{G}_-$. As presented in section 4, a

¹Strictly speaking, we consider a generalization (localization) of $\mathcal{U}(\mathcal{G})$, which contains, apart from $\mathcal{U}(\mathcal{G})$ itself, quotients $u^{-1}r$, vu^{-1} , and also u^r , $r \in \mathbb{Q}$, where $u, v \in \mathcal{U}(\mathcal{G})$, $u \neq 0$.

suitable basis for this non-linear algebra is the one made of three generators, components of the \vec{J} vector constituting an $so(3)$ part, and three other generators forming a second vector \vec{S} under the \vec{J} part, the commutation relation among the S_i 's producing a seventh generator belonging to the center of \mathcal{W} .

It is interesting to remark that, restricting to the Poincaré subalgebra the general realization so obtained, only the \vec{J} part contributes and appears to be the spin algebra directly connected to the (W_μ) Pauli-Lubanski-Wigner four vector. The usual Poincaré representations can therefore easily be recognized in this approach, the \mathcal{W} -part being related to the usual Wigner rotations. Then, realizations of the special conformal generators involve naturally the other \mathcal{W} -generators. In a rather elegant way, a (Σ_μ) four vector, conformal analogous to the (W_μ) one, can be exhibited and connected to the \vec{S} part of the \mathcal{W} -algebra.

Apart from this geometrical picture, one can recognize through the finite dimensional representation of the \mathcal{W} -algebra, the characterization of the unitary irreducible representations of the conformal algebra, as computed in [6] in case of positive energy.

It is clear, from the considered example, that the proposed method provides the same results as those given by the usual induced representation technics. This is discussed at the end of the paper.

2 \mathcal{W} -algebras as commutants

2.1 General results

We recall here the framework which leads to the main result obtained in [2], namely

Theorem

Any finite $\mathcal{W}(\mathcal{G}, \mathcal{S})$ algebra, with $\mathcal{S} = \mu sl(2)$ regular subalgebra of \mathcal{G} , can be seen as the commutant in the enveloping algebra $\mathcal{U}(\mathcal{G})$ of some \mathcal{G} -subalgebra $\tilde{\mathcal{G}}$.

Moreover, let H be the Cartan generator of the diagonal $sl(2)$ in \mathcal{S} . If we call \mathcal{G}_- , \mathcal{G}_0 and \mathcal{G}_+ the eigenspaces of respectively negative, null, and positive eigenvalues under H , then $\tilde{\mathcal{G}}$ decomposes as $\tilde{\mathcal{G}} = \mathcal{G}_- \oplus \tilde{\mathcal{G}}_0$, where $\tilde{\mathcal{G}}_0$ is a subalgebra of \mathcal{G}_0 which can be uniquely determined (see below).

We start with a simple Lie algebra \mathcal{G} , and consider an $sl(2)$ -embedding defined through the regular subalgebra \mathcal{S} in \mathcal{G} . Together with this $sl(2)$ -embedding, we get a gradation of \mathcal{G} , using the action of the $sl(2)$ Cartan generator H . We will call \mathcal{G}_- , \mathcal{G}_0 and \mathcal{G}_+ the eigenspaces of respectively negative, null, and positive eigenvalues under H . Below, we will consider the case where \mathcal{S} is a (sum of) regular subalgebra(s) $sl(2)$, which leads to Abelian \mathcal{G}_- subalgebras.

In [2], we have shown that, for integral gradations, the commutant of \mathcal{G}_- in the enveloping algebra $\mathcal{U}(\mathcal{G})$ is isomorphic to the $\mathcal{W}(\mathcal{G}, \mu sl(2))$ algebra plus a center. Moreover, we have also shown that one can get rid of the center when looking at the commutant of a wider subalgebra $\tilde{\mathcal{G}} = \mathcal{G}_- \oplus \tilde{\mathcal{G}}_0$ where $\tilde{\mathcal{G}}_0$ is a subalgebra of \mathcal{G}_0 and $\mathcal{G}_- = \mathcal{G}_{-1}$. In the case of half-integral gradations, we should use the halving procedure. However, when $\mathcal{S} = \mu sl(2)$, it can be shown that we still have $\tilde{\mathcal{G}} = \mathcal{G}_- \oplus \tilde{\mathcal{G}}_0$ with $\mathcal{G}_- = \mathcal{G}_{-1} \oplus \mathcal{G}_{-\frac{1}{2}}$.

Note that more general finite $\mathcal{W}(\mathcal{G}, \mathcal{S})$ algebras may be obtained as a commutant, but one has to extend $\tilde{\mathcal{G}}_0$ to the \mathcal{G}_+ part of \mathcal{G} (see example in [2]). The case of "affine" $\mathcal{W}(\mathcal{G}, \mathcal{S})$ algebra can also be treated with this framework [2].

As an illustration, let us consider $\mathcal{S} = \mu sl(2)$ in $\mathcal{G} = sl(n)$. The simple roots of \mathcal{G} will be noted α_i with $1 \leq i \leq n-1$. Then, one can take $\tilde{\mathcal{G}} = \bigoplus_{i=1}^{\mu} \Gamma_i$, where Γ_i is the subalgebra

$$\Gamma_i = \{E_{\beta_j}, i \leq j \leq n-i; E_{\gamma_j}, i \leq j \leq n-i-1; H_{\beta_{n-i}}\} \quad \text{with} \quad \begin{cases} \beta_j = \sum_{k=i}^j \alpha_k \\ \gamma_j = \sum_{k=i}^j \alpha_{n-k} \end{cases} \quad (2.1)$$

2.2 Explicit calculations

We start with a general element $J = J^a t_a$ in \mathcal{G} . J^a are the coordinate-functions ($J^a \in \mathcal{G}^*$), and it is known that one can provide them with a Poisson Bracket structure that mimics the commutation relations of \mathcal{G} : $[t_a, t_b] = f_{ab}^c t_c$ while $\{J^a, J^b\} = f^{ab}_c J^c$, indices being lowered via the Killing metric $\eta_{ab} = \langle t_a, t_b \rangle$, and raised by its inverse η^{ab} . Thanks to this metric (and these PB) we identify \mathcal{G}^* with \mathcal{G} so that the decompositions of \mathcal{G} (like $\mathcal{G} = \mathcal{G}_- \oplus \mathcal{G}_0 \oplus \mathcal{G}_+$) will be also valid for \mathcal{G}^* , that we will loosely write \mathcal{G} . We will also use the Cartan involution Θ defined by $\Theta(E_{\alpha_i}) = -E_{-\alpha_i}$ for any simple root α_i . It allows us to define another scalar product $(X, Y) = \langle X, \Theta(Y) \rangle$.

We consider the decomposition $\mathcal{G} = \mathcal{G}_- \oplus \mathcal{G}_0 \oplus \mathcal{G}_+$ obtained by the Cartan generator of the diagonal $sl(2)_{diag}$ in μ regular $sl(2)$. As a notation, we use latin indices a, b, c, \dots to label the generators of \mathcal{G} , and greek (resp. overlined-greek) to label generators in $\tilde{\mathcal{G}}$ (resp. in $\bar{\mathcal{G}}$, its orthogonal complementary w.r.t. the scalar product (\cdot, \cdot)). When needed, we will also use t_A, t_i and $t_{\bar{A}}$ to denote the generators of $\mathcal{G}_-, \mathcal{G}_0$, and \mathcal{G}_+ respectively. Thus, we note

$$J = J^a t_a = J^\alpha t_\alpha + J^{\bar{\alpha}} t_{\bar{\alpha}} = J^A t_A + J^i t_i + J^{\bar{A}} t_{\bar{A}} \quad \text{with} \quad \begin{cases} t_a \in \mathcal{G} \\ t_\alpha \in \tilde{\mathcal{G}} ; t_{\bar{\alpha}} \in \bar{\mathcal{G}} \\ t_A \in \mathcal{G}_- ; t_i \in \mathcal{G}_0 ; t_{\bar{A}} \in \mathcal{G}_+ \end{cases} \quad (2.2)$$

To compute the commutant of $\tilde{\mathcal{G}}$ in $\mathcal{U}(\mathcal{G})$, we consider the group transformations $J \rightarrow J^g = g J g^{-1}$ with $g \in \Theta(\tilde{G})$ where $Lie \Theta(\tilde{G}) = \Theta(\tilde{\mathcal{G}})$, \tilde{G} being the connected component of the Lie group whose algebra is $\tilde{\mathcal{G}}$. In fact, if $g = e^{x^b t_b}$, with $t_b \in \Theta(\tilde{\mathcal{G}})$, one can rewrite $g J g^{-1}$ as $exp(x_b \{J^b, \cdot\}) (J)$ where now the PB $\{J^b, \cdot\}$ acts on J^a (instead of t_a), J^b belongs to $\tilde{\mathcal{G}}^*$, and $x_b = \eta_{bc} x^c$. Then, one can show that, starting from any generic element J , there exists a unique group generator $g = e^{x^b t_b} \in \Theta(\tilde{G})$ such that $J^g = e_- + W^{\bar{\alpha}} t_{\bar{\alpha}}$, where e_- is the negative root generator of the $sl(2)_{diag}$ under consideration and $t_{\bar{\alpha}} \in \bar{\mathcal{G}}$. Let us remark that the parameters x^b depend only on the components $J^\alpha \in \tilde{\mathcal{G}}$.

In fact, decomposing $\Theta(\tilde{G})$ as $\Theta(G_-) \Theta(\tilde{G}_0)$ (in the neighborhood of the identity), it is easy to show that $\Theta(\tilde{G}_0)$ is defined as the subgroup of G_0 which transforms J into J' with

$$h \in \Theta(\tilde{G}_0) : J \rightarrow J^h = J' \quad \text{with} \quad J'|_{\mathcal{G}_-} = e_- \quad (2.3)$$

while $\Theta(G_-)$ transforms J' into $J^g = e_- + W^{\bar{\alpha}} t_{\bar{\alpha}}$:

$$g \in \Theta(\tilde{G}_-) : J' \rightarrow J^g = e_- + W^{\bar{\alpha}} t_{\bar{\alpha}} \quad (2.4)$$

This property is still true when \mathcal{G}_- is not Abelian. but in that case one has to include elements of $\Theta(G_+)$ in order for $\Theta(\tilde{G})$ to satisfy (2.3).

Altogether, the generators $W^{\bar{\alpha}}$ are the "gauge-fixed" components of J^g . Hence, they have vanishing PB with any element in $\tilde{\mathcal{G}}$, and generate the commutant of $\tilde{\mathcal{G}}$. They form a classical (PB) version of the \mathcal{W} -algebra. Note that the $W^{\bar{\alpha}}$ are linear in $J^{\bar{\beta}}$: we will come back on this property in section 3.3.

A quantum version of this algebra will be obtained by using the map $i : \mathcal{G}^* \rightarrow \mathcal{G}$ defined by $i(J^a) = \eta^{ab}t_b$ and extended to $\mathcal{U}(\mathcal{G})$ by symmetrization of the products (see [2] for details).

Explicitly, one uses a matrix representation for the t_a to get a simple expression for the group generators $g \in \Theta(\tilde{\mathcal{G}})$ which act on J . We determine g such that J^g takes the form (2.4). The coefficients $W^{\bar{\sigma}}$ of $t_{\bar{\sigma}}$ are then expressed in terms of *all* the J^a 's, contrarily to the usual Hamiltonian reduction approach, since we have not imposed any constraint on the current components. Finally, from the classical expression of the \mathcal{W} -generators, we use the above mentioned quantization procedure to obtain the commutant of $\tilde{\mathcal{G}}$ in $\mathcal{U}(\mathcal{G})$.

2.3 Example: $sl(2)$

We explicit here the results just presented in the case of $\mathcal{G} = sl(2)$. We start with a generic element

$$J = J^a t_a = \begin{pmatrix} \frac{1}{2}J^0 & J^+ \\ J^- & -\frac{1}{2}J^0 \end{pmatrix} \quad (2.5)$$

As normalization, we take

$$\begin{aligned} [t_0, t_{\pm}] &= \pm t_{\pm} \quad ; \quad [t_+, t_-] = 2t_0 \\ \eta_{00} &= \frac{1}{2} \quad ; \quad \eta_{+-} = 1 \quad \text{so that} \quad \eta^{00} = 2 \quad ; \quad \eta^{+-} = 1 \\ \{J^0, J^{\pm}\} &= \pm 2J^{\pm} \quad ; \quad \{J^+, J^-\} = J^0 \end{aligned} \quad (2.6)$$

Here the eigenspaces $\mathcal{G}_{\pm,0}$ are one-dimensional. Thus, $\tilde{\mathcal{G}}_0$ must be \mathcal{G}_0 itself, and indeed we have:

$$h = \begin{pmatrix} \sqrt{J^-} & 0 \\ 0 & \frac{1}{\sqrt{J^-}} \end{pmatrix} \quad \text{is such that} \quad hJh^{-1} = J' = \begin{pmatrix} J'^0 & J'^+ \\ 1 & -J'^0 \end{pmatrix} \quad \text{with} \quad \begin{cases} J'^0 = \frac{1}{2}J^0 \\ J'^+ = J^+J^- \end{cases}$$

Now, acting with G_+ leads to

$$gJ'g^{-1} = \begin{pmatrix} 0 & W \\ 1 & 0 \end{pmatrix} \quad \text{with} \quad W = J'^+ + (J'^0)^2 = J^+J^- + \frac{1}{4}(J^0)^2 \quad \text{and} \quad g = \begin{pmatrix} 1 & -J'^0 \\ 0 & 1 \end{pmatrix}$$

After quantization, we recover the well-known result that the Casimir operator

$$C_2 = t_0^2 + \frac{1}{2}(t_+t_- + t_-t_+) = t_0^2 + t_0 + t_-t_+ \quad (2.7)$$

commutes with t_- and t_0 . Note however that we get also the non-trivial information that the only elements in $\mathcal{U}(sl(2))$ that commute with both t_- and t_0 (not necessarily t_+) are polynomials in W .

3 \mathcal{W} -realization of Lie algebras

The above construction allows us to provide a new technic for the construction of representations of a Lie algebra \mathcal{G} starting from a given (canonical) realization of \mathcal{G} .

3.1 Regular representation of a Lie algebra \mathcal{G}

We recall here the regular representation of a Lie algebra \mathcal{G} . We start with G , the Lie group of \mathcal{G} , and consider $\mathcal{F} = \mathcal{F}(G, \mathbb{C})$ the space of smooth functions on G . One can construct two isomorphic representations of G on \mathcal{F} . They are associated to the right and left actions of G onto itself.

The left representation U_L is defined through the map

$$g \in G \rightarrow U_L(g) \quad \text{with} \quad (U_L(g)\varphi)(g') = \varphi(g^{-1}g'), \forall \varphi \in \mathcal{F}, \forall g' \in G \quad (3.1)$$

while the right representation U_R uses the map

$$g \rightarrow U_R(g) \quad \text{with} \quad (U_R(g)\varphi)(g') = \varphi(g'g), \forall \varphi \in \mathcal{F}, \forall g' \in G \quad (3.2)$$

These representations induce naturally two representations of \mathcal{G} on \mathcal{F} :

$$\begin{aligned} t \in \mathcal{G} \rightarrow T_L(t) \quad \text{with} \quad T_L(t)\varphi(g') &= \left[\frac{d}{d\epsilon} \varphi(e^{-\epsilon t} g') \right]_{\epsilon=0} \\ t \in \mathcal{G} \rightarrow T_R(t) \quad \text{with} \quad T_R(t)\varphi(g') &= \left[\frac{d}{d\epsilon} \varphi(g' e^{\epsilon t}) \right]_{\epsilon=0} \end{aligned}$$

From now on, we will focus on U_R and T_R , the way the following construction can be done for U_L and T_L being analogous.

3.2 Differential realization of \mathcal{G}

We now use the regular representation to build a differential realization (as used in [7]) adapted to the decomposition² $\mathcal{G} = \mathcal{G}_- \oplus \mathcal{G}_0 \oplus \mathcal{G}_+$ that we have used in the previous section. We first remark that $\mathcal{P} = \mathcal{G}_0 \oplus \mathcal{G}_+$ is a parabolic subalgebra of \mathcal{G} , and that any element $g \in G$ can be decomposed in a unique way as $g = pg_-$ with $p \in P$ and $g_- \in G_-$.

Now, starting from the regular representation T_R on \mathcal{F} , we can restrict it to the subspace $\overline{\mathcal{F}}$ of functions $\overline{\varphi} \in \mathcal{F}$ such that $\overline{\varphi}(pg) = \overline{\varphi}(g) \forall g \in G$ and $\forall p \in P$. We will call this realization \overline{T}_R . Note that $\overline{\mathcal{F}}$ is not empty because $\mathcal{F}(\mathcal{G}_-, \mathbb{C})$ can be canonically embedded in $\overline{\mathcal{F}}$. Thus \overline{T}_R is not the trivial representation.

Any $g \in G$ can be decomposed into $g = pg_-$ with $p \in P$ and $g_- \in G_-$, in a unique way. Moreover, the set $\{t_A\}$ being a basis of \mathcal{G}_- , we can label any $g_- \in G_-$ with $\dim \mathcal{G}_- = s$ variables x^A such that $g_- = e^{x^A t_A}$. Thus we will have

$$\overline{\varphi}(g) = \overline{\varphi}(pg_-) = \overline{\varphi}(g_-) = \overline{\varphi}(e^{x^A t_A}) \equiv \overline{\varphi}(x^1, x^2, \dots, x^s) \quad (3.3)$$

Let us look more carefully at \overline{T}_R . For $t_B \in \mathcal{G}_-$, we have:

$$\overline{T}_R(t_B)\overline{\varphi}(g) = \left[\frac{d}{d\epsilon} \overline{\varphi}(g e^{\epsilon t_B}) \right]_{\epsilon=0} = \left[\frac{d}{d\epsilon} \overline{\varphi}(pg_- e^{\epsilon t_B}) \right]_{\epsilon=0} = \left[\frac{d}{d\epsilon} \overline{\varphi}(g_- e^{\epsilon t_B}) \right]_{\epsilon=0} \quad (3.4)$$

where we have used the decomposition $g = pg_-$. Now, as \mathcal{G}_- is Abelian, we have

$$\overline{\varphi}(g e^{\epsilon t_B}) = \overline{\varphi}(e^{x^A t_A + \epsilon t_B}) = \overline{\varphi}(x^1, x^2, \dots, x^B + \epsilon, \dots, x^s)$$

²In the case of half-integral gradation $\mathcal{G}_- = \mathcal{G}_{-1} \oplus \mathcal{G}_{-\frac{1}{2}}$ we will perform a halving to get an integral one.

so that

$$\begin{aligned} \bar{T}_R(t_B)\varphi(x^1, \dots, x^s) &= \left[\frac{d}{d\epsilon} \bar{\varphi}(x^1, x^2, \dots, x^B + \epsilon, \dots, x^s) \right]_{\epsilon=0} = \frac{\partial}{\partial x^B} \bar{\varphi}(x^1, \dots, x^s) \\ \Rightarrow \bar{T}_R(t_B) &= \partial_B \end{aligned} \quad (3.5)$$

To be complete, let us add that, using

$$e^{-B} e^A e^B = e^{e^{-B} A e^B} \quad (3.6)$$

the same calculation shows that

$$\forall t_j \in \mathcal{G}_0, \bar{T}_R(t_j) = -f_{jA}{}^B x^A \partial_B \quad \text{where} \quad [t_j, t_A] = f_{jA}{}^B t_B \quad (3.7)$$

Then, it is easy to check that

$$\bar{T}_R(t_{\bar{A}}) = \frac{1}{2} f_{\bar{A}A}{}^j f_{jB}{}^C x^A x^B \partial_C \quad (3.8)$$

is solution of $[t_j, t_{\bar{A}}] = f_{j\bar{A}}{}^{\bar{B}} t_{\bar{B}}$ and $[t_A, t_{\bar{A}}] = f_{A\bar{A}}{}^i t_i$.

We have seen that one can identify $\bar{\mathcal{F}}$ with a space of functions of $\dim \mathcal{G}_-$ variables. We would like that \mathcal{G}_- elements act multiplicatively instead of acting through differential operators. Thus, we perform a Fourier transformation:

$$\begin{aligned} \bar{\varphi} \in \bar{\mathcal{F}} &\rightarrow \tilde{\varphi} \in \tilde{\mathcal{F}} \\ \bar{\varphi}(x^1, \dots, x^s) &\rightarrow \tilde{\varphi}(y_1, \dots, y_s) = \int d^s x e^{i\vec{y} \cdot \vec{x}} \bar{\varphi}(x^1, \dots, x^s) \end{aligned}$$

with $\vec{y} \cdot \vec{x} = y_A x^A$. On the new space of functions $\tilde{\mathcal{F}}$, the \mathcal{G}_- part acts multiplicatively, and the other generators of \mathcal{G} will be differential polynomials in the variables:

$$\begin{aligned} t_A \in \mathcal{G}_- &\rightarrow \tilde{T}_R(t_A) = -i y_A \\ t_j \in \mathcal{G}_0 &\rightarrow \tilde{T}_R(t_j) = f_{jA}{}^B y_B \partial^A + f_{jA}{}^A \quad \text{with} \quad \partial^B = \frac{d}{dy_B} \\ t_{\bar{A}} \in \mathcal{G}_+ &\rightarrow \tilde{T}_R(t_{\bar{A}}) = \frac{i}{2} f_{\bar{A}A}{}^j f_{jB}{}^C y_C \partial^A \partial^B + \frac{i}{2} (f_{\bar{A}A}{}^j f_{jB}{}^B + f_{\bar{A}B}{}^j f_{jA}{}^B) \partial^A \end{aligned}$$

It is this representation we will use in the following.

3.3 \mathcal{W} -realization

We use the technics of the section 2 to get a wide class of representations of \mathcal{G} , starting from the differential representation based on \mathcal{G}_- . We know that the generators of $\mathcal{W}(\mathcal{G}, \mu sl(2)) \equiv \mathcal{W}$ form the commutant of $\tilde{\mathcal{G}} = \mathcal{G}_- \dot{+} \tilde{\mathcal{G}}_0$. They can be expressed in terms of all the generators of \mathcal{G} . Moreover, we have seen that, in the differential realization, the elements of \mathcal{G}_- are represented by y_A , while the generators in $\tilde{\mathcal{G}}_0$ are represented by $y_B \partial^A$ (up to some constants). Thus, the \mathcal{W} -generators commute both with y_A and ∂^A ($\forall A$), *i.e.* they are constant on the whole differential representation. Let $w_{\bar{\alpha}}^{\bar{\alpha}}$ be these constants, where $\bar{\alpha}$ labels the generators of \mathcal{W} .

From the general results of section 2, we know that there exists a unique element $g \in \Theta(\tilde{\mathcal{G}})$ such that

$$J^g = g(J^\alpha t_\alpha + J^{\bar{\alpha}} t_{\bar{\alpha}}) g^{-1} = \epsilon_- + W^{\bar{\alpha}} t_{\bar{\alpha}} \quad \text{with} \quad g = e^{u^\alpha t_\alpha}, \quad t_\alpha \in \Theta(\tilde{\mathcal{G}}) \quad \text{and} \quad u^\alpha = u^\alpha(J^\alpha) \quad (3.9)$$

Let us rewrite this formula as

$$J = J^\alpha t_\alpha + J^{\bar{\alpha}} t_{\bar{\alpha}} = g^{-1}(e_- + W^{\bar{\alpha}} t_{\bar{\alpha}})g \quad (3.10)$$

This shows that we can write any element J^a in terms of the J^α 's and the $W^{\bar{\alpha}}$'s. We also remark that, in the same way the realization of $W^{\bar{\alpha}}$ is linear in $J^{\bar{\alpha}}$ because of (3.9), the $J^{\bar{\alpha}}$'s also depend linearly in the $W^{\bar{\alpha}}$'s because of (3.10). Then starting with a given realization of $\tilde{\mathcal{G}}$ we construct a realization of the whole algebra \mathcal{G} leaving $\tilde{\mathcal{G}}$ unchanged and expressing the other generators of \mathcal{G} in terms of those of $\tilde{\mathcal{G}}$ and the \mathcal{W} -algebra, by applying formula (3.10).

Performing the same calculation at the quantum level, we deduce that starting with a given representation of \mathcal{G} and so of $\tilde{\mathcal{G}}$ one can obtain other representations of \mathcal{G} using representations of the \mathcal{W} -algebra.

It is natural to start with the differential representation of \mathcal{G} . Then, one generalizes it into a "matrix differential representation" using a matrix representation of \mathcal{W} . More precisely, we take as representation space d copies of $\tilde{\mathcal{F}}$, where d is the dimension of the \mathcal{W} -representation one chooses. Then, \mathcal{G} will be represented by

$$\begin{aligned} t_A \in \mathcal{G}_- &\rightarrow \vec{T}_R(t_A) = -iy_A \cdot \mathbb{I} \\ t_j \in \mathcal{G}_0 &\begin{cases} t_\alpha \in \tilde{\mathcal{G}}_0 &\rightarrow \vec{T}_R(t_\alpha) = (f_{\alpha A}{}^B y_B \partial^A + f_{\alpha A}{}^A) \cdot \mathbb{I} \\ t_{\bar{\alpha}} \in (\mathcal{G}_0 \setminus \tilde{\mathcal{G}}_0) &\rightarrow \vec{T}_R(t_{\bar{\alpha}}) = (f_{\bar{\alpha} A}{}^B y_B \partial^A + f_{i A}{}^A) \cdot \mathbb{I} + R_{\bar{\alpha}\bar{\beta}}(y_A, \partial^B) \cdot (M^{\bar{\beta}} - w_0^{\bar{\beta}} \mathbb{I}) \end{cases} \\ t_{\bar{A}} \in \mathcal{G}_+ &\rightarrow \vec{T}_R(t_{\bar{A}}) = \frac{i}{2} (f_{\bar{A} A}{}^j f_{j B}{}^C y_C \partial^A \partial^B + (f_{\bar{A} A}{}^j f_{j B}{}^B + f_{\bar{A} B}{}^j f_{j A}{}^B) \partial^A) \cdot \mathbb{I} + \\ &\quad + R_{\bar{A}\bar{B}}(y_A, \partial^B) \cdot (M^{\bar{B}} - w_0^{\bar{B}} \mathbb{I}) \end{aligned}$$

where M^I is the $d \times d$ matrix representing W^I , \mathbb{I} is the $d \times d$ identity matrix, and $R_{\bar{\alpha}I}(y_A, \partial^B)$ are polynomials in the differentials. The above operators will act on vectors:

$$\vec{\varphi}(y_1, \dots, y_s) = \begin{pmatrix} \varphi_1(y_1, \dots, y_s) \\ \varphi_2(y_1, \dots, y_s) \\ \vdots \\ \varphi_d(y_1, \dots, y_s) \end{pmatrix}$$

3.4 Example: $sl(2)$

We continue the example of section 2.3. We denote an element of the group $SL(2)$ by its coordinates

$$g(x^-, x^0, x^+) = e^{x^- t_-} e^{x^0 t_0} e^{x^+ t_+} \quad (3.11)$$

so that the functions read $\varphi(x^-, x^0, x^+)$ on the regular representation. In the differential representation, we have functions $\tilde{\varphi}(y_-) \equiv \tilde{\varphi}(y)$ and we have

$$T_0(t_-) = -iy \quad ; \quad T_0(t_0) = -y\partial - 1 \quad \text{and} \quad T_0(t_+) = -iy\partial^2 - 2i\partial \quad (3.12)$$

We use the formula computed in section 2.3 to construct the \mathcal{W} -realization for $sl(2)$. We have seen that, at the quantum level, $w = t_0^2 + t_0 + t_- t_+$ is the generator of the commutant of $\{t_0, t_-\}$ (see equation (2.7)). We start with a representation T_0 corresponding to a value

w_0 of the Casimir operator. Then, we construct a new representation T keeping for t_0 and t_- the form T_0 , $T(t_0) = T_0(t_0)$ and $T(t_-) = T_0(t_-)$, and inverting the formula (2.7) for t_+ :

$$\begin{aligned} w - w_0 &= [T(t_0)^2 + T(t_0) + T(t_-)T(t_+)] - [T_0(t_0)^2 + T_0(t_0) + T_0(t_-)T_0(t_+)] \\ &= T_0(t_-)[T(t_+) - T_0(t_+)] \\ &\Rightarrow T(t_+) = T_0(t_+) + \frac{w - w_0}{T_0(t_-)} \end{aligned}$$

Using the differential representation, we get $w_0 = 0$ and

$$T(t_+) = -iy\partial^2 - 2i\partial + i\frac{w}{y} \quad \text{with } w \in \mathbb{R} \quad (3.13)$$

Thus (3.13), together with $T(t_-) = -iy$ and $T(t_0) = -y\partial - 1$ is a representation of $sl(2)$ with a value w for the Casimir operator. Note that, for $sl(2)$, a similar (up to a change of variable $y \rightsquigarrow x^2$) realization has been given in [5].

4 Application to Poincaré and $so(4, 2)$ algebras

We now apply the results of the previous sections to get realizations of the conformal $so(4, 2)$ algebra (and of its Poincaré subalgebra). We will show that the \mathcal{W} -realization of this latter provides the Pauli-Lubanski-Wigner quadrivector W^μ in a very natural way. A generalization to the whole $so(4, 2)$ algebra will exhibit two new quadrivectors which, together with P^μ and W^μ , form a basis of the Minkowski space. The scalar products of these vectors give all the physical quantities such as the mass or the spin.

4.1 Description

The $so(4, 2)$ algebra is known as the conformal algebra in four dimensions, in the Minkowski space, *i.e.* with the metric $g_{\mu\nu} = \text{diag}(1, -1, -1, -1)$. It is a non-compact form of $so(6)$, with maximal compact subalgebra $so(4) \oplus so(2)$. Note that $sl(4)$ is also a non-compact version of $so(6)$, but with maximal compact subalgebra $so(4)$. Thus, one can connect $sl(4)$ to $so(4, 2)$ through a compactification process, a property which is useful for calculations.

The algebra $so(4, 2)$ has fifteen generators:

$$\left. \begin{array}{l} \text{Four translations: } P_\mu \text{ with } \mu = 0, 1, 2, 3 \text{ which form} \\ \text{Three rotations and three boosts: } M_{ij} \text{ and } M_{0i} \text{ respectively } (i, j = 1, 2, 3) \\ \text{One dilatation: } D \\ \text{Four special conformal transformations: } K_\mu \text{ with } \mu = 0, 1, 2, 3 \text{ which form} \end{array} \right\} \begin{array}{l} \mathcal{G}_- \\ \mathcal{G}_0 \\ \mathcal{G}_+ \end{array} \quad (4.14)$$

Rotations and boosts constitute the Lorentz algebra $\mathcal{G}_0 = so(3, 1)$, and together with the translations those generators form the Poincaré algebra. In the following we will denote by $\mathcal{P} = \mathcal{G}_0 \oplus \mathcal{G}_+$ the corresponding parabolic algebra. This latter decomposition is actually induced by a natural gradation of the algebra with respect to the element $-iD$, which assigns a grade -1 to P_μ , 0 to $M_{\mu\nu}$, and $+1$ to K_μ . The commutation relations read:

$$\begin{aligned} [P_\mu, P_\nu] &= 0 & [M_{\mu\nu}, P_\rho] &= i(g_{\nu\rho}P_\mu - g_{\mu\rho}P_\nu) & [D, P_\mu] &= -iP_\mu \\ [M_{\mu\nu}, M_{\rho\sigma}] &= i(g_{\nu\rho}M_{\mu\sigma} - g_{\mu\rho}M_{\nu\sigma} + g_{\mu\sigma}M_{\nu\rho} - g_{\nu\sigma}M_{\mu\rho}) & [D, M_{\mu\nu}] &= 0 \\ [K_\mu, K_\nu] &= 0 & [M_{\mu\nu}, K_\rho] &= i(g_{\nu\rho}K_\mu - g_{\mu\rho}K_\nu) & [D, K_\mu] &= iK_\mu \\ [P_\mu, K_\nu] &= 2i(-M_{\mu\nu} + g_{\mu\nu}D) \end{aligned} \quad (4.15)$$

$so(4, 2)$ possesses three Casimir operators of degree 2, 3 and 4. We will call them C_2 , C_3 and C_4 respectively.

We will need a frame (e_0, e_1, e_2, e_3) for the Minkowski space:

$$e_0 = (e_0)_\mu = (1, \vec{0}) \quad e_j = (e_j)_\mu = (0, \vec{e}_j) \quad j = 1, 2, 3 \quad \text{with} \quad \begin{cases} \vec{e}_1 = (1, 0, 0) \\ \vec{e}_2 = (0, 1, 0) \\ \vec{e}_3 = (0, 0, 1) \end{cases} \quad (4.16)$$

which satisfies

$$e_\mu \cdot e_\nu = (\epsilon_\mu)^\rho (\epsilon_\nu)^\sigma g_{\rho\sigma} = g_{\mu\nu} \quad \text{and} \quad (e_\mu)^\rho (e_\nu)^\sigma g^{\mu\nu} = g^{\rho\sigma} \quad (4.17)$$

4.2 Momentum representation

We first construct the differential representation associated to \mathcal{G}_- (see section 3.2). We start with a coordinate representation, where functions depend on $x = (x^\mu)$ and perform a Fourier transformation:

$$\tilde{\varphi}(p) = \int d^s x e^{ip \cdot x} \varphi(x)$$

The coordinate differential representation on the space G/P :

$$\begin{aligned} P_\mu &= i \frac{\partial}{\partial x^\mu} & M_{\mu\nu} &= i(x_\mu \frac{\partial}{\partial x^\nu} - x_\nu \frac{\partial}{\partial x^\mu}) \\ D &= ix \cdot \frac{\partial}{\partial x} & K_\mu &= i(2x_\mu x \cdot \frac{\partial}{\partial x} - x^2 \frac{\partial}{\partial x^\mu}) \end{aligned} \quad (4.18)$$

is thus converted into the momentum representation:

$$\begin{aligned} P_\mu &= p_\mu & M_{\mu\nu} &= i(p_\mu \partial_\nu - p_\nu \partial_\mu) \\ D &= -i(p \cdot \partial + 4) & K_\mu &= (p_\mu \square - 2p \cdot \partial \partial_\mu - 8\partial_\mu) \end{aligned} \quad (4.19)$$

where ∂_μ stands for $\frac{\partial}{\partial x^\mu}$, p is the quadrivector (p_μ) , ∂ the quadrivector (∂_μ) , and $\square = \partial \cdot \partial = g^{\mu\nu} \partial_\mu \partial_\nu = \partial^\mu \partial_\mu$.

4.3 The \mathcal{W} -algebra

We wish to construct the commutant of the P_μ in order to build a \mathcal{W} -representation for $\mathcal{G} = so(4, 2)$. We have seen that D is the right grading operator, since $\mathcal{G}_- = \{P_\mu\}$. In the case we consider $sl(4)$ the maximally non-compact form of \mathcal{G} , this gradation corresponds to the model $\mathcal{W}(sl(4), 2sl(2))$. Then referring to the formula (2.1) we can take $\tilde{\mathcal{G}} = \mathcal{G}_- \oplus \tilde{\mathcal{G}}_0$ where

$$\tilde{\mathcal{G}}_0 = \{M_{13} - M_{01}, M_{23} - M_{02}, M_{03}, D\}. \quad (4.20)$$

The commutant of $\tilde{\mathcal{G}}$ can be seen as a compactified form³ of the algebra $\mathcal{W}(sl(4), 2sl(2))$. Indeed, calculations can be done in an easier way when using the $sl(4)$ algebra (instead of $so(4, 2)$) and then compactifying the resulting algebra. This algebra is a polynomial deformation of $\mathcal{G}_0 = so(3, 1) \oplus so(1, 1)$. It contains seven generators: three generators J_k , $k = 1, 2, 3$ of degree one in the $so(4, 2)$ elements, which form $so(3)$, a Lie subalgebra of the \mathcal{W} -algebra,

³Let us mention the recent work [8] on the real forms of \mathcal{W} -algebras.

and also four generators C_2 and S_k , $k = 1, 2, 3$ of degree two. C_2 is the second order Casimir of $so(4, 2)$:

$$C_2 = \frac{1}{2}(g^{\mu\nu}g^{\rho\sigma}M_{\mu\nu}M_{\rho\sigma} + g^{\mu\nu}(P_\mu K_\nu + K_\mu P_\nu)) - D^2 \quad (4.21)$$

For the clarity of the presentation, we postpone the expressions of the other generators in terms of the $so(4, 2)$ generators (formulae (4.37), (4.48) and (4.50)).

The commutation relations of the \mathcal{W} -algebra read:

$$\begin{aligned} [J_j, J_k] &= i\epsilon_{jkl}J_l \\ [J_j, S_k] &= i\epsilon_{jkl}S_l \\ [S_j, S_k] &= -i\epsilon_{jkl}\left(2(J_1^2 + J_2^2 + J_3^2) - C_2 - 4\right)J_l \end{aligned} \quad (4.22)$$

4.4 The \mathcal{W} -realization

Now using technics summarized in section 3.3 as well as the results of 4.3, one can determine the general expressions of the conformal algebra generators. The generators corresponding to $\tilde{\mathcal{G}}$ are unchanged with respect to the differential realization (4.19).

One remarks that the $so(4, 2)$ generators are indeed linear in the J_i ($i = 1, 2, 3$) and Z_k ($k = 0, 1, 2, 3$) -see (4.35)- which also form a basis of the \mathcal{W} -algebra.

One notes also that only the $\tilde{\mathcal{J}}$ -part of the \mathcal{W} -algebra shows up in the Poincaré part: this will be discussed in the next paragraph.

One can check that every \mathcal{W} -generator reduces to zero in the differential realization (4.19).

Finally let us emphasize that the realization given just below corresponds to values of $p^2 > 0$; therefore the quantities $p_0 + p_3$ and p^2 appearing in the denominators never vanish. Let us now give the realization that one gets from our "W-approach" (\mathbb{I} denotes the identity matrix):

$$P_\mu = p_\mu \mathbb{I} \quad \mu = 0, 1, 2, 3 \quad (4.23)$$

$$M_{12} = i(p_1\partial_2 - p_2\partial_1) \mathbb{I} + J_3 \quad (4.24)$$

$$M_{13} = i(p_1\partial_3 - p_3\partial_1) \mathbb{I} - \frac{\sqrt{p^2}}{p_0 + p_3}J_2 - \frac{p_2}{p_0 + p_3}J_3 \quad (4.25)$$

$$M_{23} = i(p_2\partial_3 - p_3\partial_2) \mathbb{I} + \frac{\sqrt{p^2}}{p_0 + p_3}J_1 + \frac{p_1}{p_0 + p_3}J_3 \quad (4.26)$$

$$M_{01} = i(p_0\partial_1 - p_1\partial_0) \mathbb{I} - \frac{\sqrt{p^2}}{p_0 + p_3}J_2 - \frac{p_2}{p_0 + p_3}J_3 \quad (4.27)$$

$$M_{02} = i(p_0\partial_2 - p_2\partial_0) \mathbb{I} + \frac{\sqrt{p^2}}{p_0 + p_3}J_1 + \frac{p_1}{p_0 + p_3}J_3 \quad (4.28)$$

$$M_{03} = i(p_0\partial_3 - p_3\partial_0) \mathbb{I} \quad (4.29)$$

$$D = -i(p \cdot \partial + 4) \mathbb{I} \quad (4.30)$$

$$\begin{aligned} K_0 = & (p_0\Box - 2p \cdot \partial\partial_0 - 8\partial_0) \mathbb{I} - \frac{2}{p_0 + p_3}Z_3 + \frac{p_0}{p^2}Z_0 + \frac{1}{(p_0 + p_3)\sqrt{p^2}}(p_1Z_1 + p_2Z_2) + \\ & - \frac{2i\sqrt{p^2}}{p_0 + p_3} \left(-\left(\frac{5}{2}\frac{p_2}{p^2} + \partial_2\right)J_1 + \left(\frac{5}{2}\frac{p_1}{p^2} + \partial_1\right)J_2 \right) - \frac{2i}{p_0 + p_3}(p_2\partial_1 - p_1\partial_2)J_3 \end{aligned} \quad (4.31)$$

$$K_1 = (p_1\Box - 2p \cdot \partial\partial_1 - 8\partial_1) \mathbb{I} + \frac{1}{\sqrt{p^2}}Z_1 + \frac{p_1}{p^2}Z_0 +$$

$$-\frac{2i\sqrt{p^2}}{p_0 + p_3} \left(\frac{5p_0 + p_3}{2p^2} + \partial_0 + \partial_3 \right) J_2 - 2i \left(\frac{p_2}{p_0 + p_3} (\partial_0 + \partial_3) - \partial_2 \right) J_3 \quad (4.32)$$

$$K_2 = (p_2 \square - 2p \cdot \partial \partial_2 - 8\partial_2) \mathbb{I} + \frac{1}{\sqrt{p^2}} Z_2 + \frac{p_2}{p^2} Z_0 + \\ + \frac{2i\sqrt{p^2}}{p_0 + p_3} \left(\frac{5p_0 + p_3}{2p^2} + \partial_0 + \partial_3 \right) J_1 + 2i \left(\frac{p_1}{p_0 + p_3} (\partial_0 + \partial_3) - \partial_1 \right) J_3 \quad (4.33)$$

$$K_3 = (p_3 \square - 2p \cdot \partial \partial_3 - 8\partial_3) \mathbb{I} + \frac{2}{p_0 + p_3} Z_3 + \frac{p_3}{p^2} Z_0 - \frac{1}{(p_0 + p_3)\sqrt{p^2}} (p_1 Z_1 + p_2 Z_2) + \\ + \frac{2i\sqrt{p^2}}{p_0 + p_3} \left(-\left(\frac{5p_2}{2p^2} + \partial_2\right) J_1 + \left(\frac{5p_1}{2p^2} + \partial_1\right) J_2 \right) + \frac{2i}{p_0 + p_3} (p_2 \partial_1 - p_1 \partial_2) J_3 \quad (4.34)$$

where

$$\begin{aligned} Z_1 &= 2S_1 + J_3 J_1 + J_1 J_3 & Z_2 &= 2S_2 + J_3 J_2 + J_2 J_3 \\ Z_3 &= S_3 - (J_1^2 + J_2^2) & Z_0 &= 2S_3 + C_2 - J_3^2 - 2(J_1^2 + J_2^2) \end{aligned} \quad (4.35)$$

4.5 The Poincaré algebra

Let us focus on the expressions of the Poincaré generators (4.23)- (4.29). We recall the expressions of the Pauli-Lubanski-Wigner quadri-vector $W^\mu = \frac{1}{2}\epsilon^{\mu\nu\rho\sigma} P_\nu M_{\rho\sigma}$ which satisfies:

$$[W_\mu, P_\nu] = 0 \quad [W_\mu, W_\nu] = i\epsilon_{\mu\nu\rho\sigma} W^\rho P^\sigma \quad W \cdot P = 0 \quad (4.36)$$

It is well-known that the irreducible representations of the Poincaré algebra are labelled by the eigenvalues of $P^2 = p^2$ and $W^2 = -s(s+1)p^2$, where s is the spin of the particle. Because of the relation $W \cdot P = 0$, the quadri-vector W^μ possesses only 3 independant components. These 3 independant components generate the spin algebra $so(3)$ when p^2 is positive [9]. This is recovered in a very natural way in our \mathcal{W} -algebra framework. Indeed, the generators J_k really play the role of the spin generators, since they can be rewritten as:

$$J_k = -\frac{1}{m} n_k \cdot W = -\frac{1}{m} (n_k)^\mu W_\mu \quad \text{and} \quad W^\mu = -m \sum_{k=1}^{k=3} (n_k)^\mu J_k \quad (\text{since } P \cdot W = 0) \quad (4.37)$$

$$\text{with} \quad m = \sqrt{p^2} \quad \text{and} \quad W \cdot W = -P^2 J^2 \quad (4.38)$$

and where we have introduced the frame of the "particle" of momentum p :

$$n_0 = (n_0)_\mu = \frac{1}{m} p = \left(\frac{p_0}{m}, \frac{p_1}{m}, \frac{p_2}{m}, \frac{p_3}{m} \right) \quad (4.39)$$

$$n_1 = (n_1)_\mu = \frac{1}{p_0 + p_3} (p_1, p_0 + p_3, 0, -p_1) \quad (4.40)$$

$$n_2 = (n_2)_\mu = \frac{1}{p_0 + p_3} (p_2, 0, p_0 + p_3, -p_2) \quad (4.41)$$

$$n_3 = (n_3)_\mu = \frac{-m}{p_0 + p_3} (1, 0, 0, -1) + \frac{1}{m} p \quad (4.42)$$

which obeys to

$$n_\mu \cdot n_\nu = (n_\mu)^\rho (n_\nu)^\sigma g_{\rho\sigma} = g_{\mu\nu} \quad (4.43)$$

and also to

$$(n_\mu)^\rho (n_\nu)^\sigma g^{\mu\nu} = g^{\rho\sigma} \quad (4.44)$$

The Lorentz transformation, denoted by $[p]$ in [9] and $L(p)$ in [6], which moves the rigid referential frame $(\epsilon_0, \epsilon_1, \epsilon_2, \epsilon_3)$ to the p -frame (n_0, n_1, n_2, n_3) :

$$(\epsilon_0, \epsilon_1, \epsilon_2, \epsilon_3) \xrightarrow{L(p)} (n_0, n_1, n_2, n_3) \quad (4.45)$$

can explicitly be written as a rotation followed by a boost

$$L(p) = B(p)R(p) \quad \text{with} \quad B(p) = \exp(-i\lambda \frac{\vec{p} \cdot \vec{B}}{\|\vec{p}\|}) \quad \text{and} \quad R(p) = \exp(-i\theta \frac{\vec{p}_\perp \cdot \vec{R}}{\|\vec{p}_\perp\|}) \quad (4.46)$$

with $sh(\lambda) = \frac{\|\vec{p}\|}{m}$, $cos(\theta) = 1 - \frac{\|\vec{p}\|^2}{(m+p_0)(p_0+p_3)}$ and $\vec{p}_\perp = (p_2, -p_1, 0)$

where we have defined $\vec{B} = (M_{01}, M_{02}, M_{03})$ and $\vec{R} = (M_{23}, M_{31}, M_{12})$. This transformation also relates the three vector (J_i) with the four vector (W_μ) :

$$mJ = (0, m\vec{J}) \xrightarrow{L(p)} W = (W_\mu) \quad (4.47)$$

Actually, $L(p)$ belongs to the set of Lorentz transformations which send the four vector $(m, 0, 0, 0)$ into $p = (p_0, p_1, p_2, p_3)$. It is through $L(p)$ that the representations of the Poincaré group can be constructed from representations of the rotation subgroup. Indeed, following [9] and [6], the Lorentz transformation Λ acting on function $\tilde{\varphi}$ of the p -variable in the U representation:

$$\Lambda \tilde{\varphi}(p) = U(\Lambda)\tilde{\varphi}(\Lambda^{-1}p)$$

writes more conveniently on the Wigner functions ψ defined by

$$v(p) = U(L(p)^{-1})\tilde{\varphi}(p)$$

as

$$(\Lambda \psi)(p) = U(L(p)^{-1}\Lambda L(\Lambda^{-1}p))\psi(\Lambda^{-1}p).$$

We recognize in the product $L(p)^{-1}\Lambda L(\Lambda^{-1}p)$ a Wigner rotation, element of the $(m, 0, 0, 0)$ -vector stabilizer, itself isomorphic to the $SO(3)$ group when $p^2 > 0$.

Let us emphasize that the transformation $L(p)$ can be chosen up to a R -rotation leaving $(m, 0, 0, 0)$ invariant. In particular, the $L(p)$ used in the computations of [6] is the pure boost $B(p)$. This of course produces a new but equivalent frame (n'_0, n'_1, n'_2, n'_3) and different expressions for the Lie algebra realization.

4.6 Generalization to $so(4, 2)$

We now look at the other generators of the conformal algebra. We proceed as for the Poincaré algebra. In the same way we have introduced the Pauli-Lubanski-Wigner vector W^μ , let us define:

$$\Sigma_\mu = -W^2 P_\mu + P^2 \left[P^\alpha M_{\alpha\mu} (D + i) - \frac{1}{2}(P_\mu P \cdot K - P^2 K_\mu) - \frac{1}{2}\epsilon_{\mu\nu\rho\sigma} W^\nu M^{\rho\sigma} \right] \quad (4.48)$$

It satisfies

$$\begin{aligned} [\Sigma_\mu, P_\nu] &= 0 & \Sigma \cdot P &= 0 \\ [\Sigma_\mu, W_\nu] &= i\epsilon_{\mu\nu\rho\sigma} \Sigma^\rho P^\sigma \\ [\Sigma_\mu, \Sigma_\nu] &= i\epsilon_{\mu\nu\rho\sigma} P^2 W^\rho P^\sigma (P^2(4 + C_2) + 2W^2) \end{aligned} \quad (4.49)$$

The generators S_i are connected to the quadrivector (Σ^μ) through

$$S_k = -\frac{1}{m^3} n_k \cdot \Sigma = -\frac{1}{m^3} (n_k)^\mu \Sigma_\mu \quad \text{and} \quad \Sigma^\mu = -m^3 \sum_{k=1}^{k=3} (n_k)^\mu S_k \quad (\text{since } P \cdot \Sigma = 0) \quad (4.50)$$

and also:

$$m^3 S = (0, m^3 \vec{S}) \xrightarrow{L(p)} \Sigma = (\Sigma_\mu) \quad (4.51)$$

Let us remark that the third and fourth Casimir operators of $so(4, 2)$ can be expressed as \mathcal{W} -polynomials:

$$\vec{J} \cdot \vec{S} = -\frac{W \cdot \Sigma}{(P^2)^2} = -\frac{1}{2} C_3 \quad (4.52)$$

$$\vec{S}^2 + \vec{J}^2 (C_2 - \vec{J}^2 + 2) = \frac{1}{(P^2)^3} \left(\Sigma^2 + P^2 W^2 (W^2 + P^2 (C_2 + 2)) \right) = -\frac{1}{4} C_4 + \frac{2}{3} C_2$$

with the following expression for the Casimir operators:

$$C_3 = -\frac{1}{4} D M_{\mu\nu} \widetilde{M}^{\mu\nu} + \frac{1}{2} (K \cdot W + P \cdot W) \quad (4.53)$$

$$C_4 = 2(W \cdot V + V \cdot W) + \frac{1}{2} (P^2 K^2 + K^2 P^2) - (P \cdot K)^2 + \frac{1}{16} (M_{\mu\nu} \widetilde{M}^{\mu\nu})^2 + \\ -2(D^2 + 2) M_{\mu\nu} M^{\mu\nu} + 4DP^\mu K^\nu M_{\mu\nu} - 8iD(C_2 - D^2) + \frac{4}{3} C_2 - 20D^2 \quad (4.54)$$

where we have introduced

$$V^\mu = \frac{1}{2} \epsilon^{\mu\nu\rho\sigma} K_\nu M_{\rho\sigma} \quad \text{and} \quad \widetilde{M}^{\mu\nu} = \epsilon^{\mu\nu\rho\sigma} M_{\rho\sigma} \quad (4.55)$$

Note that (e_0, J, S) on the one hand, and (P, W, Σ) on the other hand form three independant quadrivectors: it is natural to look for a fourth quadrivector $X = (0, \vec{X})$ such that (e_0, J, S, X) is a frame. Then, we will get a quadrivector χ such that

$$m^4 X \xrightarrow{L(p)} \chi \quad \text{and} \quad (P, W, \Sigma, \chi) \text{ frame} \quad (4.56)$$

Calculations show that

$$\begin{aligned} X_1 &= (J_2 S_3 + S_3 J_2) - (J_3 S_2 + S_2 J_3) - 3i S_1 \\ X_2 &= (J_3 S_1 + S_1 J_3) - (J_1 S_3 + S_3 J_1) - 3i S_2 \\ X_3 &= (J_1 S_2 + S_2 J_1) - (J_2 S_1 + S_1 J_2) - 3i S_3 \end{aligned} \quad \text{with} \quad \vec{X} = (X_1, X_2, X_3) \quad (4.57)$$

is such a vector. Note that the leading term of \vec{X} is a non-commutative version of the exterior product $\vec{J} \wedge \vec{S}$. From equations (4.38) and (4.52), one sees that all the "physical informations" of the Poincaré algebra as well as of the conformal algebra $so(4, 2)$ are contained in these frames.

4.7 Case $m = 0$

Up to now, the results we have presented are valid for $p^2 > 0$. One can wonder whether there exists some limiting process that could lead to the case of zero mass representations. At that point, we have to make a distinction between the $so(4, 2)$ algebra and its Poincaré subalgebra.

Indeed, in the Poincaré algebra, the mass is an invariant ($p^2 = m^2$), so that it is very natural to look at the limit point $m = 0$. In the $so(4, 2)$ algebra, this is no more the case, but the sign $\epsilon(p^2)$ is still an invariant. By sign of p^2 , we mean $\epsilon(p^2) = +, 0, -$ when p^2 is respectively positive, zero or negative. One easily guesses that a limiting process on the three points $\epsilon = +1, 0, -1$ is hard to find. In other words, in the Poincaré algebra, the (positive mass) irreducible representations are hyperboloids in the Minkowski space, while in the $so(4, 2)$ algebra, the irreducible representation $\epsilon = +1$ is the whole interior of the light cone. Then, if, in the Poincaré algebra, it is natural to look at the light cone as the limit of the family of hyperboloids when $m \rightarrow 0$, we have to face a problem of dimensional reduction of the space of representations in the $so(4, 2)$ (see [10] for a direct treatment of the case $m = 0$ in $so(4, 2)$).

Thus, we will focus below on the Poincaré algebra to study the limit $m \rightarrow 0$. The \mathcal{W} -algebra then reduces to a three-dimensional algebra since the generators \vec{S} and C_2 do not enter in the realization (4.23-4.29). Moreover, in the case $m = 0$, one cannot introduce the vector \vec{J} and the frame attached to the particle, which is now of zero mass. However, we can define:

$$Q_j = n_j \cdot W \quad \text{for } j = 1, 2 \quad \text{while} \quad Q_3 = \frac{1}{p_0 + p_3} (e_0 - e_3) \cdot W \quad (4.58)$$

which form an $E(2)$ subalgebra

$$[Q_3, Q_1] = iQ_2 \quad [Q_3, Q_2] = -iQ_1 \quad [Q_1, Q_2] = 0 \quad (4.59)$$

in accordance with the general results on zero mass particles. Moreover, (4.58) shows that for W^μ proportional to P^μ , and using the definitions (4.40-4.41), we have $Q_1 = Q_2 = 0$. Let us remind that this latter condition is the one we need to select the physical cases when $m = 0$, *i.e.* the cases of finite spin (more exactly helicity). By definition, these latter cases are the ones that correspond to finite dimensional representations of the $E(2)$ algebra. Thus, they are also natural in the \mathcal{W} -realizations, since we consider a priori matricial representations of the \mathcal{W} -algebra. In that case, the realization reads:

$$P_\mu = p_\mu \mathbb{I} \quad \mu = 0, 1, 2, 3 \quad (4.60)$$

$$M_{12} = i(p_1 \partial_2 - p_2 \partial_1) \mathbb{I} + Q_3 \quad (4.61)$$

$$M_{13} = i(p_1 \partial_3 - p_3 \partial_1) \mathbb{I} - \frac{p_2}{p_0 + p_3} Q_3 \quad (4.62)$$

$$M_{23} = i(p_2 \partial_3 - p_3 \partial_2) \mathbb{I} + \frac{p_1}{p_0 + p_3} Q_3 \quad (4.63)$$

$$M_{01} = i(p_0 \partial_1 - p_1 \partial_0) \mathbb{I} - \frac{p_2}{p_0 + p_3} Q_3 \quad (4.64)$$

$$M_{02} = i(p_0 \partial_2 - p_2 \partial_0) \mathbb{I} + \frac{p_1}{p_0 + p_3} Q_3 \quad (4.65)$$

$$M_{03} = i(p_0 \partial_3 - p_3 \partial_0) \mathbb{I} \quad (4.66)$$

where Q_3 corresponds to the helicity. Let us note that although the denominator $p_0 + p_3$ may vanish when $m = 0$, the above expressions can still be well-defined using limits $p_0 + p_3 \rightarrow 0$ and $p_j = o(p_0 + p_3)$ ($j = 1, 2$).

4.8 Comparison with induced representations

It is time to compare the above obtained \mathcal{W} -realization of the $so(4, 2)$ algebra with the usual ones constructed via the induced representation method. As could be expected, the results are

equivalent. More is given in [6] where the classification of all the unitary ray representation of the $SU(2, 2)$ group with positive energy is achieved. It will then be possible for us, using these results, to select the finite dimensional representations of the \mathcal{W} -algebra which lead to such unitary representations of the conformal one, and therefore to unitary representations of the conformal group.

Let us start by introducing a realization of the \mathcal{W} -algebra under consideration known as the Miura realization, and particularly useful in what follows. Indeed, it is the Miura realization of the \mathcal{W} -algebra which can be unearthed in [6]. Its knowledge also provides a simple algorithm for the construction of representations of the \mathcal{W} -algebra.

4.8.1 The Miura realization

The Miura transformation provides a realization of the \mathcal{W} -algebra relative to the gradation $\mathcal{G} = \mathcal{G}_{-1} \oplus \mathcal{G}_0 \oplus \mathcal{G}_{+1}$ in terms of elements of the enveloping algebra $\mathcal{U}(\mathcal{G}_0)$ [3]. There, the \mathcal{G}_0 part is the Lorentz algebra to which have been added the dilatation: $so(3, 1) \oplus so(1, 1)$, and the Miura realization reads simply:

$$\vec{J} = \vec{R} \quad \vec{S} = \vec{R} \times \vec{B} - i(D - 1)\vec{B} \quad C_2 = \vec{R}^2 - \vec{B}^2 + D(D - 4). \quad (4.67)$$

using again the notations of section 4.5 for the Lorentz generators. One immediately remarks that this realization is much simpler than the one above determined and which involves all the $so(4, 2)$ generators: (4.37) and (4.50). Thus, to each finite dimensional \mathcal{G}_0 representation can be associated, via equation (4.67), a finite dimensional \mathcal{W} representation. Labelling in a rather natural way $(j_1, j_2; d)$ an $so(3, 1) \oplus so(1, 1)$ irreducible representation, the center element C_2 in \mathcal{W} will get the scalar value:

$$c_2 = 2j_1(j_1 + 1) + 2j_2(j_2 + 1) + d(d - 4).$$

4.8.2 Unitary representations

The unitary representations with positive energy of the universal covering group \hat{G} of $G = SU(2, 2)$ are presented in [6] -see also [11]. We note the chain of isomorphisms:

$$\left(\begin{array}{c} \text{conformal group of} \\ \text{Minkowski space} \end{array} \right) \simeq SO(4, 2)/\mathbb{Z}_2 \simeq SU(2, 2)/\mathbb{Z}_4 \simeq \hat{G}/(\Gamma_1 \times \Gamma_2) \quad (4.68)$$

where $\Gamma_1 \simeq \mathbb{Z}_2$ and $\Gamma_2 \simeq \mathbb{Z}$. Such unitary representations possess a lowest weight, and are interpolation of the holomorphic discrete series. These latter are realized by considering functions on the bounded symmetric domain $SU(2, 2)/K$, where $K = SU(2) \times SU(2) \times U(1)$ is the maximal compact subgroup of $SU(2, 2)$. Actually they are induced by the finite dimensional (unitary) representations of K .

However, Mack [6] described them as induced representations by the parabolic subgroup $P = \Gamma_2 MAN$, where following his notations M, A, N are the subgroups of Lorentz, dilatation and special conformal transformations respectively. The Lie algebra of P is exactly $\mathcal{G}_0 \oplus \mathcal{G}_+$ in our language. We note that P is actually the little group of the point $x = 0$ in the compactified Minkowski space, itself isomorphic to \hat{G}/P . The induced representations on this space are then constructed in the usual way. We take finite dimensional representations D^λ of the parabolic subgroup P , where N acts trivially. They are labelled by two non-negative

(half-)integers (j_1, j_2) , associated to spinor representations of the Lorentz group $D^{j_2 j_1}$, and by d a real number associated to the dilatation:

$$D^\lambda(man) = |a|^{d-2} D^{j_2 j_1}(m).$$

It acts on a $(2j_1 + 1)(2j_2 + 1)$ dimensional vector space E^λ . From now on we will disregard the action of the discrete central part Γ , which of course will not affect the Lie algebra realization. We then consider the space of functions φ on the group G with values in E^λ which have the covariance property

$$\varphi(gman) = D^\lambda(man)^{-1} \varphi(g) \quad \forall g \in SU(2, 2).$$

The conformal group acts on these functions by the left regular representation:

$$(T(g)\varphi)(g') = \varphi(g^{-1}g').$$

Since the decomposition $g = g_- man$ is unique, functions φ can be viewed as functions $\varphi(x)$ defined on G/P , the Minkowski space. The action of $SU(2, 2)$ then takes the form:

$$(T(g)\varphi)(x) = |a|^2 D^\lambda(man)^{-1} \varphi(x')$$

where $x' \in G_-$, m, a, n are defined through the unique decomposition $g^{-1}x = x'man$. Note that this way to describe unitary representations of $SU(2, 2)$ is unusual since we induced by non-unitary representations of a parabolic subgroup which is not cuspidal [12]. By cuspidal, we mean that the Cartan subalgebra of the semi-simple part M of P is made only with compact generators, which is not the case here.

To get the action of the Lie algebra $so(4, 2)$, one has simply to take the infinitesimal form:

$$\begin{aligned} P_\mu &= i \frac{\partial}{\partial x^\mu} & M_{\mu\nu} &= i(x_\mu \frac{\partial}{\partial x^\nu} - x_\nu \frac{\partial}{\partial x^\mu} - i\Sigma_{\mu\nu}) \\ D &= i(4 - d + x \cdot \frac{\partial}{\partial x}) & K_\mu &= i((8 - 2d)x_\mu + 2x_\mu x \cdot \frac{\partial}{\partial x} - x^2 \frac{\partial}{\partial x^\mu} - 2ix^\nu \Sigma_{\mu\nu}) \end{aligned} \quad (4.69)$$

Here $\Sigma_{\mu\nu}$ is a $(2j_1 + 1)(2j_2 + 1)$ representation of the Lorentz algebra with $R_i = \frac{1}{2}\epsilon_{ijk}\Sigma^{jk}$ and $B_i = \Sigma_{0i}$.

The comparison with the \mathcal{W} -realization proceeds in several steps. First of all, we have to perform a Fourier transformation in order to get momentum representations with functions $\tilde{\varphi}(p)$:

$$\begin{aligned} P_\mu &= p_\mu & M_{\mu\nu} &= i(p_\mu \partial_\nu - p_\nu \partial_\mu - i\Sigma_{\mu\nu}) \\ D &= -i(p \cdot \partial + d) & K_\mu &= p_\mu \square - 2p \cdot \partial \partial_\mu - 2d\partial_\mu - 2i\Sigma_{\mu\nu} \partial^\nu \end{aligned} \quad (4.70)$$

Then we study the action of the conformal algebra on the functions $\psi(p)$ defined by

$$\psi(p) = D^{j_2 j_1} \left(L(p)^{-1} \right) m^{d-4} \tilde{\varphi}(p)$$

where $L(p)$ is a Lorentz transformation that transports the frame (e_0, e_1, e_2, e_3) to the frame (n_0, n_1, n_2, n_3) . If X is the form of a generator of $so(4, 2)$ acting on functions $\tilde{\varphi}(p)$ then, when acting on functions $\psi(p)$ it becomes

$$\hat{X} = D^{j_2 j_1} \left(L(p)^{-1} \right) m^{d-4} X m^{-(d-4)} D^{j_2 j_1} \left(L(p) \right)$$

Note that when restricted to the Poincaré algebra, since the mass is constant on an irreducible representation, this whole trick amounts in working on the Wigner wave function (cf section 4.5). We then recover the well-known representations of the Poincaré group induced by the group of rotations. Explicitly, Lorentz transformations are given by

$$(T(\Lambda)\psi)(p) = D^{j_2 j_1} \left(L(p)^{-1} \Lambda L(\Lambda^{-1} p) \right) \psi(\Lambda^{-1} p)$$

where $D^{j_2 j_1} (L(p)^{-1} \Lambda L(\Lambda^{-1} p))$ reduces simply to representations of spin $s = |j_1 - j_2|, \dots, j_1 + j_2$ of $so(3)$. This realization of the Poincaré algebra corresponds exactly to the form we find for the \mathcal{W} -representation (4.24)-(4.29), by identifying J_i with R_i .

For the other $so(4, 2)$ generators, to calculate \hat{X} , we first compute $m^{d-4} X m^{-(d-4)}$. Then, we perform a second order Taylor expansion of $L(p)$ around $(p_0, \vec{p}) = (m, \vec{0})$. Finally, we get the general form of \hat{X} , using the fact that K_μ transforms as a four-vector under a Lorentz transformation, and D as a scalar. We obtain for the dilatation the realization (4.30) computed in the \mathcal{W} -representation. We also get the \mathcal{W} -realization (4.31)-(4.34) for \hat{K}_μ , provided we identify

$$\vec{J} = \vec{R} \quad \vec{S} = \vec{R} \times \vec{B} - i(d-1)\vec{B} \quad C_2 = \vec{R}^2 - \vec{B}^2 + d(d-4), \quad (4.71)$$

which is precisely the Miura realization given in (4.67). Thus, we have made the link between the \mathcal{W} -realization and the induced representation technics.

As a further consequence, the unitarity conditions on j_1, j_2 and d given in [6] can be seen as conditions on the \mathcal{W} -algebra representations. In particular the class of representations of positive masses

$$\begin{aligned} d &\geq j_1 + j_2 + 2 & \text{with } j_1 j_2 &\neq 0 \\ d &> j_1 + j_2 + 1 & \text{with } j_1 j_2 &= 0 \end{aligned} \quad (4.72)$$

gives rise respectively to the C_2 conditions:

$$\begin{aligned} c_2 &\geq 2j_1(j_1 + 1) + 2j_2(j_2 + 1) + (j_1 + j_2)^2 - 4 \\ c_2 &> 3(j_1 + j_2 + 1)(j_1 + j_2 - 1) \end{aligned} \quad (4.73)$$

For $j_1 j_2 \neq 0$ and $d > j_1 + j_2 + 2$, each representation contains spins $s = |j_1 - j_2|, \dots, j_1 + j_2$, while for $d = j_1 + j_2 + 2$ or $j_1 j_2 = 0$ each representation contains only the spin $s = j_1 + j_2$.

Acknowledgements

We would like to thanks Philippe Caldero for valuable discussions about commutants in Lie algebras. We are also indebted to Raymond Stora for his interest and numerous comments on induced representations of the Poincaré and conformal algebras.

References

- [1] F. Barbarin, E. Ragoucy and P. Sorba, Nucl. Phys. **B442** (1995) 425.
- [2] F. Barbarin, E. Ragoucy and P. Sorba, *Non polynomial realizations of \mathcal{W} -algebras*, preprint ENSLAPP -AL-536/95, to appear in Int. Journ. Mod. Phys. **A**.
- [3] J. de Boer, F. Harmsze and T. Tjin. *Nonlinear finite \mathcal{W} -symmetries and applications in elementary systems*, preprint ITP-SB-95-07
- [4] L. Feher, L. O'Raiheartaigh. P. Ruelle, I. Tsutsui and A. Wipf, Phys. Rep. **222** (1992) 1, and references therein.
- [5] Leinaas and Myrheim, Int. Journ. Mod. Phys. **A8** (1993) 3649.
- [6] G. Mack, Comm. Math. Phys. **55** (1977) 1.
- [7] B. Kostant, Lecture Note in Mathematics **466** p.101, Berlin, Heidelberg, New-York: Springer 1974.
- [8] J.M. Evans and J.O. Madsen, *Real forms of non-Abelian Toda theories and their \mathcal{W} -algebras*, preprint ENSLAPP-A-592/96, DAMTP/96-49, hep-th/9605126.
- [9] P. Moussa and R. Stora, in "Methods in Subnuclear Physics", Gordon and Breach, Herceg-Novi Summer School, 1966.
- [10] G. Mack and I. Todorov, Journ. Math. Phys. **10** (1969) 2078.
- [11] W. Rühl, Comm. Math. Phys. **30** (1973) 287.
- [12] A. W. Knap and B. Speh, J. Funct. Anal. **45** (1982) 41.