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ENERGY PRINCIPLES FOR LINEAR DISSIPATIVE SYSTEMS WITH APPLICATION TO RESISTIVE MHD STABILITY

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A formalism for the construction of energy principles for dissipative systems is presented. It is shown that dissipative systems satisfy a conservation law for the bilinear Hamiltonian provided the Lagrangian is time invariant. The energy on the other hand, differs from the Hamiltonian by being quadratic and by having a negative definite time derivative (positive power dissipation). The energy is a Lyapunov functional whose definiteness yields necessary and sufficient stability criteria. The stability problem of resistive magnetohydrodynamic (MHD) is addressed: the energy principle for ideal MHD is generalized and the stability criterion by Tasso [Phys. Letters **147**, 28 (1990)] is shown to be necessary in addition to sufficient for real growth rates. An energy principle is found for the inner layer equations that yields the resistive stability criterion $D_R < 0$ in the incompressible limit, whereas the tearing mode criterion $\Delta' < 0$ is shown to result from the conservation law of the bilinear concomitant in the resistive layer.

52.30Jb, 52.65.Kj

I. INTRODUCTION

Testing the stability of a linear system usually requires the computation of the growth rate spectrum; if the most positive growth rate $\hat{\gamma}$ has a positive real part then the system is unstable. This is often a laborious task since it consists in solving the equations governing the evolution of the modes. In some cases, however, the knowledge of the growth rates is not required if one can show that stability depends upon the definiteness of a functional. For instance, one may be able to construct a positive quadratic functional which can be shown to increase in time, then the system is evidently unstable. Conversely, if this positive functional can only decrease in time then the system is stable. The problem can also be turned around; if the time derivative is negative definite then stability relies upon having the functional positive definite. Often this functional may be thought of as representing the energy of the system and the second case is more often realized since the effect of dissipation is to lower the energy level of the system.

Such a functional is called a Lyapunov functional [1]; we will derive a Lyapunov functional which possesses all the expected properties of an energy and for this reason we will refer to it as the *energy functional*. In this case we will also say that we have an *energy principle*.

The existence of an energy principle for ideal, non-resistive modes is well-known since Bernstein *et al.* [2] (henceforth referred to as BFKK), and has found numerous applications. In high-dimensional geometry, the energy principle reduces the

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complexity of determining stability to the one of analyzing the definiteness of a functional which is quadratic in the normal displacement field [see Newcomb [3] for the cylindrical case and Bineau [4] for the two-dimensional case]. The existence of an energy principle for resistive modes has been considered in works including Furth *et al.* [5], Adler *et al.* [6], Bondeson & Sobel [7], Tasso [8] and Wesson [9]. In the pressureless case of a one-dimensional plasma, Furth *et al.* [10] derived a stability criterion $\Delta' < 0$ for tearing modes with Δ' playing a role similar to the energy functional in ideal stability [5]. Adler *et al.* [6] have shown that, within the approximation of the reduced equations, Δ' is proportional to the nonlinear increase of magnetic energy in the resistive layer. [See also White [11] for a summary of the nonlinear energy balance discussion.] Bondeson & Sobel [7] extended this result by taking into account asymmetric layers and viscosity, whereas Tasso [8] derived a sufficient stability criterion against purely growing modes in arbitrary geometry by means of a functional dispersion relation.

The aim of this paper is to provide tools for the construction of the energy functional from which *necessary and sufficient* stability criteria can be extracted provided some "causality" conditions are satisfied. A large class of problems, including those which are time reversible do fulfill these conditions. However, the non-Hermitian property of some operators in resistive magnetohydrodynamic (MHD) destroys the causality conditions and therefore the stability criterion does not apply except for the restricted class of real growth rates.

We adopt the approach of field theory in § II A to derive the set of Euler-Lagrange equations that leave the action stationary. The originality of the approach resides in the fact that the equations we seek are not self-adjoint so that the variation of the action must be performed with respect to the solutions and the adjoint solutions treated as independent. Thus, the Lagrangian ought to be a bilinear form of the solutions and their adjoint rather than the quadratic form which is more common in the literature [12]. The Hamiltonian functional is also a bilinear form which is shown in § II B to be conserved; for this reason it can clearly not represent the energy of a dissipative system. The Hamiltonian is, however, not the only conserved functional; a conservation law based on the action density is derived in § II C.

In order to define the energy functional, we replace the adjoint solution in the Hamiltonian by the solution, or indeed any test function, so as to construct a *quadratic energy*. A further condition requires the energy to be real, this is achieved in § II D in the usual way by adding the Hermitian adjoint. These two conditions, which can be applied to any functional, define the notion of observable similarly to the prescription in quantum mechanics. It should be added that the arbitrariness involved in the choice of deciding which of the solutions is the adjoint and conversely can often be overcome by requiring the Lagrangian and the energy to satisfy the causal property defined in § II E, which guarantee that the energy observable decrease in time (for isolated systems) as one would expect intuitively.

Because of the dissipative nature of the system, the notion of stability is defined independently from the concept of potential energy. The system is stable if the energy remains bounded for all perturbations of the equilibrium (Lyapunov's theorem), and unstable if its energy can be released. We then find that the condition that the energy ultimately relax to the the initial energy (asymptotic stability) leads to a necessary and sufficient criterion for causal-dissipative systems in § II F based on the positive definiteness of the energy, similarly to the ideal stability criterion.

The machinery developed in § II D is used to derive the energy observable, apply the causality principle and obtain stability criteria for the resistive MHD equations. First, we focus in § III A on the full set of resistive equations which are shown to dissipate at a rate proportional to the resistivity. However, the presence of an anti-Hermitian part prevents us from obtaining a causal relation for all but real growth rates.

Tasso's stability criterion [13,8], which applies to purely growing modes, is shown in § III A to be necessary *and* sufficient without introducing a limiting scaling. We next turn our attention to the set of equations of Glasser *et al.* [14] which describes the evolution of tearing and interchange modes in toroidal plasmas. The system can be put into causal form corresponding to positive definite dissipation. However, the problem of anti-Hermiticity recurs here too; it is only in the limit of the Coppi *et al.* [15] equations that the system becomes Hermitian and that we are able to recover the resistive interchange criterion of § III B by requiring the zero-frequency energy to be positive definite. Finally, we apply in § III C the concomitant formulae of § II C to show that the well-known logarithmic jump $\Delta(Q)$ defining the inner matching index in nonideal stability studies, is positive for positive growth rates Q. The condition that the outer matching index $\Delta' < 0$ is then found to be a sufficient criterion.

II. GENERAL FORMALISM

A. Lagrangian and Euler-Lagrange equations

Consider the linear system composed of N continuous vector fields $\boldsymbol{z}_i$, $i = 1, \cdots N$. Without loss of generality, one may associate a Lagrangian functional to any linear system

$$\mathcal{L} \equiv \int_\Omega d\tau L[\mathsf{z}^+; \mathsf{z}], \tag{1}$$

which is bilinear in the vector fields

$$\mathsf{z} \equiv \begin{pmatrix} \boldsymbol{z}_1 \\ \vdots \\ \boldsymbol{z}_N \end{pmatrix}$$

and

$$\mathsf{z}^{+\dagger} \equiv (\mathsf{z}^{+*})^T \equiv \left(\boldsymbol{z}_1^+, \ \cdots, \ \boldsymbol{z}_N^+\right)^*,$$

with * denoting the complex conjugate and z^T the transpose of z. The role of the superscript $^+$ is to discriminate the adjoint solution z^+ against z, for we will be mainly interested in systems that are not self-adjoint. If the system turns out to be self-adjoint, then $\mathsf{z}^+ = \mathsf{z}$ is found *a posteriori* demonstrating the generality of (1).

We do not wish to be too specific about the components of z^+ and z, so let us assume that each component $\boldsymbol{z}_i^+$ and $\boldsymbol{z}_i$, $i = 1, \cdots N$, is a space, or three-component vector. The inner product between two such components will be denoted by $\boldsymbol{z}_i^+ \cdot \boldsymbol{z}_i$ (the $\cdot$ sign is reserved to the inner product of the space components), whereas the inner product between two "sets of vectors" by

$$\mathsf{y}^\dagger \cdot \mathsf{z} \equiv \sum_{i=1}^{N} \boldsymbol{y}_i^* \cdot \boldsymbol{z}_i.$$

The following assumptions are made regarding $L[\mathsf{z}^+; \mathsf{z}]$ in (1): firstly, $L[\mathsf{z}^+; \mathsf{z}]$ depends on time t only through the field variables (time invariance); secondly, by $[\mathsf{z}^+; \mathsf{z}]$ we mean that L is a *bilinear* combination of $\mathsf{z}^{+\dagger}$, $\boldsymbol{\nabla}\mathsf{z}^{+\dagger} \equiv \left(\boldsymbol{\nabla}\boldsymbol{z}_1^+ \ \cdots \ \boldsymbol{\nabla}\boldsymbol{z}_N^+\right)^*$, $\dot{\mathsf{z}}^{+\dagger} \equiv \left(\dot{\boldsymbol{z}}_1^+ \ \cdots \ \dot{\boldsymbol{z}}_N^+\right)^*$, $\boldsymbol{\nabla}\dot{\mathsf{z}}^{+\dagger}$, $\boldsymbol{\nabla}\mathsf{z}$, $\dot{\mathsf{z}}$ and $\boldsymbol{\nabla}\dot{\mathsf{z}}$ only, with $\dot{\mathsf{z}} \equiv \partial_t \mathsf{z}$ and excluding

higher-order derivatives so that we consider only *second-order* systems in space and time.

The equations for the z_i are the Euler-Lagrange equations that leave the action

$$\mathcal{S} \equiv \int_{\mathrm{T}} dt \mathcal{L} \tag{2}$$

stationary with respect to variations of $\mathsf{z}^{+\dagger}$:

$$\frac{\delta \mathcal{S}}{\delta \mathsf{z}^{+\dagger}} = 0.$$

This yields the set of N second-order differential equations

$$\left.\begin{aligned} -\boldsymbol{\nabla}\cdot\partial_t \tfrac{\partial L}{\partial \boldsymbol{\nabla}\dot{\boldsymbol{z}}_1^{+}\cdot} + \boldsymbol{\nabla}\cdot\tfrac{\partial L}{\partial \boldsymbol{\nabla}\boldsymbol{z}_1^{+}\cdot} + \partial_t \tfrac{\partial L}{\partial \dot{\boldsymbol{z}}_1^{+}\cdot} - \tfrac{\partial L}{\partial \boldsymbol{z}_1^{+}\cdot} \\ \vdots \\ -\boldsymbol{\nabla}\cdot\partial_t \tfrac{\partial L}{\partial \boldsymbol{\nabla}\dot{\boldsymbol{z}}_N^{+}\cdot} + \boldsymbol{\nabla}\cdot\tfrac{\partial L}{\partial \boldsymbol{\nabla}\boldsymbol{z}_N^{+}\cdot} + \partial_t \tfrac{\partial L}{\partial \dot{\boldsymbol{z}}_N^{+}\cdot} - \tfrac{\partial L}{\partial \boldsymbol{z}_N^{+}\cdot} \end{aligned}\right\} \equiv \mathsf{L}\cdot\mathsf{z} = 0 \tag{3}$$

according to the convention that the derivative of a row vector is a column vector and assuming $\delta \mathsf{z}^{+}$ to vanish at the boundaries of Ω and T. Requiring

$$\frac{\delta \mathcal{S}}{\delta \mathsf{z}} = 0,$$

leads to the Euler-Lagrange equations for z^{+}, $(\tilde{\mathsf{L}}\cdot\mathsf{z}^{+} = 0)^{\dagger}$, or

$$\left.\begin{aligned} -\boldsymbol{\nabla}\cdot\partial_t \tfrac{\partial L}{\partial \boldsymbol{\nabla}\dot{\boldsymbol{z}}_1} + \boldsymbol{\nabla}\cdot\tfrac{\partial L}{\partial \boldsymbol{\nabla}\boldsymbol{z}_1} + \partial_t \tfrac{\partial L}{\partial \dot{\boldsymbol{z}}_1} - \tfrac{\partial L}{\partial \boldsymbol{z}_1} \\ \vdots \\ -\boldsymbol{\nabla}\cdot\partial_t \tfrac{\partial L}{\partial \boldsymbol{\nabla}\dot{\boldsymbol{z}}_N} + \boldsymbol{\nabla}\cdot\tfrac{\partial L}{\partial \boldsymbol{\nabla}\boldsymbol{z}_N} + \partial_t \tfrac{\partial L}{\partial \dot{\boldsymbol{z}}_N} - \tfrac{\partial L}{\partial \boldsymbol{z}_N} \end{aligned}\right\} \equiv \tilde{\mathsf{L}}\cdot\mathsf{z}^{+} = 0 \tag{4}$$

which are the adjoint equations to (3) written as a column vector, i.e. Eqs.(4) can be derived by acting $\mathsf{z}^{+\dagger}\cdot$ onto $\mathsf{L}\cdot\mathsf{z}$, integrating the $\boldsymbol{\nabla}$ and ∂_t operators of L by parts so as to have $(\tilde{\mathsf{L}}\cdot\mathsf{z}^{+})^{\dagger}\cdot\mathsf{z}$ and disregarding the endpoint contributions.

B. Conservation laws

It is well known from Noether's theorem [16] that systems described by a Lagrangian density that does not depend explicitly on time gives rise to a conservation law for the Hamiltonian:

$$\boldsymbol{\nabla}\cdot\mathbf{S} + \partial_t H = 0, \tag{5}$$

where $\mathbf{S}$ is the Hamiltonian flux and H the Hamiltonian. We can convince ourselves that (5) is satisfied if we choose

$$\mathbf{S}[\mathsf{z}^{+};\mathsf{z}] \equiv \dot{\mathsf{z}}^{+\dagger}\cdot\frac{\partial L}{\partial \boldsymbol{\nabla}\dot{\mathsf{z}}^{+\dagger}} + \dot{\mathsf{z}}^{+\dagger}\cdot\frac{\partial L}{\partial \boldsymbol{\nabla}\mathsf{z}^{+\dagger}} + \frac{\partial L}{\partial \boldsymbol{\nabla}\dot{\mathsf{z}}}\cdot\ddot{\mathsf{z}} + \frac{\partial L}{\partial \boldsymbol{\nabla}\mathsf{z}}\cdot\dot{\mathsf{z}} \tag{6}$$

and

$$H[\mathsf{z}^{+};\mathsf{z}] = \dot{\mathsf{z}}^{+\dagger}\cdot\left(\frac{\partial L}{\partial \dot{\mathsf{z}}^{+\dagger}} - \boldsymbol{\nabla}\cdot\frac{\partial L}{\partial \boldsymbol{\nabla}\dot{\mathsf{z}}^{+\dagger}}\right) + \left(\frac{\partial L}{\partial \dot{\mathsf{z}}} - \boldsymbol{\nabla}\cdot\frac{\partial L}{\partial \boldsymbol{\nabla}\dot{\mathsf{z}}}\right)\cdot\dot{\mathsf{z}} - L, \tag{7}$$

for we have

$$\begin{aligned}\boldsymbol{\nabla}\cdot\mathbf{S}[\mathbf{z}^+;\mathbf{z}] + \partial_t H[\mathbf{z}^+;\mathbf{z}] &= \dot{\mathbf{z}}^{+\dagger}\cdot\left[-\partial_t\boldsymbol{\nabla}\cdot\frac{\partial L}{\partial\boldsymbol{\nabla}\dot{\mathbf{z}}^{+\dagger}} + \boldsymbol{\nabla}\cdot\frac{\partial L}{\partial\boldsymbol{\nabla}\mathbf{z}^{+\dagger}} + \partial_t\frac{\partial L}{\partial\dot{\mathbf{z}}^{+\dagger}}\right] \\ &+ (\boldsymbol{\nabla}\dot{\mathbf{z}}^+)^\dagger : \frac{\partial L}{\partial\boldsymbol{\nabla}\dot{\mathbf{z}}^{+\dagger}} + (\boldsymbol{\nabla}\dot{\mathbf{z}}^+)^\dagger : \frac{\partial L}{\partial\boldsymbol{\nabla}\mathbf{z}^{+\dagger}} + \ddot{\mathbf{z}}^{+\dagger}\cdot\frac{\partial L}{\partial\dot{\mathbf{z}}^{+\dagger}} \\ &+ \left[-\partial_t\boldsymbol{\nabla}\cdot\frac{\partial L}{\partial\boldsymbol{\nabla}\dot{\mathbf{z}}} + \boldsymbol{\nabla}\cdot\frac{\partial L}{\partial\boldsymbol{\nabla}\mathbf{z}} + \partial_t\frac{\partial L}{\partial\dot{\mathbf{z}}}\right]\cdot\dot{\mathbf{z}} \\ &+ \frac{\partial L}{\partial\boldsymbol{\nabla}\dot{\mathbf{z}}} : \boldsymbol{\nabla}\ddot{\mathbf{z}} + \frac{\partial L}{\partial\boldsymbol{\nabla}\mathbf{z}} : \boldsymbol{\nabla}\dot{\mathbf{z}} + \frac{\partial L}{\partial\dot{\mathbf{z}}}\cdot\ddot{\mathbf{z}} - \partial_t L[\mathbf{z}^+;\mathbf{z}] \\ &= \dot{\mathbf{z}}^{+\dagger}\cdot(\mathsf{L}\cdot\mathbf{z}) + \left(\tilde{\mathsf{L}}\cdot\mathbf{z}^+\right)^\dagger\cdot\dot{\mathbf{z}}. \end{aligned} \tag{8}$$

[Where : denotes the tensorial product between spatial tensor, e.g. $\boldsymbol{z}_1\boldsymbol{z}_2:\boldsymbol{z}_3\boldsymbol{z}_4$ is equal to $(\boldsymbol{z}_1\cdot\boldsymbol{z}_4)(\boldsymbol{z}_2\cdot\boldsymbol{z}_3)$.] Equation (8) vanishes provided $\mathsf{L}\cdot\mathbf{z} = 0$ and $\tilde{\mathsf{L}}\cdot\mathbf{z}^+ = 0$. In this formalism,

$$\frac{\partial L}{\partial\dot{\boldsymbol{z}}_i} - \boldsymbol{\nabla}\cdot\frac{\partial L}{\partial\boldsymbol{\nabla}\dot{\boldsymbol{z}}_i} \equiv \boldsymbol{\pi}_i^{+*}$$

plays the role of momentum conjugate to $\boldsymbol{z}_i$, allowing us to rewrite the Hamiltonian (7) in the more familiar form

$$H[\mathbf{z}^+;\mathbf{z}] = \sum_{i=1}^{N}\left(\boldsymbol{z}_i^{+*}\cdot\boldsymbol{\pi}_i + \boldsymbol{\pi}_i^{+*}\cdot\boldsymbol{z}_i\right) - L,$$

that is two times the kinetic energy minus the Lagrangian. Integrating (7) over an arbitrary volume Ω' which is a subset of the universe Ω, the *global Hamiltonian*

$$\mathcal{E}[\mathbf{z}^+;\mathbf{z}] \equiv \int_{\Omega'} d\tau H[\mathbf{z}^+;\mathbf{z}], \tag{9}$$

is found to be invariant except for a boundary term

$$\frac{d}{dt}\mathcal{E}[\mathbf{z}^+;\mathbf{z}] = -\int_{\partial\Omega'} d\boldsymbol{\sigma}\cdot\mathbf{S}[\mathbf{z}^+;\mathbf{z}]. \tag{10}$$

We must be careful not to interpret (9) as the energy of the system and (10) as the corresponding radiation losses, for it is clear that the conservation property (5) is the consequence of time invariance. We shall come back to the question of constructing the "proper energy" in § II D. Equation (5) may, however, yield important information about the time behaviour of the solution knowing the adjoint solution and *vice versa*. We see that if one component, say $\mathbf{z}^+$, is exponentially growing in time and the other component, $\mathbf{z}$, is decaying at the same rate then this will give rise to a time invariant Hamiltonian. On the other hand, the situation where $\mathbf{z}^+$ and $\mathbf{z}$ possess identical growth rates cannot be excluded on the grounds that the different contributions forming the Hamiltonian may cancel. This will be the case for non-dissipative systems in particular where kinetic and potential energies are balanced.

The conservation of the global momentum follows the invariance property of $L[\mathbf{z}^+;\mathbf{z}]$ with respect to translation of the space coordinates. Provided $L[\mathbf{z}^+;\mathbf{z}]$ depends on the space coordinates *only* through $\mathbf{z}^+$ and $\mathbf{z}$, we can show by following a similar path to (5) and (8) that the momentum $\int_\Omega d\tau\mathbf{p}[\mathbf{z}^+;\mathbf{z}]$ with

$$\mathbf{p}[\mathbf{z}^+;\mathbf{z}] \equiv (\boldsymbol{\nabla}\mathbf{z}^+)^\dagger\cdot\left(-\boldsymbol{\nabla}\cdot\frac{\partial L}{\partial\boldsymbol{\nabla}\dot{\mathbf{z}}^{+\dagger}} + \frac{\partial L}{\partial\dot{\mathbf{z}}^{+\dagger}}\right) + \left(-\boldsymbol{\nabla}\cdot\frac{\partial L}{\partial\boldsymbol{\nabla}\dot{\mathbf{z}}} + \frac{\partial L}{\partial\dot{\mathbf{z}}}\right)\cdot\boldsymbol{\nabla}\mathbf{z} \tag{11}$$

and

$$\mathsf{T}[\mathsf{z}^+;\mathsf{z}] = (\boldsymbol{\nabla}\dot{\mathsf{z}}^+)^\dagger\cdot\frac{\partial L}{\partial\boldsymbol{\nabla}\dot{\mathsf{z}}^{+\dagger}} + (\boldsymbol{\nabla}\mathsf{z}^+)^\dagger\cdot\frac{\partial L}{\partial\boldsymbol{\nabla}\mathsf{z}^{+\dagger}} + \frac{\partial L}{\partial\boldsymbol{\nabla}\dot{\mathsf{z}}}\cdot\boldsymbol{\nabla}\dot{\mathsf{z}} + \frac{\partial L}{\partial\boldsymbol{\nabla}\mathsf{z}}\cdot\boldsymbol{\nabla}\mathsf{z} - L\,\mathsf{I} \quad (12)$$

are conserved,

$$\boldsymbol{\nabla}\cdot\mathsf{T}[\mathsf{z}^+;\mathsf{z}] + \partial_t\mathbf{p}[\mathsf{z}^+;\mathsf{z}] = (\mathsf{L}\cdot\mathsf{z})^\dagger\cdot\boldsymbol{\nabla}\mathsf{z}^+ + \left(\tilde{\mathsf{L}}\cdot\mathsf{z}^+\right)^T\cdot\boldsymbol{\nabla}\mathsf{z},$$

where T is the stress tensor and I the identity tensor. Equations (6), (7), (11) and (12) form the 16 components of the stress-energy tensor [17].

C. The concomitant invariant

We introduce here quantities whose physical interpretation may appear at first sight obscure but which will be of interest in the subsequent stability analysis as performed in § III C. Let

$$U_+[\mathsf{z}^+;\mathsf{z}] \equiv \mathsf{z}^{+\dagger}\cdot\left(\frac{\partial L}{\partial\dot{\mathsf{z}}^{+\dagger}} - \boldsymbol{\nabla}\cdot\frac{\partial L}{\partial\boldsymbol{\nabla}\dot{\mathsf{z}}^{+\dagger}}\right) \quad (13)$$

and

$$U_-[\mathsf{z}^+;\mathsf{z}] \equiv -\left(\frac{\partial L}{\partial\dot{\mathsf{z}}} - \boldsymbol{\nabla}\cdot\frac{\partial L}{\partial\boldsymbol{\nabla}\dot{\mathsf{z}}}\right)\cdot\mathsf{z} \quad (14)$$

be two bilinear densities, with

$$\mathbf{P}_+[\mathsf{z}^+;\mathsf{z}] \equiv \dot{\mathsf{z}}^{+\dagger}\cdot\frac{\partial L}{\partial\boldsymbol{\nabla}\dot{\mathsf{z}}^{+\dagger}} + \mathsf{z}^{+\dagger}\cdot\frac{\partial L}{\partial\boldsymbol{\nabla}\mathsf{z}^{+\dagger}} \quad (15)$$

and

$$\mathbf{P}_-[\mathsf{z}^+;\mathsf{z}] \equiv -\frac{\partial L}{\partial\boldsymbol{\nabla}\dot{\mathsf{z}}}\cdot\dot{\mathsf{z}} - \frac{\partial L}{\partial\boldsymbol{\nabla}\mathsf{z}}\cdot\mathsf{z} \quad (16)$$

being the fluxes corresponding to (13) and (14). Equations (13) and (15) result from integrating by parts $(\mathsf{z}^+)^\dagger\cdot(\mathsf{L}\cdot\mathsf{z})$ of (3) in space and time, and (14) and (16) from applying a similar procedure onto $-(\tilde{\mathsf{L}}\cdot\mathsf{z}^+)^\dagger\cdot\mathsf{z}$, which yields

$$\begin{aligned}\mathsf{z}^{+\dagger}\cdot(\mathsf{L}\cdot\mathsf{z}) = {}& \boldsymbol{\nabla}\cdot\mathbf{P}_+[\mathsf{z}^+;\mathsf{z}] + \partial_t U_+[\mathsf{z}^+;\mathsf{z}] - \mathsf{z}^{+\dagger}\cdot\frac{\partial L}{\partial\mathsf{z}^{+\dagger}} \\ & - (\boldsymbol{\nabla}\mathsf{z}^+)^\dagger\!:\!\frac{\partial L}{\partial\boldsymbol{\nabla}\mathsf{z}^{+\dagger}} - \dot{\mathsf{z}}^{+\dagger}\cdot\frac{\partial L}{\partial\dot{\mathsf{z}}^{+\dagger}} - (\boldsymbol{\nabla}\dot{\mathsf{z}}^+)^\dagger\!:\!\frac{\partial L}{\partial\boldsymbol{\nabla}\dot{\mathsf{z}}^{+\dagger}} \end{aligned} \quad (17)$$

$$\begin{aligned}-(\tilde{\mathsf{L}}\cdot\mathsf{z}^+)^\dagger\cdot\mathsf{z} = {}& \boldsymbol{\nabla}\cdot\mathbf{P}_-[\mathsf{z}^+;\mathsf{z}] + \partial_t U_-[\mathsf{z}^+;\mathsf{z}] + \frac{\partial L}{\partial\mathsf{z}}\cdot\mathsf{z} \\ & + \frac{\partial L}{\partial\boldsymbol{\nabla}\mathsf{z}}\!:\!\boldsymbol{\nabla}\mathsf{z} + \frac{\partial L}{\partial\dot{\mathsf{z}}}\cdot\dot{\mathsf{z}} + \frac{\partial L}{\partial\boldsymbol{\nabla}\dot{\mathsf{z}}}\!:\!\boldsymbol{\nabla}\dot{\mathsf{z}}. \end{aligned} \quad (18)$$

Since the right-hand side of (17) and (18) is equal to $\boldsymbol{\nabla}\cdot\mathbf{P}_+ + \partial_t U_+ - L[\mathsf{z}^+;\mathsf{z}]$ and $\boldsymbol{\nabla}\cdot\mathbf{P}_- + \partial_t U_+ + L[\mathsf{z}^+;\mathsf{z}]$ respectively, adding (17) and (18) leads to a third conservation law

$$\boldsymbol{\nabla}\cdot\mathbf{P}[\mathsf{z}^+;\mathsf{z}] + \partial_t U[\mathsf{z}^+;\mathsf{z}] = \mathsf{z}^{+\dagger}\cdot(\mathsf{L}\cdot\mathsf{z}) - (\tilde{\mathsf{L}}\cdot\mathsf{z}^+)^\dagger\cdot\mathsf{z}, \quad (19)$$

with

$$U[\mathsf{z}^+;\mathsf{z}] \equiv U_+[\mathsf{z}^+;\mathsf{z}] + U_-[\mathsf{z}^+;\mathsf{z}] \tag{20}$$

and

$$\mathbf{P}[\mathsf{z}^+;\mathsf{z}] \equiv \mathbf{P}_+[\mathsf{z}^+;\mathsf{z}] + \mathbf{P}_-[\mathsf{z}^+;\mathsf{z}]. \tag{21}$$

Equation (19) is a direct consequence of the bilinearity of L, i.e. the linearity of Eqs.(3) and (4). To grasp the significance of (19), suppose we take $\mathsf{z}^+ = \mathsf{z}$ with $\mathsf{L}\cdot\mathsf{z} = 0$, then (19) is a measure of the non-self-adjointness of L. Comparing Eqs.(7) with (13) and (14) and assuming a single Fourier mode dependence of $\mathsf{z} \propto \exp\gamma t$ and $\mathsf{z}^+ \propto \exp\gamma_+ t$, we find that the Hamiltonian density

$$H[\mathsf{z}^+;\mathsf{z}] = \gamma_+^* \mathcal{U}_+[\mathsf{z}^+;\mathsf{z}] - \gamma \mathcal{U}_-[\mathsf{z}^+;\mathsf{z}] \tag{22}$$

is proportional to $\mathcal{U}$ provided $\gamma = -\gamma_+^*$, which defines in § II E the subclass of dissipative systems which are time-reversible. Equation (22) also points to $\mathcal{U}$ having the dimensions of an "action density", which appears plays an important role in the Vlasov-Maxwell theory of wave propagation Kull *et al.* [18].

On the other hand, choose an L that is self-adjoint and let z^+ and z be two solutions of $\mathsf{L}\cdot\mathsf{y} = \tilde{\mathsf{L}}\cdot\mathsf{y} = 0$, then $\int_\Omega d\tau U[\mathsf{z}^+;\mathsf{z}]$ can be seen from (19) to be a constant of motion. Because of the minus sign occurring in (19), $\int_\Omega d\tau U[\mathsf{z}^+;\mathsf{z}]$ is also a constant of motion if z^+ and z are linearly dependent. Equations (20) and (21) generalize what is called the *bilinear concomitant* [17].

The importance of the concomitant concept arises also for self-adjoint operators acting on the space coordinates. We see in particular that, when integrating (19) over the finite volume $\Omega' \subset \Omega$ while assuming that $U = 0$ (i.e. the Lagrangian density does not contain any $\dot{\mathsf{z}}$ nor $\dot{\mathsf{z}}^+$ dependence), $\mathbf{P}$ may contribute at the endpoints $\partial\Omega'$ even though $\mathsf{L} = \tilde{\mathsf{L}} = \mathsf{L}^\dagger$, violating the relation $\int_{\Omega'} d\tau \mathsf{z}_i^\dagger\cdot(\mathsf{L}\cdot\mathsf{z}_j) = \int_{\Omega'} d\tau(\mathsf{L}\cdot\mathsf{z}_i)^\dagger\cdot\mathsf{z}_j$ which defines Hermiticity. For homogeneous Dirichlet boundary conditions $\mathbf{P}$ vanishes at $\partial\Omega'$. These are the zero-flux boundary conditions

$$\left[\dot{\mathsf{z}}^{+\dagger}\cdot\frac{\partial L}{\partial\boldsymbol{\nabla}\dot{\mathsf{z}}^+} + \mathsf{z}^{+\dagger}\cdot\frac{\partial L}{\partial\boldsymbol{\nabla}\mathsf{z}^+} - \frac{\partial L}{\partial\boldsymbol{\nabla}\dot{\mathsf{z}}}\cdot\dot{\mathsf{z}} - \frac{\partial L}{\partial\boldsymbol{\nabla}\mathsf{z}}\cdot\mathsf{z}\right]_{\partial\Omega'} = 0. \tag{23}$$

We will refer to (23) as the zero-flux boundary conditions in subsequent sections with the terminology motivated by the similarity exhibited by Eqs.(15) and (16) as compared to (6). In the following sections we will concentrate on systems satisfying the zero-flux boundary condition with exception of § III C.

D. Energy and other observables

The distinction between Hamiltonian and energy is intrinsic to non-conservative systems, where the Hamiltonian is conserved [c.f. (8)] but the energy is in general not. It is also clear that the Hamiltonian defined by (9) cannot correspond to any physical reality, being merely a mathematical trick which allowed us to treat dissipative systems on the same footing as conservative systems; the reason being that the state of a system must be defined from z, and z alone, or alternatively from z^+ only *but not from both* since they are independent functions of t. Therefore, it is natural in a linear theory to postulate that only functionals which depend *quadratically* in z can give rise to measurable quantities which we refer to as *observables.*

Furthermore, we shall require these observables to be *real*. This is achieved in quantum mechanics by associating observables with Hermitian operators. However, we find it preferable to use the following, more general prescription; let

$$\mathcal{A}[\mathbf{z}_i;\mathbf{z}_j] \equiv (\mathbf{z}_i, \mathsf{A}\cdot\mathbf{z}_j)$$

be any bilinear functional, where

$$(\mathbf{z}_i,\mathbf{z}_j) \equiv \int_{\Omega'} d\tau \mathbf{z}_i^\dagger\cdot\mathbf{z}_j$$

defines the inner product in the volume $\Omega' \subseteq \Omega$ of two states $\mathbf{z}_i$ and $\mathbf{z}_j$ in the Hilbert space $\mathcal{H}$ with positive norm $\| \mathbf{z} \|^2 \equiv (\mathbf{z},\mathbf{z}) > 0$, $\mathbf{z} \in \mathcal{H}$. A functional $\mathcal{A}$ is *symmetric* if

$$\mathcal{A}[\mathbf{z}_i;\mathbf{z}_j] \equiv \mathcal{A}[\mathbf{z}_j;\mathbf{z}_i]^* \quad \forall\ \mathbf{z}_i,\mathbf{z}_j \in \mathcal{H}. \tag{24}$$

Such a functional satisfies the requirements of an observable which we denote using the subscript r: $\mathcal{A}_r[\mathbf{z}_i;\mathbf{z}_j] \equiv \mathcal{A}[\mathbf{z}_i;\mathbf{z}_j]$. However, if (24) is not fulfilled we construct the symmetric form

$$\begin{aligned}\mathcal{A}_r[\mathbf{z}_i;\mathbf{z}_j] &\equiv \tfrac{1}{2}(\mathbf{z}_i,\mathsf{A}\cdot\mathbf{z}_j) + \tfrac{1}{2}(\mathsf{A}\cdot\mathbf{z}_i,\mathbf{z}_j)\\ &\equiv \tfrac{1}{2}(\mathbf{z}_i,\mathsf{A}\cdot\mathbf{z}_j) + \text{h.c.},\end{aligned} \tag{25}$$

where h.c. denotes the Hermitian conjugate of the matrix $\mathcal{A}[\mathbf{z}_i;\mathbf{z}_j]$, in order that $\mathcal{A}_r[\mathbf{z},\mathbf{z}]$ be real. The lack of symmetry may be due to $\mathsf{A} \neq \mathsf{A}^\dagger$ [here † denotes the Hermitian adjoint in the domain Ω: $\int_\Omega d\tau \mathbf{z}_i^\dagger\cdot(\mathsf{A}\cdot\mathbf{z}_j) \equiv \int_\Omega d\tau (\mathsf{A}^\dagger\cdot\mathbf{z}_i)^\dagger\cdot\mathbf{z}_j$ with $\mathbf{z}_i$ and $\mathbf{z}_j$ satisfying homogeneous boundary conditions at $\partial\Omega$] or may result from inhomogeneous boundary conditions of $\mathbf{z}_i$ and $\mathbf{z}_j$ at $\partial\Omega'$ indicating the interaction of the system with a "reservoir" for instance.

E. Causality condition

According to (25), the energy must take the form

$$\mathcal{E}_r[\mathbf{z}_i;\mathbf{z}_j] \equiv \tfrac{1}{2}\int_{\Omega'} d\tau H[\mathbf{z}_i;\mathbf{z}_j] + \text{h.c.} \tag{26}$$

and the energy flux

$$\mathbf{S}_r[\mathbf{z}_i;\mathbf{z}_j] \equiv \tfrac{1}{2}\mathbf{S}[\mathbf{z}_i;\mathbf{z}_j] + \text{h.c.}$$

so that, from (8),

$$\begin{aligned}\frac{d}{dt}\mathcal{E}_r[\mathbf{z}_i;\mathbf{z}_j] &= \tfrac{1}{2}\int_{\Omega'} d\tau \{\boldsymbol{\nabla}\cdot\mathbf{S}_r[\mathbf{z}_i;\mathbf{z}_j] + \partial_t H_r[\mathbf{z}_i;\mathbf{z}_j]\}\\ &= \tfrac{1}{2}\left(\dot{\mathbf{z}}_i, \left[\mathsf{L} + \tilde{\mathsf{L}}\right]\cdot\mathbf{z}_j\right) + \text{h.c.}\end{aligned} \tag{27}$$

We see from (27) that dissipation is a consequence of the non-self-adjointness of L ($\mathsf{L}\cdot\mathbf{z}_j = 0$). Expressing L in the generic form of the damped oscillator

$$\mathsf{L} \equiv \mathsf{M}\partial_t^2 + \mathsf{D}\partial_t + \mathsf{V}, \tag{28}$$

with the adjoint operator to L being

$$\tilde{\mathsf{L}} \equiv \mathsf{M}^\dagger \partial_t^2 - \mathsf{D}^\dagger \partial_t + \mathsf{V}^\dagger, \tag{29}$$

and writing

$$\begin{aligned} \mathsf{M}_H &\equiv \tfrac{1}{2}(\mathsf{M} + \mathsf{M}^\dagger), \ \mathsf{D}_H \equiv \tfrac{1}{2}(\mathsf{D} + \mathsf{D}^\dagger), \ \mathsf{V}_H \equiv \tfrac{1}{2}(\mathsf{V} + \mathsf{V}^\dagger), \\ \mathsf{M}_A &\equiv \tfrac{1}{2}(\mathsf{M} - \mathsf{M}^\dagger), \ \mathsf{D}_A \equiv \tfrac{1}{2}(\mathsf{D} - \mathsf{D}^\dagger), \ \mathsf{V}_A \equiv \tfrac{1}{2}(\mathsf{V} - \mathsf{V}^\dagger), \end{aligned}$$

in terms of Hermitian and anti-Hermitian parts respectively, which involve differential operators acting on space variables only, we find from (27)

$$\frac{d}{dt}\mathcal{E}_r[\mathbf{z};\mathbf{z}] = (\dot{\mathbf{z}}, \mathsf{M}_H\cdot\ddot{\mathbf{z}} + \mathsf{V}_H\cdot\mathbf{z}) + \text{h.c.},$$

which can be integrated over t to give (up to a constant of integration)

$$\mathcal{E}_r[\mathbf{z};\mathbf{z}] = (\dot{\mathbf{z}}, \mathsf{M}_H\cdot\dot{\mathbf{z}}) + (\mathbf{z}, \mathsf{V}_H\cdot\mathbf{z}), \tag{30}$$

provided the concomitant of (21) vanishes at $\partial\Omega'$, that is (28) obeys boundary conditions (23). Equation (30) can be shown to be consistent with definition (26).

It is our intention to discuss now the various degrees of freedom one is confronted with as one goes through the procedure of constructing the Lagrangian density L. It is well-known that for conservative systems $L = T - V$ where T is the kinetic energy and V the potential energy. As T is a positive definite quantity, the sign of the action (2), and therefore the sign L is determined by $T > 0$, this regardless of the fact that the Euler-Lagrange equations (3) and (4) are invariant under $L \mapsto -L$. We will see that for a number of dissipative systems the definiteness of D determines which of $\mathbf{z}^+$ and $\mathbf{z}$ should be chosen as representing the physical state of the system in the sense of observable as defined in § II D.

For dissipative systems, we shall arrange the Euler-Lagrange equations, whenever possible, in such a way that the kinetic contribution be positive definite. That is the term quadratic in $\dot{\mathbf{z}}$ in (30),

$$(\dot{\mathbf{z}}, \mathsf{M}_H\cdot\dot{\mathbf{z}}) \geq 0$$

$\forall\ \dot{\mathbf{z}} \in \mathcal{H}$. We then postulate that the physically relevant states are such that the the sign of dissipation be positive, which corresponds to the decrease of $\mathcal{E}_r$ in time,

$$\frac{d}{dt}\mathcal{E}_r[\mathbf{z};\mathbf{z}] = -2\,(\dot{\mathbf{z}}, \mathsf{D}_H\cdot\dot{\mathbf{z}}) - [(\dot{\mathbf{z}}, \mathsf{M}_A\cdot\ddot{\mathbf{z}}) + (\dot{\mathbf{z}}, \mathsf{V}_A\cdot\mathbf{z}) + \text{c.c.}] \leq 0 \tag{31}$$

and which is obtained from (27) by writing $\tilde{\mathsf{L}} = \mathsf{L} - 2\mathsf{M}_A\partial_t^2 - 2\mathsf{D}_H\partial_t - 2\mathsf{V}_H$ for $\mathbf{z}$ such that $\mathsf{L}\cdot\mathbf{z} = 0$. The first term on the right-hand side of (31) being quadratic, we write the *causality condition* as

$$(\mathbf{z}, \mathsf{D}_H\cdot\mathbf{z}) \geq 0 \qquad \forall\ \mathbf{z} \in \mathcal{H} \tag{32}$$

which expresses the fact that the energy is bound to decrease provided $(\dot{\mathbf{z}}, \mathsf{M}_A\cdot\ddot{\mathbf{z}}) + (\dot{\mathbf{z}}, \mathsf{V}_A\cdot\mathbf{z}) + \text{c.c.}$ vanishes. If either $\mathsf{M}_A \neq 0$ or $\mathsf{V}_A \neq 0$, then the causality condition (31) only holds in general for real growth rates. Note that the condition M_A and V_A vanishing is in particular satisfied for time reversible systems where $\mathbf{z}^+$ is the "advanced" solution and $\mathbf{z}$ the "retarded" solution with $\mathbf{z}^+(t) = \mathbf{z}(-t)$.

F. Stability criteria

The issue of stability for dissipative systems will be addressed in this section using the causality condition. For conservative systems, it is well known since the work of

BFKK that stability is given by the positive definiteness of the potential energy. It is, however, far from obvious how this criterion generalizes to the dissipative case. One reason is that the concept of potential energy becomes meaningless in the dissipative case. Thus, one is led to use a more general definition of stability, which is consistent with the linear perturbation approximation: *a system is unstable when there exists the possibility for the system to lower its energy exponentially.* That is, an equilibrium is stable when it is a minimum energy state. Suppose the zero-energy state is the equilibrium state, any infinitesimal perturbation that elevates the energy gives rise to a positive energy and, conversely, any perturbation that lowers the energy produces a negative energy. As the system evolves in time, the energy is bound to decrease for both positive and negative energies if the causality principle (32) holds. This situation is exposed in Fig. 1. Thus, negative energies depart exponentially from the zero-energy state, $\mathcal{E}_r = 0$ as time passes by, whereas positive energies have the choice between relaxing to $\mathcal{E}_r = 0$, or else crossing the $\mathcal{E}_r = 0$ axis and joining the negative energy states. Since we have not specified the origin of time $t = 0$, the latter case corresponds in essence to a negative initial energy which is translated in time (such that the energy is positive for $t < 0$). Thus, requiring

$$\mathcal{E}_r[\mathbf{z};\mathbf{z}] \geq 0 \quad \forall \ \mathbf{z} \in \mathcal{H} \tag{33}$$

for causal systems [satisfying (32)] provides a *necessary* and *sufficient* criterion to ensure stability; if (33) is satisfied, $\mathcal{E}_r$ is necessarily positive at all times and this is sufficient to prohibit the system from diverging exponentially from $\mathcal{E}_r = 0$ as $t \to \infty$.

Criterion (33) is obviously consistent with having the dominant growth rate $\mathrm{Re}\hat{\gamma} < 0$ since

$$\frac{d}{dt}\mathcal{E}_r = 2\mathrm{Re}\{\hat{\gamma}\}\mathcal{E}_r$$

as $t \to \infty$.

Since (33) is valid for all initial perturbations, including those with $\dot{\mathbf{z}}$ initially zero, we can rewrite (33) as

$$W[\mathbf{z};\mathbf{z}] \equiv (\mathbf{z}, \mathsf{V}_H \cdot \mathbf{z}) \geq 0 \quad \forall \ \mathbf{z} \in \mathcal{H}. \tag{34}$$

without loss of generality *if* $(\mathbf{z}, \mathsf{M}_H \cdot \mathbf{z})$ is positive definite. Criterion (34) is equivalent to (33) in that $\mathcal{E}_r[\mathbf{z};\mathbf{z}]$ is positive definite if $W[\mathbf{z};\mathbf{z}]$ is positive definite, and there always exists an $\mathcal{E}_r < 0$ if $W < 0$.

Equations (33) and (34) are the main results of this section. The procedure to test stability is the following: first we must ensure that the equations can be put in causal form, meaning that (32) is satisfied for all states $\mathbf{z} \in \mathcal{H}$. We speculate that most physical systems can be put in causal form since intuitively all systems are dissipative (the complex conjugate in the definition of the inner product ensuring that no reactive power can alter the causal behaviour). This is the case of all three examples considered in §§III A – III C. When either of M_A or V_A is non-vanishing, as it is the case for the unreduced set of resistive equations (see § III A), then the energy principle (33) applies to perturbations having real growth rates only.

III. RESISTIVE MHD

A. Resistivity in the plasma bulk

The resistive model with $\eta = \text{const}$ is considered here in arbitrary geometry [c.f. Tasso [8] to generalize this case to $\eta \neq \text{const}$]. The Lagrangian density takes the

form

$$L = \tfrac{1}{2}\left[\rho\dot{\boldsymbol{\xi}}^{+}\cdot\dot{\boldsymbol{\xi}} - 2w(\boldsymbol{\xi}^{+},\boldsymbol{\xi}) - \boldsymbol{\nabla}\times\mathbf{a}^{+}\cdot\boldsymbol{\nabla}\times\mathbf{a} + \frac{1}{2\eta}\left(\dot{\mathbf{a}}^{+}\cdot\mathbf{a} - \mathbf{a}^{+}\cdot\dot{\mathbf{a}}\right)\right.$$
$$\left. - \mathbf{Q}^{+}\cdot\boldsymbol{\nabla}\times\mathbf{a} - \boldsymbol{\nabla}\times\mathbf{a}^{+}\cdot\mathbf{Q} + \boldsymbol{\xi}^{+}\cdot\mathbf{J}\times(\boldsymbol{\nabla}\times\mathbf{a})\right] \tag{35}$$

for the motion of the virtual displacement field $\boldsymbol{\xi}$ and the potential vector $\mathbf{a}$. This choice of variable is appropriate for taking the limit of $\eta \to 0$, for $\mathbf{a}$ represents the correction to the ideal vector potential $\boldsymbol{\xi}\times\mathbf{B}$; the total magnetic field perturbation being

$$\mathbf{b} \equiv \mathbf{Q} + \boldsymbol{\nabla}\times\mathbf{a},$$

where

$$\mathbf{Q} \equiv \boldsymbol{\nabla}\times(\boldsymbol{\xi}\times\mathbf{B}).$$

In (35), $\mathbf{J} \equiv \boldsymbol{\nabla}\times\mathbf{B}$ is the equilibrium current density and

$$2w(\boldsymbol{\xi}^{+},\boldsymbol{\xi}) \equiv (\mathbf{Q}^{+} + \xi_n^{+}\mathbf{J}\times\mathbf{n})\cdot(\mathbf{Q} + \xi_n\mathbf{J}\times\mathbf{n}) + \boldsymbol{\nabla}\cdot\boldsymbol{\xi}^{+}\Gamma p\boldsymbol{\nabla}\cdot\boldsymbol{\xi} - 2U\xi^{+}\xi \tag{36}$$

represents the (symmetric) ideal potential energy when integrated over the volume of the plasma. Here, ψ is the poloidal magnetic flux coordinate, $\xi_n \equiv \boldsymbol{\xi}\cdot\mathbf{n}$ and $\xi \equiv \boldsymbol{\xi}\cdot\boldsymbol{\nabla}\psi = \xi_n|\boldsymbol{\nabla}\psi|$, p is equilibrium pressure and $\Gamma = 5/3$ the ratio of specific heats. For a complete description of the terms involved in the Hermitian operator U in (36) we refer to Dewar & Pletzer [19].

Following § II B, we form the vector

$$\mathsf{z} \equiv \begin{pmatrix} \boldsymbol{\xi} \\ \mathbf{a} \end{pmatrix}$$

allowing us to write the linear, resistive equations in the form of (28), with

$$\mathsf{M} = \tfrac{1}{2}\begin{pmatrix} \rho & 0 \\ 0 & 0 \end{pmatrix}, \quad \mathsf{D} = \tfrac{1}{2}\begin{pmatrix} 0 & 0 \\ 0 & 1/\eta \end{pmatrix}, \tag{37}$$

$$\mathsf{V}_H = \tfrac{1}{2}\begin{pmatrix} -\mathsf{F} & +\mathbf{B}\times[\boldsymbol{\nabla}\times(\boldsymbol{\nabla}\times\mathsf{I})] - \tfrac{1}{2}\mathbf{J}\times(\boldsymbol{\nabla}\times\mathsf{I}) \\ -\boldsymbol{\nabla}\times[\boldsymbol{\nabla}\times(\mathbf{B}\times\mathsf{I}) + \tfrac{1}{2}\boldsymbol{\nabla}\times(\mathbf{J}\times\mathsf{I}) & \boldsymbol{\nabla}\times(\boldsymbol{\nabla}\times\mathsf{I}) \end{pmatrix} \tag{38}$$

and

$$\mathsf{V}_A = \tfrac{1}{2}\begin{pmatrix} 0 & -\tfrac{1}{2}\mathbf{J}\times(\boldsymbol{\nabla}\times\mathsf{I}) \\ -\tfrac{1}{2}\boldsymbol{\nabla}\times(\mathbf{J}\times\mathsf{I}) & 0 \end{pmatrix},$$

where

$$\mathsf{F}\cdot\boldsymbol{\xi} = -\mathbf{B}\times[\boldsymbol{\nabla}\times(\mathbf{Q} + \xi_n\mathbf{J}\times\mathbf{n})] + \boldsymbol{\nabla}[\Gamma p\boldsymbol{\nabla}\cdot\boldsymbol{\xi}] - \mathbf{n}\mathbf{J}\times\mathbf{n}\cdot(\mathbf{Q} + \xi_n\mathbf{J}\times\mathbf{n}) + 2U\boldsymbol{\xi}\boldsymbol{\nabla}\psi \tag{39}$$

is the force operator of BFKK [see for instance Bineau [4] for a derivation of the present expression]. The cross product of a vector and the unit dyadic, e.g. $\mathbf{J}\times\mathsf{I}$ in (38), is by definition the dyadic $\mathbf{J}\times\mathsf{I} \equiv \sum_i \mathbf{J}\times\mathbf{e}_i\mathbf{e}_i$ where $\{\mathbf{e}_i\}$ is any orthonormal basis.

The adjoint equation (29) can be derived by means of (19), or alternatively from (4). Using (6), we find

$$\mathbf{S}_r[\mathbf{z}_i;\mathbf{z}_j] = \tfrac{1}{4}\Big[(\mathbf{Q}+\xi_n\mathbf{J}\times\mathbf{n})_i^*\cdot\mathbf{B}\dot{\boldsymbol{\xi}}_j - \mathbf{B}(\mathbf{Q}+\xi_n\mathbf{J}\times\mathbf{n})_i^*\cdot\dot{\boldsymbol{\xi}}_j - \boldsymbol{\nabla}\cdot\boldsymbol{\xi}_i^*\Gamma p\dot{\boldsymbol{\xi}}_j$$
$$- \mathbf{B}\boldsymbol{\nabla}\times\mathbf{a}_i^*\cdot\dot{\boldsymbol{\xi}}_j + \boldsymbol{\nabla}\times\mathbf{a}_i^*\cdot\mathbf{B}\dot{\boldsymbol{\xi}}_j + (\boldsymbol{\nabla}\times\mathbf{a} + \mathbf{Q} - \tfrac{1}{2}\boldsymbol{\xi}\times\mathbf{J})_i^*\times\dot{\mathbf{a}}_j\Big]$$
$$+\text{h.c.} \tag{40}$$

and, using (30),

$$\mathcal{E}_r[\mathbf{z}_i;\mathbf{z}_j] = \tfrac{1}{2}\left(\rho\dot{\boldsymbol{\xi}}_i,\dot{\boldsymbol{\xi}}_j\right) + \tfrac{1}{2}\left(\boldsymbol{\nabla}\times\mathbf{a}_i,\boldsymbol{\nabla}\times\mathbf{a}_j\right)$$
$$+ \tfrac{1}{2}\int_{\Omega'} d\tau\left[2w(\boldsymbol{\xi}_i^*,\boldsymbol{\xi}_j) + \boldsymbol{\nabla}\times\mathbf{a}_i^*\cdot(\mathbf{Q}_j - \tfrac{1}{2}\boldsymbol{\xi}_j\times\mathbf{J}) + (\mathbf{Q}_i - \tfrac{1}{2}\boldsymbol{\xi}_i\times\mathbf{J})^*\cdot\boldsymbol{\nabla}\times\mathbf{a}_j\right], \tag{41}$$

which is in agreement with (7). The zero-flux boundary conditions (23) become

$$\begin{aligned}\left[(\mathbf{Q}+\xi_n\mathbf{J}\times\mathbf{n}+\boldsymbol{\nabla}\times\mathbf{a}_i)_i\cdot\mathbf{B}\xi_j\right]_{\partial\Omega'} &= 0\\ \left[\boldsymbol{\nabla}\psi\cdot(\boldsymbol{\nabla}\times\mathbf{a}+\mathbf{Q}^+ - \tfrac{1}{2}\boldsymbol{\xi}^+\times\mathbf{J})_i\times(\mathbf{a}_j)\right]_{\partial\Omega'} &= 0\end{aligned} \tag{42}$$

where $\partial\Omega'$ is a ψ = const surface, i.e. $\mathbf{B}\cdot\boldsymbol{\nabla}\psi = 0$. The boundary conditions (42) correspond to the presence of an infinitely conducting wall at $\partial\Omega$, but can be easily extended to incorporate an interface between plasma and vacuum.

Substituting (37) into (32), we find

$$\frac{d}{dt}\mathcal{E}_r = -\frac{1}{\eta}\parallel\dot{\mathbf{a}}\parallel^2 - (\mathsf{V}_A\cdot\mathbf{z},\dot{\mathbf{z}}) - (\mathbf{z},\mathsf{V}_A\cdot\dot{\mathbf{z}}), \tag{43}$$

which is causal (negative-definite) provided the analysis is restricted to *real* growth rates. From the second Lagrange equation (Ohm's law),

$$\mathbf{a} = \left(1-\frac{\eta}{\gamma}\nabla^2\right)^{-1}\frac{\eta}{\gamma}\nabla^2\boldsymbol{\xi}\times\mathbf{B} = \left(1+\frac{\eta}{\gamma}\nabla^2+\frac{\eta^2}{\gamma^2}\nabla^4\cdots\right)\frac{\eta}{\gamma}\nabla^2\boldsymbol{\xi}\times\mathbf{B}$$

it is found that $\mathbf{a}\sim\eta$ and thus, the rate of dissipation $\parallel\mathbf{a}\parallel^2/\eta$ is proportional to η to lowest order in η ($\eta\nabla^2/\gamma << 1$ except near the rational surfaces). It can be shown that the momentum conjugate to $\mathbf{a}$ is $\partial L/\partial\dot{\mathbf{a}}^+ = \mathbf{a}/2\eta$, that is the momentum is of order η^0 and proportional to $\mathbf{a}$ itself. This is characteristic of a diffusion equation, which cannot for this reason be put into Hamiltonian form. Note also that expression (41) for $\mathcal{E}_r$ is invariant under the gauge transformations $\mathbf{a}\mapsto\mathbf{a}+\boldsymbol{\nabla}\phi$ since $\mathcal{E}_r$ depends only upon the magnetic field perturbation $\boldsymbol{\nabla}\times\mathbf{a}$ [20].

As M is positive definite, we may apply the necessary and sufficient stability criterion (34),

$$(\mathbf{z},\mathsf{V}_H\cdot\mathbf{z}) = \int_{\Omega'} d\tau\left[w(\boldsymbol{\xi}^*,\boldsymbol{\xi}) + \tfrac{1}{2}|\boldsymbol{\nabla}\times\mathbf{a}+\mathbf{Q}-\tfrac{1}{2}\boldsymbol{\xi}\times\mathbf{J}|^2 - \tfrac{1}{2}|\mathbf{Q}-\tfrac{1}{2}\boldsymbol{\xi}\times\mathbf{J}|^2\right] \geq 0 \tag{44}$$

for purely growing/decaying modes γ, $\mathbf{z}\propto\exp\gamma t$. Equation (44) identifies $\mathbf{Q}-\frac{1}{2}\boldsymbol{\xi}\times\mathbf{J} = \mathbf{B}\cdot(\boldsymbol{\nabla}\boldsymbol{\xi}) - (\boldsymbol{\nabla}\cdot\boldsymbol{\xi})\mathbf{B} - \frac{1}{2}\boldsymbol{\xi}\cdot(\boldsymbol{\nabla}\mathbf{B}) - \frac{1}{2}(\boldsymbol{\nabla}\mathbf{B})\cdot\boldsymbol{\xi}$ as the driving term of the instability. In the vicinity of rational surfaces where $\mathbf{B}\cdot\boldsymbol{\nabla}$ vanishes, the main contributions come from field line bending and compressibility.

Criterion (44) is in essence equivalent to the criterion of Tasso [8] which derives from the dispersion relation

$$(\mathbf{z},\mathsf{M}\cdot\ddot{\mathbf{z}}) + (\mathbf{z},\mathsf{D}\cdot\dot{\mathbf{z}}) + (\mathbf{z},\mathsf{V}\cdot\mathbf{z}) = 0. \tag{45}$$

As a consequence of the positive definiteness of the kinetic term $\gamma^2(\mathbf{z},\mathsf{M}\cdot\mathbf{z})$ and since the dissipative term $\gamma(\mathbf{z},\mathsf{D}\cdot\mathbf{z})$ has the sign of the growth rate, it is readily

seen that (45) does not admit any positive root $\gamma > 0$ if $(\mathbf{z}, \mathsf{V}\cdot\mathbf{z})$ is positive definite. Clearly, Tasso's criterion is only sufficient since, following his argument, the case where all the roots γ are negative (stable) with $(\mathbf{z}, \mathsf{V}\cdot\mathbf{z})$ being indefinite, or even possibly negative definite cannot *a priori* be excluded. On the other hand, the existence of a negative definite Lyapunov functional $\frac{d}{dt}\mathcal{E}_r = 2\gamma\mathcal{E}_r$ given by (43) shows independently that a positive γ corresponds to a negative $\mathcal{E}_r$ and conversely, hence demonstrating that requiring $\mathcal{E}_r = (\dot{\mathbf{z}}, \mathsf{M}\cdot\dot{\mathbf{z}}) + (\mathbf{z}, \mathsf{V}_H\cdot\mathbf{z})$ to be positive definite is a necessary and sufficient criterion for stability. The positive definiteness of $(\dot{\mathbf{z}}, \mathsf{M}\cdot\dot{\mathbf{z}})$ reduces then this criterion to the zero-frequency form (44) without loss of generality.

The limit of zero-resistivity is straightforward, setting $\mathbf{a} = 0$ we find the familiar energy principle of BFKK

$$W[\boldsymbol{\xi};\boldsymbol{\xi}] = -\tfrac{1}{2}(\boldsymbol{\xi}, \mathsf{F}\cdot\boldsymbol{\xi}) = \int_\Omega d\tau w(\boldsymbol{\xi},\boldsymbol{\xi}) \geq 0, \tag{46}$$

$\forall\ \boldsymbol{\xi} \in \mathcal{H}$, which is the global stability criterion against ideal modes.

B. Interchange modes

Our prime interest, of course, is to investigate the existence of criteria for the case where η is small but non-zero. The criterion against fast resistive interchange modes appears suitable for this task, for these modes are known to be localized; they do not need to be matched to outer region solutions and thus we may apply the simpler formalism developed in § II D which is only valid for solutions satisfying Hermitian, or zero-flux boundary conditions. The case where energy fluxes play a dominant role, as for tearing modes, will be discussed in § III C. The corresponding Lagrangian density is

$$\begin{aligned} L = \tfrac{1}{4}\Big\{ & -\Psi^+\dot{\Psi} + \dot{\Psi}^+\Psi - \Psi^+ X\dot{\Xi} + \dot{\Psi}^+ X\Xi + \Psi^+ H\Upsilon' - \Psi^{+\prime} H\Upsilon \\ & - 2\Psi^{+\prime}\Psi' + 2\dot{\Xi}^{+\prime}\dot{\Xi}' - \Xi^+ X\dot{\Psi} + \dot{\Xi}^+ X\Psi - \Xi^+ X^2\dot{\Xi} + \dot{\Xi}^+ X^2\Xi \\ & - \Xi^+ H\Psi' + \Xi^{+\prime} H\Psi + 2\Xi^+(E+F)\Upsilon - \dot{\Upsilon}^+ KH\dot{\Psi}' + \dot{\Upsilon}^{+\prime} KH\dot{\Psi} \\ & + 2\dot{\Upsilon}^+(KE-G)\dot{\Xi} + 2\dot{\Upsilon}^+(G+KF)\dot{\Upsilon} - \Upsilon^{+\prime}\dot{\Upsilon}' + \dot{\Upsilon}^{+\prime}\Upsilon' - 2\Upsilon^+ X\Psi - 2\Upsilon^+ X^2\Upsilon\Big\}. \end{aligned} \tag{47}$$

Constructing the solution vector

$$\mathbf{z} \equiv \begin{pmatrix} \Psi \\ \Xi \\ \Upsilon \end{pmatrix},$$

(47) yields $\mathsf{L}\mathbf{z} = 0$ with L given by (28). The matrix operators of (28) are

$$\mathsf{M} = \tfrac{1}{2}\begin{pmatrix} 0 & 0 & 0 \\ 0 & -\partial_X^2 & 0 \\ -KH\partial_X & KE-G & G+KF \end{pmatrix}, \quad \mathsf{D} = \tfrac{1}{2}\begin{pmatrix} 1 & X & 0 \\ X & X^2 & 0 \\ 0 & 0 & -\partial_X^2 \end{pmatrix} \tag{48}$$

and

$$\mathsf{V} = \tfrac{1}{2}\begin{pmatrix} -\partial_X^2 & 0 & -H\partial_X \\ H\partial_X & 0 & -E-F \\ X & 0 & X^2 \end{pmatrix}. \tag{49}$$

We find from (48) that

$$2\,(\mathbf{z}, \mathsf{D}\cdot\mathbf{z}) = \int_{-\infty}^{\infty} dX\,\left(|\Psi - X\Xi|^2 + |\Upsilon'|^2\right) \geq 0 \tag{50}$$

is causal in the sense defined by (32). The fact, however, that V and M are non-Hermitian means that the stability considerations must be restricted, as in § III A, to real growth rates in spite of the existence of overstable modes [14]. Although M can be made Hermitian by introducing another dependent variable [21], the resulting equations would violate the causality condition. The stability analysis can, however, be pursued provided the additional ordering $\Gamma p/B^2 \sim 1/G << 1$ is taken, which gives $\Upsilon = \Xi$ to leading order, or

$$\mathbf{z} \equiv \begin{pmatrix} \Psi \\ \Xi \end{pmatrix}, \tag{51}$$

$$\mathsf{M} = \tfrac{1}{2}\begin{pmatrix} 0 & 0 \\ 0 & -\partial_X^2 \end{pmatrix}, \quad \mathsf{D} = \tfrac{1}{2}\begin{pmatrix} 1 & X \\ X & X^2 \end{pmatrix} \tag{52}$$

and

$$\mathsf{V} = \tfrac{1}{2}\begin{pmatrix} -\partial_X^2 & -H\partial_X \\ H\partial_X & -E - F \end{pmatrix}. \tag{53}$$

These equations exhibit the desired symmetry and possess asymptotic solutions with exponentially small behaviour as $X \to \pm\infty$ [22]. The energy observable is given from (30),

$$W[\mathbf{z};\mathbf{z}] = \tfrac{1}{2}\int_{-\infty}^{\infty} dX\,\left(|\Psi' + H\Xi|^2 - D_R|\Xi|^2\right) \tag{54}$$

where $D_R = E + F + H^2$ is the Mercier resistive index. Thus, stability relies upon having

$$D_R < 0, \tag{55}$$

the sufficient criterion towards interchange modes. This result has been derived for a set of equations similar to Eqs.(52)-(53) in Johnson *et al.* [22] by means of a quadratic functional that bears resemblance to the energy functional (54). The fact that a functional possessing the properties of an energy has been used to obtain this result is new.

C. Tearing modes

In the previous section, we excluded the possibility for $\mathbf{z}$ to be non-vanishing at $\partial\Omega'$. Inhomogeneous boundary conditions can be perceived as being the consequence of energy fluxes interacting between the resistive layer and the outer region. The Johnson *et al.* [22] equations also admit the power-like asymptotic solutions with leading exponents given by $\Psi \sim X^{\frac{1}{2}\pm\sqrt{\frac{1}{4}-E-F-H}}$ and $\Xi \sim X^{-\frac{1}{2}\pm\sqrt{\frac{1}{4}-E-F-H}}$.

Let us give for convenience the Lagrangian density,

$$\begin{aligned} L = \tfrac{1}{2}\Big\{ & \tfrac{1}{2}\left(\dot{\Psi}^+ + X\dot{\Xi}^+\right)(\Psi + X\Xi) - \tfrac{1}{2}\left(\Psi^+ + X\Xi^+\right)\left(\dot{\Psi} + X\dot{\Xi}\right) \\ & - \Psi^{+\prime} H\Xi - \Xi^+ H\Psi' + \dot{\Xi}^{+\prime}\dot{\Xi}' - \Psi^{+\prime}\Psi' + \Xi^+(E+F)\Xi\Big\}. \end{aligned} \tag{56}$$

As for § III B, we seek a quadratic form which involves the solutions Ψ and Ξ at $X \to \pm\infty$. To do so, we may use the conservation law for the concomitant (17),

$$\partial_X P_+[\mathbf{z};\mathbf{z}] + \partial_t U_+[\mathbf{z};\mathbf{z}] = L[\mathbf{z};\mathbf{z}] \tag{57}$$

or alternatively, one could also take (18),

$$\partial_X P_-[\mathbf{z};\mathbf{z}] + \partial_t U_-[\mathbf{z};\mathbf{z}] = \left(\dot{\Psi}^+ + X\dot{\Xi}^+\right)(\Psi + X\Xi) - L[\mathbf{z};\mathbf{z}],$$

assuming $\mathsf{L}\mathbf{z} = 0$ with L given by (52)–(53). Taking the former choice, we find

$$P_+[\mathbf{z};\mathbf{z}] = \tfrac{1}{2}\left(\dot{\Xi}^*\dot{\Xi}' - \Psi^*\Psi' - \Psi^* H\Xi\right) \tag{58}$$

and

$$U_+[\mathbf{z};\mathbf{z}] = \tfrac{1}{2}\left(\tfrac{1}{2}|\Psi + X\Xi|^2 - \Xi^*\dot{\Xi}''\right) \tag{59}$$

from (15) and (13). Inserting Eqs.(58) and (59) into (57) and integrating over the layer, we then get

$$\tfrac{1}{2}\left[\Psi^*\Psi' + \Psi^* H\Xi + \Xi^*\ddot{\Xi}'\right]_{-\infty}^{\infty} = W[\mathbf{z};\mathbf{z}] + \tfrac{1}{2}\int_{-\infty}^{\infty} dX\left\{\Xi'^*\ddot{\Xi}' + (\Psi + X\Xi)^*\left(\dot{\Psi} + X\dot{\Xi}\right)\right\}, \tag{60}$$

with $W[\mathbf{z};\mathbf{z}]$ being the zero-frequency energy (34). Considering real growth rates $Q^* = Q$, the right hand side of (60) is composed of a positive definite term by assumption of (55), and a term that possesses the sign of the growth rate. Equation (60) shows that, in the constant-Ψ approximation where

$$\left.\begin{array}{rcl} \Psi &=& \Psi_0 + \epsilon X\Psi_1 + \cdots \\ \Xi &=& \Xi_0/(\epsilon X) + \Xi_1 + \cdots \end{array}\right\} \tag{61}$$

($\epsilon << 1$ measures the thickness of the layer), the endpoint contribution on the left hand side of (60) is

$$[\Psi^*\Psi']_{-\infty}^{\infty} = \Psi_0^2\Delta(Q). \tag{62}$$

Equating the right-hand sides of (60) and (60), we readily find that the inner matching index $\Delta(Q)$ is positive for $Q > 0$ and zero in the limit of $Q \to 0$ at the marginal point of slow interchange stability $W[\mathbf{z};\mathbf{z}] = 0$. On the other hand, $\Delta < 0$ is a sufficient condition to ensure that $Q < 0$. Equation (60) with (61) and (60) can be regarded as the counterpart of Furth *et al*'s [5] expression for the outer matching index $\epsilon\Delta' = \Delta(Q)$, which can be obtained by multiplying the Euler-Lagrange equation in the ideal, infinite conductivity region by the solution and integrating by parts to form a symmetric, energy-like functional.

IV. CONCLUSIONS

The focus of the present paper is on the derivation of energy principles. Energy principles are desirable not only because the equations need not be solved directly, but also because they provide an insight into the driving mechanisms of instabilities. In some cases, the complexity could be substantially reduced by performing a minimization of the energy functional [23], which removes modes that tend to numerically pollute the spectrum [24] without affecting stability.

Although there has been some previous attempts to find energy principles for the resistive MHD equations, they did not yield stability criteria that are quite as stringent and useful as in the ideal case. The present work shows that Tasso's criterion can be made necessary but this criterion only applies to real growth rates because of the anti-Hermitian part of the zero-frequency energy. This anti-Hermitian contribution, however, vanishes in the vicinity of the rational surfaces if the plasma motion is taken to be incompressible and field line bending is neglected. Resistive interchange modes have real growth rates and can be deduced from the zero-frequency energy in elegant and straightforward way for incompressible plasma motions.

In addition to the Hamiltonian conservation law, a conservation law for the bilinear concomitant has been used to derive the sufficient tearing mode stability criterion $\Delta' < 0$. Although (62) is finite only in the zero pressure gradient limit $p' \to 0$ where $\Psi \sim 1$ and X to leading order as $X \to \pm\infty$ [$E = F = H = 0$ in (48) and (49)], it is well known that Δ' is analytic as $p' \to 0$ so that $\frac{1}{2}\Psi_0^2\Delta(Q) = W[\mathbf{z};\mathbf{z}] + \frac{1}{2}\int_{-\infty}^{\infty} dX\{Q^2|\Xi|^2 + Q|\Psi + X\Xi|^2\}$ must remain approximately valid for small p' as well. This expression is instructive in revealing the stabilizing effect of $W[\mathbf{z};\mathbf{z}] > 0$ when $D_R < 0$, and the importance of the term $\Psi + X\Xi$ that allows the magnetic field to slip from the ideal, infinite conductivity motion.

Finally, it may be worthwhile to note that the energy principle formalism can be applied to derive useful properties of the equations. The causality principle which ensures that the dissipative power is positive definite is an example. The numerical finite element method which reduces the problem to the one of finding the minimum of the Lagrangian, is known to be also affected by these properties [25].

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APPENDIX A: EULER-LAGRANGE EQUATIONS

In this appendix, intermediate steps for the derivation of (28) are presented. To do so the Lagrangian density (35) must be rewritten in a form including only inner products of $\boldsymbol{\xi}^+$, $\mathbf{a}^+$, $\boldsymbol{\nabla}\boldsymbol{\xi}^+$ and $\boldsymbol{\nabla}\mathbf{a}^+$ [or inner products of $\boldsymbol{\xi}$, $\mathbf{a}$, $\boldsymbol{\nabla}\boldsymbol{\xi}$ and $\boldsymbol{\nabla}\mathbf{a}$ to derive (29)]. For instance, we write $\mathbf{Q} = \boldsymbol{\nabla}\times(\boldsymbol{\xi}\times\mathbf{B})$ as $\mathbf{B}\cdot(\boldsymbol{\nabla}\boldsymbol{\xi}) - \mathsf{I}:(\boldsymbol{\nabla}\boldsymbol{\xi})\mathbf{B} - \boldsymbol{\xi}\cdot(\boldsymbol{\nabla}\mathbf{B})$, where I is the identity tensor: $\mathsf{I}\cdot\mathbf{f} = \mathbf{f}$. Using the standard Gibb's notation, we write

$$\boldsymbol{\nabla}\times\mathbf{f} = \sum_i \boldsymbol{\nabla}\times\mathbf{e}_i\mathbf{e}_i\cdot\mathbf{f} \equiv \boldsymbol{\nabla}\times\mathsf{I}\cdot\mathbf{f}, \tag{A1}$$

where $\mathbf{f}$ is any vector and $\{\mathbf{e}_i\}$ is an orthonormal basis so that $\mathsf{I} = \sum_i \mathbf{e}_i\mathbf{e}_i$. To express $\boldsymbol{\nabla}\times\mathbf{f}$ in terms of $\boldsymbol{\nabla}\mathbf{f}$, we introduce the triadic $\mathbf{e}$ whose elements: $e_{ijk} = +1$ if $i,j,k \in \{1,2,3\}$ are cyclic, $e_{jik} = e_{ikj} = e_{kji} = -1$, or $e_{iik} = e_{iji} = e_{ijj} = 0$ otherwise. Equation (A1) becomes

$$\boldsymbol{\nabla}\times\mathbf{f} = \mathbf{e}:\boldsymbol{\nabla}\mathbf{f} \equiv \sum_i\sum_j\sum_k e_{ijk}\partial_j(\mathbf{f}\cdot\mathbf{e}_k)\mathbf{e}_i,$$

allowing us to write (35) as

$$L = \tfrac{1}{2}\Big\{\rho\dot{\boldsymbol{\xi}}^+\cdot\dot{\boldsymbol{\xi}} - [\mathbf{B}\cdot(\nabla\boldsymbol{\xi}^+) - \mathsf{I}:(\nabla\boldsymbol{\xi}^+)\mathbf{B} - \boldsymbol{\xi}^+\cdot(\nabla\mathbf{B}) + \boldsymbol{\xi}^+\cdot\mathbf{n}\mathbf{J}\times\mathbf{n}]\cdot(\mathbf{Q} + \xi_n\mathbf{J}\times\mathbf{n})$$
$$- \mathsf{I}:(\nabla\boldsymbol{\xi}^+)\Gamma p\nabla\cdot\boldsymbol{\xi} + 2U\boldsymbol{\xi}^+\cdot(\nabla\psi)\xi - \mathbf{e}:(\nabla\mathbf{a}^+)\cdot\nabla\times\mathbf{a} + \frac{1}{2\eta}\left(\dot{\mathbf{a}}^+\cdot\mathbf{a} - \mathbf{a}^+\cdot\dot{\mathbf{a}}\right)$$
$$- [\mathbf{B}\cdot(\nabla\boldsymbol{\xi}^+) - \mathsf{I}:(\nabla\boldsymbol{\xi}^+)\mathbf{B} - \boldsymbol{\xi}^+\cdot(\nabla\mathbf{B})]\cdot\nabla\times\mathbf{a} - \mathbf{e}:(\nabla\mathbf{a}^+)\cdot\mathbf{Q} + \boldsymbol{\xi}^+\cdot\mathbf{J}\times(\nabla\times\mathbf{a})\Big\} \quad \text{(A2)}$$

so that

$$\frac{\partial L}{\partial\nabla\boldsymbol{\xi}^+} = \tfrac{1}{2}\{\mathbf{B}\cdot(\mathbf{Q} + \xi_n\mathbf{J}\times\mathbf{n})\,\mathsf{I} - \mathbf{B}(\mathbf{Q} + \xi_n\mathbf{J}\times\mathbf{n}) - \mathsf{I}\,\Gamma p\nabla\cdot\boldsymbol{\xi} + \mathsf{I}\,\mathbf{B}\cdot\nabla\times\mathbf{a} - \mathbf{B}\nabla\times\mathbf{a}\}$$
$$\frac{\partial L}{\partial\nabla\mathbf{a}^+} = -\tfrac{1}{2}\mathbf{e}\cdot(\nabla\times\mathbf{a} + \mathbf{Q}) \quad \text{(A3)}$$

and

$$-\frac{\partial L}{\partial\boldsymbol{\xi}^+} = \tfrac{1}{2}\{-(\nabla\mathbf{B})\cdot(\mathbf{Q} + \xi_n\mathbf{J}\times\mathbf{n}) + \mathbf{n}\mathbf{J}\times\mathbf{n}\cdot(\mathbf{Q} + \xi_n\mathbf{J}\times\mathbf{n})$$
$$-2U\nabla\psi\xi - \mathbf{J}\times\nabla\times\mathbf{a} - (\nabla\mathbf{B})\cdot\nabla\times\mathbf{a}\}$$
$$-\frac{\partial L}{\partial\mathbf{a}^+} = \frac{1}{4\eta}\dot{\mathbf{a}}. \quad \text{(A4)}$$

The divergence of (A3) is taken using $\nabla\cdot(\mathbf{B}\cdot\mathbf{Q}\,\mathsf{I}) = (\nabla\mathbf{B})\cdot\mathbf{Q} + (\nabla\mathbf{Q})\cdot\mathbf{B}$, yielding (28).

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[1] T. L. Saaty and J. Bram, *Nonlinear Mathematics* (Dover, ADDRESS, 1964).
[2] I. B. Bernstein, E. A. Frieman, M. D. Kruskal, and R. M. Kulsrud, Proc. Roy. Soc. **A244**, 17 (1958).
[3] W. A. Newcomb, Ann. Phys. **10**, 232 (1960).
[4] M. Bineau, Nucl. Fusion **2**, 130 (1962).
[5] H. P. Furth, P. H. Rutherford, and H. Selberg, Phys. Fluids **16**, 1054 (1973).
[6] A. E. Adler, R. M. Kulsrud, and R. B. White, Phys. Fluids **23**, 1375 (1980).
[7] A. Bondeson and J. R. Sobel, Phys. Fluids **27**, 2028 (1984).
[8] H. Tasso, Phys. Letters **147**, 28 (1990).
[9] J. A. Wesson, Report JET-P(91)39, JET (unpublished).
[10] H. P. Furth, J. Killeen, and M. N. Rosenbluth, Phys. Fluids **6**, 459 (1963).
[11] R. B. White, Rev. Modern Physics **58**, 183 (1986).
[12] C. Itsykson and J.-P. Zuber, *Quantum Field Theory* (McGraw-Hill, ADDRESS, 1980).
[13] H. Tasso, Phys. Letters **94 A**, 217 (1983).
[14] A. H. Glasser, J. M. Greene, and J. L. Johnson, Phys. Fluids **18**, 875 (1975).
[15] B. Coppi, J. Greene, and J. Johnson, Nucl. Fusion **6**, 101 (1966).
[16] L. Fonda and G. C. Ghirardi, *Symmetry Principles in Quantum Physics* (Marcel Dekker Inc., ADDRESS, 1970).
[17] P. M. Morse and H. Feshbach, *Methods of Theoretical Physics* (McGraw-Hill, New York, 1953).
[18] H. . Kull, H. . Berk, and P. . Morrison, Phys. Fluids B 1, 55 (1989).
[19] R. L. Dewar and A. Pletzer, J. Plasma Phys. **43**, 291 (1990).
[20] M. S. Chu, M. S. Chance, J. M. Greene, and T. H. Jensen, Phys. Fluids B **2**, 97 (1990).
[21] J. M. Greene, Private Communication, 1992.
[22] J. L. Johnson, J. M. Greene, and B. Coppi, Phys. Fluids **6**, 1169 (1963).
[23] R. C. Grimm, R. L. Dewar, and J. Manickam, J. Comp. Phys. **49**, 94 (1983).

[24] X. Llobet, K. Appert, A. Bondeson, and J. Vaclavik, Comput. Phys. Commun. **59**, 199 (1990).
[25] A. Pletzer, Comput. Phys. Commun. **91**, 1 (1996).

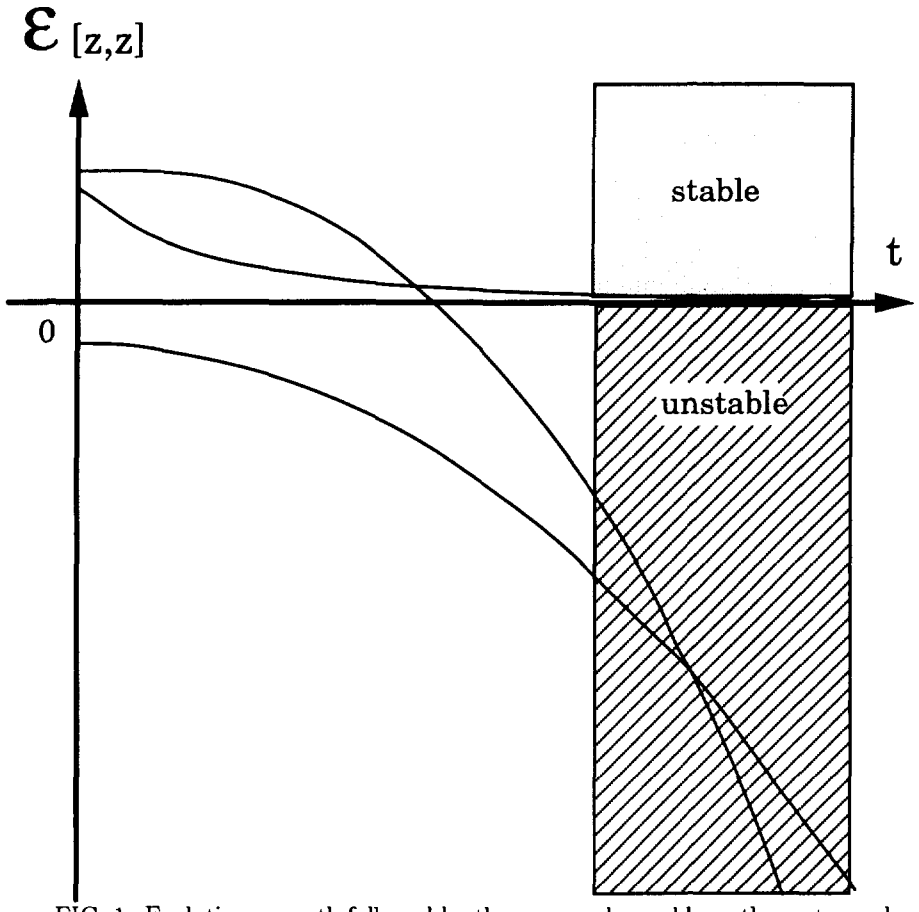

FIG. 1. Evolutionary path followed by the energy observable as the system relaxes. For stable modes, the system relaxes to its anterior state, whereas for unstable modes, the energy departs exponentially from the initial state as $t \to \infty$.