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O.V. Man'ko and N.V. Tcherniega
International Centre for Theoretical Physics, Trieste, Italy
and
Lebedev Physical Institute,
Leninsky Prospekt, 53, Moscow 117924, Russian Federation.

ABSTRACT

New time-dependent integrals of motion are found for stimulated Raman scattering. Explicit formula for the photon-number probability distribution as a function of the laser-field intensity and the medium parameters is obtained in terms of Hermite polynomials of two variables.

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E-mail addresses: omanko@sci.lpi.msk.su; tchera@sci.lpi.msk.su

1 Introduction

Since the discovery of stimulated Raman scattering in 1962 [1] this phenomenon has been intensively investigated both theoretically and experimentally [2–7]. Quantum-mechanical description of stimulated Raman scattering can be done in the framework of different equations, namely, Heisenberg–Langevin equation [7], Maxwell–Bloch equation [8], and Fokker–Planck equation [9]. Quantum-statistical properties of stimulated Raman scattering were treated using a quadratic Hamiltonian [10]. For studying properties of stimulated Raman scattering taking into account intermolecular interaction, a cubic Hamiltonian was used [11].

Nonclassical properties of stimulated Raman scattering such as squeezing and sub-Poissonian statistics were considered in a number of papers [12–16]. The purpose of our work is to study the photon distribution function for the Stocks wave in the framework of linear integrals of motion for systems with quadratic Hamiltonians [17–19] using the results of [20–26]. General formulas for matrix elements of the Gaussian density operator for the multimode oscillator in the Fock basis were calculated explicitly in [20]. For one-mode light described by the Wigner function of a generic Gaussian form with five real parameters describing the quadrature means, variances, and covariance, the photon distribution function was obtained explicitly in terms of Hermite polynomials of two variables [21]. In [22], the temperature dependence of oscillations of the photon distribution function for squeezed states was investigated. It was shown that oscillations of the photon distribution function for squeezed and correlated light were decreasing if the temperature increased.

The photon distribution function for N-mode mixed state of light described by the Wigner function of the generic Gaussian form was calculated explicitly in terms of Hermite polynomials of $2N$ variables in [23, 24], and parameters of the photon distribution function were determined through the dispersion matrix and mean values of quadrature components of the light. The photon distribution for two-mode squeezed vacuum was investigated in [25] where its dependence on four parameters (two squeezing parameters, the relative phase between the two oscillators, and their spatial orientations) was shown. In [26], the case of two-mode squeezed coherent states was considered, and the photon distribution function for the states was expressed both through four-variable and two-variable Hermite polynomials dependent on two squeezing parameters, the relative phase between the two oscillators, their spatial orientation, and four-dimensional shift in the phase space of the electromagnetic-field oscillator.

As an application of the theoretical consideration, the linear optical transformer of photon statistics for multimode light was suggested [27]. The transformation coefficient was obtained explicitly in terms of multivariable Hermite polynomials.

In this paper, we find new integrals of motion for the process of stimulated Raman scattering. We describe stimulated Raman scattering with the help of a simple model of a two-dimensional oscillator, the photons of the Stock mode being described by the one mode of the oscillator and the phonons of the medium being described by the another mode of the oscillator. The interaction of the photons and phonons is taken to be quadratic in creation and annihilation operators of the photons and phonons. We obtain the photon–phonon probability distribution function for the Stocks and phonon modes after the interaction of the laser field with the medium in terms of Hermite polynomials of four variables. The photon distribution function for the Stocks mode is found explicitly and expressed both in terms of Hermite polynomials of two variables with zero arguments and in terms of Legendre polynomials. The mean photon number and its dispersion in the Stocks mode are expressed as functions of the medium parameter (temperature), the laser frequency, and a parameter of interaction (coupling constant).

2 Integrals of Motion

The simplest phenomenological Hamiltonian, which can be used for the description of one-mode Stocks-wave excitation, can be written [7, 28, 29]

$$\hat{H} = \hbar\omega_S \hat{a}^\dagger \hat{a} + \hbar\omega_{31} \hat{b}^\dagger \hat{b} + \hbar\kappa \left[e^{-i\omega_L t} \hat{a}^\dagger \hat{b}^\dagger + e^{i\omega_L t} \hat{b}\hat{a} \right], \tag{1}$$

where $\hat{a}$ and ω_S are the annihilation operator and the frequency of the Stocks photon, $\hat{b}$ and ω_{31} are the annihilation operator and the frequency of the phonon, ω_L is the laser frequency, and κ is the coupling constant. The laser field is considered as a classical one and its frequency is determined by the condition

$$\omega_L = \omega_{31} + \omega_S. \tag{2}$$

The damping and depletion of the laser light wave are neglected. Antistocks-mode excitation is also neglected and excitation of only one phonon mode is taken into consideration.

We will show that there exist time-dependent integrals of motion for the model of Stocks-wave excitation. Let us construct four nonhermitian operators

$$\begin{aligned}
\hat{a}(t) &= \hat{a}e^{i\omega_S t}\cosh(\kappa t) + i\hat{b}^\dagger e^{-i\omega_{31}t}\sinh(\kappa t)\,; \\
\hat{a}^\dagger(t) &= \hat{a}^\dagger e^{-i\omega_S t}\cosh(\kappa t) - i\hat{b}e^{i\omega_{31}t}\sinh(\kappa t)\,; \\
\\
\hat{b}(t) &= \hat{b}e^{i\omega_{31}t}\cosh(\kappa t) + i\hat{a}^\dagger e^{-i\omega_S t}\sinh(\kappa t)\,; \\
\hat{b}^\dagger(t) &= \hat{b}^\dagger e^{-i\omega_{31}t}\cosh(\kappa t) - i\hat{a}e^{i\omega_S t}\sinh(\kappa t)\,.
\end{aligned} \tag{3}$$

In view of the commutation relations between the photon creation and annihilation operators $\hat{a}$, $\hat{a}^\dagger$ and the phonon creation and annihilation operators $\hat{b}$, $\hat{b}^\dagger$, one can check that the operators constructed above satisfy boson commutation relations

$$[\hat{a}(t), \hat{a}^\dagger(t)] = 1\,; \qquad [\hat{b}(t), \hat{b}^\dagger(t)] = 1\,,$$

and the operators $\hat{a}(t)$, $\hat{b}(t)$, and their hermitian conjugates commute as

$$[\hat{a}(t), \hat{b}(t)] = 0\,; \qquad [\hat{a}(t), \hat{b}^\dagger(t)] = 0\,;$$

$$[\hat{a}^\dagger(t), \hat{b}(t)] = 0\,; \qquad [\hat{a}^\dagger(t), \hat{b}^\dagger(t)] = 0\,.$$

It can be shown, that the total time derivatives

$$\begin{aligned}
\frac{d\hat{a}(t)}{dt} &= \frac{\partial \hat{a}(t)}{\partial t} + \frac{i}{\hbar}[\hat{H}, \hat{a}(t)]\,; \\
\frac{d\hat{b}(t)}{dt} &= \frac{\partial \hat{b}(t)}{\partial t} + \frac{i}{\hbar}[\hat{H}, \hat{b}(t)]
\end{aligned}$$

of the operators (3) are equal to zero, i.e.,

$$\frac{d\hat{a}(t)}{dt} = 0\,; \qquad \frac{d\hat{a}^\dagger(t)}{dt} = 0\,;$$

$$\frac{d\hat{b}(t)}{dt} = 0\,; \qquad \frac{d\hat{b}^\dagger(t)}{dt} = 0\,.$$

Consequently, the operators $\hat{a}(t)$, $\hat{a}^\dagger(t)$, $\hat{b}(t)$, and $\hat{b}^\dagger(t)$ are the integrals of motion (linear with respect to the photon and phonon creation and annihilation operators) for Stocks-mode excitation in the framework of the model with the Hamiltonian (1). The operators (3) are equal to standard photon and phonon creation and annihilation operators at the initial time moment and their commutators are time-independent at all time moments.

Let us introduce quadrature components of photon and phonon creation and annihilation operators

$$\hat{p}_a = \frac{\hat{a} - \hat{a}^\dagger}{i\sqrt{2}}\,; \qquad \hat{q}_a = \frac{\hat{a} + \hat{a}^\dagger}{\sqrt{2}}\,;$$

$$\hat{p}_b = \frac{\hat{b} - \hat{b}^\dagger}{i\sqrt{2}}\,; \qquad \hat{q}_b = \frac{\hat{b} + \hat{b}^\dagger}{\sqrt{2}}\,.$$

For Stocks-mode excitation, one can write four additional integrals of motion using the properties of the integrals of motion [18, 19]

$$\begin{aligned}
\hat{p}_a(t) &= \hat{p}_a \cosh\kappa t\, \cos\omega_S t + \hat{q}_a \cosh\kappa t\, \sin\omega_S t \\
&\quad - \hat{p}_b \sinh\kappa t\, \sin\omega_{31} t + \hat{q}_b \sinh\kappa t\, \cos\omega_{31} t\,; \\
\hat{q}_a(t) &= -\hat{p}_a \cosh\kappa t\, \cos\omega_S t + \hat{q}_a \cosh\kappa t\, \cos\omega_S t \\
&\quad - \hat{p}_b \sinh\kappa t\, \cos\omega_{31} t + \hat{q}_b \sinh\kappa t\, \sin\omega_{31} t\,; \\
\hat{p}_b(t) &= -\hat{p}_a \sinh\kappa t\, \sin\omega_S t + \hat{q}_a \sinh\kappa t\, \cos\omega_S t \\
&\quad + \hat{p}_b \cosh\kappa t\, \cos\omega_{31} t + \hat{q}_b \cosh\kappa t\, \sin\omega_{31} t\,; \\
\hat{q}_b(t) &= -\hat{p}_a \sinh\kappa t\, \cos\omega_S t + \hat{q}_a \sinh\kappa t\, \sin\omega_S t \\
&\quad - \hat{p}_b \cosh\kappa t\, \cos\omega_{31} t + \hat{q}_b \cosh\kappa t\, \cos\omega_{13} t\,.
\end{aligned} \tag{4}$$

The physical meaning of the invariants (4) is that their eigenvalues determine the initial values of classical quadrature components in the phase space of mean values $\langle p_a\rangle$, $\langle p_b\rangle$, $\langle q_a\rangle$, and $\langle q_b\rangle$. The number of photons does not conserve in the process of stimulated Raman scattering. But since any function of integrals of motion is the integral of motion [18, 19], one can find some time-dependent combinations of the photon and phonon numbers, which are integrals of motion. Thus, the observable

$$\begin{aligned} N_a(t) &= \hat{a}^\dagger(t)\hat{a}(t) \\ &= \hat{a}^\dagger\hat{a}\cosh^2(\kappa t) + \left(\hat{b}^\dagger\hat{b}+1\right)\sinh^2(\kappa t) \\ &+\frac{i}{2}\left\{\hat{a}^\dagger\hat{b}^\dagger\exp\left[-it\left(\omega_S+\omega_{31}\right)\right] - \hat{a}\hat{b}\exp\left[it\left(\omega_S+\omega_{31}\right)\right]\right\}\sinh\left(2\kappa t\right) \end{aligned}$$

is the integral of motion, which has the physical meaning of the initial number of photons in the system state. The observable

$$\begin{aligned} N_b(t) &= \hat{b}^\dagger(t)\hat{b}(t) \\ &= \hat{b}^\dagger\hat{b}\cosh^2(\kappa t) + \left(\hat{a}^\dagger\hat{a}+1\right)\sinh^2(\kappa t) \\ &+\frac{i}{2}\left\{\hat{a}^\dagger\hat{b}^\dagger\exp\left[-it\left(\omega_S+\omega_{31}\right)\right] - \hat{a}\hat{b}\exp\left[it\left(\omega_S+\omega_{31}\right)\right]\right\}\sinh\left(2\kappa t\right) \end{aligned}$$

is the integral of motion, which has the physical meaning of the initial number of phonons in the system state. The difference of the two integrals of motion

$$N_a(t) - N_b(t) = \hat{a}^\dagger\hat{a} - \hat{b}^\dagger\hat{b}$$

is the time-dependent integral of motion, which has the physical meaning of the difference of photon and phonon numbers in the system, which is constant of the motion for the phenomenological Hamiltonian (1). Thus, for stimulated Raman scattering we have found new integrals of motion.

3 Photon–Phonon Probability Distribution Function

In this section, we obtain an explicit expression for the photon distribution function of the Stocks mode. Let us introduce the vector column constructed from quadrature components of photon and phonon creation and annihilation operators in the medium at the initial time moment

$$\hat{\mathbf{Q}} = (\hat{p}_a, \hat{p}_b, \hat{q}_a, \hat{q}_b)$$

and the vector column, constructed from the integrals of motion

$$\hat{\mathbf{I}}\,(t) = (\hat{p}_a(t), \hat{p}_b(t), \hat{q}_a(t), \hat{q}_b(t))\,.$$

Then the relation between the integrals of motion (4) and the initial quadrature components is of the form

$$\hat{\mathbf{I}}\,(t) = \Lambda(t)\hat{\mathbf{Q}}\,,$$

where the real symplectic matrix $\Lambda(t)$ is determined by the equation

$$\Lambda(t) = \begin{pmatrix} \cosh\kappa t\cos\omega_S t & -\sinh\kappa t\sin\omega_{13}t & \cosh\kappa t\sin\omega_S t & \sinh\kappa t\cos\omega_{13}t \\ -\sinh\kappa t\sin\omega_S t & \cosh\kappa t\cos\omega_{13}t & \sinh\kappa t\cos\omega_S t & \cosh\kappa t\sin\omega_{13}t \\ -\cosh\kappa t\cos\omega_S t & -\sinh\kappa t\cos\omega_{13}t & \cosh\kappa t\cos\omega_S t & \sinh\kappa t\sin\omega_{13}t \\ -\sinh\kappa t\cos\omega_S t & -\cosh\kappa t\cos\omega_{13}t & \sinh\kappa t\sin\omega_S t & \cosh\kappa t\cos\omega_{13}t \end{pmatrix}. \quad (5)$$

Let us introduce the dispersion matrix of quadrature components

$$\sigma(0) = \begin{pmatrix} \sigma_{p_a^2} & \sigma_{p_ap_b} & \sigma_{p_aq_a} & \sigma_{p_aq_b} \\ \sigma_{p_ap_b} & \sigma_{p_b^2} & \sigma_{p_bq_a} & \sigma_{p_bq_b} \\ \sigma_{p_aq_a} & \sigma_{q_ap_b} & \sigma_{q_a^2} & \sigma_{q_aq_b} \\ \sigma_{p_aq_b} & \sigma_{q_ap_b} & \sigma_{q_aq_b} & \sigma_{q_b^2} \end{pmatrix},$$

where the matrix elements are determined through the density matrix as follows

$$\begin{aligned} \sigma_{p_ip_j} &= \mathrm{Tr}\,\hat{\rho}\hat{p}_i\hat{p}_j - \langle\hat{p}_i\rangle\langle\hat{p}_j\rangle; \\ \sigma_{q_iq_j} &= \mathrm{Tr}\,\hat{\rho}\hat{q}_i\hat{q}_j - \langle\hat{q}_i\rangle\langle\hat{q}_j\rangle; \\ \sigma_{p_iq_j} &= \frac{1}{2}\mathrm{Tr}\,\hat{\rho}\,(\hat{q}_j\hat{p}_i + \hat{p}_i\hat{q}_j) - \langle\hat{p}_i\rangle\langle\hat{q}_j\rangle \end{aligned}$$

(indices i and j can be equal to a and b). The dispersion matrix at the initial time moment t can be expressed through the initial dispersion matrix of quadrature components of medium photons and phonons in the form of matrix equation

$$\sigma(t) \;= \Lambda^{-1}\sigma(0)\Sigma\Lambda^T\Sigma\,, \quad (6)$$

where the 4×4-block matrix Σ consists of 2×2-zero matrices and unity matrices I_2,

$$\Sigma = \begin{pmatrix} 0 & I_2 \\ I_2 & 0 \end{pmatrix}.$$

If the medium photons are in the ground state and phonons are in the state of thermodynamical equilibrium with temperature T at the initial time moment, then

$$\sigma_{p_a^2}(0) = \frac{1}{2}; \qquad \sigma_{q_a^2}(0) = \frac{1}{2};$$

$$\sigma_{p_b^2}(0) = \frac{1}{2}\coth\frac{\beta}{2}; \qquad \sigma_{q_b^2}(0) = \frac{1}{2}\coth\frac{\beta}{2}; \qquad \beta = \frac{\hbar\omega_{31}}{T},$$

and the matrix elements of the matrix $\sigma(t)$ can be found in the explicit form

$$\begin{aligned}
\sigma_{p_a^2}(t) &= \frac{1}{2}\left(\cosh^2\kappa t + \coth\frac{\beta}{2}\sinh^2\kappa t\right); \\
\sigma_{p_b^2}(t) &= \frac{1}{2}\left(\sinh^2\kappa t + \coth\frac{\beta}{2}\cosh^2\kappa t\right); \\
\sigma_{p_a p_b}(t) &= \frac{1}{4}\left(1 + \coth\frac{\beta}{2}\right)\sinh 2\kappa t\,\sin\omega_L t; \\
\sigma_{q_a^2}(t) &= \cosh^2\kappa t\,\cos\omega_S t + \frac{1}{2}\sinh^2\kappa t\,\coth\frac{\beta}{2}; \\
\sigma_{q_b^2}(t) &= \frac{1}{2}\sinh^2\kappa t + \cosh^2\kappa t\,\cos^2\omega_{13}t\,\coth\frac{\beta}{2}; \qquad (7) \\
\sigma_{q_a q_b}(t) &= \frac{1}{4}\sinh 2\kappa t\,[\cos\omega_S t\,(\cos\omega_{13}t - \sin\omega_{13}t) \\
&\quad + \coth\frac{\beta}{2}\cos\omega_{13}t\,(\cos\omega_S t - \sin\omega_S t)]; \\
\sigma_{p_a q_a}(t) &= \frac{\sinh^2\kappa t}{2}\sin 2\omega_S t\,\coth\frac{\beta}{2} + \frac{\cosh^2\kappa t}{2}\cos\omega_S t\,(\cos\omega_S t - \sin\omega_S t); \\
\sigma_{p_b q_b}(t) &= \frac{\sinh^2\kappa t}{2}\sin 2\omega_{13}t + \frac{\cosh^2\kappa t}{2}\coth\frac{\beta}{2}\cos\omega_{13}t\,(\cos\omega_{13}t - \sin\omega_{13}t); \\
\sigma_{p_a q_b}(t) &= \frac{1}{4}\sinh 2\kappa t\left(\cos(\omega_S - \omega_{13})t + \cos\omega_{13}t\,\coth\frac{\beta}{2}(\sin\omega_S t - \cos\omega_S t)\right); \\
\sigma_{p_b q_a}(t) &= \frac{1}{4}\sinh 2\kappa t\left(\cos(\omega_S - \omega_{13})t\,\coth\frac{\beta}{2} + \cos\omega_S t\,(\sin\omega_{13}t - \cos\omega_{13}t)\right).
\end{aligned}$$

One can see that dispersions of the photon and phonon quadratures become larger in the process of stimulated Raman scattering. Being initially noncorrelated the quadratures become statistically-dependent observables, since the Hamiltonian (1) is quadratic one.

After interacting with the laser field, the state of the system can be described by the Wigner function of the Gaussian type

$$W(\mathrm{Q}) = \frac{1}{\sqrt{\det \sigma(t)}} \exp\left(-\frac{1}{2}\mathrm{Q}\sigma^{-1}(t)\mathrm{Q}\right), \tag{8}$$

where matrix $\sigma(t)$ is determined by formulas (7). We can express the inverse matrix $\sigma^{-1}(t)$ at the time moment t through the initial inverse matrix $\sigma^{-1}(0)$ using (6) and the known property of symplectic matrices, namely,

$$\sigma^{-1}(t) \;=\; \Lambda^T\sigma^{-1}(0)\Lambda. \tag{9}$$

The matrix elements of the inverse dispersion matrix $\sigma^{-1}(t)$ (9) have the explicit form

$$\begin{aligned}
\sigma^{-1}_{p_a^2}(t) &= 4\cosh^2\kappa t\,\cos^2\omega_S t + 2\tanh\frac{\beta}{2}\,\sinh^2\kappa t\,; \\
\sigma^{-1}_{p_b^2}(t) &= 4\tanh\frac{\beta}{2}\,\cosh^2\kappa t\,\cos^2\omega_{13}t + 2\sinh^2\kappa t\,; \\
\sigma^{-1}_{p_a p_b}(t) &= \sinh 2\kappa t\Big[\cos\omega_S t\,(\cos\omega_{13}t - \sin\omega_{13}t) \\
&\quad +\cos\omega_{13}t\,\tanh\frac{\beta}{2}\,(\cos\omega_S t - \sin\omega_S t)\Big]; \\
\sigma^{-1}_{p_a q_a}(t) &= 2\cosh^2\kappa t\,\cos\omega_S t\,(\sin\omega_S t - \cos\omega_S t) - 2\tanh\frac{\beta}{2}\,\sinh^2\kappa t\,\sin 2\omega_S t\,; \\
\sigma^{-1}_{p_a q_b}(t) &= \sinh 2\kappa t\left[\cos\omega_S t\,\cos\omega_{13}t\left(1-\tanh\frac{\beta}{2}\right)\right. \\
&\quad \left. -\sin\omega_{13}\left(\tanh\frac{\beta}{2}\,\sin\omega_S t + \cos\omega_S t\right)\right]; \\
\sigma^{-1}_{p_b q_a}(t) &= \sinh 2\kappa t\left[\cos\omega_{13}t\,\cos\omega_S t\left(\tanh\frac{\beta}{2} - 1\right)\right. \\
&\quad \left. -\sin\omega_S t\left(\sin\omega_{13}t + \tanh\frac{\beta}{2}\,\cos\omega_{13}t\right)\right]; \\
\sigma^{-1}_{p_b q_b}(t) &= -2\sinh^2\kappa t\,\sin 2\omega_{13}t + \tanh\frac{\beta}{2}\,\cosh^2\kappa t\left(\sin 2\omega_{13}t - 2\cos^2\omega_{13}t\right);
\end{aligned} \tag{10}$$

$$\sigma^{-1}_{q_a^2}(t) = 2\left(\cosh^2\kappa t + \tanh\frac{\beta}{2}\sinh^2\kappa t\right);$$

$$\sigma^{-1}_{q_b^2}(t) = 2\left(\sinh^2\kappa t + \tanh\frac{\beta}{2}\cosh^2\kappa t\right);$$

$$\sigma^{-1}_{q_aq_b}(t) = \sinh 2\kappa t\,\sin\left(\omega_S + \omega_{13}\right)t\left(1 + \tanh\frac{\beta}{2}\right).$$

We determine the photon–phonon probability distribution function P_{nm} as the probability to obtain n photons in the Stocks mode and m phonons in the phonon mode after the interaction of the laser field with the medium. The function P_{nm} is the matrix element of the density matrix of the system in the Fock basis

$$P_{nm} = \langle n, m \mid \hat{\rho} \mid n, m\rangle,$$

where $\mid n, m\rangle$ is the eigenstate of the set of the photon and phonon number operators $\hat{a}^\dagger\hat{a}$ and $\hat{b}^\dagger\hat{b}$

$$\hat{a}^\dagger\hat{a} \mid n, m\rangle = n \mid n, m\rangle;$$

$$\hat{b}^\dagger\hat{b} \mid n, m\rangle = m \mid n, m\rangle.$$

Using the scheme of calculations developed in [20–26] for multimode coupled oscillators we can obtain the expression for the photon–phonon probability distribution function P_{nm} in terms of Hermite polynomials of four variables with zero arguments

$$P_{nm} = \frac{H^{\{R\}}_{nmnm}(0, 0, 0, 0)}{\left[\det\left(\sigma(t) + I_4/2\right)\right]^{1/2} n!\, m!}, \qquad (11)$$

where the matrix R is expressed through the dispersion matrix at the time moment t

$$R = U^\dagger\left(I_4 - 2\sigma(t)\right)\left(I_4 + 2\sigma(t)\right)^{-1} U^*,$$

the matrix U is

$$U = \frac{1}{\sqrt{2}}\begin{pmatrix} -i & 0 & i & 0 \\ 0 & -i & 0 & i \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix},$$

and the matrix I_4 is four-dimensional unity matrix.

4 Photon Probability Distribution Function

Usually in experiments the photon number in the Stocks mode is measured. So, it is interesting to average the photon–phonon probability distribution function over the phonon mode and to obtain the probability to have n photons in the Stocks mode.

The photon–phonon probability distribution function can be described by the Wigner function (8). One has to integrate the Wigner function over the variables q_b, p_b in order to obtain the Wigner function (averaged over the phonon mode) describing the photon state

$$\begin{aligned} W_{\rm ph}(q_a,p_a) &= \frac{1}{2\pi}\int\int_{-\infty}^{\infty} W(\mathbf{Q})\, dq_b\, dp_b \\ &= \frac{1}{2\pi\sqrt{\det\sigma(t)}}\int\int_{-\infty}^{\infty} \exp\left(-\frac{1}{2}\mathbf{Q}\sigma^{-1}(t)\mathbf{Q}\right) dq_b\, dp_b\,, \end{aligned} \tag{12}$$

where $\sigma^{-1}(t)$ is determined by (9). The Wigner function $W_{\rm ph}(q_a,p_a)$ (12) describes the photon mode.

It is convenient to change the places of quadrature components. For this purpose, we introduce a vector $\mathbf{X} = P\mathbf{Q}$, where the matrix P is of the form

$$P = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Then, in view of (9), for the argument of exponential function in (12) one has the equality

$$\mathbf{Q}\sigma^{-1}(t)\mathbf{Q} = \mathbf{X}P\Lambda^T\sigma^{-1}(0)\Lambda P\mathbf{X}\,.$$

By introducing the block matrix

$$A = \frac{1}{2}P\Lambda^T\sigma^{-1}(0)\Lambda P = \begin{pmatrix} a & b \\ c & d \end{pmatrix},$$

the integral in (12) can be rewritten in the form

$$\int\int_{-\infty}^{\infty} \exp\left(-\mathbf{X}A\mathbf{X}\right) dy, \qquad \mathbf{y} = (p_b, q_b).$$

One can easily see that the integral in (12) has the form of the Gaussian integral calculated in the Appendix. We obtain, that the Wigner function of the photon state of the Stocks mode is described in the explicit form by the formula

$$W_{\text{ph}}(p_a, q_a) = \frac{1}{2\sqrt{\det\sigma(t)\det d}} \exp\left[-\frac{1}{2}\begin{pmatrix} p_a & q_a \end{pmatrix} \sigma_{\text{ph}}^{-1} \begin{pmatrix} p_a \\ q_a \end{pmatrix}\right], \tag{13}$$

where

$$\sigma_{\text{ph}}^{-1} = 2a - \frac{1}{2}\left(c^T + b\right) d^{-1} \left(b^T + c\right) \tag{14}$$

with the matrix elements

$$\sigma_{\text{ph}}^{-1} = \begin{pmatrix} s_{11} & s_{12} \\ s_{21} & s_{22} \end{pmatrix}$$

expressed through the matrix elements of the inverse photon–phonon matrix $\sigma^{-1}(t)$ in the following form

$$\begin{aligned}
s_{11} &= -\sigma_{p_a^2}^{-1} - \frac{\sigma_{q_b^2}^{-1}\left(\sigma_{p_a p_b}^{-1}\right)^2 + \sigma_{p_b^2}^{-1}\left(\sigma_{p_a q_b}^{-1}\right)^2 - 2\sigma_{p_a p_b}^{-1}\sigma_{p_a q_b}^{-1}\sigma_{p_b q_b}^{-1}}{2\left[\sigma_{p_b^2}^{-1}\sigma_{q_b^2}^{-1} - \left(\sigma_{p_b q_b}^{-1}\right)^2\right]}; \\
s_{22} &= -\sigma_{q_a^2}^{-1} - \frac{\sigma_{q_b^2}^{-1}\left(\sigma_{p_b p_a}^{-1}\right)^2 + \sigma_{p_b^2}^{-1}\left(\sigma_{q_a q_b}^{-1}\right)^2 - 2\sigma_{q_a q_b}^{-1}\sigma_{p_b q_a}^{-1}\sigma_{p_b q_b}^{-1}}{2\left[\sigma_{p_b^2}^{-1}\sigma_{q_b^2}^{-1} - \left(\sigma_{p_b q_b}^{-1}\right)^2\right]}; \\
s_{12} &= s_{21} \\
&= -\sigma_{p_a q_a}^{-1} \\
&\quad - \frac{\sigma_{q_b^2}^{-1}\sigma_{p_a p_b}^{-1}\sigma_{p_b q_a}^{-1} + \sigma_{p_b^2}^{-1}\sigma_{p_a q_b}^{-1}\sigma_{q_a q_b}^{-1} - \sigma_{p_a q_b}^{-1}\sigma_{p_b q_a}^{-1}\sigma_{p_b q_b}^{-1} - \sigma_{p_a p_b}^{-1}\sigma_{p_b q_b}^{-1}\sigma_{q_a q_b}^{-1}}{2\left[\sigma_{p_b^2}^{-1}\sigma_{q_b^2}^{-1} - \left(\sigma_{p_b q_b}^{-1}\right)^2\right]}.
\end{aligned} \tag{15}$$

The averaged probability distribution function of photons in the Stocks mode can be expressed through the Hermite polynomials of two variables with zero arguments using the scheme developed in [20–27]

$$P_n = \left[\det\left(\sigma_{\text{ph}} + \frac{I_2}{2}\right)\right]^{-1/2} \frac{H_{nn}^{\{\tilde{R}\}}(0,0)}{n!}, \tag{16}$$

where the matrix $\tilde{R}$ is given by the formula

$$\tilde{R} = U^{+}(I_2 - 2\sigma_{\text{ph}})(I_2 + 2\sigma_{\text{ph}})^{-1}U^{*}$$

with

$$U = \frac{1}{\sqrt{2}}\begin{pmatrix} -i & i \\ 1 & 1 \end{pmatrix}.$$

One can write the photon probability distribution function as a function of Legendre polynomials using the relations between Hermite and Legendre polynomials [19].

The symmetric matrix $\tilde{R}$ has the form

$$\tilde{R} = \begin{pmatrix} r_{11} & r_{12} \\ r_{12} & r_{22} \end{pmatrix}.$$

Using the relation [19]

$$H_{nn}^{\{\tilde{R}\}}(0,0) = (r_{11}r_{22})^{n/2} H_{nn}^{\{\beta\}}(0,0),$$

where the matrix

$$\beta = \begin{pmatrix} 0 & r \\ r & 0 \end{pmatrix}$$

has the matrix element

$$r = (r_{11}r_{22})^{-1/2} r_{12},$$

we get

$$P_n = \left[\det\left(\sigma_{\text{ph}} + \frac{I_2}{2}\right)\right]^{-1/2} (-1)^n \left(r_{12}^2 - r_{11}r_{22}\right)^{n/2} L_n\left(\frac{r}{\sqrt{r^2-1}}\right), \qquad (17)$$

where L_n are Legendre polynomials.

The mean number of photons in the Stocks mode is the function of temperature and the coupling constant (of the laser field with the Stocks mode) and it can be calculated in the explicit form

$$\begin{aligned} \langle n \rangle &= \frac{1}{2}\left(\sigma_{p_a p_a} + \sigma_{q_a q_a} - 1\right) \\ &= \frac{1}{2}\left[\cosh^2 \kappa t \left(\frac{1}{2} + \cos^2 \omega_S t\right) + \sinh^2 \kappa t \coth\frac{\beta}{2} - 1\right]. \qquad (18) \end{aligned}$$

The dispersion of mean photon number in the Stocks mode is

$$\begin{aligned}\sigma_{n^2}(t) &= \frac{1}{2}\cosh^4\kappa t\left[\frac{1}{4}+\cos^4\omega_S t+\cos^2\omega_S t\left(\frac{1}{2}-\sin\omega_S t\right)\right]\\ &+\frac{1}{4}\sinh^4\kappa t\coth\frac{\beta}{2}\left(1+\sin^2 2\omega_S t\right)+\frac{1}{2}\sinh^2\kappa t\sin 2\omega_S t\coth\frac{\beta}{2}\\ &+\frac{1}{8}\sinh^2 2\kappa t\coth\frac{\beta}{2}\left(\frac{1}{2}+\cos^2\omega_S t\right)\\ &-\frac{1}{2}\cosh^2\kappa t\cos\omega_S t\left(\cos\omega_S t-\sin\omega_S t\right)-\frac{1}{4}\,. \end{aligned} \quad (19)$$

5 Conclusion

We have shown that in the framework of a simple quadratic model there exist new time-dependent integrals of motion for the stimulated Raman scattering process. The linear (in photon and phonon quadratures) integrals of motion describe the initial values of the mean quadratures of the system trajectory in the phase space. The quadratic (in photon and phonon creation and annihilation operators) integrals of motion describe the initial numbers of photons and phonons in the system state.

Another result of our work is the calculated photon distribution function, which may be expressed either in terms of Hermite polynomials of two variables or in terms of Legendre polynomials.

The dependence of the photon number distribution function on the parameters of the laser field and the coupling constant shows the possibility of processing the statistics of stimulated Raman scattering by varying the laser field and medium parameters (for example, temperature). So, the stimulated Raman scattering can be used for production of nonclassical light. An analogous method of investigation can be applied for studying the stimulated Brillouin scattering.

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Appendix

Here we calculate the Gaussian integral used in Section 4. We consider the integral in (12)

$$\int \exp\left[-\mathbf{X}A\mathbf{X}\right] dy_1 \cdots dy_m ,$$

where

$$\mathbf{X} = (\mathbf{x}, \mathbf{y}) = \begin{pmatrix} x_1 \\ \vdots \\ x_n \\ y_1 \\ \vdots \\ y_m \end{pmatrix}$$

and the matrix A has four blocks

$$A \doteq \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

We introduce the notation

$$\mathbf{X}A\mathbf{X} = \sum_{k=1}^{n}\sum_{l=1}^{n} x_k a_{kl} x_l + \sum_{s=1}^{m}\sum_{p=1}^{m} y_s d_{sp} y_p + \sum_{k=1}^{n}\sum_{s=1}^{m} x_k \left(b + c^T\right)_{ks} y_s .$$

The matrix c^T is the transposed matrix c.

Using the general formula for N-dimensional Gaussian integral (see, for example, [19])

$$\int \cdots \int \exp\left(-\mathbf{Z}M\mathbf{Z} + \mathbf{K}\mathbf{Z}\right) dz_1 \cdots dz_N = \frac{\left(\sqrt{\pi}\right)^N}{\sqrt{\det M}} \exp\left(\frac{1}{4}\mathbf{K}M^{-1}\mathbf{K}\right),$$

we get

$$\int \exp\left(-\mathbf{X}A\mathbf{X}\right) dy = \frac{\left(\sqrt{\pi}\right)^m}{\sqrt{\det d}} \exp\left(-\mathbf{x}g\mathbf{x}\right), \tag{20}$$

where the matrix g is

$$g = a - \frac{1}{4}\left(c^T + b\right) d^{-1} \left(b^T + c\right)$$

and

$$dy = dy_1 \cdots dy_m.$$

Using the result obtained one can formulate the following theorem.

Given an arbitrary quantum state of a composed system with a Gaussian Wigner function $W\left(\mathbf{Q}_1, \mathbf{Q}_2\right)$ depending on the set of $2N$ phase-space variables $\mathbf{Q}_1$ describing the first subsystem and on the set of $2M$ phase-space variables $\mathbf{Q}_2$ describing the second subsystem. Then the Wigner function of the first subsystem

$$W_1\left(\mathbf{Q}_1\right) = \langle W\left(\mathbf{Q}_1, \mathbf{Q}_2\right)\rangle_2,$$

which is the given Wigner function of the composed system averaged over variables of the second subsystem $\mathbf{Q}_2$, has the Gaussian form.

The statement takes place for both finite and infinite numbers of the degrees of freedom N and M of the first and second subsystems.

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