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(4,0) SUPERSYMMETRIC SIGMA-MODEL AND T-DUALITY

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ABSTRACT

The conserved supercurrents J^{++} and J_{--} are deduced for the (4,0) supersymmetric sigma model on harmonic superspace with arbitrary background gauge connection. These are introduced in the Lagrangian density of the model by their couplings to the analytic gauge superfields Γ_{--} and Γ^{++}. The T-duality transformations are obtained by integrating out the analytic gauge superfields. Finally the (4,0) supersymmetric anomaly is derived.

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1 Introduction

Various target space T-duality transformations [1] connecting two apparently different sigma-models or string backgrounds are playing important roles nowadays. These are important examples of string symmetries showing the equivalence of strings propagating on background space-times with different geometry. The duality transformations were originally formulated in the σ-model description of the conformal field theory underlying string theory [1]. In some recent papers [2] it was argued that from the point of view of the low energy effective action, the original and T-dual theories do not share the same number of space-time supersymmetries. Indeed, this is not suitable since duality transformation is a canonical transformation [3] and the symmetries of the original theory should be preserved although they may become non-local [4]. On the other hand, there are general theorems [5] relating space-time supersymmetries to symmetries on the world-sheet and if the manipulations involved in carrying out a T-duality transformation preserve them, then space-time supersymmetry is expected to be maintained.

However, duality transformation for a general (1,0) heterotic sigma-model [6] has been worked out in Ref. [7] with arbitrary connection and background gauge field. They found that in order to not have a world-sheet non-local T-dual action due to anomalies which appear when duality transformations are implemented, the right moving fermions must be transformed under the isometry. This yields to a non trivial transformation of the background gauge field under T-duality.

In the present work we will discuss the T-duality transformation for the (4,0) supersymmetric sigma-model [8], with arbitrary background gauge connection, on the harmonic superspace by following the procedure of Ref.[7]. These models are ultra-violet finite [9] and arise naturally in heterotic string compactifications. The T-dual theories of (4,0) supersymmetric sigma-models are expected to also be ultra-violet finite. On the other hand, it is known that the target space of (4,0) supersymmetric sigma models admits three complex structures that obey the algebra of imaginary unit quaternions. When the torsion vanishes the target space is hyper-Kähler with respect to the metric. In the presence of torsion, the geometry of the target space of (4,0) supersymmetric sigma models may not be hyper-Kähler and new geometry arises.

The outline of this paper is as follows. In Section 2, we recall the basic concepts of the construction of (4,0) supersymmetric model in harmonic superspace. In Section 3, we present the (4,0) supersymmetric sigma model in harmonic superspace with arbitrary background gauge connection. In Section 4, we work out the duality transformation for the (4,0) supersymmetric sigma-model on harmonic superspace. The conserved (4,0) supercurrents are derived and introduced in the Lagrangian density via their couplings to the (4,0) analytic gauge superfields Γ^{++} and Γ_{--}. In Section 5 we obtain the manifest (4,0) supersymmetric anomaly and its cancellation leads to assume that the spin and gauge connections match in the original theory. Finally, in Section 6 we present our conclusion and in the Appendix we give some general formulas concerning the action of the isometries on forms on the supermanifold.

2 Two-Dimensional (4,0) Supersymmetric Model

In this section we recall the basic concepts of the construction of two-dimensional (4,0) supersymmetric model in harmonic superspace [8]. In (1 + 1) dimensions, it is useful to choose the light cone coordinates σ^{++} and σ^{--} where $\sigma^{++} = \frac{1}{\sqrt{2}}(\tau + \sigma)$ and $\sigma^{--} = \frac{1}{\sqrt{2}}(\tau - \sigma)$. In this basis, two-dimensional vectors V_n and Majorana spinors S_α; $\alpha = +, -$; are reducible under Lorentz group transformations which is a remarkable feature. Indeed, each component $\theta^i_\pm$; $i = 1, \ldots, N$; of a Majorana spinor θ_α is real and one learns that two-dimensional (4,0) supersymmetric theories are basically real. The large real automorphism group is $SO(4) \simeq SU(2) \otimes SU(2)$ and the associated (4,0) superspace in the central basis is

$$(\sigma^{++}, \sigma^{--}, \theta^i_-) \tag{2.1}$$

The idea of projecting symmetries has been fruitful in the formulation of the $N = 2$, $D = 4$ off-shell supersymmetic theories which preserve manifestly the SU(2) automorphism group [10]. The harmonic variables are used to convert any SO(4) vector θ^i in a $SU(2) \otimes U(1)$ objects $\theta^{\pm\alpha}$ as follows:

$$\theta^{\pm\beta}_- = \theta^{\alpha\beta}_- U^\pm_\alpha \tag{2.2}$$

and the two-dimensional (4,0) harmonic superspace is $(\sigma^{\pm\pm}, \theta^{\pm\beta}_-, U^\pm_\alpha)$. It was realized that the following subspace $(\sigma^{--}, \sigma^{++} + \theta^+_-\theta^-_-, \theta^+_-, U^\pm)$ is closed under (4,0) supersymmetric transformations [8]. Such subspace is called analytic (chiral) subspace and is stable under the combined conjugation ($\bar{*}$) where — is the complex conjugation and $*$ the Cartan-Weyl charge conjugation [8].

We know that the two-dimensional (4,0) supersymmetric model is described, in a first step, in terms of the left superfield Ω which is real, analytic, Cartan-Weyl and Lorentz scalar as follows [8]:

$$\mathcal{L}^{++}_{--,LM} = iD^{++}\Omega\partial_{--}\Omega \tag{2.3}$$

Its θ^+_- expansion reads as:

$$\Omega = \omega + \sqrt{2}\theta^+_-\zeta^-_+ + \theta^+_-\theta^+_- f^{--}_{++} \tag{2.4}$$

and satisfy the following consistency condition

$$D^{++2}\Omega = 0 \ , \tag{2.5}$$

with

$$D^{++} = \partial^{++} + i\theta^+_-\theta^+_-\partial_{++} \tag{2.6}$$

$$\partial^{++} = U^{+\alpha}\frac{\partial}{\partial U^{-\alpha}}$$

the harmonic derivative, which means that all the components of Ω are auxiliary, except for the lowest component in the harmonic expansion of $\omega(\sigma, u)$ and $\zeta^-_+(\sigma, u)$ fields. In fact the consistency condition (2.5) is equivalent to:

$$\partial^{++2}\omega = 0$$

$$\partial^{++2}\zeta^-_+ = 0 \tag{2.7}$$

$$\partial^{++} f^{--}_{++} = -2i\partial_{++}\omega$$

which imply that

$$\omega = \frac{1}{\sqrt{3}}\omega^0 + \frac{1}{\sqrt{6}}\omega^{(\alpha\beta)}U^+_\alpha U^-_\beta \tag{2.8}$$

$$\zeta^-_+ = \zeta^\alpha_+ U^-_\alpha$$

and

$$\partial^{++}\zeta^-_+ = \zeta^+_+$$

The (4,0) supersymmetric action for the left multiplet Ω is given from (2.3) after integrating with respect to θ^+_- by:

$$I^{(4,0)}_{LM} = i\int d^2\sigma du \left\{2(\partial_{--}\omega)(\partial^{++} f^{--}_{++}) + 2\zeta^+_+\partial_{--}\zeta^-_+ + i(\partial_{++}\omega)(\partial_{--}\omega)\right\} \tag{2.9}$$

By replacing the f^{--}_{++} field in terms of ω and by integrating with respect to the harmonic variables using the techniques of Ref. [10] we obtain:

$$I^{(4,0)}_{LM} = \int d^2\sigma \left(\partial_{++}\omega^0)(\partial_{--}\omega^0) + \frac{1}{2}(\partial_{++}\omega^{(\alpha\beta)})(\partial_{--}\omega_{(\alpha\beta)}) + 2i\zeta^\alpha_+\partial_{--}\zeta_{+\alpha}\right\} \tag{2.10}$$

Such action is equivalent to the action of the (4,0) SM-II (scalar multiplet two) theory given by S.J. Gates and L. Rana [11] in order to discuss the ADHM instanton construction but without using the harmonic variables.

In a second step, the (4,0) supersymmetric model is also described by the right (4,0) multiplet $\psi_-(\sigma, \theta_-^+, u)$ which is a real minus spinor with a vanishing Cartan Weyl charge. For the same reasons ψ_- satisfies the consistency condition:

$$D^{++2}\psi_- = 0 \tag{2.11}$$

Its θ_-^+ expansion is given by

$$\psi_- = \lambda_- + \theta_-^+\chi^- + \theta_-^+\theta_-^+\eta_+^{--} \tag{2.12}$$

and the consistency condition is equivalent to

$$\begin{aligned} \partial^{++2}\lambda_- &= 0 \\ \partial^{++2}\chi^- &= 0 \\ \partial^{++2}\eta_+^{--} &= 2i\partial^{++}\partial_{++}\lambda_- \end{aligned} \tag{2.13}$$

which imply

$$\begin{aligned} \lambda_- &= \tfrac{1}{\sqrt{2}}\lambda_-^0 + \lambda_-^{(\alpha\beta)}U_\alpha^+U_\beta^- \\ \chi^- &= \chi^\alpha U_\alpha^- \\ \partial^{++}\chi^- &= \chi^+ \end{aligned} \tag{2.14}$$

The corresponding (4,0) supersymmetric Lagrangian density is given by [8]:

$$\mathcal{L}^{++}_{--,RM} = \psi_- D^{++}\psi_- \tag{2.15}$$

and its integration with respect to the θ_-^+ variables leads to the following (4,0) supersymmetric action namely:

$$I_{RM}^{(4,0)} = \int d^2\sigma dU \left\{ i\lambda_-\partial_{++}\lambda_- + \chi^-\chi^+ \right\} \tag{2.16}$$

Finally the integration of the harmonic variables allows to obtain

$$I_{RM}^{(4,0)} = \int d^2\sigma \left\{ \frac{i}{2}\lambda_-^0\partial_{++}\lambda_-^0 + i\lambda_-^{(\alpha\beta)}\partial_{++}\lambda_{-(\alpha\beta)} + \chi^\alpha\chi_\alpha \right\} \tag{2.17}$$

which is equivalent also to the action of the (4,0) SM IV (scalar multiplet four) theory given in Ref. [11]. The main purpose of the remaining sections is to exploit the power of these representations in order to study the (4,0) supersymmetric sigma model and its associated (4,0) T-duality.

3 The (4,0)-Supersymmetric Sigma Model

We consider a supermanifold $\mathcal{M}$ with metric G_{ab} $a = 1, \ldots, D$, antisymmetric tensor B_{ab} determining a supersymmetric generalized Wess-Zumino term [12] and a background gauge connection $V_a{}^A{}_B$ associated to a gauge group G. The fields ζ_+^{-a} and λ_-^A have opposite world-sheet chirality and ω^a are the fields embedding the world-sheet in the target space $\mathcal{M}$. The lagrangian density of the (4,0) supersymmetric sigma model with target space $\mathcal{M}$ is then given by:

$$L = \int d^2\theta_-^+ \left\{ i(G_{ab} + B_{ab}) D^{++}\Omega^a \partial_{--}\Omega^b + \delta_{AB}\psi_-^A \mathcal{D}^{++}\psi_-^B \right\} \tag{3.1}$$

with

$$\mathcal{D}^{++}\psi_-^A = D^{++}\psi_-^A + V_{aB}^A(\Omega) D^{++}\Omega^a \psi_-^B \tag{3.2}$$

The fermionic superfields ψ_-^A, $A = 1, \ldots, n$, take values in some vector bundle over the sigma model supermanifold $\mathcal{M}$ with n-dimensional fibres. The metric on the fibres used to raise and lower indices $A, B, \ldots$ is taken to be δ_{AB} [6]. $V_B^A(\Omega)$ is a connection on the vector bundle so that the fermionic action

$$I_F^{(4,0)} = \int d^2\sigma dU d^2\theta_-^+ \delta_{AB}\psi_-^A \mathcal{D}^{++}\psi_-^B \tag{3.3}$$

is invariant under local rotations

$$\delta\psi_-^A = \Lambda_B^A(\Omega)\psi_-^B \ , \tag{3.4}$$

where Λ_B^A is a generator of the structure group G of the vector bundle. As only the antisymmetric part of V_{AB} appears in (3.3) we take

$$V_{AB} = -V_{BA}$$

without loss of generality.

The expansion of the (4,0) supersymmetric Lagrangian (3.1) in terms of components is given by

$$\begin{aligned} L \;=\; & (G_{ab} + B_{ab})\partial_{++}\omega^a \partial_{--}\omega^b + 4iG_{ab}\zeta_+^{+a}\mathcal{D}_{--}\zeta_+^{-b} \\ & + i\lambda_-^A \mathcal{D}_{++}\lambda_{-A} + \chi^{-A}[\chi_A^+ + 2\omega^a V_{a,CA}\chi^{+C}] \\ & + 2F_{ab,CA}\lambda_-^A \zeta_+^{-b}\zeta_+^{+a}\lambda_-^C \end{aligned} \tag{3.5}$$

where the covariant derivatives $\mathcal{D}_{--}$ and $\mathcal{D}_{++}$ are given by

$$\mathcal{D}_{++}\lambda_-^A = \partial_{++}\lambda_-^A - 2V_{aC}^A \partial_{++}\omega^a \lambda_-^C$$

$$\mathcal{D}_{--}\zeta_{+}^{-b} = \partial_{--}\zeta_{+}^{-b} + (\Gamma^{b}_{de} + H^{b}_{de})\partial_{--}\omega^{d}\zeta_{+}^{-e} \tag{3.6}$$

and

$$F_{ab} = \partial_a V_b - \partial_b V_a + [V_a, V_b] \tag{3.7}$$

Γ^{b}_{de} is the usual christoffel connection and H^{b}_{de} is a totally antisymmetric torsion. In this expansion we have used the fact that

$$D^{++}\Omega^{a} = \partial^{++}\omega^{a} + \sqrt{2}\theta^{+}_{-}\partial^{++}\zeta_{+}^{-a} + \theta^{+}_{-}\theta^{+}_{-}(\partial^{++}f_{++}^{--a} + i\partial_{++}\omega^{a})$$

and the equations (2.7) and (2.13) which imply that

$$\eta_{+}^{--} = \frac{i}{2}\partial_{++}\partial^{--}\lambda_{-}$$

$$f_{++}^{--} = \frac{-i}{2}\partial_{++}\partial^{--}\omega$$

since $\partial^{--} = U^{-\alpha}\frac{\partial}{\partial U^{+\alpha}}$ and

$$[\partial^{++}, \partial^{--}] = 2\partial^{0} = 2\left(U^{+\alpha}\frac{\partial}{\partial U^{+\alpha}} - U^{-\alpha}\frac{\partial}{\partial U^{-\alpha}}\right)$$

The non linear sigma model with manifest (4,0) supersymmetry which we have constructed on the harmonic superspace generalize the (1,0) and (2,0) supersymmetric sigma models given by C.M. Hull and E. Witten [6] and where they discuss their geometry and their relevance to compactifications of the heterotic superstring. In the next section we will analyze T-duality for the (4,0) supersymmetric sigma model by using the approach of Ref. [7].

4 T-Duality and (4,0) Sigma Model

Following the procedure outlined in refs. [7,13], in order to carry out a duality transformation in (3.1) we have to assume, in a manifestly (4,0) supersymmetric way, that the metric has an isometry under which (3.1) and (3.5) are invariant. In fact, the first term of the lagrangian density (3.1) is invariant under the isometry [14]

$$\delta\Omega^{a} = \varepsilon K^{a} \ , \tag{4.1}$$

where ε are infinitesimal constant parameters, if the set of vector superfield K^a satisfy the Killing equations

$$K^{c}\partial_{c}G_{ab} + G_{cb}\partial_{a}K^{c} + G_{ac}\partial_{b}K^{c} = 0 \tag{4.2}$$

and

$$K^c \partial_c B_{ab} + B_{cb} \partial_a K^c + B_{ac} \partial_b K^c = 0 \tag{4.3}$$

The constraint (4.3) allows to obtain the following equation

$$K^b H_{bde} = \partial_d (B_{eb} K^b) - \partial_e (B_{db} K^b) \tag{4.4}$$

which implies that $K^b H_{bde}$ is an exact two-form (or closed), so that there is a locally defined one-form v_a such that

$$K^b H_{bde} = \partial_d v_e - \partial_e v_d \tag{4.5}$$

with

$$v_a = K^b B_{ab} \tag{4.6}$$

K^a are Killing vectors and generate a subgroup of the isometry group of $\mathcal{M}$. Since the isometries act on forms on the manifold $\mathcal{M}$ through the lie derivative (see Appendix) H is lie invariant and the variation of B_{ab} is as follows

$$\delta_K B_{ab} = \partial_a \left[K^c B_{cb} + v_b \right] - \partial_b \left[K^c B_{ca} \right] \tag{4.7}$$

The conserved (4,0) supercurrent for the first term of (3.1) associated with the isometry (4.1) is obtained by using the Noether theorem. In fact the variation of the first term $\mathcal{L}_B$ of the lagrangian (3.1) is given by

$$\delta \mathcal{L}_B = 0 = \frac{\partial \mathcal{L}_B}{\partial \Omega^a} \delta \Omega^a + \frac{\partial \mathcal{L}_B}{\partial (\partial_{--} \Omega^a)} \delta (\partial_{--} \Omega^a) + \frac{\partial \mathcal{L}_B}{\partial (D^{++} \Omega^a)} \delta (D^{++} \Omega^a)$$

The use of the Euler-Lagrange equation leads to

$$\partial_{--} J^{++} + D^{++} J_{--} = 0 \tag{4.8}$$

with

$$J^{++} = i K^a (G_{ab} - B_{ab}) D^{++} \Omega^b \tag{4.9}$$

and

$$J_{--} = i K^a (G_{ab} + B_{ab}) \partial_{--} \Omega^b \tag{4.10}$$

The expression (4.6) and the fact that

$$K^a G_{ab} = K_b$$

allow to rewrite the supercurrents (4.9) and (4.10) as:

$$J^{++} = i (K_b + v_b) D^{++} \Omega^b$$

$$J_{--} = i(K_b - v_b)\partial_{--}\Omega^b \tag{4.11}$$

We remark that J^{++} and J_{--} are not conserved separately as in the (1,0) supersymmetric case [7]. Furthermore, to obtain the dual (4,0) supersymmetric model we introduce (4,0) analytic gauge superfields Γ^{++} and Γ_{--} which can be expanded in terms of θ^+_- as follows:

$$\begin{aligned}\Gamma^{++} &= U^{++} + \theta^+_- f^+_+ + \theta^+_-\theta^+_- v_{++}\\ \Gamma_{--} &= h_{--} + \theta^+_- g^-_- + \theta^+_-\theta^+_- q^{--}\end{aligned} \tag{4.12}$$

Then we consider the following term

$$\mathcal{L}_B = i(G_{ab} + B_{ab})D^{++}\Omega^a\partial_{--}\Omega^b + J^{++}\Gamma_{--} + J_{--}\Gamma^{++} \tag{4.13}$$

and we will see if such term is gauge invariant with respect to the local isometry gauge transformation namely:

$$\delta_\varepsilon\Omega^a(\sigma,\theta^\pm,U) = \varepsilon(\sigma,\theta^\pm,U)K^a(\Omega) \tag{4.14}$$

where $\varepsilon(\sigma,\theta^\pm,U)$ is the gauge superparameter. In fact the variations of the supercurrent J^{++} and J_{--} with respect to (4.14) are given by

$$\begin{aligned}\delta J^{++} &= i\varepsilon\left[K^a\partial_c G_{ab} + G_{ab}\partial_c K^a + B_{ba}\partial_c K^a + K^a\partial_c B_{ba}\right]K^c D^{++}\Omega^b\\ &\quad + i(K_b + v_b)K^b D^{++}\varepsilon + i(K_b + v_b)\varepsilon\partial_c K^b D^{++}\Omega^c\end{aligned} \tag{4.15.1}$$

and

$$\begin{aligned}\delta J_{--} &= i\varepsilon\left[G_{ab}\partial_c K^a + K^a\partial_c G_{ab} - \partial_c K^a B_{ba} - K^a\partial_c B_{ba}\right]K^c\partial_{--}\Omega^b\\ &\quad + i(K_b - v_b)\varepsilon\partial_c K^b\partial_{--}\Omega^c + i(K_b - v_b)K^b\partial_{--}\varepsilon\end{aligned} \tag{4.15.2}$$

Consequently, by using the Killing equations (4.2) and (4.3) the variation of the Lagrangian density (4.13) with respect to (4.14) is given by

$$\begin{aligned}\delta\mathcal{L}_B &= J_{--}[D^{++}\varepsilon + \delta\Gamma^{++}] + J^{++}[\partial_{--}\varepsilon + \delta\Gamma_{--}]\\ &\quad + i(K_b + v_b)K^b D^{++}\varepsilon\Gamma_{--} + i(K_b - v_b)K^b\partial_{--}\varepsilon\Gamma^{++}\end{aligned} \tag{4.16}$$

We remark, as for the (1,0) supersymmetric case [7], that full gauge invariance can be achieved if we add the following term

$$iK^2\Gamma^{++}\Gamma_{--} \tag{4.17}$$

The variation of such term is given by

$$\begin{aligned}\delta(iK^2\Gamma^{++}\Gamma_{--}) &= iK^2\delta\Gamma^{++}\Gamma_{--} + iK^2\Gamma^{++}\delta\Gamma_{--} +\\ &\quad iK_a\delta K^a\Gamma^{++}\Gamma_{--} + i\delta K_a K^a\Gamma^{++}\Gamma_{--}\end{aligned} \tag{4.18}$$

In fact

$$K_a\delta K^a + \delta K_a K^a = \varepsilon[K_a K^c\partial_c K^a + K^a K^c\partial_c K_a]$$

with

$$K_a = G_{ab}K^b$$

and the Killing equation we show that

$$K_a\delta K^a + \delta K_a K^a = K^a K^b[K^c\partial_c G_{ab} + G_{cb}\partial_a K^c + G_{ac}\partial_b K^c] = 0$$

Consequently

$$\delta(iK^2\Gamma^{++}\Gamma_{--}) = iK^2\Gamma_{--}\delta\Gamma^{++} + iK^2\Gamma^{++}\delta\Gamma_{--}$$

Then if we take

$$\begin{aligned} \delta\Gamma^{++} &= -D^{++}\varepsilon \\ \delta\Gamma_{--} &= -\partial_{--}\varepsilon \end{aligned} \qquad (4.19)$$

we obtain

$$\delta(iK^2\Gamma^{++}\Gamma_{--}) = -iK^2\Gamma_{--}D^{++}\varepsilon - iK^2\Gamma^{++}\partial_{--}\varepsilon$$

and the variation of the Lagrangian density (4.16) after adding (4.17) becomes

$$\delta\mathcal{L}_B = iv_b K^b[\Gamma_{--}D^{++}\varepsilon - \Gamma^{++}\partial_{--}\varepsilon]$$

we have

$$v_b = K^a B_{ba}$$

which implies that

$$v_b K^b = 0 \qquad (4.20)$$

Therefrom, the bosonic (4,0) supersymmetric Lagrangian density which is gauge invariant is

$$\begin{aligned} \mathcal{L}_B^{inv} &= i(G_{ab} + B_{ab})D^{++}\Omega^a\partial_{--}\Omega^b + J^{++}\Gamma_{--} + \\ &\quad J_{--}\Gamma^{++} + iK^2\Gamma^{++}\Gamma_{--} \end{aligned} \qquad (4.21)$$

Concerning the (4,0) supersymmetric Lagrangian density of the spinorial multiplet namely:

$$\mathcal{L}_F = \psi_-[D^{++} + V_a D^{++}\Omega^a]\psi_- \qquad (4.22)$$

the invariance under the global transformation (4.14) is manifest when the isometry variation can be compensated by a gauge transformation. In fact if we take

$$\delta\psi_- = -\varepsilon k\psi_-$$

$$\delta\Omega^a = \varepsilon K^a(\Omega) \; , \tag{4.23}$$

then the variation of the Lagrangian density (4.22) is given by

$$\delta\mathcal{L}_F = -\psi_-\varepsilon \left\{\partial_a k + [V_a, k] - K^b\partial_b V_a - V_b\partial_a K^b\right\} D^{++}\Omega^a\psi_- \tag{4.24}$$

where we have used the fact that

$$\delta V_a = \partial_b V_a \delta\Omega^b = \varepsilon\partial_b V_a K^b$$

$$D^{++}k = \partial_b k D^{++}\Omega^b \tag{4.25}$$

From (4.24) we note that the invariance of $\mathcal{L}_F$ under (4.23) leads to the following constraint

$$\partial_a k + [V_a, k] - K^b\partial_b V_a - V_b\partial_a K^b = 0 \tag{4.26}$$

We set

$$\delta_k V_a = \partial_a k + [V_a, k] \equiv \mathcal{D}_a k \tag{4 27}$$

Furthermore, if we define

$$\mu = k - K^a V_a \quad . \tag{4.28}$$

Its covariant derivative is given by

$$\begin{aligned} \mathcal{D}_b\mu \;&=\; \partial_b k + [V_b, k] - V_a\partial_b K^a - K^a\partial_a V_b + \\ & \quad K^a\left\{\partial_a V_b - \partial_b V_a + [V_a, V_b]\right\} \end{aligned}$$

By using the constraint (4.26) we obtain

$$\mathcal{D}_b\mu = K^a F_{ab} \tag{4.29}$$

with

$$F_{ab} = \partial_a V_b - \partial_b V_a + [V_a, V_b] \tag{4.30}$$

Now we make ε a function of (4,0) harmonic superspace coordinates, and after some algebra we obtain

$$\delta[\psi_-\mathcal{D}^{++}\psi_-] = -\mathcal{D}^{++}\varepsilon\psi_-\mu\psi_- \tag{4.31}$$

In order to achieve gauge invariance of (4.22) we must add the coupling

$$\Gamma^{++}\psi_-\mu\psi_- \tag{4.32}$$

because $\psi_-\mu\psi_-$ is gauge invariant. In fact

$$\delta(\psi_-\mu\psi_-) = \varepsilon\psi_-\left\{K^b\partial_b\mu + [k, \mu]\right\}$$

since

$$K^b K^a F_{ab} = 0$$

which implies that

$$K^b \partial_b \mu + [k, \mu] = 0 \tag{4.33}$$

Thereby $\delta(\psi_- \mu \psi_-) = 0$ and

$$\delta[\Gamma^{++} \psi_- \mu \psi_-] = -D^{++} \epsilon \psi_- \mu \psi_- \tag{4.34}$$

Finally the full gauge invariant Lagrangian density is

$$\begin{aligned} \mathcal{L} &= i(G_{ab} + B_{ab}) D^{++} \Omega^a \partial_{--} \Omega^b + J^{++} \Gamma_{--} + J_{--} \Gamma^{++} + \\ & iK^2 \Gamma^{++} \Gamma_{--} + \psi_- \mathcal{D}^{++} \psi_- - \Gamma^{++} \psi_- \mu \psi_- \end{aligned} \tag{4.35}$$

Following the procedure outlined in [7,13] to obtain the (4,0) supersymmetric dual model we add the Lagrange multiplier term namely

$$i\Lambda(D^{++}\Gamma_{--} - \partial_{--}\Gamma^{++}) \tag{4.36}$$

and if we integrate over Λ, leading to the following constraint

$$D^{++}\Gamma_{--} - \partial_{--}\Gamma^{++} = 0 \tag{4.37}$$

which implies that

$$\begin{aligned} \Gamma_{--} &= \partial_{--} \alpha \\ \Gamma^{++} &= D^{++} \alpha \end{aligned} , \tag{4.38}$$

and using the invariance of (4.35) we obtain the original theory. The constraint (4.37) means that the corresponding superfield strength is vanishing [15]. Classical duality is obtained by integrating out the gauge superfields Γ^{++}, Γ_{--} and by using their equations of motion we obtain the (4,0) supersymmetric dual lagrangian density in harmonic superspace. In fact

$$\begin{aligned} \frac{\partial \mathcal{L}_T}{\partial \Gamma_{--}} &= J^{++} + iK^2 \Gamma^{++} - iD^{++}\Lambda = 0 \\ \frac{\partial \mathcal{L}_T}{\partial \Gamma^{++}} &= J_{--} + iK^2 \Gamma_{--} - \psi_- \mu \psi_- + i\partial_{--}\Lambda = 0 \end{aligned} \tag{4.39}$$

with

$$\mathcal{L}_T = \mathcal{L} + i\Lambda(D^{++}\Gamma_{--} - \partial_{--}\Gamma^{++}) \tag{4.40}$$

The equations (4.39) allow to determine the gauge superfields

$$\Gamma^{++} = \frac{-i}{K^2}(iD^{++}\Lambda - J^{++})$$

$$\Gamma_{--} = \frac{-i}{K^2}(-i\partial_{--}\Lambda - J_{--} + \psi_- \mu \psi_-) \qquad (4.41)$$

If we replace (4.41) in the Lagrangian density (4.40) and if we use the expressions (4.11) of the supercurrents J^{++} and J_{--} we obtain the (4,0) supersymmetric dual model namely:

$$\begin{aligned}
\tilde{\mathcal{L}}_T \;=\;& i(G_{ij} + B_{ij})D^{++}\Omega^i\partial_{--}\Omega^j - \tfrac{i}{K^2}(K_i + v_i)(K_j - v_j)D^{++}\Omega^i\partial_{--}\Omega^j \\
& -\tfrac{i}{K^2}(K_i + v_i)D^{++}\Omega^i\partial_{--}\Lambda + \tfrac{i}{K^2}(K_i - v_i)\partial_{--}\Omega^i D^{++}\Lambda + \\
& +\tfrac{i}{K^2}(\partial_{--}\Lambda)(D^{++}\Lambda) - \tfrac{1}{K^2}(K_i + v_i)D^{++}\Omega^i\psi_-\mu\psi_- + \psi_- D^{++}\psi_- \\
& +\psi_- V_i D^{++}\Omega^i\psi_- - \tfrac{1}{K^2}(D^{++}\Lambda)\psi_-\mu\psi_-
\end{aligned} \qquad (4.42)$$

where we have used the locally gauge choice $\Omega^0 = 0$. If we set $\Lambda = \tilde{\Theta}$ the (4,0) supersymmetric dual model can be rewritten in terms of $\tilde{G}$, $\tilde{B}$ and $\tilde{V}$ as:

$$\begin{aligned}
\tilde{\mathcal{L}}_T \;=\;& i(\tilde{G}_{ab} + \tilde{B}_{ab})D^{++}\tilde{\Omega}^a\partial_{--}\tilde{\Omega}^b + \\
& \psi_-[D^{++} + \tilde{V}_a D^{++}\tilde{\Omega}^a]\psi_-
\end{aligned} \qquad (4.43)$$

where

$$\begin{aligned}
\tilde{G}_{00} &= \tfrac{1}{K^2}, \quad \tilde{G}_{0i} = -\tfrac{v_i}{K^2}, \quad \tilde{B}_{0i} = -\tfrac{K_i}{K^2} \\
\tilde{G}_{ij} &= G_{ij} - \tfrac{1}{K^2}(K_iK_j - v_iv_j) \\
\tilde{B}_{ij} &= B_{ij} - \tfrac{1}{K^2}(v_iK_j - K_iv_j) \\
\tilde{V}_0 &= -\tfrac{\mu}{K^2} \\
\tilde{V}_i &= V_i + \tfrac{1}{K^2}(K_i + v_i)\mu
\end{aligned} \qquad (4.44)$$

and $\tilde{\Omega}^a = (\tilde{\Theta}, \Omega^i)$. We note that the expressions (4.44) are the generalization of the (1,0) expressions [7] to the (4,0) supersymmetric case and can also be equivalent to Buscher's formulae [16], since in adapted coordinates we can choose $v_i = -B_{0i}$, with a non trivial transformation of the background gauge connection under T-duality. The expressions (4.44) are obtained by using only classical manipulations. In general, as for the (1,0) supersymmetric case [7], there will be anomalies and the (4,0) supersymmetric dual action may not have the same properties as the original one. This will be the subject of the next section.

5 (4,0) Supersymmetric Anomaly

Depending on the gauge group and the choice for μ (4.28) the dual theory (4.43) may be afflicted with anomalies. In this case the models (4.43) and (4.35) are not equivalent. Following the procedure outlined in Ref. [7] equivalence may exist if we include Wess–Zumino terms [17] generated by the quantum measure. Furthermore, if we want the local

(4,0) supersymmetric lagrangian densities (4.43) and (4.35) to be equivalent we must find conditions on G, B, V, μ in order to cancel the anomalies. As for the (1,0) supersymmetric case [7], in order to understand the origin of the anomalies, we first take the simpler case where $B_{ab} = 0$ and we ignore manifest (4,0) supersymmetry by considering the kinetic terms for fermions. The generalization to the (4,0) supersymmetric case can be deduced immediately. In fact, the kinetic term for the left spinorial field ζ_+^{-a} is given [see (3.5) and (3.6)] by

$$4iG_{ab}\zeta_+^{+a}\mathcal{D}_{--}\zeta_+^{-b} \tag{5.1}$$

with the covariant derivative $\mathcal{D}_{--}$ completed by

$$\mathcal{D}_{--}\zeta_+^{-b} = [\partial_{--} + \Gamma^b_{de}\partial_{--}\omega^d - \Omega^b_e h_{--}]\zeta_+^{-e} \tag{5.2}$$

where

$$\Omega_{be} = \frac{1}{2}(\nabla_b K_a - \nabla_a K_b) \tag{5.3}$$

For the right spinorial field λ_- the quadratic term is given by

$$\lambda_-\mathcal{D}_{++}\lambda_- \tag{5.4}$$

where the covariant derivative $\mathcal{D}_{++}$ is modified by

$$\mathcal{D}_{++}\lambda_- = [\partial_{++} - 2V_a\partial_{++}\omega^a - v_{++}\mu]\lambda_- \tag{5.5}$$

If we define the derivative

$$\mathcal{D}_{++}\zeta_+^{-b} = \partial_{++}\zeta_+^{-b} + v_{++}\Omega^b_e\zeta^{-e} \tag{5.6}$$

and calculate the commutator of $\mathcal{D}_{++}$ and $\mathcal{D}_{--}$ we obtain

$$[\mathcal{D}_{++}, \mathcal{D}_{--}]^a{}_b = R^a_{bk\ell}\partial_{++}\omega^k\partial_{--}\omega^\ell - F_{++,--}\Omega^a{}_b \tag{5.7}$$

with

$$F_{++,--} = \partial_{++}h_{--} - \partial_{--}v_{++} \tag{5.8}$$

On the other hand, if we work in orthonormal frames:

$$\delta_{nm}E^n_a E^m_b = G_{ab} \tag{5.9}$$

the kinetic term of the spinorial field ζ_+^{-b} becomes

$$\delta_{nm}\zeta_+^{+n}\mathcal{D}_{--}\zeta_+^{-m} = \zeta^+_{+n}(\partial_{--}\delta^n_m + \sigma^n{}_{--m} - h_{--}\Omega^n_m)\zeta_+^{-m} \tag{5.10}$$

where

$$\sigma^n_{--m} = \sigma^n_{am}\partial_{--}\omega^a$$

$$\zeta_+^{\mp n} = E^n_a \zeta_+^{\mp a} \tag{5.11}$$

From (4.14) one finds that the variation of $\zeta_+^{\pm n}$ under the gauged isometry is given by

$$\delta\zeta_+^{\pm n} = -(K_L \zeta_+^{\pm})^n \tag{5.12}$$

with

$$K_L = \varepsilon(K^a\sigma_a + \Omega)$$

If we define the effective gauge field

$$V^n_{--m} = \sigma^n_{--m} - h_{--}\Omega^n_m \tag{5.13}$$

the kinetic term (5.10) is invariant under

$$\delta\zeta_+^{\pm} = -K_L\zeta_+^{\pm}$$

$$\delta V_{--} = \partial_{--}K_L + [V_{--}, K_L] \tag{5.14}$$

The effective action for the left spinorial field [18] is given by

$$\Gamma^L_{eff}[V_{--}] = -Ln\int[d\zeta]e^{-I[V_{--},\zeta]}$$

$$= Ln\det{}^{1/2}\mathcal{D}_{--} \tag{5.15}$$

where $I[V_{--}, \zeta]$ is the fermionic action corresponding to the Lagrangian density (5.10). Such effective action is anomalous under (5.12) and in terms of the Wess-Zumino-Witten Lagrangian for the field g defined by

$$V_{--} = g^{-1}\partial_{--}g$$

the variation of (5.15) under (5.12) is as follows:

$$\delta\Gamma^L_{eff}[V_{--}] = -\frac{1}{4\pi}\int T_r V_{--}\partial_{++}K_L \tag{5.16}$$

A similar computation can be carried out for the right spinorial field λ_- which leads to

$$\delta\Gamma^R_{eff}[V_{++}] = -\frac{1}{4\pi}\int T_r V_{++}\partial_{--}K_R \tag{5.17}$$

with

$$V_{++} = 2V_a\partial_{++}\omega^a + v_{++}\mu$$

$$K_R = \varepsilon(K^a V_a + k) \tag{5.18}$$

Adding the variations of the two effective actions (5.16) and (5.17) we arrive at the following result:

$$\delta\Gamma^{R+L}_{eff} = -\frac{1}{4\pi}\int d^2\sigma dU T_r\left\{(\sigma_a\partial_{--}\omega^a - h_{--}\Omega)\partial_{++}[\varepsilon(K^b\sigma_b + \Omega)]\right\}$$
$$-\frac{1}{4\pi}\int d^2\sigma dU T_r\left\{(2V_a\partial_{++}\omega^a + v_{++}\mu)\partial_{--}[\varepsilon(K^b V_b + k)]\right\} \tag{5.19}$$

Finally the (4,0) supersymmetric extension of the variation (5.19) with $B_{ab} \neq 0$ follows straightforwardly and is gien by

$$\delta\Gamma^{(4,0)}_{eff} = -\frac{1}{4\pi}\int d\mu^{--}_{++} T_r\left\{\sigma_a\partial_{--}\Omega^a - \Gamma_{--}\Omega\right\} D^{++}\left[\varepsilon(K^b\sigma_b + \Omega)\right]$$
$$-\frac{1}{4\pi}\int d\mu^{--}_{++} T_r\left\{2V_a D^{++}\Omega^a - \Gamma^{++}\Omega\right\}\partial_{--}\left[\varepsilon(K^b V_b + \mu)\right] \tag{5.20}$$

In the same way as the (1,0) supersymmetric case [7] we remark that the usual way to cancel the (4,0) supersymmetric anomaly (5.2) is to assume that the spin and gauge connections match in the original theory. However, for the consistency of the (4,0) supersymmetric dual theory we must also have the matching of the dual spinorial and gauge connections [19,7]. This is important for the consistency of the model with respect to global world-sheet and target-space anomalies [19] and it implies that if the original theory is superconformally invariant so is the dual theory. Furthermore, having determined the two-dimensional (4,0) supersymmetric dual σ-model in harmonic superspace, we can now clarify the interpretation of the generalized duality as a symmetry of the (4,0) superconformal field theory.

6 Conclusion

In this paper we presented the two-dimensional (4,0) supersymmetric model in harmonic superspace. This model is described in terms of the left Ω and right ψ_- multiplets and is equivalent to the (4,0) supersymmetric models given in Ref. [11]. Furthermore the Lagrangian density of the (4,0) supersymmetric sigma model with metric G_{ab} antisymmetric tensor B_{ab} and a background gauge connection V_a is given on the harmonic superspace. The metric G_{ab} has an isometry under which the (4,0) supersymmetric lagrangian density, corresponding to the sigma model with background gauge connection V_a, is invariant. The conserved (4,0) supercurrents for the model described in terms of the multiplet Ω are derived. In order to obtain the dual (4,0) supersymmetric model, analytic gauge superfields Γ^{++} and Γ_{--} are introduced via their coupling to the supercurrents J_{--} and J^{++}

respectively. Classical duality is obtained by integrating out the gauge superfields Γ^{++} and Γ_{--}. Therefore, the T-duality transformations are obtained and are equivalent to Buscher's formulae with a non trivial transformation of the background gauge connection. Moreover, in order to simplify we consider the particular case $B_{ab} = 0$ and we calculate the anomaly for the right and left spinorial fields ζ_+^- and λ_-. The generalisation of this to the (4,0) supersymmetric case is given. Finally, it will be of interest to generalize duality as a symmetry of the (4,0) superconformal field theory.

Appendix

The isometries act on forms T on the manifold $\mathcal{M}$ through the Lie derivative [20]

$$\delta T = \mathcal{L}_{\varepsilon^P K_p} T = \varepsilon^P \mathcal{L}_P T \qquad (A.1)$$

The Killing vectors K_p^a can be taken to generate some n-dimensional isometry group satisfying:

$$[K_p, K_q] = \mathcal{L}_p K_q = f_{pq}^m K_m \qquad (A.2)$$

where f_{pq}^m are the structure constants and $\mathcal{L}_p$ denotes the Lie derivative with respect to K_p. An important identity is

$$\mathcal{L}_p = i_p d + d i_p \qquad (A.3)$$

The map i_p from n-forms on $\mathcal{M}$ to lie algebra valued $(n-1)$ forms is given by

$$i_p T = \frac{1}{(n-1)!} K_p^a T_{ab_1 \ldots b_{n-1}} d\Omega^{b_1} \ldots d\Omega^{b_{n-1}} \qquad (A.4)$$

General p-rank antisymmetric tensor $T_{a_1 \ldots a_p}$ on $\mathcal{M}$ corresponds to the p-form

$$T = \frac{1}{p!} T_{a_1 \ldots a_p} d\Omega^{a_1} \ldots d\Omega^{a_p} \qquad (A.5)$$

The exterior derivative is given by

$$dT = \frac{1}{p!} \partial_{a_0} T_{a_1 \ldots a_p} d\Omega^{a_0} d\Omega^{a_1} \ldots d\Omega^{a_p} \qquad (A.6)$$

H is a three form and

$$\begin{aligned} i_p H &= \tfrac{1}{2!} (K_p^a H_{abc}) d\Omega^b d\Omega^c \\ &= \tfrac{1}{2} [\partial_b v_c - \partial_c v_b] d\Omega^b d\Omega^c \qquad (A.7) \\ &= dv_p \end{aligned}$$

for some set of one-forms v_p. By using (A.3) the previous equation implies that H is Lie invariant (H closed).

$$\begin{aligned} \mathcal{L}_p H &= 0 \\ H &= dB \end{aligned} \qquad (A.8)$$

This allows to

$$\mathcal{L}_p B = df \qquad (A.9)$$

On the other hand from (A.7) we see that

$$\begin{aligned} i_p H &= dv_p \Rightarrow i_p(dB) = \mathcal{L}_p B - d(i_p B) = dv_p \\ \mathcal{L}_p B &= d(v_p + i_p B) = df \end{aligned}$$

Consequently

$$f = v_p + i_p B \tag{A.10}$$

which is a one form. Furthermore

$$i_p B = K_p^a B_{ab} d\Omega^b$$

$$(i_p B)_b = K_p^a B_{ab}$$

and

$$\mathcal{L}_p B_{ab} = \partial_a f_b - \partial_b f_a$$

$$\mathcal{L}_p B_{ab} = \partial_a [v_b + K^c B_{cb}] - \partial_b [v_a + K^c B_{ca}]$$

which allows to deduce that

$$\delta_K B_{ab} = \partial_a [K^c B_{cb} + v_b] - \partial_b [K^c B_{ca}] \tag{A.11}$$

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