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ABSTRACT

A unified approach to geometric, symbol and deformation quantizations on a generalized flag manifold endowed with an invariant pseudo-Kähler structure is proposed. In particular cases we arrive at Berezin's quantization via covariant and contravariant symbols.

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1. Introduction

In the series of papers [13],[4],[5],[6] a modern approach to quantization on Kähler manifolds was proposed which combines together geometric quantization [11],[15], symbol quantization [3] and deformation quantization [2]. The main idea of this approach can be formulated for quantization on a general symplectic manifold $\mathcal{M}$ as follows.

- To give a geometric realization of a family of Hilbert spaces $H_\hbar$ over the manifold $\mathcal{M}$, parametrized by a small parameter $\hbar$ which plays a role of Planck constant (by means of geometric quantization or of its generalizations).
- To describe a geometric construction of a symbol mapping from functions on $\mathcal{M}$ to operators in $H_\hbar$ (the construction of operator symbols).
- To choose appropriate algebras $\mathcal{A}_\hbar$ of symbols such that the symbol mapping provides a representation of $\mathcal{A}_\hbar$ in $H_\hbar$.
- To find the deformation quantization which controls the asymptotic expansion of the symbol product as $\hbar \to 0$ in the same geometric framework.

The aim of this paper is to carry out this quantization program on a generalized flag manifold, a homogeneous space of a compact semisimple Lie group, endowed with an invariant pseudo-Kähler structure.

In [8] it was shown that the theory of spherical Harish-Chandra modules provides a natural algebraic construction of "mixed" symbol algebras on a generalized flag manifold, which in particular cases are algebras of Berezin's covariant and contravariant symbols.

In the present paper we consider an alternative, geometric construction of such an algebra $\mathcal{A}$ on a flag manifold $\mathcal{M}$, which involves some invariant pseudo-Kähler structure on $\mathcal{M}$. This construction was independently introduced in [1] and [9]. Then we describe a natural geometric representation of the algebra $\mathcal{A}$ in sheaf cohomology of the quantum line bundle on $\mathcal{M}$. The description is based on the Bott–Borel–Weil theorem. It turns out that the algebras of Berezin's covariant and contravariant symbols correspond to totally positive and totally negative Kähler structures respectively.

Then we consider $\hbar$-parametrized families of symbol algebras for which the asymptotic expansion of the symbol product as $\hbar \to 0$ leads to deformation quantization with separation of variables (see [7]).

2. Equivariant families of functions on homogeneous manifolds

Let K be a real Lie group, k_r its real Lie algebra, k_r^* the real dual to k_r. For $X \in k_r$, $F \in k_r^*$ denote their pairing by $< F, X >$.

Let $\mathcal{M}$ be a homogeneous K-manifold. Denote by T_k the shift operator by $k \in K$ in $C^\infty(\mathcal{M})$, $T_k f(x) = f(k^{-1}x)$, $x \in \mathcal{M}$, $f \in C^\infty(\mathcal{M})$. We call a family of real smooth functions $\{f_X\}$, $X \in k_r$, on $\mathcal{M}$ a *K-equivariant family* if $k_r \ni X \mapsto f_X$ is a linear mapping from k_r to $C^\infty(\mathcal{M})$, K-equivariant with respect to the adjoint action on k_r and the shift action on $C^\infty(\mathcal{M})$, so that for all $k \in K$, $X \in k_r$ holds $T_k f_X = f_{Ad(k)X}$.

For $X \in k_r$ denote by v_X the corresponding fundamental vector field on $\mathcal{M}$. For $k \in K$ holds the relation $T_k v_X T_k^{-1} = v_{Ad(k)X}$, $X \in k_r$, where v_X is treated as a differential operator in $C^\infty(\mathcal{M})$.

For a K-equivariant family $\{f_X\}$ on $\mathcal{M}$ and for all $X, Y \in k_r$ holds the relation $v_X f_Y = f_{[X,Y]}$.

Given a K-equivariant family $\{f_X\}$ on $\mathcal{M}$, define a "moment" mapping $\gamma : \mathcal{M} \to k_r^*$ K-equivariant with respect to the shift action on $\mathcal{M}$ and coadjoint action on k_r^*, such that for all $x \in \mathcal{M}$, $X \in k_r$ holds $< \gamma(x), X >= f_X(x)$.

Since $\mathcal{M}$ is homogeneous, the image of γ is a single coadjoint orbit $\Omega = \gamma(\mathcal{M}) \subset k_r^*$. For $X \in k_r$ denote by v_X^Ω the corresponding fundamental vector field on Ω. It is Hamiltonian with respect to the K-invariant symplectic structure given by the Kirillov symplectic form ω^Ω on Ω. The function $f_X^\Omega(F) =< F, X >$, $F \in \Omega$, is its Hamiltonian function, i.e., $df_X^\Omega = -i(v_X^\Omega)\omega^\Omega$. For $X, Y \in k_r$ holds $\omega^\Omega(v_X^\Omega, v_Y^\Omega) = f_{[X,Y]}^\Omega$.

Remark. K-equivariant families $\{f_X\}$ on $\mathcal{M}$ are in one-to-one correspondence with K-equivariant mappings $\gamma : \mathcal{M} \to k_r^*$. To a given γ there corresponds the coadjoint orbit $\Omega = \gamma(\mathcal{M})$ and the family $\{f_X\}$ such that $f_X = \gamma^* f_X^\Omega = f_X^\Omega \circ \gamma$.

Fix a point $x_0 \in \mathcal{M}$ and denote by $K_0 \subset K$ the isotropy subgroup of the point x_0. The mapping γ and thus the K-equivariant family $\{f_X\}$ itself are completely determined by the image point $\gamma(x_0)$ which is an arbitrary K_0-stable point in k_r^*.

For two K-equivariant families $\{f_X^{(1)}\}$ and $\{f_X^{(2)}\}$ their linear combination $\{\alpha f_X^{(1)} + \beta f_X^{(2)}\}$ is a K-equivariant family as well. Therefore the set of all K-equivariant families $\{f_X\}$ on $\mathcal{M}$ is a vector space which can be identified with the subspace $(k_r^*)^{K_0}$ of all K_0-stable points in k_r^*.

Denote by ω the pullback of the form ω^Ω by γ. Then ω is a closed (but not necessarily nondegenerate) K-invariant form on $\mathcal{M}$ such that for $X, Y \in k_r$ holds $\omega(v_X, v_Y) = f_{[X,Y]}$ and $df_X = -i(v_X)\omega$. We say that ω is associated to the K-equivariant family $\{f_X\}$.

The form ω is nondegenerate iff the tangent mapping to $\gamma : \mathcal{M} \to \Omega$ at any point $x \in \mathcal{M}$ is an isomorphism of the tangent spaces $T_x\mathcal{M}$ and $T_{\gamma(x)}\Omega$ or, equivalently, if γ is a covering mapping.

3. Modules of functions on complex homogeneous manifolds

Let K be a real Lie group with the real Lie algebra k_r. Denote by g_c the complexification of k_r, $g_c = k_r \otimes \mathbf{C}$, by g_r the realification of g_c, and by J the corresponding operator of complex structure in g_r, so that (g_r, J) is isomorphic to g_c.

Let the group K act transitively and holomorphically on a complex manifold $\mathcal{M}$.

Then for $X \in k_r$ the fundamental vector field v_X on $\mathcal{M}$ decomposes into the sum of holomorphic and antiholomorphic vector fields ξ_X and η_X respectively, $v_X = \xi_X + \eta_X$. Therefore $\eta_X = \bar{\xi}_X$ and for arbitrary $X, Y \in k_r$ ξ_X commutes with η_Y, $[\xi_X, \xi_Y] = \xi_{[X,Y]}$ and $[\eta_X, \eta_Y] = \eta_{[X,Y]}$. For $X \in k_r$, $k \in K$ the following relations hold, $T_k \xi_X T_k^{-1} = \xi_{Ad(k)X}$ and $T_k \eta_X T_k^{-1} = \eta_{Ad(k)X}$.

For $Z = X + JY \in g_r$, $X, Y \in k_r$, set $\xi_Z = \xi_X + i\xi_Y$ and $\eta_Z = \eta_X - i\eta_Y$. Now $(g_r, J) \ni Z \mapsto \xi_Z$ is a C-linear homomorphism from (g_r, J) to the Lie algebra of holomorphic vector fields on $\mathcal{M}$, and $\eta_Z = \bar{\xi}_Z$. We get that g_r acts on $\mathcal{M}$ by real vector fields $v_Z = \xi_Z + \eta_Z = (\xi_X + \eta_X) + i(\xi_Y - \eta_Y)$ which respect the holomorphic structure on $\mathcal{M}$.

We call a mapping $k_r \ni X \mapsto s_X$ of the Lie algebra k_r to $End(C^\infty(\mathcal{M}))$ K-equivariant if it is K-equivariant with respect to the adjoint action of K in k_r and the shift action in $C^\infty(\mathcal{M})$. This means that $T_k s_X T_k^{-1} = s_{Ad(k)X}$, $X \in k_r, k \in K$.

Now we shall define a special (g_r, K)-module structure on $C^\infty(\mathcal{M})$. Let K act in $C^\infty(\mathcal{M})$ by the shifts T_k, $k \in K$, and g_r act by real differential operators $m_Z = v_Z + \varphi_Z$, $Z \in g_r$, where φ_Z is a real smooth function on $\mathcal{M}$, so that the actions of K and g_r agree in the usual sense. This means that the actions of the algebra k_r as of a subalgebra of g_r and as of the Lie algebra of K coincide, $m_X = v_X$ for $X \in k_r$, and $T_k m_Z T_k^{-1} = m_{Ad(k)Z}$ for $k \in K$, $Z \in g_r$. In particular, for $X \in k_r$ holds $\varphi_X = 0$ and $T_k \varphi_Z = \varphi_{Ad(k)Z}$. Then we say that there is given an *s-module* on $\mathcal{M}$.

For $X, Y \in k_r$ set $Z(X, Y) = 1/2(X - iJX + Y + iJY) \in g_r \otimes \mathbf{C}$. The mapping $k_r \times k_r \ni (X, Y) \mapsto Z(X, Y)$ is a Lie algebra homomorphism from $k_r \times k_r$ to $g_r \otimes \mathbf{C}$ (moreover, it extends by C-linearity to an isomorphism of the complex Lie algebras $(g_r, J) \times (g_r, -J)$ and $g_r \otimes \mathbf{C}$).

For $X \in k_r$ introduce a function $f_X = (-1/2)\varphi_{JX}$ on $\mathcal{M}$. It is easy to check that the functions f_X, $X \in k_r$, form a K-equivariant family. For $X \in k_r$ set $l_X = \xi_X + if_X$, $r_X = \eta_X - if_X$. Notice that $\eta_X = \bar{\xi}_X$ and that the mappings $k_r \ni X \mapsto l_X$ and $k_r \ni X \mapsto r_X$ are K-equivariant.

A straightforward calculation shows that $m_{Z(X,Y)} = l_X + r_Y$, where the mapping $g_r \ni Z \mapsto m_Z$ is extended to $g_r \otimes \mathbf{C}$ by C-linearity. Taking into account that $(X, 0)$ commutes with $(0, Y)$ in $k_r \times k_r$, we get the following lemma.

Lemma 1. *The mappings $k_r \ni X \mapsto l_X$ and $k_r \ni X \mapsto r_X$ are commuting K-equivariant complex conjugate representations of k_r in $C^\infty(\mathcal{M})$.*

Suppose there is given a representation of k_r in $C^\infty(\mathcal{M})$ of the form $k_r \ni X \mapsto l_X = \xi_X + if_X$, where $\{f_X\}$ is a K-equivariant family on $\mathcal{M}$ (which is equivalent to $X \mapsto l_X$ being K-equivariant). Then there exists an s-module on $\mathcal{M}$ to which the representation $k_r \ni X \mapsto l_X$ is associated.

Lemma 2. *Let $k_r \ni X \mapsto l_X = \xi_X + if_X$ be a K-equivariant representation of k_r in $C^\infty(\mathcal{M})$. For $Z = X + JY \in g_r$, $X, Y \in k_r$, define the function $\varphi_Z = -2f_Y$ on $\mathcal{M}$. Then* *the mapping $g_r \ni Z \mapsto m_Z = v_Z + \varphi_Z$ is a representation of g_r in $C^\infty(\mathcal{M})$. Together with the shift action of K in $C^\infty(\mathcal{M})$ it defines an s-module on $\mathcal{M}$.*

Proof. Consider the complex conjugate representation $k_r \ni X \mapsto r_X = \bar{l}_X = \eta_X - if_X$ to the representation $X \mapsto l_X$ of k_r. Then for $X, Y \in k_r$ and $Z(X,Y) = 1/2(X - iJX + Y + iJY)$ we have as above $m_{Z(X,Y)} = l_X + r_Y$. In order to show that the mapping $g_r \ni Z \mapsto m_Z$ is a representation of g_r in $C^\infty(\mathcal{M})$, it is enough to show that the representations $X \mapsto l_X$ and $X \mapsto r_X$ of k_r commute or, equivalently, that $\xi_X f_Y + \eta_Y f_X = 0$. We get from the identity $[l_X, l_Y] = l_{[X,Y]}$ that $f_{[X,Y]} = \xi_X f_Y - \xi_Y f_X$. Since $v_Y f_X = f_{[Y,X]}$, we get that $f_{[X,Y]} = -\xi_Y f_X - \eta_Y f_X$. Equating the two expressions for $f_{[X,Y]}$ we obtain the desired identity. The rest of the proof is straightforward.

It follows from Lemma 2 that any s-module is completely determined by some K-equivariant family of functions $\{f_X\}$ on $\mathcal{M}$ for which the mapping $k_r \ni X \mapsto l_X = \xi_X + if_X$ is a representation of k_r, or, equivalently, such that for $X, Y \in k_r$ holds the relation $f_{[X,Y]} = \xi_X f_Y - \xi_Y f_X$. Since this relation is linear with respect to the family $\{f_X\}$, the set S of all s-modules on $\mathcal{M}$ is naturally identified with a linear subspace of the vector space of K-equivariant families of functions on $\mathcal{M}$.

It turns out that one can give a simple characterization of those K-equivariant families of functions which give rise to s-modules. It is given in terms of the closed 2-form ω on $\mathcal{M}$ associated to the K-equivariant family.

Theorem 1. *A K-equivariant family $\{f_X\}$ on $\mathcal{M}$ corresponds to an s-module on $\mathcal{M}$ iff the 2-form ω associated to $\{f_X\}$ is of the type $(1,1)$ with respect to the complex structure on $\mathcal{M}$.*

Proof. We have to prove that the form ω is of the type $(1,1)$ iff the relation $f_{[X,Y]} = \xi_X f_Y - \xi_Y f_X$ holds for all $X, Y \in k_r$. We can rewrite this relation in terms of ω using that $df_X = -i(v_X)\omega$ and $\omega(v_X, v_Y) = f_{[X,Y]}$ as follows, $\omega(v_X, v_Y) = \omega(\xi_X, v_Y) - \omega(\xi_Y, v_X)$. Since $v_X = \xi_X + \eta_Y$, it is equivalent to the relation

$$\omega(\xi_X, \xi_Y) = \omega(\eta_X, \eta_Y) \text{ for all } X, Y \in k_r. \tag{1}$$

If ω is of the type $(1,1)$, the both sides of (1) vanish. Suppose now that (1) is true. For $Z = X + iY \in g_c = k_r \otimes \mathbf{C}$, $X, Y \in k_r$, set $v_Z = v_X + iv_Y$, $\xi_Z = \xi_X + i\xi_Y$ and $\eta_Z = \eta_X + i\eta_Y$. It follows from (1) that

$$\omega(\xi_Z, \xi_{Z'}) = \omega(\eta_Z, \eta_{Z'}) \text{ for all } Z, Z' \in g_c. \tag{2}$$

Fix a point $x \in \mathcal{M}$. Since $\mathcal{M}$ is K-homogeneous, the vectors v_X, $X \in k_r$, at the point x span the real tangent space $T_x\mathcal{M}$ and thus the vectors v_Z, $Z \in g_c$, span $T_x\mathcal{M} \otimes \mathbf{C}$. For arbitrary vectors $\xi, \xi' \in T_x\mathcal{M} \otimes \mathbf{C}$ of the type $(1,0)$ one can find $Z, Z' \in g_c$ such that $v_Z = \xi$ and $v_{Z'} = \xi'$ at the point x. Therefore, at the point x, $\xi_Z = \xi$, $\xi_{Z'} = \xi'$ and $\eta_Z = \eta_{Z'} = 0$. It follows from (2) that $\omega(\xi, \xi') = 0$ at the point $x \in \mathcal{M}$ for arbitrary vectors ξ, ξ' of the type (1,0). Since ω is real, it follows that it is of the type (1,1), which completes the proof.

We say that an s-module is nondegenerate if the corresponding 2-form ω is nondegenerate. It follows from Theorem 1 that the 2-form ω associated to a nondegenerate s-module is a pseudo-Kähler form.

The set of nondegenerate s-modules is either empty or it is a dense open conical (i.e. invariant with respect to the multiplication by non-zero constants, $\mathbf{s} \mapsto t \cdot \mathbf{s}$, $\mathbf{s} \in S, t \in \mathbf{R}\backslash\{0\}$) subset of S.

Example. Let $\Omega \subset k_r^*$ be a coadjoint orbit of the group K, endowed with an invariant pseudo-Kähler polarization. This means that there is given an invariant complex structure on Ω such that the Kirillov form ω^Ω is of the type (1,1). The fundamental vector field v_X^Ω, $X \in k_r$, decomposes into the sum of a holomorphic and antiholomorphic vector fields ξ_X^Ω and η_X^Ω respectively, $v_X^\Omega = \xi_X^\Omega + \eta_X^\Omega$. Then the functions f_X^Ω, $X \in k_r$, form a K-equivariant family that corresponds to an s-module on Ω. In particular, the mappings $k_r \ni X \mapsto l_X = \xi_X^\Omega + i f_X^\Omega$ and $k_r \ni X \mapsto r_X = \eta_X^\Omega - i f_X^\Omega$ are two commuting K-equivariant representations of k_r in $C^\infty(\Omega)$.

We are going to associate to each s-module on $\mathcal{M}$ an associative algebra $\mathcal{A}$ whose elements are smooth functions on $\mathcal{M}$.

Extend the representations $k_r \ni X \mapsto l_X$ and $k_r \ni X \mapsto r_X = \bar{l}_X$ to $g_c = k_r \otimes \mathbf{C}$ by $\mathbf{C}$-linearity. Then, extending them further to the universal enveloping algebra $\mathcal{U}(g_c)$ of g_c, one obtains two commuting K-equivariant representations of $\mathcal{U}(g_c)$ in $C^\infty(\mathcal{M})$, $u \mapsto l_u$ and $u \mapsto r_u$, $u \in \mathcal{U}(g_c)$ (K acts on $\mathcal{U}(g_c)$ by the properly extended adjoint action).

Let $u \mapsto \check{u}$ denote the standard anti-automorphism of $\mathcal{U}(g_c)$ which maps $X \in g_c$ to $-X$.

Lemma 3. *For $u \in \mathcal{U}(g_c)$ the following relation holds, $l_u 1 = r_{\check{u}} 1$.*

Proof. We prove the Lemma for the monomials $u_n = X_1 \dots X_n$, $X_j \in g_c$, using induction over n. One checks directly that for $X \in \mathcal{U}(g_c)$ holds $l_X 1 = i f_X = r_{\check{X}} 1$. Assume that $l_{u_{n-1}} 1 = r_{\check{u}_{n-1}} 1$ holds. Then $l_{u_n} 1 = l_{u_{n-1}} l_{X_n} 1 = l_{u_{n-1}} r_{\check{X}_n} 1 = r_{\check{X}_n} l_{u_{n-1}} 1 = r_{\check{X}_n} r_{\check{u}_{n-1}} 1 = r_{\check{u}_n} 1$. The Lemma is proved.

For $u \in \mathcal{U}(g_c)$ denote $\sigma_u = l_u 1$ and let $\mathcal{A}$ denote the image of the mapping $\sigma : u \mapsto \sigma_u$ from $\mathcal{U}(g_c)$ to $C^\infty(\mathcal{M})$. Notice that the mapping σ is K-equivariant with respect to the adjoint action on $\mathcal{U}(g_c)$ and the action by shifts on $C^\infty(\mathcal{M})$.

Lemma 4. *The kernel I of the mapping $\sigma : \mathcal{U}(g_c) \to C^\infty(\mathcal{M})$ is a two-sided ideal in $\mathcal{U}(g_c)$ and thus $\mathcal{A}$ inherits the algebra structure from the quotient algebra $\mathcal{U}(g_c)/I$.*

Proof. It follows from the relation $l_u 1 = 0$ that I is a left ideal, while $r_{\check{u}} 1 = 0$ shows that I is a right ideal since $u \mapsto \check{u}$ is an anti-homomorphism. The Lemma is proved.

We shall denote the associative product in $\mathcal{A}$ by $*$. It follows from Lemma 3 that for $u \in \mathcal{U}(g_c)$, $f \in \mathcal{A}$ holds $l_u f = \sigma_u * f$ and $r_{\check{u}} f = f * \sigma_u$.

Remark. As a subspace of $C^\infty(\mathcal{M})$ the algebra $\mathcal{A}$ is the spherical (g_r, K)-submodule of the s-module it is associated to, generated by the constant function 1, which is a spherical (K-invariant) vector.

Denote by $\mathcal{Z}(g_c)$ the center of $\mathcal{U}(g_c)$. The elements of $\mathcal{Z}(g_c)$ are stable under the adjoint action of K. Since the mapping σ is K-equivariant, σ maps the central elements of $\mathcal{U}(g_c)$ to constants in $\mathcal{A}$. Thus the restriction of the mapping σ to $\mathcal{Z}(g_c)$ defines a central character $\psi : \mathcal{Z}(g_c) \to \mathbf{C}$ of the algebra $\mathcal{U}(g_c), \psi(z) = \sigma_z$, $z \in \mathcal{Z}(g_c)$ (here we identify the constant functions in $\mathcal{A}$ with the corresponding complex constants).

4. Holomorphic differential operators on hermitian line bundles

Let $\pi : L \to \mathcal{M}$ be a holomorphic hermitian line bundle over $\mathcal{M}$ with hermitian metrics h. Denote by $L^\bullet$ the bundle L with the zero section removed. It is a $\mathbf{C}^\bullet$-principal bundle. A local holomorphic trivialization of L is given by a pair (U, s) where U is an open chart on $\mathcal{M}$ and $s : U \to L^\bullet$ is a nonvanishing local holomorphic section of L.

We are going to define a pushforward of holomorphic differential operators on L to the base space $\mathcal{M}$.

A holomorphic differential operator A on L is a global geometric object given locally, for a holomorphic trivialization (U_α, s_α), by a holomorphic differential operator A_α on U_α. On the intersection of two charts U_α and U_β the operators A_α and A_β must satisfy the relation $A_\alpha \varphi_{\alpha\beta} = \varphi_{\alpha\beta} A_\beta$ where $\varphi_{\alpha\beta}$ is a holomorphic transition function on $U_\alpha \cap U_\beta$ such that $\varphi_{\alpha\beta} s_\alpha = s_\beta$. In this relation we consider $\varphi_{\alpha\beta}$ as a multiplication operator.

Holomorphic differential operators on L act on the sheaf of local holomorphic sections of L and form an algebra.

On each chart (U_α, s_α) introduce a real function $\Phi_\alpha = -\log h \circ s_\alpha$.

Lemma 5. *On the intersection of two charts U_α and U_β the following equality holds,* $\Phi_\alpha - \Phi_\beta = \log |\varphi_{\alpha\beta}|^2$.

Proof. We have $\Phi_\beta = -\log h \circ s_\beta = -\log h \circ (\varphi_{\alpha\beta} s_\alpha) = -\log(|\varphi_{\alpha\beta}|^2 h \circ s_\alpha) = \Phi_\alpha - \log |\varphi_{\alpha\beta}|^2$. The Lemma is proved.

Given a global holomorphic differential operator A on L, consider differential operators $\check{A}_\alpha = e^{-\Phi_\alpha} A_\alpha\, e^{\Phi_\alpha}$ on each chart U_α.

Lemma 6. *The operators $\check{A}_\alpha$ define a global differential operator $\check{A}$ on $\mathcal{M}$.*

Proof. We have to check that on the intersection of two charts U_α and U_β holds the equality $\check{A}_\alpha = \check{A}_\beta$. It is equivalent to $e^{-\Phi_\alpha} A_\alpha\, e^{\Phi_\alpha} = e^{-\Phi_\beta} A_\beta\, e^{\Phi_\beta}$ or $A_\alpha e^{\Phi_\alpha - \Phi_\beta} = e^{\Phi_\alpha - \Phi_\beta} A_\beta$. Applying Lemma 5 we get an equivalent equality $A_\alpha \varphi_{\alpha\beta} \bar{\varphi}_{\alpha\beta} = \varphi_{\alpha\beta} \bar{\varphi}_{\alpha\beta} A_\beta$. The assertion of the Lemma follows now from the fact that the holomorphic differential operator A_β commutes with the multiplication by the antiholomorphic function $\bar{\varphi}_{\alpha\beta}$.

We call $\check{A}$ the pushforward of the holomorphic differential operator A on the line bundle L to the base space $\mathcal{M}$. It is clear that the pushforward mapping $A \mapsto \check{A}$ is an injective homomorphism of the algebra of holomorphic differential operators on L into the algebra of differential operators on $\mathcal{M}$.

Let ∇ denote the canonical holomorphic connection of the hermitian line bundle (L, h). For a local holomorphic trivialization (U_α, s_α) a local expression of ∇ on U_α is $\nabla = d - \partial\Phi_\alpha$. The curvature ω of ∇ has a local expression $\omega = i\partial\bar{\partial}\Phi_\alpha$.

Let the Lie group K act on the line bundle $\pi : L \to \mathcal{M}$ by holomorphic line bundle automorphisms which respect the hermitian metrics h.

The metrics h can be considered as a function on L, $L \ni q \mapsto h(q)$. To each local section s of L over an open set $U \subset \mathcal{M}$ relate a function ψ_s on $\pi^{-1}(U) \cap L^*$ such that $\psi_s(q)q = s(\pi(q))$. For $t \in \mathbf{C}^*$ holds $h(tq) = |t|^2 h(q)$ and $\psi_s(tq) = t^{-1}\psi_s(q)$.

Any element X of the Lie algebra k_r of K acts on L^* by a real vector field v_X^L which is the sum of holomorphic and antiholomorphic vector fields ξ_X^L and $\eta_X^L = \bar{\xi}_X^L$ respectively, $v_X^L = \xi_X^L + \eta_X^L$. The vector fields v_X^L, ξ_X^L and η_X^L are homogeneous of order 0 with respect to the action of $\mathbf{C}^*$ on L^*. Let v_X, ξ_X and η_X denote their projections to $\mathcal{M}$, so $v_X = \xi_X + \eta_X$.

The action of ξ_X^L on the functions ψ_s on L^* can be transferred to the action on the corresponding local holomorphic sections s of L, which defines a global holomorphic differential operator A_X on L. The object of interest to us will be its pushforward $\check{A}_X$ to the base space $\mathcal{M}$.

First, consider a local trivialization of L^* by a local section $s_0 : U \to L^*$, which identifies $(x, v) \in U \times \mathbf{C}^*$ with $s_0(x)v \in L^*|_U$. Then, locally, $\xi_X^L = \xi_X - a_X v\partial/\partial v$ for some holomorphic function a_X on U. To push forward a holomorphic differential operator from $L|_U$ to U we use the function $\Phi = -\log h \circ s_0$ on U. The metrics h at the point $(x, v) \in U \times \mathbf{C}^*$ can be expressed as follows, $h(x, v) = e^{-\Phi}|v|^2$. Since the metrics h is K-invariant, we have $v_X^L h = 0$. A simple calculation shows then that

$$(\xi_X + \bar{\xi}_X)\Phi = -a_X - \bar{a}_X. \tag{3}$$

Introduce a function $f_X = -i(a_X + \xi_X\Phi)$ on U. Then (3) means that f_X is real.

Proposition 1. *The holomorphic differential operator A_X on L and its pushforward $\check{A}_X$ to the base space $\mathcal{M}$ can be expressed as follows, $A_X = \nabla_{\xi_X} + if_X$ and $\check{A}_X = \xi_X + if_X$. The mapping $k_r \ni X \mapsto \check{A}_X$ is a K equivariant representation of k_r in $C^\infty(\mathcal{M})$. The function f_X is globally defined on $\mathcal{M}$ and satisfies the relations $h(\xi_X^L h^{-1}) = if_X \circ \pi$ and $df_X = -i(v_X)\omega$.*

Proof. Fix a trivialization of L over an open subset $U \subset \mathcal{M}$, $L|_U \approx U \times \mathbf{C}$, and consider a local section of L over U, $s : x \mapsto s(x) = (x, \varphi_s(x)) \in U \times \mathbf{C}$. The function ψ_s corresponding to s is defined by the equality $\psi_s(q)q = s(x)$ for $q \in L^*$, $x = \pi(q)$. At the point $q = (x, v) \in U \times \mathbf{C}^*$ we have $(x, \psi_s(x, v)v) = (x, \varphi_s(x))$, whence $\psi_s(x, v) = \varphi_s(x)v^{-1}$. To find the local expression of the operator A_X calculate its action on $\psi_s(x, v)$, $(\xi_X - a_X v\partial/\partial v)(\varphi_s v^{-1}) = (\xi_X\varphi_s + a_X\varphi_s)v^{-1}$. Thus, locally, $A_X = \xi_X + a_X = (\xi_X -$

$\xi_X\Phi) + (\xi_X\Phi + a_X) = \nabla_{\xi_X} + if_X$. Pushing it forward to U we get $\check{A}_X = e^{-\Phi}(\xi_X + a_X)e^{\Phi} = \xi_X + (\xi_X\Phi + a_X) = \xi_X + if_X$. The K-equivariance of the mapping $k_r \ni X \mapsto \check{A}_X$ follows from the fact that K acts on the hermitian line bundle (L, h) by the line bundle automorphisms which preserve the metrics h. We have locally that $h(\xi_X^L h^{-1}) = e^{-\Phi}|v|^2((\xi_X - a_X v\partial/\partial v)e^{\Phi}|v|^{-2}) = \xi_X\Phi + a_X = if_X \circ \pi$. To prove the last relation of the Proposition we notice that $i(\xi_X)\omega$ is of the type $(0,1)$ and $i(\eta_X)\omega$ is of the type $(1,0)$. We have to show that $\bar{\partial} f_X = -i(\xi_X)\omega$ and $\partial f_X = -i(\eta_X)\omega$. These equalities are complex conjugate, so we prove the former one. Let $\{z^k\}$ be local holomorphic coordinates on U and $\xi_X = a^k(z)\partial/\partial z^k$. Then $\bar{\partial} f_X = -i\bar{\partial}(a_X + \xi_X\Phi) = -i\bar{\partial}\xi_X\Phi = -ia^k(z)(\partial^2\Phi/\partial z^k\partial\bar{z}^l)d\bar{z}^l$. Taking into account that $\omega = i(\partial^2\Phi/\partial z^k\partial\bar{z}^l)dz^k \wedge d\bar{z}^l$ we immediately obtain the desired equality, which completes the proof.

It follows from Proposition 1 that to a hermitian line bundle $(L, h) \to \mathcal{M}$ on which the group K acts by holomorphic automorphisms which respect the metrics h there corresponds an s-module s on $\mathcal{M}$. The relation $df_X = -i(v_X)\omega$ implies that $\omega(v_X, v_Y) = f_{[X,Y]}$, therefore the (1,1)-form corresponding to the s-module s is the curvature ω of the canonical connection ∇ on L. The pushforward of the operator A_X to $\mathcal{M}$ coincides with the operator l_X, associated to s, $\check{A}_X = l_X$. The mapping $k_r \ni X \mapsto A_X$ can be extended to the homomorphism of the algebra $\mathcal{U}(g_c)$ to the algebra of holomorphic differential operators on L, $\mathcal{U}(g_c) \ni u \mapsto A_u$. Since the pushforward mapping is a homomorphism of the algebra of holomorphic differential operators on L into the algebra of differential operators on $\mathcal{M}$, we get the following corollary of Proposition 1.

Corollary. *To a hermitian line bundle $(L, h) \to \mathcal{M}$ on which the group K acts by holomorphic automorphisms which respect the metrics h there corresponds an s-module* s *on $\mathcal{M}$ such that for any $u \in \mathcal{U}(g_c)$ the pushforward of the operator A_u from L to $\mathcal{M}$ coincides with the operator l_u, associated to* s, *$\check{A}_u = l_u$.*

Denote by L_{can} the canonical line bundle of $\mathcal{M}$, i.e., the top exterior power of the holomorphic cotangent bundle $T^{*\prime}\mathcal{M}$ of $\mathcal{M}$, $L_{can} = \wedge^m T^{*\prime}\mathcal{M}$, where $m = \dim_{\mathbb{C}}\mathcal{M}$. Its local holomorphic sections are the local holomorphic m-forms on $\mathcal{M}$. Let μ be a global positive volume form on $\mathcal{M}$. One can associate to it a hermitian metrics h_μ on L_{can} such that for an arbitrary local holomorphic m-form α on $\mathcal{M}$ $h_\mu(\alpha) = \alpha \wedge \bar{\alpha}/\mu$.

Recall that the divergence of a vector field ξ with respect to the volume form μ is given by the formula $\mathrm{div}_\mu\xi = \mathcal{L}_\xi\mu/\mu$, where $\mathcal{L}_\xi$ is the Lie derivative corresponding to ξ.

Let the volume form μ on $\mathcal{M}$ be K-invariant. Then for $X \in k_r$ $\mathrm{div}_\mu v_X = \mathrm{div}_\mu\xi_X + \mathrm{div}_\mu\eta_X = 0$. Since μ is real, it follows that $\mathrm{div}_\mu\xi_X$ and $\mathrm{div}_\mu\eta_X$ are complex conjugate and thus pure imaginary. For $X \in k_r$ introduce a real function $f_X^{can} = -i\mathrm{div}_\mu\xi_X$.

The natural geometric action of K on the hermitian line bundle (L_{can}, h_μ) by holomorphic line bundle automorphisms preserves the metrics h_μ. The corresponding infinitesimal action of an element $X \in k_r$ on the local holomorphic m-forms on $\mathcal{M}$ by the Lie derivative

$\mathcal{L}_{\xi_X}$, defines a global holomorphic differential operator A_X on L_{can}.

Proposition 2. *The holomorphic differential operator A_X on L_{can} is given by the formula $A_X = \nabla_{\xi_X} + \mathrm{div}_\mu \xi_X = \nabla_{\xi_X} + i f_X^{can}$.*

Proof. Let $U \subset \mathcal{M}$ be a local coordinate chart with holomorphic coordinates $\{z^k\}$. The form $\alpha_0 = dz^1 \wedge \ldots \wedge dz^m$ is a local holomorphic trivialization of L_{can}. Set $\Phi = -\log h(\alpha_0)$. Then locally on (U, α_0) $\mu = e^\Phi \alpha_0 \wedge \bar{\alpha}_0$ and $\nabla = d - \partial\Phi$. Since ξ_X is a holomorphic vector field and $\bar{\alpha}_0$ is an anti-holomorphic form, we get $\mathcal{L}_{\xi_X} \bar{\alpha}_0 = 0$. Therefore, $\mathcal{L}_{\xi_X} \mu = (\xi_X \Phi) e^\Phi \alpha_0 \wedge \bar{\alpha}_0 + e^\Phi (\mathcal{L}_{\xi_X} \alpha_0) \wedge \bar{\alpha}_0$. On the other hand, $\mathcal{L}_{\xi_X} \mu = (\mathrm{div}_\mu \xi_X) \mu = (\mathrm{div}_\mu \xi_X) e^\Phi \alpha_0 \wedge \bar{\alpha}_0$. Therefore, $(\mathrm{div}_\mu \xi_X) \alpha_0 = (\xi_X \Phi) \alpha_0 + \mathcal{L}_{\xi_X} \alpha_0$. Let $\alpha = f \alpha_0$ be a holomorphic m-form on U. The holomorphic function f represents the local holomorphic section α of L_{can} in the trivialization (U, α_0). Now $\mathcal{L}_{\xi_X} \alpha = \mathcal{L}_{\xi_X}(f \alpha_0) = (\xi_X f) \alpha_0 + f \mathcal{L}_{\xi_X} \alpha_0 = ((\xi_X - \xi_X \Phi) f) \alpha_0 + (\mathrm{div}_\mu \xi_X) \alpha = (\nabla_{\xi_X} + \mathrm{div}_\mu \xi_X) \alpha$. The Proposition is proved.

Applying Proposition 1 we obtain the following

Corollary. *The pushforward of the operator A_X to $\mathcal{M}$ is $\check{A}_X = \xi_X + \mathrm{div}_\mu \xi_X = \xi_X + i f_X^{can}$. The mapping $k_r \ni X \mapsto \xi_X + i f_X^{can}$ is a K-equivariant representation of k_r in $C^\infty(\mathcal{M})$.*

This means that if there exists a K-invariant measure μ on $\mathcal{M}$, we get an s-module on $\mathcal{M}$. It is easy to check that if we replace μ by an arbitrary K-invariant measure $c \cdot \mu$, $c \in \mathbf{R}_+$, we will get the same functions f_X^{can}, $X \in k_r$, and thus the same s-module. This s-module will be called *canonical* and denoted $\mathbf{s}_{can}$. If the set of nondegenerate s-modules on $\mathcal{M}$ is non-empty, then there exists K-invariant symplectic (pseudo-Kähler) form ω on $\mathcal{M}$ associated to a nondegenerate s-module. The corresponding symplectic volume is K-invariant as well, and therefore gives rise to the canonical s-module.

Suppose μ is a K-invariant measure on $\mathcal{M}$, and $k_r \ni X \mapsto l_X = \xi_X + i f_X$ is a K-equivariant representation of k_r in $C^\infty(\mathcal{M})$, which corresponds to the s-module $\mathbf{s} \in S$. For a differential operator A in $C^\infty(\mathcal{M})$, denote by A^t its formal transpose with respect to the measure μ, so that for all $\phi, \psi \in C_0^\infty(\mathcal{M})$ holds $\int (A\phi)\psi d\mu = \int \phi (A^t \psi) d\mu$. Consider the K-equivariant representation $k_r \ni X \mapsto (l_{-X})^t = \xi_X - i f_X + \mathrm{div}_\mu \xi_X$. It corresponds to the s-module which we call dual to $\mathbf{s}$ and denote by $\mathbf{s}'$. Since the canonical module $\mathbf{s}_{can}$ corresponds to the K- equivariant family $\{-i \mathrm{div}_\mu \xi_X\}$, we get $\mathbf{s}' = -\mathbf{s} + \mathbf{s}_{can}$.

5. Deformation quantizations with separation of variables

Recall the definition of deformation quantization on a symplectic manifold $\mathcal{M}$ introduced in [2].

Definition. Formal differentiable deformation quantization on a symplectic manifold $\mathcal{M}$ is a structure of associative algebra in the space of all formal series $C^\infty(\mathcal{M})[[\nu]]$. The product $\star$ of two elements $f = \sum_{r \geq 0} \nu^r f_r$, $g = \sum_{r \geq 0} \nu^r g_r$ of $C^\infty(\mathcal{M})[[\nu]]$ is given by the following formula,

$$f \star g = \sum_r \nu^r \sum_{i+j+k=r} C_i(f_j, g_k), \tag{4}$$

where $C_r(\cdot,\cdot)$, $r = 0, 1, \ldots$, are bidifferential operators such that for smooth functions φ, ψ on $\mathcal{M}$ holds $C_0(\varphi,\psi) = \varphi\psi$ and $C_1(\varphi,\psi) - C_1(\psi,\varphi) = i\{\varphi,\psi\}$. Here $\{\cdot,\cdot\}$ is the Poisson bracket on $\mathcal{M}$, corresponding to the symplectic structure.

Then the product $\star$ is called a star-product. The star-product can be extended by the same formula (4) to the space $\mathcal{F} = C^\infty(\mathcal{M})[\nu^{-1}, \nu]]$ of formal Laurent series with a finite polar part.

Since the star-product is given by bidifferential operators, it is localizable, that is, it can be restricted to any open subset $U \subset \mathcal{M}$. For $U \subset \mathcal{M}$ denote $\mathcal{F}(U) = C^\infty(U)[\nu^{-1}, \nu]]$ and for $f, g \in \mathcal{F}(U)$ let L_f and R_g denote the left star-multiplication operator by f and the right star-multiplication operator by g in $\mathcal{F}(U)$ respectively, so that $L_f g = f \star g = R_g f$. The operators L_f and R_g commute for all $f, g \in \mathcal{F}(U)$. Let $\mathcal{L}(U)$ and $\mathcal{R}(U)$ denote the algebras of left and right star-multiplication operators in $\mathcal{F}(U)$ respectively. It is important to notice that both left and right star-multiplication operators are formal Laurent series of differential operators with a finite polar part (i.e., with finitely many terms of negative degree of the formal parameter ν).. We call such operators formal differential operators.

Let $\mathcal{M}$ be a complex manifold endowed with a pseudo-Kähler form ω_0. This means that ω_0 is a real closed nondegenerate form of the type $(1,1)$. Then $\mathcal{M}$ is a pseudo-Kähler manifold. The form ω_0 defines a symplectic structure on $\mathcal{M}$.

A formal deformation of pseudo-Kähler form ω_0 is a formal series $\omega = \omega_0 + \nu\omega_1 + \ldots$, where ω_r, $r > 0$, are closed, possibly degenerate forms of the type $(1,1)$ on $\mathcal{M}$. On any contractible chart $U \subset \mathcal{M}$ there exists a formal potential $\Phi = \Phi_0 + \nu\Phi_1 + \ldots$ of ω, which means that $\omega_r = i\partial\bar{\partial}\Phi_r$, $r \geq 0$.

Definition. Deformation quantization on a pseudo-Kähler manifold $\mathcal{M}$ is called quantization with separation of variables if for any open $U \subset \mathcal{M}$ and any holomorphic function $a(z)$ and antiholomorphic function $b(\bar{z})$ on U left $\star$-multiplication by a and right $\star$-multiplication by b are point-wise multiplications, i.e., $L_a = a$ and $R_b = b$.

We call the corresponding $\star$-product a $\star$-product with separation of variables.

In [7] a complete description of all deformation quantizations with separation of variables on an arbitrary Kähler manifold was given. It was shown that such quantizations are parametrized by the formal deformations of the original Kähler form. The results obtained in [7] are trivially valid for pseudo-Kähler manifolds as well.

Theorem 2. [7] *Deformation quantizations with separation of variables on a pseudo-Kähler manifold $\mathcal{M}$ are in one-to-one correspondence with formal deformations of the pseudo-Kähler form ω_0. If there is given a quantization with separation of variables on*

$\mathcal{M}$ corresponding to a formal deformation ω of the form ω_0, U is a contractible coordinate chart on $\mathcal{M}$ with holomorphic coordinates $\{z^k\}$, and Φ is a formal potential of ω, then the algebra $\mathcal{L}(U)$ of the left $\star$-multiplication operators consists of those formal differential operators on U which commute with all $\bar{z}^l$ and $\partial\Phi/\partial\bar{z}^l + \nu\partial/\partial\bar{z}^l$. Similarly, the algebra $\mathcal{R}(U)$ of the right $\star$-multiplication operators on U consists of those formal differential operators which commute with all z^k and $\partial\Phi/\partial z^k + \nu\partial/\partial z^k$.

Remark. Given the algebra $\mathcal{L}(U)$, one can recover the $\star$-product $f \star g$ for $f, g \in \mathcal{F}(U)$ as follows. One finds a unique operator $A \in \mathcal{L}(U)$ such that $A1 = f$. Obviously, $A = L_f$, whence $f \star g = L_f g$.

Let $(\mathcal{F}, \star)$ denote the deformation quantization with separation of variables on $\mathcal{M}$ corresponding to a formal deformation $\omega = \omega_0 + \nu\omega_1 + \ldots$ of a pseudo-Kähler form ω_0. Then for $f, g \in \mathcal{F}$ $f \star g = \sum_r \nu^r C_r(f, g)$ for bidifferential operators $C_r(\cdot, \cdot)$. Later we shall meet the product $\tilde{\star}$ on $\mathcal{F}$, opposite to the $\star$-product $\star$. This means that for $f, g \in \mathcal{F}$ $f \tilde{\star} g = g \star f = \sum_r \nu^r C_r(g, f)$, whence it is straightforward that $\tilde{\star}$ is the $\star$-product corresponding to a formal deformation quantization on the symplectic manifold $(\mathcal{M}, -\omega_0)$.

Denote by $\tilde{\mathcal{L}}, \tilde{\mathcal{R}}$ the algebras of left and right star-multiplication operators of the deformation quantization $(\mathcal{F}, \tilde{\star})$, and by $\tilde{L}_f, \tilde{R}_f$ the operators of left and right star-multiplication by an element $f \in \mathcal{F}$ respectively. It is clear that $\tilde{L}_f = R_f, \tilde{R}_f = L_f, \tilde{\mathcal{L}} = \mathcal{R}, \tilde{\mathcal{R}} = \mathcal{L}$. If a, b are, respectively, a holomorphic and antiholomorphic functions on an open subset $U \subset \mathcal{M}$, then $\tilde{L}_b = b$ and $\tilde{R}_a = a$. This means that the product $\tilde{\star}$ is a $\star$-product with separation of variables on the complex manifold $\bar{\mathcal{M}}$, opposite to $\mathcal{M}$ (i.e., with the opposite complex structure).

Let U be a contractible coordinate chart on $\mathcal{M}$ with holomorphic coordinates $\{z^k\}$, and Φ a formal potential of ω, then the algebra $\tilde{\mathcal{L}}(U) = \mathcal{R}(U)$ consists of formal operators, commuting with all z^k and $\partial\Phi/\partial z^k + \nu\partial/\partial z^k$. Since on $\bar{\mathcal{M}}$ holomorphic and antiholomorphic coordinates are swapped, the formal (1,1)-form on $\bar{\mathcal{M}}$, corresponding to the quantization $(\mathcal{F}, \tilde{\star})$ is $i\bar{\partial}\partial\Phi = -\omega$. This (1,1)-form is a formal deformation of the pseudo-Kähler form $-\omega_0$ on $\bar{\mathcal{M}}$.

Let $\mathcal{M}$ be a K-homogeneous complex manifold, $\mathbf{s}_0, \mathbf{s}_1, \ldots$ be s-modules on $\mathcal{M}$, $\{f_X^n\}$ and ω_n be the K-equivariant family and the (1,1)-form on $\mathcal{M}$, respectively, associated to $\mathbf{s}_n$. Then $df_X^n = -i(v_X)\omega_n$. Assume that $\mathbf{s}_0$ is nondegenerate, i.e., ω_0 is a pseudo-Kähler form.

Denote by $(\mathcal{F}, \star)$ the deformation quantization with separation of variables on $\mathcal{M}$ corresponding to the formal deformation $\omega = \omega_0 + \nu\omega_1 + \ldots$ of the pseudo-Kähler form ω_0. Since all the (1,1)-forms ω_n are K-invariant, the $\star$-product $\star$ is invariant under K-shifts.For $X \in k_r$ denote $f_X^{(\nu)} = f_X^0 + \nu f_X^1 + \ldots$. Introduce a formal operator $l_X^{(\nu)} = \xi_X + (i/\nu)f_X^{(\nu)}$.

Proposition 3. *The mapping $k_r \ni X \mapsto l_X^{(\nu)}$ is a Lie algebra homomorphism of k_r*

to the algebra $\mathcal{L}(\mathcal{M})$ of the left $\star$-multiplication operators of the deformation quantization $(\mathcal{F}, \star)$. It is K-equivariant with respect to the coadjoint action on k_r and the conjugation by shift operators in $\mathcal{L}(\mathcal{M})$.

Proof. The mapping $k_r \ni X \mapsto l_X^{(\nu)}$ is a K-equivariant Lie algebra homomorphism to the Lie algebra of formal operators on $\mathcal{M}$ if and only if $\xi_X f_Y^{(\nu)} - \xi_Y f_X^{(\nu)} = f_{[X,Y]}^{(\nu)}$ and $T_k f_X^{(\nu)} = f_{Ad(k)X}^{(\nu)}$ for all $X, Y \in k_r$ and $k \in K$. These relations follow immediately from the corresponding relations for the functions f_X^n. Theorem 2 tells that in order to show that $l_X^{(\nu)} \in \mathcal{L}(\mathcal{M})$ one has to check that for a formal potential Φ of ω on any contractible coordinate chart U with holomorphic coordinates $\{z^k\}$ the formal operator $l_X^{(\nu)} = \xi_X + (i/\nu) f_X^{(\nu)}$ commutes with all $\bar{z}^l$ and $\partial\Phi/\partial\bar{z}^l + \nu\partial/\partial\bar{z}^l$. Thus we have to check the equality

$$\xi_X(\partial\Phi/\partial\bar{z}^l) = i\partial f_X^{(\nu)}/\partial\bar{z}^l. \tag{5}$$

Taking into account that $\omega = i\partial\bar{\partial}\Phi = i(\partial^2\Phi/\partial z^k\partial\bar{z}^l)dz^k \wedge d\bar{z}^l$ and writing down the local expression for ξ_X, $\xi_X = a^k(z)\partial/\partial z^k$, we rewrite the left hand side of (5) as follows, $a^k(z)\partial^2\Phi/\partial z^k\partial\bar{z}^l$. On the other hand, $i\partial f_X^{(\nu)}/\partial\bar{z}^l = i < -i(v_X)\omega, \partial/\partial\bar{z}^l >= -i\omega(v_X, \partial/\partial\bar{z}^l) = a^k(z)\partial^2\Phi/\partial z^k\partial\bar{z}^l$, which proves (5) and completes the proof of the Proposition.

Extend the mapping $k_r \ni X \mapsto l_X^{(\nu)}$ to the homomorphism $\mathcal{U}(g_c) \ni u \mapsto l_u^{(\nu)}$ from $\mathcal{U}(g_c)$ to $\mathcal{L}(\mathcal{M})$ and set $\sigma_u^{(\nu)} = l_u^{(\nu)}1$.

Corollary. *The mapping $\mathcal{U}(g_c) \ni u \mapsto \sigma_u^{(\nu)}$ is a homomorphism from $\mathcal{U}(g_c)$ to the algebra $(\mathcal{F}, \star)$, K-equivariant with respect to the adjoint action on $\mathcal{U}(g_c)$ and the shift action on $\mathcal{F}$.*

It follows that the mapping $\mathcal{U}(g_c) \ni u \mapsto \sigma_u^{(\nu)}$ maps the elements of the center $\mathcal{Z}(g_c)$ of $\mathcal{U}(g_c)$ to formal series with constant coefficients.

Lemma 7. *For $z \in \mathcal{Z}(g_c)$ the operator $l_z^{(\nu)}$ is scalar and is equal to $\sigma_z^{(\nu)}$.*

Proof. If $A \in \mathcal{L}(\mathcal{M})$ then $A = L_f$ for $f = A1 \in \mathcal{F}$. Therefore $l_z^{(\nu)} = L_{\sigma_z^{(\nu)}}$. Let B denote the multiplication operator by the formal series with constant coefficients $\sigma_z^{(\nu)}$. It commutes with all formal differential operators and therefore $B \in \mathcal{L}(\mathcal{M})$. Since $B1 = \sigma_z^{(\nu)}$ we get that $l_z^{(\nu)} = B$. The Lemma is proved.

It was shown in Section 3 that for a given s-module $\mathbf{s}$ on $\mathcal{M}$ the function $\sigma_z = l_z 1$, $z \in \mathcal{Z}(g_c)$, is scalar and is equal to the value $\psi(z)$ of the central character ψ associated to $\mathbf{s}$. Yet it does not mean that for $z \in Z$ the corresponding operator l_z is scalar. We shall use deformation quantization to prove the following proposition.

Proposition 4. *Let $\mathbf{s}_1$ be an arbitrary s-module on $\mathcal{M}$, l_u, $u \in \mathcal{U}(g_c)$, and ψ be the associated operators and the central character of $\mathcal{U}(g_c)$ respectively. If the set of nondegenerate s-modules on $\mathcal{M}$ is non-empty then for $z \in \mathcal{Z}(g_c)$ holds $l_z = \psi(z) \cdot \mathbf{1}$.*

(We denote by $\mathbf{1}$ the identity operator.)

Proof. Choose a nondegenerate s-module $\mathbf{s}_0$. Denote by $\{f_X^j\}$ and by ω_j the K-equivariant family and the (1,1)-form associated to $\mathbf{s}_j$, $j = 0, 1$, respectively. Consider a parameter dependent s-module $\mathbf{s}(t) = t\mathbf{s}_0 + \mathbf{s}_1$. The K-equivariant family $\{f_X\}$ associated to $\mathbf{s}(t)$ is such that $f_X = tf_X^0 + f_X^1$. Thus for $X \in k_r$ the operator $l_X(t)$ associated to $\mathbf{s}(t)$ is given by the formula $l_X(t) = \xi_X + i(tf_X^0 + f_X^1)$. When $t = 0$ the operator $l_X(t)$ reduces to the operator $l_X = \xi_X + if_X^1$ associated to the (possibly degenerate) s-module $\mathbf{s}_1$. If we replace the parameter t in $l_X(t)$ by $1/\nu$ we will get the operator $l_X^{(\nu)} = \xi_X + (i/\nu)(f_X^0 + \nu f_X^1)$ of the deformation quantization with separation of variables $(\mathcal{F}, \star)$ which corresponds to the formal (1,1)-form $\omega = \omega_0 + \nu\omega_1$. For $z \in \mathcal{Z}(g_c)$ the operator $l_z(t)$ is polynomial in t. If we replace t by $1/\nu$ in $l_z(t)$ we will get the operator $l_z^{(\nu)}$ which is scalar by Lemma 7. Therefore $l_z(t)$ is scalar as well. Taking $t = 0$ we get that the operator $l_z(0) = l_z$ associated to the s-module $\mathbf{s}_1$ is scalar. Since $l_z 1 = \sigma_z = \psi(z)$ it follows that $l_z = \psi(z) \cdot \mathbf{1}$. This completes the proof.

Theorem 3. *Let $(L, h) \to \mathcal{M}$ be a hermitian line bundle on $\mathcal{M}$ on which the group K acts by holomorphic automorphisms which preserve the metrics h. The algebra $\mathcal{U}(g_c)$ acts on (L, h) by holomorphic differential operators A_u, $u \in \mathcal{U}(g_c)$. Let $\mathbf{s}$ be the corresponding s-module on $\mathcal{M}$ and ψ be the central character of $\mathcal{U}(g_c)$ associated to $\mathbf{s}$. If the set of nondegenerate s-modules on $\mathcal{M}$ is non-empty, the center $\mathcal{Z}(g_c)$ of $\mathcal{U}(g_c)$ acts on the sheaf of local holomorphic sections of L by scalar operators $A_z = \psi(z) \cdot \mathbf{1}$, $z \in \mathcal{Z}(g_c)$.*

Proof. It follows from Proposition 3 and Corollary to Proposition 1 that for $z \in \mathcal{Z}(g_c)$ the pushforward of the holomorphic differential operator A_z from L to $\mathcal{M}$ is scalar and is equal to $\psi(z) = \sigma_z$. Now the theorem is a consequence of the fact that the pushforward mapping $A \mapsto \check{A}$ is injective.

6. s-modules on flag manifolds

We are going to apply the results obtained above to the case of K being a compact semisimple Lie group. The general facts from the theory of semisimple Lie groups mentioned below may be found in [16].

Let g_c be a complex semisimple Lie algebra, h_c its Cartan subalgebra, h_c^* the dual of h_c, W the Weyl group of the pair (g_c, h_c), $\Delta, \Delta^+, \Delta^-, \Sigma \subset h_c^*$ the sets of all nonzero, positive, negative and simple roots respectively, δ the half-sum of positive roots. For each $\alpha \in \Delta$ choose weight elements $X_\alpha \in g_c$ such that $[H_\alpha, X_{\pm\alpha}] = \pm 2X_{\pm\alpha}$ for $H_\alpha = [X_\alpha, X_{-\alpha}]$.

An element $\lambda \in h_c^*$ is called dominant if $\lambda(H_\alpha) \geq 0$ for all $\alpha \in \Sigma$, and is a weight if $\lambda(H_\alpha) \in \mathbf{Z}$ for all $\alpha \in \Sigma$. Denote by $\mathcal{W}$ the set of all weights in h_c^* (the weight lattice).

Fix an arbitrary subset Θ of Σ and denote by $< \Theta >$ the set of roots which are linear combinations of elements of Θ. Then $\Pi =< \Theta > \cup \Delta^-$ is a parabolic subset of Δ. Denote by g_c^Θ the Levi subalgebra of g_c generated by h_c and X_α, $\alpha \in < \Theta >$, and by q_c the parabolic subalgebra generated by h_c and X_α, $\alpha \in \Pi$.

Denote by g_r, q_r, g_r^Θ the realifications of g_c, q_c, g_c^Θ respectively, and by J the complex structure in g_r inherited from g_c.

Let $k_r \subset g_r$ denote the compact form of g_c generated by $JH_\alpha, X_\alpha - X_{-\alpha}, J(X_\alpha + X_{-\alpha})$, $\alpha \in \Delta$. Define $k_r^\Theta = k_r \cap g_c^\Theta = k_r \cap q_c$. It is generated by JH_α, $\alpha \in \Delta$, and $X_\alpha - X_{-\alpha}, J(X_\alpha + X_{-\alpha})$, $\alpha \in <\Theta>$.

Introduce the real Lie algebra $t_r = h_c \cap k_r$, the Lie algebra of a maximal torus in K. It is generated by JH_α, $\alpha \in \Delta$.

Let G be a complex connected simply connected Lie group with the Lie algebra g_r, G^Θ and Q the Levi and parabolic subgroups of G with the Lie algebras g_r^Θ and q_r respectively. In the rest of this paper K will denote maximal compact subgroup of G with the Lie algebra k_r, and $K^\Theta = K \cap G^\Theta = K \cap Q$. It is known that K^Θ is the centralizer of a torus and is connected, and that $G/Q = K/K^\Theta$ is a complex compact homogeneous manifold (a generalized flag manifold). Denote it by $\mathcal{M}$.

Denote by x_0 the class of the unit element of K in $\mathcal{M}$ (the "origin" of $\mathcal{M}$) and by $\mathcal{E}$ the set of all K^Θ-invariant points of k_r. The set of K-equivariant mappings $\gamma : \mathcal{M} \to k_r$ is parametrized by $\mathcal{E}$ so that γ corresponds to $E = \gamma(x_0) \in \mathcal{E}$. Since the group K^Θ is connected, the set $\mathcal{E}$ is the centralizer of k_r^Θ. It is easy to check that $\mathcal{E} = \{H \in t_r | \alpha(H) = 0$ for all $\alpha \in <\Theta>\}$.

Denote by $(\cdot,\cdot)$ the Killing form on g_c. It is C-linear, and its restriction to k_r is negative-definite.

Identify the dual k_r^* of the Lie algebra k_r with k_r via the Killing form. We are going to show that any K-equivariant mapping $\gamma : \mathcal{M} \to k_r$ (or the K-equivariant family defined by γ) corresponds to an s-module on $\mathcal{M}$. Let $\Omega \subset k_r$ be the orbit of the point $E = \gamma(x_0) \in \mathcal{E}$, ω^Ω be the Kirillov 2-form on Ω, and v_X^Ω, $X \in k_r$, the fundamental vector fields on Ω. Then the 2-form ω on $\mathcal{M}$ corresponding to γ equals $\gamma^*\omega^\Omega$. It is known that at the point $E \in \Omega$ for $X, Y \in k_r$ holds $\omega^\Omega(v_X^\Omega, v_Y^\Omega) = (E, [X,Y])$. Thus at the point $x_0 \in \mathcal{M}$ $\omega(v_X, v_Y) = (E, [X,Y])$. The tangent space $T_{x_0}\mathcal{M}$ to the complex manifold $\mathcal{M}$ carries the natural complex structure $\tilde{J}$. In view of Theorem 1 in order to show that the mapping γ corresponds to an s-module it is enough to check that the form ω on the tangent space $T_{x_0}\mathcal{M}$ is of the type (1,1) or, equivalently, that for any $v_1, v_2 \in T_{x_0}\mathcal{M}$ holds $\omega(v_1, v_2) = \omega(\tilde{J}v_1, \tilde{J}v_2)$. We can identify $T_{x_0}\mathcal{M}$ as a real vector space with the subspace of k_r generated by the basis consisting of the elements $X_\alpha - X_{-\alpha}, J(X_\alpha + X_{-\alpha})$, $\alpha \in \Delta^+ \setminus <\Theta>$. Since $g_r/q_r = k_r/k_r^\Theta$, we get that $\tilde{J}(X_\alpha - X_{-\alpha}) = J(X_\alpha + X_{-\alpha})$ for $\alpha \in \Delta^+ \setminus <\Theta>$. The tangent space $T_{x_0}\mathcal{M}$ can be represented as the direct sum of 2-dimensional real subspaces spanned by the vectors $X_\alpha - X_{-\alpha}, J(X_\alpha + X_{-\alpha})$, $\alpha \in \Delta^+ \setminus <\Theta>$. These subspaces are mutually orthogonal with respect to the skew-symmetric form $(E, [\cdot,\cdot])$. Now, for $\alpha \in \Delta^+ \setminus <\Theta>$ we have $(E, [\tilde{J}(X_\alpha - X_{-\alpha}), \tilde{J}J(X_\alpha + X_{-\alpha})]) = (E, [J(X_\alpha + X_{-\alpha}), -(X_\alpha - X_{-\alpha})]) = (E, [X_\alpha - X_{-\alpha}, J(X_\alpha + X_{-\alpha})]) = i(E, [X_\alpha - X_{-\alpha}, X_\alpha + X_{-\alpha}]) = 2i(E, H_\alpha)$. Thus the form ω is of the type (1,1). For $\alpha \in \Delta\setminus <\Theta>$ the linear functional $\mathcal{E} \ni H \mapsto \alpha(H)$ is nonzero, therefore the set $\mathcal{E}_{reg} = \{H \in \mathcal{E} | (H, H_\alpha) \neq 0 \text{ for all } \alpha \in \Delta\setminus <\Theta>\}$ is a dense open subset of $\mathcal{E}$. The form ω is nondegenerate iff $E \in \mathcal{E}_{reg}$.

It is known that under the adjoint action of the compact group K on k_r the isotropy subgroup of any element of k_r is connected. Now if ω is nondegenerate, the isotropy subgroup of $E = \gamma(x_0)$ coincides with K^Θ and thus the mapping $\gamma : \mathcal{M} \to \Omega$ is a bijection.

Define a sesquilinear form $<\cdot,\cdot>$ on $(T\mathcal{M}, \tilde{J})$ by the formula $<v_1, v_2> = \omega(v_1, \tilde{J}v_2) - i\omega(v_1, v_2)$. If ω is nondegenerate and thus pseudo-Kähler, the form $<\cdot,\cdot>$ is the corresponding pseudo-Kähler metrics on $\mathcal{M}$. The vectors $X_\alpha - X_{-\alpha}$, $\alpha \in \Delta^+\setminus <\Theta>$, form a basis in the complex vector space $(T_{x_0}\mathcal{M}, \tilde{J})$. They are orthogonal with respect to the form $<\cdot,\cdot>$. We have $<X_\alpha - X_{-\alpha}, X_\alpha - X_{-\alpha}> = (E, [X_\alpha - X_{-\alpha}, J(X_\alpha + X_{-\alpha})]) = i(E, [X_\alpha - X_{-\alpha}, X_\alpha + X_{-\alpha}]) = 2i(E, H_\alpha) = 2(E, JH_\alpha)$. (Notice that since $E, JH_\alpha \in t_r$, (E, JH_α) is real.) Thus we have proved the following theorem.

Theorem 4. *To an arbitrary K-equivariant mapping $\gamma : \mathcal{M} \to k_r$ there corresponds an s-module* s *on $\mathcal{M}$. It is nondegenerate iff for $E = \gamma(x_0)$ and all $\alpha \in \Delta\setminus <\Theta>$ holds $(E, H_\alpha) \neq 0$. The set of nondegenerate s-modules on $\mathcal{M}$ is non-empty. For a nondegenerate* s *the associated mapping $\gamma : \mathcal{M} \to \Omega = \gamma(\mathcal{M})$ is a bijection and the pseudo-Kähler structure on $\mathcal{M}$, pushed forward to the orbit Ω defines a pseudo-Kähler polarization on it. The index of inertia of the corresponding pseudo-Kähler metrics $<\cdot,\cdot>$ on $\mathcal{M}$ (i.e. the number of minuses in the signature) equals $\#\{\alpha \in \Delta^+\setminus <\Theta> | (E, JH_\alpha) < 0\}$.*

7. Convergent star-products on flag manifolds

We are going to extend the class of convergent star-products on generalized flag manifolds introduced in [4], using results from [8]. We retain the notations of Section 6. In particular, the group K is compact semisimple and $\mathcal{M}$ is a generalized flag manifold.

A representation of the group K in a vector space V is called K-finite if any vector $v \in V$ is K-finite, i.e., the set $\{kv\}$, $k \in K$, is contained in a finite dimensional subspace of V. If this is the case, V splits into the direct sum of isotypic components. For a dominant weight $\zeta \in \mathcal{W}$ denote by V^ζ the component isomorphic to a multiple of irreducible representation of K with highest weight ζ.

For a K-homogeneous manifold M denote by $F(M)$ the space of continuous functions on M K-finite with respect to the shift action. Since K is compact, it follows from the Frobenius theorem that each isotypic component $F(M)^\zeta$ is finite dimensional.

Let $\Omega \subset k_r$ be a K-orbit. A function on Ω is called regular if it is the restriction of a polynomial function on k_r. It is easy to show that the set of all regular functions on Ω coincides with $F(\Omega)$ (see, e.g.,[8]).

Let d be a nonnegative integer. Denote by $\mathcal{U}_d$ the subspace of $\mathcal{U}(g_c)$, generated by all monomials of the form $X_1 \dots X_k$, where $X_1, \dots, X_k \in g_c$ and $k \le d$. The subspaces $\{\mathcal{U}_d\}$ determine the canonical filtration on $\mathcal{U}(g_c)$.

The symmetric algebra $S(g_c)$ can be identified with the space of polynomials on k_r, so that the element $X \in g_c$ corresponds to the linear functional on k_r, $\tilde{X}(Y) = (X, Y)$, $Y \in k_r$. Let $S^d(g_c)$ be the space of homogeneous polynomials on k_r of degree d. The graded algebra, associated with the canonical filtration on $\mathcal{U}(g_c)$ is canonically isomorphic to $S(g_c)$, so that $\mathcal{U}_d/\mathcal{U}_{d-1}$ corresponds to $S^d(g_c)$. For $u \in \mathcal{U}_d$ let $\underline{u}^{(d)}$ denote the corresponding element of $S^d(g_c)$. If $k \le d$ and $u = X_1 \dots X_k \in \mathcal{U}_d$, then $\underline{u}^{(d)} = 0$ for $k < d$ and $\underline{u}^{(d)} = \tilde{X}_1 \dots \tilde{X}_d$ for $k = d$.

We say that a parameter dependent vector $v(\hbar)$ in a vector space V depends rationally on a real parameter $\hbar$ if $v(\hbar)$ can be represented in a form $v(\hbar) = \sum_j a_j(\hbar) v_j$ for a finite number of elements $v_j \in V$ and rational functions $a_j(\hbar)$, i.e., $v(\hbar) \in \mathbf{C}((\hbar)) \otimes V$, where $\mathbf{C}((\hbar))$ is the field of rational functions of $\hbar$. Denote by $O(\hbar) \subset \mathbf{C}((\hbar))$ the ring of rational functions of $\hbar$ regular at $\hbar = 0$. Vector $v(\hbar)$ is called regular at $\hbar = 0$ if $v(\hbar) \in O(\hbar) \otimes V$.

Let $v(\hbar) = \sum_r \hbar^r v_r$, $v_r \in V$, be the Laurent expansion of $v(\hbar)$ at $\hbar = 0$. Since $v(\hbar)$ depends rationally on $\hbar$, its Laurent expansion has a finite polar part. Denote by $\Psi(v(\hbar))$ the corresponding formal Laurent series, $\Psi(v(\hbar)) = \sum_r \nu^r v_r$.

The set S of s-modules on $\mathcal{M}$ is a finite dimensional vector space. Thus we can consider an s-module $\mathbf{s}(\hbar)$ on $\mathcal{M}$ depending rationally on $\hbar$ and regular at $\hbar = 0$. Denote by $\omega(\hbar)$ the (1,1)-form associated to $\mathbf{s}(\hbar)$. It is clear that $\omega(\hbar)$ also depends rationally on $\hbar$ and is regular at $\hbar = 0$. Moreover, $\Psi(\mathbf{s}(\hbar)) = \sum_{r \ge 0} \nu^r \mathbf{s}_r$ for some $\mathbf{s}_r \in S$ and $\Psi(\omega(\hbar)) = \sum_{r \ge 0} \nu^r \omega_r$ where ω_r is the (1,1)-form associated to $\mathbf{s}_r$.

Denote by $\gamma(\hbar)$, $\gamma_r : \mathcal{M} \to k_r$ the K-equivariant mappings and by $\{f_X^{(\hbar)}\}$, $\{f_X^r\}$ the K-equivariant families corresponding to $\mathbf{s}(\hbar)$, $\mathbf{s}_r$ respectively. For $X \in k_r$ the function $f_X^{(\hbar)}$ on $\mathcal{M}$ depends rationally on $\hbar$, is regular at $\hbar = 0$ and $\Psi(f_X^{(\hbar)}) = \sum_{r \ge 0} \nu^r f_X^r$.

The K-equivariant family $\{(1/\hbar) f_X^{(\hbar)}\}$ corresponds to the s-module $(1/\hbar)\mathbf{s}(\hbar)$.

It will be convenient to us to denote by $l_u^{(\hbar)}$, $u \in \mathcal{U}(g_c)$, the operators on $\mathcal{M}$ associated to the s-module $(1/\hbar)\mathbf{s}(\hbar)$ (rather than to $\mathbf{s}(\hbar)$) and set $\sigma_u^{(\hbar)} = l_u^{(\hbar)} 1$. In particular, for $X \in k_r$ $l_X^{(\hbar)} = \xi_X + (i/\hbar) f_X^{(\hbar)}$ and $\sigma_X^{(\hbar)} = (i/\hbar) f_X^{(\hbar)}$.

Let $\mathcal{D}_\hbar$ denote the algebra of differential operators on $\mathcal{M}$ depending rationally on $\hbar$.

Lemma 8. *For $u \in \mathcal{U}_d$ the differential operator $\hbar^d l_u^{(\hbar)}$ belongs to $\mathcal{D}_\hbar$. It is regular at $\hbar = 0$ and $\lim_{\hbar \to 0} \hbar^d l_u^{(\hbar)}$ is a multiplication operator by the function $i^d\ \underline{u}^{(d)} \circ \gamma_0$. In particular, the function $\hbar^d \sigma_u^{(\hbar)}$ depends rationally on $\hbar$, is regular at $\hbar = 0$ and $\lim_{\hbar \to 0} \hbar^d \sigma_u^{(\hbar)} = i^d\ \underline{u}^{(d)} \circ \gamma_0$.*

Proof. The function $f_X^{(\hbar)}$ equals $f_X^0 = \underline{X}^{(1)} \circ \gamma_0$ at $\hbar = 0$. Let $u = X_1 \dots X_k$, $X_j \in k_r$, for $k \le d$. Then $u \in \mathcal{U}_d$. We have $\hbar^d l_u^{(\hbar)} = \hbar^d l_{X_1}^{(\hbar)} \dots l_{X_k}^{(\hbar)} = \hbar^{d-k}(\hbar \xi_{X_1} + i f_{X_1}^{(\hbar)}) \dots (\hbar \xi_{X_k} +$

$if^{(\hbar)}_{X_k})$. Thus the limit $\lim_{\hbar\to 0}\hbar^d l^{(\hbar)}_u$ equals zero if $k < d$ and equals $i^d f^0_{X_1}\dots f^0_{X_d} = i^d\, \underline{u}^{(d)}\circ\gamma_0$ if $k = d$, whence the Lemma follows immediately.

For the rest of the section denote $\Omega = \gamma_0(\mathcal{M})$ and assume that the s-module $\mathbf{s}_0$ is nondegenerate, so that ω_0 is pseudo-Kähler. Then Theorem 4 implies that the K-equivariant mapping $\gamma_0 : \mathcal{M} \to \Omega$ is a bijection.Thus $\gamma_0^* : F(\Omega) \to F(\mathcal{M})$ is an isomorphism of K-modules.

Proposition 5. *If the s-module $\mathbf{s}_0$ is nondegenerate, then for any $f \in F(\mathcal{M})$ there exist elements $u_j \in \mathcal{U}_{d(j)}$ for some numbers $d(j)$ and rational functions $a_j(\hbar)$ regular at $\hbar = 0$, such that $f = \sum_j \hbar^{d(j)} a_j(\hbar)\sigma^{(\hbar)}_{u_j}$ for all but a finite number of values of $\hbar$.*

Proof. Fix a dominant weight $\zeta \in \mathcal{W}$. The subspace $\mathcal{U}_d \subset \mathcal{U}(g_c)$ is invariant under the adjoint action of the group K, and is finite dimensional. The mapping $\mathcal{U}_d \ni u \mapsto \underline{u}^{(d)} \in S^d(g_c)$ is K-equivariant, therefore it maps $\mathcal{U}^\zeta_d$ to $F(\Omega)^\zeta$. Since the space $F(\Omega)$ coincides with the space of regular functions on Ω and is isomorphic to $F(\mathcal{M})$, one can choose elements $u_j \in \mathcal{U}^\zeta_{d(j)}$ for some numbers $d(j)$ such that the functions $f_j = i^{d(j)}\underline{u_j}^{(d(j))}\circ\gamma_0$ form a basis $\{f_j\}$ in $F(\mathcal{M})^\zeta$. Since the function $\tilde{f}_j = \hbar^{d(j)}\,\sigma^{(\hbar)}_{u_j}$ on $\mathcal{M}$ depends rationally on $\hbar$ and is regular at $\hbar = 0$, the elements of the matrix $(b_{jk}(\hbar))$ such that $\tilde{f}_j = \sum_k b_{jk}(\hbar) f_k$ are rational functions of $\hbar$ regular at $\hbar = 0$. It follows from Lemma 8 that the matrix $(b_{jk}(\hbar))$ coincides with the identity matrix at $\hbar = 0$. Thus the elements of the inverse matrix $(a_{kj}(\hbar)) = (b_{jk}(\hbar))^{-1}$ such that $f_k = \sum_j a_{kj}(\hbar)\tilde{f}_j = \sum_j a_{kj}(\hbar)\hbar^{d(j)}\,\sigma^{(\hbar)}_{u_j}$ are also rational functions of $\hbar$ regular at $\hbar = 0$. Now the Proposition follows from the fact that the space $F(\mathcal{M})$ is a direct sum of the subspaces $F(\mathcal{M})^\zeta$.

Let $(\mathcal{A}_\hbar, *_\hbar)$ denote the algebra of functions on $\mathcal{M}$ associated to the s-module $(1/\hbar)\mathbf{s}(\hbar)$. Any function $f \in F(\mathcal{M})$ can be represented in the form $f = \sum_j \hbar^{d(j)} a_j(\hbar)\sigma^{(\hbar)}_{u_j}$ for some $u_j \in \mathcal{U}_{d(j)}$. Thus $f \in \mathcal{A}_\hbar$ for all but a finite number of values of $\hbar$.

For a function $g \in \mathcal{A}_\hbar$ holds $f *_\hbar g = \sum_j \hbar^{d(j)} a_j(\hbar) l^{(\hbar)}_{u_j} g$. We get from Lemma 8 the following corollary to Proposition 5.

Corollary. *Any functions $f, g \in F(\mathcal{M})$ are elements of the algebra $(\mathcal{A}_\hbar, *_\hbar)$ for all but a finite number of values of $\hbar$. The product $f *_\hbar g$ as a function on $\mathcal{M}$ depends rationally on $\hbar$ and is regular at $\hbar = 0$, i.e., $f *_\hbar g \in O(\hbar)\otimes F(\mathcal{M})$.*

Remark. It is easy to show that extending the multiplication $*_\hbar$ by $O(\hbar)$-linearity we obtain the associative algebra $(O(\hbar)\otimes F(\mathcal{M}), *_\hbar)$ over the ring $O(\hbar)$ of rational functions of $\hbar$ regular at $\hbar = 0$.

Denote by ω the formal (1,1)-form $\Psi(\omega(\hbar)) = \omega_0 + \nu\omega_1 + \dots$. It is a formal deformation of the pseudo-Kähler form ω_0. Denote by $(\mathcal{F}, \star)$ the deformation quantization with separation of variables on $\mathcal{M}$ corresponding to ω.

Set $f^{(\nu)}_X = \Psi(f^{(\hbar)}_X) = f^0_X + \nu f^1_X + \dots$. Then $\Psi(l^{(\hbar)}_X) = \xi_X + (i/\nu) f^{(\nu)}_X$. It follows from Proposition 3 that for $X \in k_r$ the operator $l^{(\nu)}_X = \xi_X + (i/\nu) f^{(\nu)}_X$ belongs to the algebra $\mathcal{L}$ of left $\star$-multiplication operators of the deformation quantization $(\mathcal{F}, \star)$. It is easy to check that the mapping $\mathcal{D}_\hbar \ni A \mapsto \Psi(A)$ is a homomorphism from $\mathcal{D}_\hbar$ to the algebra of formal differential operators on $\mathcal{M}$, therefore for $u \in \mathcal{U}(g_c)$ $\Psi(l_u^{(\hbar)}) = l_u^{(\nu)} \in \mathcal{L}$.

Represent a function $f \in F(\mathcal{M})$ in the form $f = \sum_j \hbar^{d(j)} a_j(\hbar)\sigma_{u_j}^{(\hbar)}$ for some $u_j \in \mathcal{U}_{d(j)}$ and consider the operator $A = \sum_j \hbar^{d(j)} a_j(\hbar) l_{u_j}^{(\hbar)} \in \mathcal{D}_\hbar$. It follows from Lemma 8 that A is regular at $\hbar = 0$. It is straightforward that $\Psi(A) \in \mathcal{L}$ and $A1 = f$, whence one can easily obtain that $\Psi(A)1 = f$ and therefore $\Psi(A) = L_f$.

For $g \in F(\mathcal{M})$ the product $f *_\hbar g = Ag$ is a function on $\mathcal{M}$ which depends rationally on $\hbar$ and is regular at $\hbar = 0$. Therefore the product $f *_\hbar g$ expands to the uniformly and absolutely convergent Taylor series in $\hbar$ at $\hbar = 0$. Finally, $\Psi(f *_\hbar g) = \Psi(Ag) = L_f g = f \star g$. Thus we have proved the following theorem.

Theorem 5. *Let $\mathbf{s}(\hbar)$ be an s-module on $\mathcal{M}$ which depends rationally on the parameter $\hbar$ and is regular at $\hbar = 0$, and $\omega(\hbar)$ be the associated (1,1)-form. Then $\Psi(\mathbf{s}(\hbar)) = \sum_{r \geq 0} \nu^r \mathbf{s}_r$ for some $\mathbf{s}_r \in S$. Assume that the s-module $\mathbf{s}_0$ is nondegenerate and denote by $(\mathcal{A}_\hbar, *_\hbar)$ the algebra of functions associated to the s-module $(1/\hbar)\mathbf{s}(\hbar)$. Any functions $f, g \in F(\mathcal{M})$ belong to $\mathcal{A}_\hbar$ for all but a finite number of values of $\hbar$. The product $f *_\hbar g$ expands to the uniformly and absolutely convergent Taylor series in $\hbar$ at the point $\hbar = 0$,*
$f *_\hbar g = \sum_{r \geq 0} \hbar^r C_r(f, g)$, where $C_r(\cdot,\cdot), r = 0, 1, \ldots$, are bidifferential operators which define the deformation quantization with separation of variables on $\mathcal{M}$ corresponding to the formal deformation $\omega = \Psi(\omega(\hbar)) = \omega_0 + \nu\omega_1 + \ldots$ of the pseudo-Kähler form ω_0.

8. Characters associated to s-modules on flag manifolds

Since the group K is compact, there exists the K-invariant measure μ of the total volume 1 on the flag manifold $\mathcal{M}$. Let $\mathbf{s}$ be an s-module on $\mathcal{M}$ and $\mathcal{A}$ the corresponding algebra of functions on $\mathcal{M}$.

It is known that $\mathcal{U}(g_c) = \mathcal{Z}(g_c) \oplus [\mathcal{U}(g_c), \mathcal{U}(g_c)]$ (see [16]). Let $\mathcal{U}(g_c) \ni u \mapsto u^0$ denote the corresponding projection of $\mathcal{U}(g_c)$ onto $\mathcal{Z}(g_c)$. Recall the following

Definition (see [16]). A linear form $\kappa : \mathcal{U}(g_c) \to \mathrm{C}$ is called a character of g_c if:

(1) $\kappa(uv) = \kappa(vu)$, $\kappa(1) = 1$;

(2) $\kappa(u^0 v) = \kappa(u)\kappa(v) \quad (u, v \in \mathcal{U}(g_c))$.

Thus one has $\kappa(u) = \kappa(u^0)$ for all $u \in \mathcal{U}(g_c)$. Moreover, κ is then a homomorphism of $\mathcal{Z}(g_c)$ into C, a central character of g_c. This central character determines κ completely.

Fix an s-module $\mathbf{s}$ on $\mathcal{M}$ and let l_u, $u \in \mathcal{U}(g_c)$, be the operators on $\mathcal{M}$, associated to $\mathbf{s}$, $\sigma_u = l_u 1$ and ψ be the corresponding central character of $\mathcal{U}(g_c)$, $\psi(z) = \sigma_z$, $z \in \mathcal{Z}(g_c)$.

Proposition 6. *A linear form $\kappa(u) = \int_{\mathcal{M}} \sigma_u \, d\mu$, $u \in \mathcal{U}(g_c)$, on $\mathcal{U}(g_c)$ is a character of g_c. For $z \in \mathcal{Z}(g_c)$ $\quad \kappa(z) = \psi(z)$.*

Proof. Since the mapping $u \mapsto \sigma_u = l_u 1$ is K-equivariant and the measure μ is K-invariant, for $k \in K$ holds $\kappa(Ad(k)u) = \kappa(u)$ or, infinitesimally, for $X \in k_r$ $\kappa(Xu -$

$uX) = 0$, therefore $\kappa(uv) = \kappa(vu)$. The measure μ is of the total volume 1 and for $z \in \mathcal{Z}(g_c)$ $\sigma_z = \psi(z)$ is scalar, therefore $\kappa(z) = \psi(z)$. In particular, $\kappa(1) = 1$. Thus (1) is proved. Now, using that for $u \in \mathcal{U}(g_c)$ holds $u^0 \in \mathcal{Z}(g_c)$ and $l_{u^0} = \sigma_{u^0}$ is scalar, we get $\kappa(u^0 v) = \int l_{u^0 v} 1 \, d\mu = \int l_{u^0} l_v 1 \, d\mu = \sigma_{u^0} \int l_v 1 \, d\mu = \kappa(u^0)\kappa(v)$. This completes the proof of the Proposition.

Remark. Proposition 6 implies that to any s-module $\mathbf{s}$ on $\mathcal{M}$ there corresponds a character κ of the Lie algebra g_c. The central character $\mathcal{Z}(g_c) \ni z \mapsto \kappa(z)$ of $\mathcal{U}(g_c)$ coincides with the central character ψ associated to $\mathbf{s}$. Moreover, the mapping $\mathcal{A} \ni f \mapsto t(f) = \int_{\mathcal{M}} f \, d\mu$ is a trace on the algebra $\mathcal{A}$, i.e., $t(f * g) = t(g * f)$ for $f, g \in \mathcal{A}$.

Let τ be an n-dimensional irreducible representation of g_c. Since for $z \in \mathcal{Z}(g_c)$ $\tau(z)$ is scalar, it is straightforward that $\kappa_\tau = (1/n)\mathrm{tr}\ \tau$ is a character of g_c. In particular, $\mathcal{Z}(g_c) \ni z \mapsto \kappa_\tau(z)$ is the central character of the representation τ.

Proposition 7. *Let τ be an n-dimensional irreducible representation of $\mathcal{U}(g_c)$ in the vector space V and $\mathbf{s}$ be an s-module on $\mathcal{M}$. If the central character ψ associated to $\mathbf{s}$ coincides with the central character of the representation τ, then there exists a representation ρ of the algebra $\mathcal{A}$ in the same vector space V such that $\tau = \rho \circ \sigma$.*

Proof. Since the characters $\kappa_\tau = (1/n)\mathrm{tr}\ \tau$ and $\kappa = \int_{\mathcal{M}} \sigma \, d\mu$ of g_c coincide on $\mathcal{Z}(g_c)$, they coincide identically. We have that for $u, v \in \mathcal{U}(g_c)$

$$\mathrm{tr}(\tau(u)\tau(v)) = \mathrm{tr}\ \tau(uv) = n \int \sigma_{uv} \, d\mu = n \int \sigma_u * \sigma_v \, d\mu. \tag{6}$$

Assume that $\sigma_v = 0$. Then the last expression in (6) is zero for all $u \in \mathcal{U}(g_c)$. Since τ is irreducible, $\tau(u)$ is an arbitrary endomorphism of the representation space V, therefore $\tau(v) = 0$. Thus the representation τ factors through the mapping $\sigma : \mathcal{U}(g_c) \to \mathcal{A}$, $\tau = \rho \circ \sigma$. The Proposition is proved.

9. Holomorphic line bundles on flag manifolds

The Levi subgroup $G^\Theta \subset G$ is reductive. The pair (g_c^Θ, h_c) has a root system $< \Theta >$. Induce the ordering on $< \Theta >$ from Δ. Denote by W^Θ the Weyl group of the pair (g_c^Θ, h_c), by δ_Θ the half-sum of positive roots from $< \Theta >$, and set $\delta'_\Theta = \delta - \delta_\Theta$. The one-dimensional holomorphic representations (the holomorphic characters) of G^Θ are parametrized by the set $\mathcal{W}^\Theta$ of W^Θ-invariant weights from $\mathcal{W}$. The parabolic group Q is a semi-direct product of G^Θ and of the unipotent radical R of Q. For $\lambda \in \mathcal{W}^\Theta$ denote by χ_λ the holomorphic character of Q, which is trivial on R, and whose restriction to G^Θ is the character of G^Θ parametrized by λ. For $H \in h_c$ $\chi_\lambda(\exp H) = \exp \lambda(H)$.

Denote by $\mathbf{C}_\lambda$ a one-dimensional complex vector space with the action of Q given by χ_λ. Consider the holomorphic line bundle $L_\lambda = G \times_Q \mathbf{C}_\lambda$. It is the coset space of $G \times \mathbf{C}_\lambda$ under the equivalence $(gq, v) = (g, \chi_\lambda(q)v)$, $g \in G, q \in Q, v \in \mathbf{C}_\lambda$. The group G acts on L_λ as follows, $G \ni g_0 : (g, v) \mapsto (g_0 g, v)$. Since $G/Q = K/K^\Theta$, one has an alternative description of L_λ, $L_\lambda = K \times_{K^\Theta} \mathbf{C}_\lambda$. Using that description one can define a K-invariant hermitian metrics h on L_λ setting $h(k,v) = |v|^2$. It follows from Iwasawa decomposition that each element $g \in G$ can be (non-uniquely) represented as a product $g = kq$ for some $k \in K, q \in Q$. Thus for $g = kq$ we get $h(g,v) = h(kq,v) = h(k, \chi_\lambda(q)v) = |\chi_\lambda(q)v|^2$.

It follows from the results obtained in Sect. 4, that to the hermitian line bundle (L_λ, h) there corresponds an s-module on $\mathcal{M}$. Denote it $\mathbf{s}_\lambda$. Let $\{f_X^\lambda\}$, $X \in k_r$, be the corresponding K-equivariant family which defines the mapping $\gamma : \mathcal{M} \to k_r$ such that $(\gamma(x), X) = f_X^\lambda(x)$ for all $x \in \mathcal{M}$, $X \in k_r$. We are going to apply Theorem 4 to the s-modules $\mathbf{s}_\lambda$, $\lambda \in \mathcal{W}^\Theta$. Calculate the element $E^\lambda = \gamma(x_0)$. Since $E^\lambda \in t_r$, in order to determine E^λ it is enough to consider only the pairing (E^λ, JH_α) for all $\alpha \in \Delta$. For $Z \in g_r$ let v_Z^L be the fundamental vector field on L_λ, then $\xi_Z^L = (1/2)(v_Z^L - i v_{JZ}^L)$ is its holomorphic component. For $\varphi \in C^\infty(L_\lambda)$ $v_Z\varphi(g,v) = (d/dt)\varphi(\exp(-tZ)g, v)|_{t=0}$, $g \in G, v \in \mathbf{C}_\lambda$.

Using Proposition 1 and taking into account K-invariance of the metrics h we get $if_X^\lambda \circ \pi = h(\xi_X^L h^{-1}) = (-i/2)h(v_{JX}^L h^{-1})$ for $X \in k_r$. Thus $f_{JH_\alpha}^\lambda(x_0) = (1/2)h(v_{H_\alpha}h^{-1}) =$ $(1/2)h(e,v)(d/dt)(h(\exp(-tH_\alpha), v))^{-1}|_{t=0} = (1/2)(d/dt)(|\chi_\lambda(\exp(-tH_\alpha))|^{-2})|_{t=0} =$ $(1/2)(d/dt)(|\exp\lambda(tH_\alpha)|^2)|_{t=0} = \lambda(H_\alpha)$. Now $E^\lambda = \gamma(x_0)$ is the element of t_r such that $(E^\lambda, JH_\alpha) = \lambda(H_\alpha)$ for all $\alpha \in \Delta$. The following proposition is a direct consequence of Theorem 4.

Proposition 8. *The s-module $\mathbf{s}_\lambda$ corresponding to the holomorphic hermitian line bundle (L_λ, h), $\lambda \in \mathcal{W}^\Theta$, is nondegenerate iff for all $\alpha \in \Delta \setminus < \Theta >$ holds $\lambda(H_\alpha) \neq 0$. In this case the index of inertia of the corresponding pseudo-Kähler metrics on $\mathcal{M}$ equals $\#\{\alpha \in \Delta^+ \setminus < \Theta > | \lambda(H_\alpha) < 0\}$.*

Lemma 9. *The canonical line bundle L_{can} on $\mathcal{M}$ is isomorphic to the bundle L_λ for $\lambda = -2\delta'_\Theta$. The canonical s-module $\mathbf{s}_{can}$ on $\mathcal{M}$ coincides with $\mathbf{s}_\lambda$ for $\lambda = -2\delta'_\Theta$.*

Proof. The isotropy subgroup $Q \subset G$ of the point $x_0 \in \mathcal{M} = G/Q$ acts on the fibers of G-bundles at x_0. The fiber of the line bundle L_λ at x_0 is isomorphic as a Q-module to $\mathbf{C}_\lambda$. On the other hand, the holomorphic tangent space of $\mathcal{M}$ at x_0, $T'_{x_0}\mathcal{M}$, is isomorphic as a Q-module to g_c/q_c under the adjoint action. For $H \in h_c$ the operator $ad(H)$ on g_c/q_c is diagonal in the basis $\{X_\alpha + q_c\}$, $\alpha \in \Delta \setminus \Pi$, and takes the eigenvalue $\alpha(H)$ on $X_\alpha + q_c$. The element $H \in h_c$ acts on $\wedge^m(g_c/q_c)$ by the scalar $2\delta'_\Theta(H)$, where $m = \dim_{\mathbf{C}}\mathcal{M}$. Therefore the element $q \in Q$ acts on $\wedge^m(g_c/q_c)$ by $\chi_\lambda(q)$ for $\lambda = 2\delta'_\Theta$. The Lemma follows from the fact that the fiber of the canonical line bundle L_{can} at x_0 is dual to $\wedge^m(g_c/q_c)$ as a Q-module.

Now we shall use a particular case of the Bott-Borel-Weil theorem concerning cohomological realizations of finite dimensional irreducible holomorphic representations of the group G in the sheaf cohomologies of line bundles over $\mathcal{M} = G/Q$ (see [10]). Let $H^i(\mathcal{M}, \mathcal{S}L_\lambda)$ denote the space of i-dimensional cohomology with coefficients in the sheaf of germs of holomorphic sections of the line bundle L_λ. The action of the group G on L_λ

gives rise to the action of G on the local holomorphic sections of L_λ, which induces the action of G in the cohomology spaces $H^i(\mathcal{M}, SL_\lambda)$.

Theorem 6. (Bott-Borel-Weil) *Let $\lambda \in \mathcal{W}^\Theta$, $k = \#\{\alpha \in \Delta^+ | (\lambda+\delta)(H_\alpha) < 0\}$. If $(\lambda+\delta)(H_\alpha) = 0$ for some $\alpha \in \Delta$ then $H^i(\mathcal{M}, SL_\lambda) = 0$ for all i. If $(\lambda+\delta)(H_\alpha) \neq 0$ for all $\alpha \in \Delta$ one can choose $w \in W$ so that $w(\lambda+\delta)$ is dominant. Then $\zeta = w(\lambda+\delta) - \delta$ is dominant as well. For all $i \neq k$ $H^i(\mathcal{M}, SL_\lambda) = 0$. The representation of the group G in $H^k(\mathcal{M}, SL_\lambda)$ is isomorphic to the irreducible finite dimensional holomorphic representation of G with highest weight ζ.*

Assume that an irreducible finite dimensional holomorphic representation τ of the group G is realized in the cohomology space $H^k(\mathcal{M}, SL_\lambda)$ as in Theorem 6. Retain the same notation for the representations of the Lie algebra g_c and of its universal enveloping algebra $\mathcal{U}(g_c)$ which correspond to τ. The action of the Lie algebra g_c on L_λ by holomorphic differential operators can be extended to the action of $\mathcal{U}(g_c)$. For $u \in \mathcal{U}(g_c)$ denote by A_u the corresponding holomorphic differential operator on L_λ. It induces the representation operator $\tau(u)$ in $H^k(\mathcal{M}, SL_\lambda)$.

According to Theorem 3, for $z \in \mathcal{Z}(g_c)$ the holomorphic operator A_z on L_λ is scalar and is equal to the value $\psi(z)$ of the central character ψ associated to the s-module $\mathbf{s}_\lambda$. It follows immediately that the central character of the representation τ coincides with ψ. As in the proof of Proposition 7 we obtain that for $u \in \mathcal{U}(g_c)$ hoids the equality

$$n \int_{\mathcal{M}} \sigma_u \, d\mu = \mathrm{tr}\ \tau(u), \tag{7}$$

where $n = \dim \tau$. According to Proposition 7, there exists a representation ρ of the algebra $\mathcal{A}$ of functions on $\mathcal{M}$ associated to the s-module $\mathbf{s}_\lambda$ in the space $H^k(\mathcal{M}, SL_\lambda)$, such that $\tau = \rho \circ \sigma$.

Let $\{f_X^\lambda\}$, $X \in k_r$, be the K-equivariant family associated to $\mathbf{s}_\lambda$. Since $\sigma_X = i f_X^\lambda$ for $X \in k_r$, the algebra $\mathcal{A}$ contains the functions f_X^λ, $X \in k_r$, and is generated by them. The algebra k_r acts on L_λ by the holomorphic differential operators $A_X = \nabla_{\xi_X} + i f_X^\lambda$, $X \in k_r$, due to Proposition 1. The operator A_X induces in $H^k(\mathcal{M}, SL_\lambda)$ the representation operator $\rho(i f_X^\lambda)$.

Remark. For $X \in k_r$ consider the operator $Q_X = \nabla_{v_X} + i f_X^\lambda$. The operator A_X differs from Q_X by the anti-holomorphic operator ∇_{η_X}, which annihilates the local holomorphic sections of L_λ and thus induces the trivial action on the sheaf cohomology. Therefore the operator Q_X also induces in $H^k(\mathcal{M}, SL_\lambda)$ the operator $\rho(i f_X^\lambda)$. If the s-module $\mathbf{s}_\lambda$ is nondegenerate, the curvature form ω of the connection ∇ is symplectic and the function f_X^λ is a Hamiltonian of the fundamental vector field v_X on $\mathcal{M}$. Then the operator Q_X is the operator of geometric quantization corresponding to the function f_X^λ.

We see that the Bott-Borel-Weil theorem provides a natural geometric representation of the algebra $\mathcal{A}$ in the sheaf cohomology space of the line bundle L_λ.

Theorem 7. *Let $\lambda \in \mathcal{W}^{\Theta}$ be such that $(\lambda + \delta)(H_\alpha) \neq 0$ for all $\alpha \in \Delta$, $\mathcal{A}$ and $\{f_X^\lambda\}$, $X \in k_r$, be the algebra of functions on $\mathcal{M}$ and the K-equivariant family associated to the s-module $\mathbf{s}_\lambda$ respectively. The algebra $\mathcal{A}$ is generated by its elements f_X^λ, $X \in k_r$. Set $k = \#\{\alpha \in \Delta^+ | (\lambda+\delta)(H_\alpha) < 0\}$. There exists a unique finite dimensional irreducible representation ρ of the algebra $\mathcal{A}$ in the space $H^k(\mathcal{M}, \mathcal{S}L_\lambda)$ such that for all $X \in k_r$ the representation operator $\rho(if_X^\lambda)$ is induced from the holomorphic differential operator $\nabla_{\xi_X} + if_X^\lambda$ on L_λ. There exists an element $w \in W$ such that $\zeta = w(\lambda + \delta) - \delta$ is a dominant weight of the Lie algebra g_c. The representation $\tau = \rho \circ \sigma$ of $g_c$in $H^k(\mathcal{M}, \mathcal{S}L_\lambda)$ is irreducible with highest weight ζ.*

Denote by w_0 and w_0^Θ the elements of the maximal reduced length in the Weyl groups W and W^Θ respectively. Let τ be the irreducible finite dimensional representation of the algebra g_c with highest weight ζ. It is known that the dual representation τ' has the highest weight $\zeta' = -w_0\zeta$.

Lemma 10. *Let $\lambda \in \mathcal{W}^\Theta$ and $w \in W$ be such that $\zeta = w(\lambda + \delta) - \delta$ is a dominant weight. Then $\lambda' = -\lambda - 2\delta'_\Theta \in \mathcal{W}^\Theta$ and there exists an element $w' \in W$ such that $\zeta' = w'(\lambda'+\delta) - \delta$. If $(\lambda+\delta)(H_\alpha) \neq 0$ for all $\alpha \in \Delta$ and $k = \#\{\alpha \in \Delta^+ | (\lambda+\delta)(H_\alpha) < 0\}$ then $(\lambda' + \delta)(H_\alpha) \neq 0$ for all $\alpha \in \Delta$ and $\#\{\alpha \in \Delta^+ | (\lambda' + \delta)(H_\alpha) < 0\} = m - k$, where $m = \dim_{\mathbf{C}}\mathcal{M}$.*

Proof. For $\alpha \in \Theta$ the reflection $s_\alpha \in W^\Theta$ maps α to $-\alpha$ and preserves both $\Delta^+ \backslash \{\alpha\}$ and $< \Theta >$. It follows that the group W^Θ preserves the set $\Delta^+ \backslash < \Theta >$, whence $-2\delta'_\Theta \in \mathcal{W}^\Theta$ and therefore $\lambda' = -\lambda - 2\delta'_\Theta \in \mathcal{W}^\Theta$. The element w_0^Θ maps $< \Theta >^+$ to $< \Theta >^-$ and preserves $\Delta^+ \backslash < \Theta >$, whence $w_0^\Theta \delta_\Theta = -\delta_\Theta$. Take $w' = w_0 w w_0^\Theta$, then $w'(\lambda' + \delta) = w'(-\lambda - \delta'_\Theta + \delta_\Theta) = w_0 w w_0^\Theta(-\lambda - \delta'_\Theta + \delta_\Theta) = w_0 w(-\lambda - \delta'_\Theta - \delta_\Theta) = w_0 w(-\lambda - \delta) = w_0(-\zeta - \delta) = -w_0\zeta + \delta = \zeta' + \delta$, thus $\zeta' = w'(\lambda' + \delta) - \delta$. For $\alpha \in \Delta$ we have $(\lambda' + \delta)(H_\alpha) = (-\lambda - \delta'_\Theta + \delta_\Theta)(H_\alpha) = (w_0^\Theta(-\lambda - \delta'_\Theta + \delta_\Theta))(H_{w_0^\Theta \alpha}) = (-\lambda - \delta'_\Theta - \delta_\Theta)(H_{w_0^\Theta \alpha}) = (-\lambda - \delta)(H_{w_0^\Theta \alpha})$. Now it is clear that if $(\lambda + \delta)(H_\alpha) \neq 0$ for all $\alpha \in \Delta$ then $(\lambda' + \delta)(H_\alpha) \neq 0$ for all $\alpha \in \Delta$. Since $m = \#(\Delta^+ \backslash < \Theta >)$, $\lambda(H_\alpha) = \lambda'(H_\alpha) = 0$ for $\alpha \in < \Theta >$ and $\delta(H_\alpha) > 0$ for $\alpha \in \Delta^+$, we get $\#\{\alpha \in \Delta^+ | (\lambda' + \delta)(H_\alpha) < 0\} = \#\{\alpha \in \Delta^+ \backslash < \Theta > | (\lambda' + \delta)(H_\alpha) < 0\} = \#\{\alpha \in \Delta^+ \backslash < \Theta > | (-\lambda - \delta)(H_{w_0^\Theta \alpha}) < 0\} = \#\{\alpha \in \Delta^+ \backslash < \Theta > | (\lambda + \delta)(H_\alpha) > 0\} = m - k$. The Lemma is proved.

Retain the notations of Lemma 10 and assume that $(\lambda + \delta)(H_\alpha) \neq 0$ for all $\alpha \in \Delta$. It follows from Theorem 6 and Lemma 10 that the dual representations τ and τ' of the algebra g_c with highest weights ζ and ζ' are realized in the cohomology spaces $H^k(\mathcal{M}, \mathcal{S}L_\lambda)$ and $H^{m-k}(\mathcal{M}, \mathcal{S}L_{\lambda'})$ respectively. The spaces $H^k(\mathcal{M}, \mathcal{S}L_\lambda)$ and $H^{m-k}(\mathcal{M}, \mathcal{S}L_{\lambda'})$ are dual. This is, in fact, the Kodaira–Serre duality.

According to Theorem 7 and Lemma 9, the s-modules $\mathbf{s}_\lambda$ and $\mathbf{s}_{\lambda'}$ are dual and the associated function algebras $\mathcal{A}$ and $\mathcal{A}'$ on $\mathcal{M}$ have representations ρ and ρ' in $H^k(\mathcal{M}, \mathcal{S}L_\lambda)$ and $H^{m-k}(\mathcal{M}, \mathcal{S}L_{\lambda'})$, such that $\tau = \rho \circ \sigma$ and $\tau' = \rho' \circ \sigma'$ respectively (here all the notations have their usual meaning). Let $n = \dim \tau$.

Proposition 9. *For $f \in \mathcal{A}$ and $g \in \mathcal{A}'$ holds the equality*

$$n \int_{\mathcal{M}} fg \, d\mu = \operatorname{tr}(\rho(f)(\rho'(g))^t).$$

Proof. Choose $u, v \in \mathcal{U}(g_c)$ such that $f = \sigma_u$, $g = \sigma'_v$. Then, using Eq. (7), one gets $n \int fg \, d\mu = n \int \sigma_u \sigma'_v \, d\mu = n \int \sigma_u l'_v 1 \, d\mu = n \int (l'_v)^t \sigma_u \, d\mu = n \int l_{\check{v}} \sigma_u \, d\mu = n \int \sigma_{\check{v}u} \, d\mu =$ $\operatorname{tr} \tau(\check{v}u) = \operatorname{tr}((\tau'(v))^t \tau(u)) = \operatorname{tr}(\rho(f)(\rho'(g))^t)$. The Proposition is proved.

10. Covariant and contravariant symbols on flag manifolds

Assume that $\lambda \in \mathcal{W}^\Theta$ is such that the (finite dimensional) space $\mathcal{H} = H^0(\mathcal{M}, \mathcal{S}L_\lambda)$ of global holomorphic sections of L_λ is nontrivial. According to Theorem 6, this is the case iff $(\lambda + \delta)(H_\alpha) > 0$ for all $\alpha \in \Delta^+$ or, equivalently, iff λ is a dominant weight.

For any elements q, q' of the same fiber of L^*_λ denote by $h(q, q')$ their K-invariant hermitian scalar product such that $h(q, q) = h(q)$. Let $L^2(\mathcal{M}, L_\lambda)$ denote the Hilbert space of sections of L_λ, square integrable with respect to the K-invariant Hilbert norm $|| \cdot ||$ given by the formula $||s||^2 = \int_{\mathcal{M}} h(s) \, d\mu$, s a section of L_λ. Denote the corresponding hermitian scalar product in $L^2(\mathcal{M}, L_\lambda)$ by $< \cdot, \cdot >$.

We introduce coherent states in $\mathcal{H}$ in a geometrically invariant fashion, following [13]. For $q \in L^*_\lambda$ the corresponding coherent state e_q is a unique element in $\mathcal{H}$ such that the relation $< s, e_q > q = s \circ \pi(q)$ holds for all $s \in \mathcal{H}$. It is known that the coherent states e_q exist for all $q \in L^*_\lambda$ and the mapping $L^*_\lambda \ni q \mapsto e_q \in \mathcal{H}$ is antiholomorphic. For $c \in \mathbf{C}$ holds $e_{cq} = \bar{c}^{-1} e_q$.

The group K acts on the sections of the line bundle L_λ as follows, $(ks)(x) = k(s(k^{-1}x))$ for $k \in K$, $x \in \mathcal{M}$ and s a section of L_λ. This action is unitary with respect to the scalar product $< \cdot, \cdot >$. For any holomorphic section s of L_λ we have $< ks, e_{kq} > kq = (ks)(kx) =$ $k(s(x))$, therefore $< ks, e_{kq} > q = s(x)$. On the other hand, $< ks, ke_q > = < s, e_q > = s(x)$, whence $ke_q = e_{kq}$. The function $||e_q||^2 h(q)$ is homogeneous of order 0 with respect to $\mathbf{C}^\bullet$-action and K-invariant. Thus it is identically constant. Set $||e_q||^2 h(q) = C$.

Let A be an operator on $\mathcal{H}$. It is easy to check that the function $\tilde{f}(q) =$ $< Ae_q, e_q > / < e_q, e_q >$ on the bundle L^*_λ is constant on the fibers. Therefore there exists a function f_A on $\mathcal{M}$ such that $f_A \circ \pi = \tilde{f}$.

Definition. Berezin's covariant symbol of an operator A on $\mathcal{H}$ is the function f_A on $\mathcal{M}$ given by the formula $f_A(x) = < Ae_q, e_q > / < e_q, e_q >$ for any $q \in L^*_\lambda$ such that $\pi(q) = x \in \mathcal{M}$.

The operator—symbol mapping $A \mapsto f_A$ is injective and thus induces an algebra structure on the set of all covariant symbols. The algebra of covariant symbols is isomorphic to $End(\mathcal{H})$.

Let A be a holomorphic differential operator on L_λ. Fix a local holomorphic trivialization (U, s_0) of L_λ and let A_0 denote the local expression of the operator A on U. Then for $x, y \in U$ we have $< Ae_{s_0(y)}, e_{s_0(x)} > s_0(x) = Ae_{s_0(y)}(x) = s_0(x)A_0(e_{s_0(y)}(x)/s_0(x))$. The function $e_{s_0(y)}(x)/s_0(x)$ on $U \times U$ is holomorphic in x and antiholomorphic in y. Set $S(x) =< e_{s_0(x)}, e_{s_0(x)} >= e_{s_0(x)}(x)/s_0(x)$. Let f be the covariant symbol of the operator A. Since A_0 is a holomorphic differential operator on U, we get for $q = s_0(x)$ that $f(x) =< Ae_q, e_q > / < e_q, e_q >= (< Ae_{s_0(y)}, e_{s_0(x)} > |_{y=x})/S(x) = (A_0(e_{s_0(y)}(x)/s_0(x)))|_{y=x}/S(x) = A_0S(x)/S(x)$.

Introduce the function $\Phi = -\log h \circ s_0$ on U. We have $S(x) = ||e_{s_0(x)}||^2 = C \exp \Phi$, whence $f(x) = A_0S(x)/S(x) = e^{-\Phi}(A_0 \ e^{\Phi}) = \check{A}1$, where $\check{A}$ is the pushforward of the operator A to $\mathcal{M}$. The formula $f = \check{A}1$ holds globally on $\mathcal{M}$.

For $u \in \mathcal{U}(g_c)$ the pushforward to $\mathcal{M}$ of the operator A_u on L_λ coincides with the operator l_u related to the s-module $\mathbf{s}_\lambda$, $\check{A}_u = l_u$. Therefore the covariant symbol f_u of the operator A_u on $\mathcal{H}$ can be expressed by the formula $f_u = \check{A}_u 1 = l_u 1 = \sigma_u$. We have proved the following theorem.

Theorem 8. *Let $\lambda \in \mathcal{W}^\Theta$ be dominant. Then the space $\mathcal{H} = H^0(\mathcal{M}, SL_\lambda)$ of global holomorphic sections of L_λ is nontrivial. Endow it with the Hilbert space structure via the norm $|| \cdot ||$ such that $||s||^2 = \int_\mathcal{M} h(s) \, d\mu$, $s \in \mathcal{H}$. Then for $u \in \mathcal{U}(g_c)$ the covariant symbol of the operator A_u on $\mathcal{H}$ equals σ_u, where $\sigma : \mathcal{U}(g_c) \to C^\infty(\mathcal{M})$ is the mapping associated to the s-module $\mathbf{s}_\lambda$.*

According to Theorem 7, the representation ρ of the algebra $\mathcal{A} = \sigma(\mathcal{U}(g_c))$ in $\mathcal{H}$ is irreducible (the representation $\tau = \rho \circ \sigma$ of the Lie algebra g_c is irreducible with highest weight λ). Therefore any operator on $\mathcal{H}$ can be represented as A_u for some $u \in \mathcal{U}(g_c)$. Thus we get the following corollary.

Corollary. *The algebra $\mathcal{A}$ associated to the s-module $\mathbf{s}_\lambda$ coincides with the algebra of Berezin's covariant symbols of the operators on $\mathcal{H}$.*

Let $P : L^2(\mathcal{M}, L_\lambda) \to \mathcal{H}$ be the orthogonal projection operator. For a measurable function f on $\mathcal{M}$ let M_f denote the multiplication operator by f. Introduce the operator $\hat{f} = PM_fP$ on $\mathcal{H}$.

Definition. A measurable function f on $\mathcal{M}$ is called a contravariant symbol of an operator A on $\mathcal{H}$ if $A = \hat{f}$.

Let s_1, s_2 be holomorphic sections of L_λ. Calculate the covariant symbol of the rank one operator $A_0 = s_1 \otimes s_2^*$ in $\mathcal{H}$,

$$f_{A_0} = \frac{< A_0 e_q, e_q >}{< e_q, e_q >} = \frac{< s_1, e_q >< e_q, s_2 >}{< e_q, e_q >} = \frac{(s_1/q)\overline{(s_2/q)}}{||e_q||^2} = \frac{h(s_1, s_2)}{||e_q||^2 h(q)}.$$

Since $||e_q||^2 h(q) = C$, we obtain $f_{A_0} = h(s_1, s_2)/C$. For any measurable function g on $\mathcal{M}$ $\mathrm{tr}(A_0\hat{g}) =< \hat{g}s_1, s_2 >=< gs_1, s_2 >= \int h(gs_1, s_2) \, d\mu = \int gh(s_1, s_2) \, d\mu = C \int f_{A_0} g \, d\mu$. Therefore for any operator A on $\mathcal{H}$ holds $\mathrm{tr}(A\hat{g}) = C \int f_A g \, d\mu$. Taking $A = \mathbf{1}$, $g = 1$ we immediately obtain that $C = n = \dim \mathcal{H}$.

Proposition 10. *A measurable function g on $\mathcal{M}$ is a contravariant symbol of an operator B on $\mathcal{H}$ iff for any operator A on $\mathcal{H}$ holds the formula* $\mathrm{tr}(AB) = n \int f_A g \, d\mu$.

The proof is straightforward.

Let $\lambda \in \mathcal{W}^\Theta$ be dominant. Set $\lambda' = -\lambda - 2\delta'_\Theta$. For the s-module $\mathbf{s}_{\lambda'}$ dual to $\mathbf{s}_\lambda$ let $\sigma' : \mathcal{U}(g_c) \to C^\infty(\mathcal{M})$ denote the mapping associated to $\mathbf{s}_{\lambda'}$, ρ' be the corresponding representation of the algebra $\mathcal{A}' = \sigma'(\mathcal{U}(g_c))$ in $H^m(\mathcal{M}, \mathcal{S}L_{\lambda'})$ The spaces $\mathcal{H} = H^0(\mathcal{M}, \mathcal{S}L_\lambda)$ and $H^m(\mathcal{M}, \mathcal{S}L_{\lambda'})$ are dual as representation spaces of the group G. The following theorem is a direct consequence of Propositions 9,10 and Theorem 8.

Theorem 9. *A function $f \in \mathcal{A}'$ is a contravariant symbol of the operator $(\rho'(f))^t$ in $\mathcal{H}$.*

11. Quantization on flag manifolds

Now we are ready to put together various results obtained above to give examples of quantization on a generalized flag manifold $\mathcal{M}$ endowed with a pseudo-Kähler metrics.

Let $\lambda \in \mathcal{W}^\Theta$ be such that for all $\alpha \in \Delta \backslash < \Theta >$ holds $\lambda(H_\alpha) \neq 0$. According to Proposition 8, in this case the s-module $\mathbf{s}_\lambda$ on $\mathcal{M}$ is nondegenerate. Denote by ω the 2-form associated to $\mathbf{s}_\lambda$. This form is pseudo-Kähler, and the index of inertia of the corresponding pseudo-Kähler metrics equals $l = \#\{\alpha \in \Delta^+ \backslash < \Theta > | \lambda(H_\alpha) < 0\}$. Denote by $\mathcal{A}_\hbar$ the algebra of functions on $\mathcal{M}$ associated to the s-module $(1/\hbar)\mathbf{s}_\lambda$. It follows from Theorem 5 that any functions $f, g \in F(\mathcal{M})$ belong to $\mathcal{A}_\hbar$ for all but a finite number of values of $\hbar$. The product $f *_\hbar g$ expands to the uniformly and absolutely convergent Taylor series in $\hbar$ at the point $\hbar = 0$, $f *_\hbar g = \sum_{r \geq 0} \hbar^r C_r(f, g)$, where $C_r(\cdot, \cdot), r = 0, 1, \ldots$, are bidifferential operators which define the deformation quantization with separation of variables on $\mathcal{M}$ corresponding to the (non-deformed) pseudo-Kähler form ω.

For $n \in \mathcal{N}$ holds $n\lambda \in \mathcal{W}^\Theta$. Theorem 7 implies that for $\hbar = 1/n$ the algebra $\mathcal{A}_\hbar$ has a natural geometric representation $\rho_\hbar$ in the sheaf cohomology space of the line bundle $L_{n\lambda} = (L_\lambda)^n$, $\mathcal{H}_\hbar = H^{k_n}(\mathcal{M}, \mathcal{S}L_{n\lambda})$, where $k_n = \#\{\alpha \in \Delta^+ | (n\lambda + \delta)(H_\alpha) < 0\}$. Since for $\alpha \in \Delta^+$ holds $\delta(H_\alpha) > 0$, only those $\alpha \in \Delta^+$ contribute to k_n for which $\lambda(H_\alpha) < 0$. Therefore $k_n = l$ for $n >> 0$. In other words, for sufficiently small values of $\hbar = 1/n$ the dimension of the sheaf cohomology the representation $\rho_\hbar$ is realized in is equal to the index of inertia l of the pseudo-Kähler metrics on $\mathcal{M}$ corresponding to the (1,1)-form ω.

We have obtained pseudo-Kähler quantization on a generalized flag manifold.

Now assume $\lambda \in \mathcal{W}^\Theta$ is dominant in the rest of the paper. Then the metrics corresponding to the (1,1)-form ω on $\mathcal{M}$ is positive definite, i.e., ω is a Kähler form, and Theorem 8 implies that for $\hbar = 1/n$ the space $\mathcal{H}_\hbar = H^0(\mathcal{M}, \mathcal{S}L_{n\lambda})$ is the space of global holomorphic sections of the line bundle $L_{n\lambda} = (L_\lambda)^n$ and $\mathcal{A}_\hbar$ is the corresponding algebra of Berezin's covariant symbols on $\mathcal{M}$. Thus we arrive at Berezin's Kähler quantization on $\mathcal{M}$ and identify the associated formal deformation quantization obtained in [12], [4] with the quantization with separation of variables, corresponding to the non-deformed Kähler form ω.

Consider the s-module $\mathbf{s}(\hbar) = -\mathbf{s}_\lambda + \hbar \mathbf{s}_{can}$. It depends rationally on $\hbar$ and is regular at $\hbar = 0$. Denote by ω_{can} the (1,1)-form associated to the canonical s-module $\mathbf{s}_{can}$ on $\mathcal{M}$. Then the form, associated to $\mathbf{s}(\hbar)$ is $-\omega + \hbar\omega_{can}$. The s-module $(1/\hbar)\mathbf{s}(\hbar)$ is dual to $(1/\hbar)\mathbf{s}_\lambda$. Denote by $(\mathcal{A}'_\hbar, *'_\hbar)$ the algebra of functions on $\mathcal{M}$ associated to the s-module $(1/\hbar)\mathbf{s}(\hbar)$. For any functions $f, g \in F(\mathcal{M})$ the product $f *'_\hbar g$ depends rationally on $\hbar$ and is regular at $\hbar = 0$. The asymptotic expansion of the product $f *'_\hbar g$ gives rise to the deformation quantization with separation of variables $(\mathcal{F}, \star')$ corresponding to the formal deformation of the negative-definite Kähler form $-\omega$, $\omega' = -\omega + \nu\omega_{can}$.

If $\hbar = 1/n$ then the algebra $\mathcal{A}'_\hbar$ has a representation $\rho'_\hbar$ in the space $H^m(\mathcal{M}, SL_{\lambda'_n})$, where $m = \dim_{\mathbf{C}}\mathcal{M}$ and $\lambda'_n = -n\lambda - 2\delta'_\Theta$. The space $H^m(\mathcal{M}, SL_{\lambda'_n})$ is dual to $\mathcal{H}_\hbar$ and Theorem 9 implies that any function $f \in \mathcal{A}'_\hbar$ is a contravariant symbol of the operator $(\rho'_\hbar(f))^t$ in the space $\mathcal{H}_\hbar$. The mapping $\mathcal{A}'_\hbar \ni f \mapsto (\rho'_\hbar(f))^t$ is an anti-homomorphism. Thus, in order to obtain quantization on $\mathcal{M}$ by contravariant symbols (it is usually called Berezin-Toeplitz quantization, see [14]), we have to consider the algebras $(\tilde{\mathcal{A}}_\hbar, \tilde{*}_\hbar)$, opposite to $(\mathcal{A}'_\hbar, *'_\hbar)$. Then $\tilde{\mathcal{A}}_\hbar \ni f \mapsto (\rho'_\hbar(f))^t$ will be a representation of the algebra $\tilde{\mathcal{A}}_\hbar$. The corresponding deformation quantization $(\mathcal{F}, \tilde{\star})$ is opposite to $(\mathcal{F}, \star')$. As it was shown in Section 5, this quantization is also a quantization with separation of variables, though with respect to the opposite complex structure on $\mathcal{M}$. It corresponds to the formal (1,1)-form $-\omega' = \omega - \nu\omega_{can}$ on the opposite complex manifold $\bar{\mathcal{M}}$. This form is a formal deformation of the Kähler form ω on $\bar{\mathcal{M}}$. (The metrics on $\bar{\mathcal{M}}$, corresponding to the (1,1)-form ω is a negative-definite Kähler metrics.)

It would be interesting to compare the deformation quantization associated to Berezin-Toeplitz quantization on a general compact Kähler manifold in [14] with deformation quantization with separation of variables.

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