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Andrija Rašin

INTERNATIONAL ATOMIC ENERGY AGENCY

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Andrija Rašin
International Centre for Theoretical Physics, Trieste, Italy.

ABSTRACT

I discuss some general aspects of diagonalizing the quark mass matrices and list all possible parametrizations of the Cabibbo-Kobayashi-Maskawa matrix (CKM) in terms of three rotation angles and a phase. I systematically study the relation between the rotations needed to diagonalize the Yukawa matrices and various parametrizations of the CKM.

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A. Introduction

The most general Cabibbo-Kobayashi-Maskawa matrix (CKM) [1,2] can be written as a function of three angles and one phase. Various parametrizations of CKM exist today [2–7] in which these three angles and one phase appear in various places. These four physical parameters along with the six quark masses are derived from the up and down quark mass matrices, which are in general arbitrary complex matrices with 9 real parameters and 9 imaginary parameters each.

Usually general arguments are used to do the counting of parameters and some "trial and error" to set up a particular parametrization for the CKM matrix. However, to a model builder who predicts a particular pattern for the Yukawa matrices it may prove useful to see how exactly one obtains the parametrization from the original quark mass matrices. In the literature there exist many examples of arguing in favor of a particular parametrization of the CKM for some pattern of quark mass matrices [7–9]. However, a systematic study of an arbitrary case is still lacking.

In this paper I lay out a procedure of diagonalizing the quark mass matrices and I numerate all possible parametrizations of the CKM matrix in terms of three rotation angles and one phase. I first start in Section B with some preliminaries about diagonalizing arbitrary complex matrices and offer another way of counting the number of parameters. In Section C I explicitly parametrize arbitrary 2×2 and 3×3 unitary matrices in terms of rotation angles and phases. The CKM parametrizations follow trivially by a phase redefinition and I list all possible parametrizations in a Table. Next, I turn to explicit relations between the parameters of the CKM and the elements of the quark mass matrices. The case of two generations is studied in Sections D (diagonalization) and E (CKM). This case serves as a proof of some simple results that become useful in studying the case of three generations in Sections F and G. I show how different parametrizations of the CKM come about and how some of them are more suitable for some quark mass matrix structures.

As inspired by the difference in the quark masses, I will assume in this paper that all eigenvalues of the mass matrices are different, although the discussion is easily generalized to the case of degenerate masses. Also the discussion easily generalizes to an arbitrary number of generations.

B. Generalities on the diagonalization of an arbitrary complex square matrix

An arbitrary $n \times n$ complex matrix $\mathbf{h}$ is diagonalized by two unitary matrices $\mathbf{U}$ and $\mathbf{V}$

$$\mathbf{m} = \mathbf{U}^\dagger \mathbf{h} \mathbf{V} \tag{1}$$

where $\mathbf{m}$ is a diagonal matrix with real and nonnegative entries on the diagonal. The matrices $\mathbf{U}$ and $\mathbf{V}$ diagonalize the following products of $\mathbf{h}$

$$\mathbf{m}^2 = \mathbf{U}^\dagger \mathbf{h}\mathbf{h}^\dagger \mathbf{U}\ ,\ \mathbf{m}^2 = \mathbf{V}^\dagger \mathbf{h}^\dagger \mathbf{h} \mathbf{V} \tag{2}$$

It is known that an arbitrary unitary matrix has $n(n-1)/2$ angles and $n(n+1)/2$ phases. Let us see how this counting works for $\mathbf{U}$ and $\mathbf{V}$ and how many of these parameters are fixed by matrix $\mathbf{h}$, the matrix which we are diagonalizing, and how many are left completely arbitrary. For example, let us look at the product $\mathbf{h}^\dagger\mathbf{h}$. It has $n(n+1)/2$ angles and $n(n-1)/2$ phases *. Of the $n(n+1)/2$ angles, n describe the n diagonal entries of $\mathbf{m}^2$ so that $n(n-1)/2$ angles must enter $\mathbf{V}$. Similarly, all the $n(n-1)/2$ phases must enter $\mathbf{V}$, since $\mathbf{m}$ is real. Notice however that $\mathbf{V}$ is not completely determined by (2). It can be multiplied from the right by an arbitrary phase rotation which contains n phases that do not depend on $\mathbf{h}$

$$\mathbf{V} \to \mathbf{V} \begin{pmatrix} e^{i\alpha_1} & & \\ & \dots & \\ & & e^{i\alpha_n} \end{pmatrix} \tag{3}$$

giving a total of $n(n+1)/2$ phases for $\mathbf{V}$. Similar counting is valid for $\mathbf{U}$.

Finally, inverting the first relation (1)

$$\mathbf{h} = \mathbf{U}\mathbf{m}\mathbf{V}^\dagger \tag{4}$$

we can count the number of phases and angles on both sides. On the left-hand side (lhs) we have n^2 angles and n^2 phases. This should be matched by the right-hand side (rhs). Let us count first the number of angles. The matrices

*In this article I will use the terms "angles" and "phases" instead of real and imaginary parameters, respectively.

U and V have each $n(n-1)/2$ angles together with n real parameters in $\mathbf{m}$ give $2(n(n-1)/2)+n=n^2$ which checks. For the phases, U and V have $n(n+1)/2$ each, of which $n(n-1)/2$ in each matrix is already fixed by the elements of h from (2). But then notice that in (4) of the remaining pair of arbitrary n phases in U and V only the difference of the phases appears and gets fixed by elements of h. Thus the number of phases on the rhs is $2(n(n-1)/2)+n=n^2$ which checks again.

Thus, we conclude that U and V are unitary matrices with $n(n-1)/2$ angles $n(n+1)/2$ phases each, of which there are altogether n phases left that are *not* fixed by h. These n arbitrary phases can be chosen only among the phases that sit on the far right in a product of matrices representing U or V.

Let us now turn to the counting of parameters in CKM. Quark masses come from the following terms in the Lagrangian

$$u^c \mathbf{h}^u Q + d^c \mathbf{h}^d Q \tag{5}$$

Each of the matrices $\mathbf{h}^{u,d}$ is diagonalized by the procedure outlined above

$$\mathbf{h}^{u,d} = \mathbf{U}^{u,d}\mathbf{m}\mathbf{V}^{u,d\dagger} \tag{6}$$

so that the CKM matrix is given by

$$\mathbf{K} = \mathbf{V}^{u\dagger}\mathbf{V}^d \tag{7}$$

K is a unitary matrix and can also in principle have $n(n-1)/2$ angles and $n(n+1)/2$ phases. However, some of the phases can be rotated away. From the discussion above it follows that we can choose n arbitrary phases in $\mathbf{V}^u$ and n arbitrary phases in $\mathbf{V}^d$. These phases sit on the far right of each of $\mathbf{V}^i$ so that K has n arbitrary phases on the left and n arbitrary phases on the right. Now all of these $2n$ phases can be used to redefine phases in K, except one. It is the phase proportional to unity that must be set to zero†. Thus the CKM has $n(n+1)/2-2n+1=(n-1)(n-2)/2$ phases.

C. General form of a unitary matrix

In this Section, I will write out the general forms of $n\times n$ unitary matrices for $n=2,3$ with the obvious generalization to the case of arbitrary n. They follow from the definition of a unitary $n\times n$ matrix V

$$\mathbf{V}\mathbf{V}^\dagger = \mathbf{V}^\dagger\mathbf{V} = 1 \tag{8}$$

General 2×2 unitary matrix

Equations (8) can be written out in terms of the elements of V

$$|V_{11}|^2+|V_{12}|^2=1 \tag{9}$$
$$V_{11}V_{21}^*+V_{12}V_{22}^*=0 \tag{10}$$
$$V_{21}V_{11}^*+V_{22}V_{12}^*=0 \tag{11}$$
$$|V_{21}|^2+|V_{22}|^2=1 \tag{12}$$
$$|V_{11}|^2+|V_{21}|^2=1 \tag{13}$$
$$V_{11}^*V_{12}+V_{21}^*V_{22}=0 \tag{14}$$
$$V_{12}^*V_{11}+V_{22}^*V_{21}=0 \tag{15}$$
$$|V_{12}|^2+|V_{22}|^2=1 \tag{16}$$

From (9) we can introduce an angle θ so that $|V_{11}|=\cos\theta\equiv c$ and $|V_{12}|=\sin\theta\equiv s$. From (12)-(13) also $|V_{21}|=s$ and $|V_{22}|=c$. So far V is of the form

$$\mathbf{V} = \begin{pmatrix} ce^{i\alpha_{11}} & se^{i\alpha_{12}} \\ se^{i\alpha_{21}} & ce^{i\alpha_{22}} \end{pmatrix} \tag{17}$$

and we have to determine the phases from the remaining equations. For example from (10)

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†This freedom corresponds to the familiar baryon number of the standard model.

$$cse^{i(\alpha_{11}-\alpha_{21})} + sce^{i(\alpha_{12}-\alpha_{22})} = 0 \tag{18}$$

The other equations will not introduce any new constraint. We can eliminate one phase, for example α_{21}, so that the most general form for an arbitrary unitary matrix is

$$\mathbf{V} = \begin{pmatrix} ce^{i\alpha_{11}} & se^{i\alpha_{12}} \\ -se^{i(\alpha_{11}-\alpha_{12}+\alpha_{22})} & ce^{i\alpha_{22}} \end{pmatrix} = \begin{pmatrix} 1 & \\ & e^{i(\alpha_{22}-\alpha_{12})} \end{pmatrix} \begin{pmatrix} c & s \\ -s & c \end{pmatrix} \begin{pmatrix} e^{i\alpha_{11}} & \\ & e^{i\alpha_{12}} \end{pmatrix} \tag{19}$$

Thus, we obtain the familiar result that an arbitrary 2×2 unitary matrix is described by one angle and three phases. Notice the ambiguity in placing the three phases. They could have been in any of the four diagonal positions in the left and right matrices on the rhs.

General 3×3 unitary matrix

In the case of $n = 3$ we see from (8) that the elements of a unitary matrix $\mathbf{V}$ satisfy

$$|V_{11}|^2 + |V_{12}|^2 + |V_{13}|^2 = 1 \tag{20}$$
$$V_{11}V_{21}^* + V_{12}V_{22}^* + V_{13}V_{23}^* = 0 \tag{21}$$
$$V_{11}V_{31}^* + V_{12}V_{32}^* + V_{13}V_{33}^* = 0 \tag{22}$$
$$V_{21}V_{11}^* + V_{22}V_{12}^* + V_{23}V_{13}^* = 0 \tag{23}$$
$$|V_{21}|^2 + |V_{22}|^2 + |V_{23}|^2 = 1 \tag{24}$$
$$V_{21}V_{31}^* + V_{22}V_{32}^* + V_{23}V_{33}^* = 0 \tag{25}$$
$$V_{31}V_{11}^* + V_{32}V_{12}^* + V_{33}V_{13}^* = 0 \tag{26}$$
$$V_{31}V_{21}^* + V_{32}V_{22}^* + V_{33}V_{23}^* = 0 \tag{27}$$
$$|V_{31}|^2 + |V_{32}|^2 + |V_{33}|^2 = 1 \tag{28}$$
$$|V_{11}|^2 + |V_{21}|^2 + |V_{31}|^2 = 1 \tag{29}$$
$$V_{11}^*V_{12} + V_{21}^*V_{22} + V_{31}^*V_{32} = 0 \tag{30}$$
$$V_{11}^*V_{13} + V_{21}^*V_{23} + V_{31}^*V_{33} = 0 \tag{31}$$
$$V_{12}^*V_{11} + V_{22}^*V_{21} + V_{32}^*V_{31} = 0 \tag{32}$$
$$|V_{12}|^2 + |V_{22}|^2 + |V_{32}|^2 = 1 \tag{33}$$
$$V_{12}^*V_{13} + V_{22}^*V_{23} + V_{32}^*V_{33} = 0 \tag{34}$$
$$V_{13}^*V_{11} + V_{23}^*V_{21} + V_{33}^*V_{31} = 0 \tag{35}$$
$$V_{13}^*V_{12} + V_{23}^*V_{22} + V_{33}^*V_{32} = 0 \tag{36}$$
$$|V_{13}|^2 + |V_{23}|^2 + |V_{33}|^2 = 1 \tag{37}$$

I now proceed similarly to case $n = 2$. I will first parametrize the "diagonal" equations (20),(24),(28), (29),(33) and (37). This is the place where various parametrizations come in. Let us first do the "standard" parametrization [4,5]. We introduce the first angle θ_{13} such that [‡]

$$|V_{13}| = \sin\theta_{13} \equiv s_{13} \tag{38}$$

The various parametrizations will introduce this first angle at a different element (see the Table later). It follows from (20) and (37) that

$$|V_{11}|^2 + |V_{12}|^2 = c_{13}^2\,,\; |V_{23}|^2 + |V_{33}|^2 = c_{13}^2 \tag{39}$$

This motivates us to introduce two more angles θ_{12} and θ_{23} such that

$$\begin{aligned} |V_{11}|^2 = c_{12}^2c_{13}^2\,,\; |V_{12}|^2 = s_{12}^2c_{13}^2 \\ |V_{23}|^2 = s_{23}^2c_{13}^2\,,\; |V_{33}|^2 = c_{23}^2c_{13}^2 \end{aligned} \tag{40}$$

[‡] As will be obvious later, the choice of the two indices on the angle represent the plane in which a rotation is made. One might think of using a nicer notation with one index only, (i.e. θ_2) to denote the axis of rotation, but this will make less transparent the meaning of the angles and not possible to generalize to more generations. And it would certainly draw criticism (θ_3 would be the Cabibbo angle?!).

This way of introducing additional angles is the only one that allows the interpretation of these angles as rotation angles§.

Since of the 6 diagonal equations only 4 are independent, it is enough to introduce just one more parameter, to take care of all the diagonal equations. I choose a phase δ such that

$$|V_{21}|^2 = |s_{12}c_{23} + c_{12}s_{23}s_{13}e^{i\delta}|^2 \, , \, |V_{22}|^2 = |c_{12}c_{23} - s_{12}s_{23}s_{13}e^{i\delta}|^2$$
$$|V_{31}|^2 = |s_{12}s_{23} - c_{12}c_{23}s_{13}e^{i\delta}|^2 \, , \, |V_{32}|^2 = |c_{12}s_{23} + s_{12}c_{23}s_{13}e^{i\delta}|^2 \qquad (41)$$

Now we are ready to look at the off-diagonal equations, and determine the phases of elements of $\mathbf{V}$. Let us define the yet unknown phases as $V_{11} = c_{12}c_{13}e^{i\alpha_{11}}$, $V_{21} = (s_{12}c_{23} + c_{12}s_{23}s_{13}e^{i\delta})e^{i\alpha_{21}}$, *etc*, and $V_{13} = s_{13}e^{i(\alpha_{13}-\delta)}$. From (21) we have

$$\begin{aligned} & c_{12}c_{13}(s_{12}c_{23} + c_{12}s_{23}s_{13}e^{-i\delta})e^{i(\alpha_{11}-\alpha_{21})} \\ & + s_{12}c_{13}(c_{12}c_{23} - s_{12}s_{23}s_{13}e^{-i\delta})e^{i(\alpha_{12}-\alpha_{22})} \\ & + \quad s_{13}s_{23}c_{13}e^{-i\delta}e^{i(\alpha_{13}-\alpha_{23})} = 0 \end{aligned} \qquad (42)$$

This should be valid for any θ_{12}, θ_{13}, θ_{23} and δ. For $\theta_{12} = 0$ or for $\theta_{13} = 0$ we get respectively

$$e^{i(\alpha_{11}-\alpha_{21})} = -e^{i(\alpha_{13}-\alpha_{23})} \, , \, e^{i(\alpha_{11}-\alpha_{21})} = -e^{i(\alpha_{12}-\alpha_{22})} \qquad (43)$$

Similarly, from (22)

$$e^{i(\alpha_{11}-\alpha_{31})} = e^{i(\alpha_{13}-\alpha_{33})} \, , \, e^{i(\alpha_{11}-\alpha_{31})} = -e^{i(\alpha_{12}-\alpha_{32})} \qquad (44)$$

All other equations are not independent and introduce no new constraints. Thus, out of the 9 α_{ij}, there will be 5 independent ones. We choose α_{11}, α_{12}, α_{13}, α_{23} and α_{33}. Then the most general unitary matrix $\mathbf{V}$ is given by

$$\mathbf{V} = \begin{pmatrix} 1 & & \\ & e^{i(\alpha_{23}-\alpha_{13})} & \\ & & e^{i(\alpha_{33}-\alpha_{13})} \end{pmatrix} \begin{pmatrix} c_{12}c_{13} & s_{12}c_{13} & s_{13}e^{-i\delta} \\ -s_{12}c_{23} - c_{12}s_{23}s_{13}e^{i\delta} & c_{12}c_{23} - s_{12}s_{23}s_{13}e^{i\delta} & s_{23}c_{13} \\ s_{12}s_{23} - c_{12}c_{23}s_{13}e^{i\delta} & -c_{12}s_{23} - s_{12}c_{23}s_{13}e^{i\delta} & c_{23}c_{13} \end{pmatrix} \begin{pmatrix} e^{i\alpha_{11}} & & \\ & e^{i\alpha_{12}} & \\ & & e^{i\alpha_{13}} \end{pmatrix} \qquad (45)$$

Notice that $\mathbf{V}$ can be broken up into a product of matrices

$$\begin{aligned} \mathbf{V} = & \begin{pmatrix} 1 & & \\ & e^{i(\alpha_{23}-\alpha_{13})} & \\ & & e^{i(\alpha_{33}-\alpha_{13})} \end{pmatrix} \\ & \times \begin{pmatrix} 1 & & \\ & c_{23} & s_{23} \\ & -s_{23} & c_{23} \end{pmatrix} \begin{pmatrix} e^{-i\delta} & & \\ & 1 & \\ & & 1 \end{pmatrix} \begin{pmatrix} c_{13} & & s_{13} \\ & 1 & \\ -s_{13} & & c_{13} \end{pmatrix} \begin{pmatrix} e^{i\delta} & & \\ & 1 & \\ & & 1 \end{pmatrix} \begin{pmatrix} c_{12} & s_{12} & \\ -s_{12} & c_{12} & \\ & & 1 \end{pmatrix} \\ & \times \begin{pmatrix} e^{i\alpha_{11}} & & \\ & e^{i\alpha_{12}} & \\ & & e^{i\alpha_{13}} \end{pmatrix} \end{aligned} \qquad (46)$$

or in an obvious symbolic notation

$$\mathbf{V} = (1\alpha_4\alpha_5)\mathbf{R}_{23}(-\delta 11)\mathbf{R}_{13}(\delta 11)\mathbf{R}_{12}(\alpha_1\alpha_2\alpha_3) \qquad (47)$$

where

$$(\alpha_1\alpha_2\alpha_3) = diag(e^{i\alpha_1}, e^{i\alpha_2}, e^{i\alpha_3}) \, , \, (1\alpha_4\alpha_5) = diag(1, e^{i\alpha_4}, e^{i\alpha_5}) \qquad (48)$$

§An alternative choice would be that we choose another element of $\mathbf{V}$ to be a single angle, for example $V_{12} = s_{12}$ [10]. However, I view this as unattractive since then V_{11} is forced to be a $\sqrt{c_{13}^2 - s_{12}^2}$ and it certainly does not allow an interpretation as a rotation angle, so that s_{12} is just doubling of the name for V_{12}.

and I renamed αs: $\alpha_1 = \alpha_{11}$, $\alpha_2 = \alpha_{12}$, *etc.* $\mathbf{V}$ has 3 angles and 6 phases, as it should be for an arbitrary 3×3 unitary matrix. Notice again the ambiguity in placing the 5 αs among the six diagonal entries of the far left and far right matrices on the rhs. Also notice that the angle δ could have been placed at different places in the middle matrix, by an appropriate redefinition of αs. However, what is important to notice is that one combination of phases cannot be pulled out between the three rotations.

An important note is in order here regarding the Cabibbo-Kobayashi-Maskawa matrix. The CKM matrix is an arbitrary unitary matrix with five phases rotated away through the phase redefinition of the left-handed up and down quark fields. Notice that the middle part of the rhs of (45) is exactly the "standard" parametrization [4–6]

$$\mathbf{K} = \mathbf{R}_{23}(-\delta 11)\mathbf{R}_{13}(\delta 11)\mathbf{R}_{12} \tag{49}$$

As we saw in the previous section, it is exactly the freedom of the far right phases in $\mathbf{V}^{\mathbf{u}}$ and $\mathbf{V}^{\mathbf{d}}$ that allows us to eliminate all the phases α_i, but not the phase δ.

Now that we have one parametrization explicitly worked out it is easy to obtain any other parametrization. One just starts by picking one element of the unitary matrix $\mathbf{V}$ to be a single sine or a cosine of an angle, and proceeds by first looking at the diagonal equations to parametrize the other two angles and the phase δ. Then, from the off diagonal equations, one fixes all the other phases in terms of five phases. I list all possibilities in Table I.

	K	J
1	$\mathbf{R}_{23}(-\delta 11)\mathbf{R}_{13}(\delta 11)\mathbf{R}_{12} = \begin{pmatrix} c_{12}c_{13} & s_{12}c_{13} & s_{13}e^{-i\delta} \\ -s_{12}c_{23} - c_{12}s_{23}s_{13}e^{i\delta} & c_{12}c_{23} - s_{12}s_{23}s_{13}e^{i\delta} & s_{23}c_{13} \\ s_{12}s_{23} - c_{12}c_{23}s_{13}e^{i\delta} & -c_{12}s_{23} - s_{12}c_{23}s_{13}e^{i\delta} & c_{23}c_{13} \end{pmatrix}$	$s_{12}s_{13}s_{23}c_{12}c_{13}^2c_{23}s_\delta$
2	$\mathbf{R}_{23}(-\delta 11)\mathbf{R}_{12}(\delta 11)\mathbf{R}_{13} = \begin{pmatrix} c_{12}c_{13} & s_{12}e^{-i\delta} & c_{12}s_{13} \\ -s_{13}s_{23} - c_{13}c_{23}s_{12}e^{i\delta} & c_{12}c_{23} & c_{13}s_{23} - s_{13}c_{23}s_{12}e^{i\delta} \\ -s_{13}c_{23} + c_{13}s_{23}s_{12}e^{i\delta} & -c_{12}s_{23} & c_{13}c_{23} + s_{13}s_{23}s_{12}e^{i\delta} \end{pmatrix}$	$s_{12}s_{13}s_{23}c_{12}^2c_{13}c_{23}s_\delta$
3	$(-\delta 11)\mathbf{R}_{13}(\delta 11)\mathbf{R}_{23}\mathbf{R}_{12} = \begin{pmatrix} c_{12}c_{23} + s_{12}s_{13}s_{23}e^{-i\delta} & s_{12}c_{13} - c_{12}s_{23}s_{13}e^{-i\delta} & s_{13}c_{23}e^{-i\delta} \\ -s_{12}c_{23} & c_{12}c_{23} & s_{23} \\ -c_{12}s_{13}e^{i\delta} + s_{12}s_{23}c_{13} & -s_{12}s_{13}e^{i\delta} - c_{12}s_{23}c_{13} & c_{13}c_{23} \end{pmatrix}$	$s_{12}s_{13}s_{23}c_{12}c_{13}c_{23}^2s_\delta$
4	$(-\delta 11)\mathbf{R}_{12}(\delta 11)\mathbf{R}_{23}\mathbf{R}'_{12} = \begin{pmatrix} c_{12}c'_{12} - s'_{12}s_{12}c_{23}e^{-i\delta} & c_{12}s'_{12} + s_{12}c_{23}c'_{12}e^{-i\delta} & s_{12}s_{23}e^{-i\delta} \\ -s_{12}c'_{12}e^{i\delta} - s'_{12}c_{12}c_{23} & -s_{12}s'_{12}e^{i\delta} + c_{12}c'_{12}c_{23} & c_{12}s_{23} \\ s'_{12}s_{23} & -c'_{12}s_{23} & c_{23} \end{pmatrix}$	$s_{12}s'_{12}s_{23}^2c_{12}c'_{12}c_{23}s_\delta$
5	$(-\delta 11)\mathbf{R}_{13}(\delta 11)\mathbf{R}_{23}\mathbf{R}'_{13} = \begin{pmatrix} c_{13}c'_{13} - s'_{13}s_{13}c_{23}e^{-i\delta} & -s_{13}s_{23}e^{-i\delta} & c_{13}s'_{13} + s_{13}c_{23}c'_{13}e^{-i\delta} \\ -s'_{13}s_{23} & c_{23} & c'_{13}s_{23} \\ -s_{13}c'_{13}e^{i\delta} - s'_{13}c_{13}c_{23} & -c_{13}s_{23} & -s_{13}s'_{13}e^{i\delta} + c_{13}c'_{13}c_{23} \end{pmatrix}$	$s_{13}s'_{13}s_{23}^2c_{13}c'_{13}c_{23}s_\delta$
6	$(11-\delta)\mathbf{R}_{23}(11\delta)\mathbf{R}_{12}\mathbf{R}'_{23} = \begin{pmatrix} c_{12} & s_{12}c'_{23} & s_{12}s'_{23} \\ -s_{12}c_{23} & c_{12}c_{23}c'_{23} - s_{23}s'_{23}e^{i\delta} & c_{12}c_{23}s'_{23} + s_{23}c'_{23}e^{i\delta} \\ s_{12}s_{23}e^{-i\delta} & -c_{12}c'_{23}s_{23}e^{-i\delta} - s'_{23}c_{23} & -c_{12}s'_{23}s_{23}e^{-i\delta} + c_{23}c'_{23} \end{pmatrix}$	$s_{12}^2s_{23}s'_{23}c_{12}c_{23}c'_{23}s_\delta$
7	$(11-\delta)\mathbf{R}_{13}(11\delta)\mathbf{R}_{12}\mathbf{R}'_{13} = \begin{pmatrix} c_{13}c'_{13}c_{12} - s_{13}s'_{13}e^{i\delta} & c_{13}s_{12} & c_{13}c_{12}s'_{13} + s_{13}c'_{13}e^{i\delta} \\ -s_{12}c'_{13} & c_{12} & -s_{12}s'_{13} \\ -s_{13}c_{12}c'_{13}e^{-i\delta} - c_{13}s'_{13} & -s_{13}s_{12}e^{-i\delta} & -s_{13}s'_{13}c_{12}e^{-i\delta} + c_{13}c'_{13} \end{pmatrix}$	$s_{12}^2s_{13}s_{13}c_{12}c_{13}c'_{13}s_\delta$
8	$(11-\delta)\mathbf{R}_{23}(11\delta)\mathbf{R}_{13}\mathbf{R}'_{23} = \begin{pmatrix} c_{13} & -s_{13}s'_{23} & s_{13}c'_{23} \\ -s_{13}s_{23}e^{-i\delta} & c_{23}c'_{23} - s_{23}s'_{23}c_{13}e^{i\delta} & c_{23}s'_{23} + s_{23}c_{13}c'_{23}e^{i\delta} \\ -s_{13}c_{23} & -s_{23}c'_{23}e^{-i\delta} - c_{23}c_{13}s'_{23} & -s_{23}s'_{23}e^{-i\delta} + c_{13}c_{23}c'_{23} \end{pmatrix}$	$s_{13}^2s_{23}s'_{23}c_{13}c_{23}c'_{23}s_\delta$
9	$(1-\delta 1)\mathbf{R}_{12}(1\delta 1)\mathbf{R}_{13}\mathbf{R}'_{12} = \begin{pmatrix} c_{12}c_{13}c'_{12} - s_{12}s'_{12}e^{i\delta} & c_{12}c_{13}s'_{12} + s_{12}c'_{12}e^{i\delta} & c_{12}s_{13} \\ -s_{12}c_{13}c'_{12}e^{-i\delta} - s'_{12}c_{12} & -s_{12}s'_{12}c_{13}e^{-i\delta} + c_{12}c'_{12} & -s_{12}s_{13}e^{-i\delta} \\ -s_{13}c'_{12} & -s_{13}s'_{12} & c_{13} \end{pmatrix}$	$s_{12}s'_{12}s_{13}^2c_{12}c'_{12}c_{13}s_\delta$

(50)

TABLE I. Parametrizations of the Cabibbo-Kobayashi-Maskawa matrix K in terms of three rotation angles and one phase. $\mathbf{R}_{12}$ is the rotation in the 1-2 plane by an angle θ_{12}, $(\delta 11) = \mathrm{diag}(e^{i\delta}, 1, 1)$, etc. Three further parametrizations can be obtained by taking the hermitian conjugate of the first three parametrizations. J is the Jarlskog invariant $J = |\mathrm{Im}(K_{ij}K_{kl}K^*_{il}K^*_{kj})|$. An arbitrary unitary matrix V is given by $\mathbf{V} = \mathrm{diag}(1, e^{i\alpha_4}, e^{i\alpha_5})\,\mathbf{K}\,\mathrm{diag}(e^{i\alpha_1}, e^{i\alpha_2}, e^{i\alpha_3})$ with αs arbitrary.

Three more parametrizations can be achieved by simply taking the hermitian conjugate of the first three entries in Table I. As in the example above, five phases will always multiply from the far left and far right, and I call them α_i (i=1,..,5). Table I lists the middle matrix, **K**, which is related to **V** by

$$\mathbf{V} = (1\alpha_4\alpha_5)\mathbf{K}(\alpha_1\alpha_2\alpha_3) \tag{51}$$

The Table lists also the Jarlskog invariant [11] $J = |\mathrm{Im}(K_{ij}K_{kl}K^*_{il}K^*_{kj})|$.

One angle is always between the rotations and I call it δ. Because of the free phases, δ can be moved within the rotations. Here I adopt the following convention (although any other parametrization is obtained by simple multiplication of phases from left or right). δ and $-\delta$ are always around one rotation through which they do not "go through", i.e. they cancel only if the rotation is trivial. In the example worked out and matrix 1 in Table I, the middle rotation is $\mathbf{R}_{13}$ so δ can be in the $1-1$ or $3-3$ entry. Notice that we positioned the phase δ so that the matrix containing $-\delta$ can always be pulled out of **K** and reabsorbed into one of the α_i. Notice that the presence of both δ and $-\delta$ matrices in **K** is useful in seeing how the complex phase δ explicitly dissapears when the rotation angle between vanishes (for example in matrix 1 of the Table, when $s_{13}=0$, δ also dissapears, and there is no CP violation) [12,13]. This convention however still leaves some freedom of where δ exactly appears in **K**. The physical observables, of course do not depend on the choice of its position.

Each representation starts with choosing one entry to be just a single sine or cosine of an angle. In the example above $|V_{13}| \equiv s_{13}$. I choose the convention if the starting entry is off diagonal we choose a sine, and if it is diagonal we choose a cosine.

As I said before, the matrix K actually represents the most general CKM since the phases α_i are arbitrary and can be set to zero**. Some of the representations for the CKM already appeared in the literature (up to a possible moving of the phase δ by phase multiplications from left and right that are absorbed in α_is and sign ambiguity in sinuses of the angles). Matrix 1 is the standard parametrization of Maiani and Chau and Keung [4,5]. Matrix 2 was proposed by Maiani in [3]. Matrix 4 is the parametrization obtained in [7]. Matrix 6 is the Kobayashi-Maskawa parametrization [2] (which in our notation would be $(11,-1)\mathbf{R}_{23}(11\delta)\mathbf{R}^T_{12}\mathbf{R}'_{23}$).

Now we have everything ready for the construction of CKM directly from the quark mass matrices. I first start with the two generation case, building up for the case of three generations which involves more complicated matrix manipulations.

D. Case of two generations

Here I do the diagonalization of a 2×2 matrix. Let us get for example the form for **V** in terms of elements from **h**. We see that

$$\mathbf{h}^\dagger\mathbf{h} = \begin{pmatrix} |h_{11}|^2+|h_{21}|^2 & h^*_{11}h_{12}+h^*_{21}h_{22} \\ h_{11}h^*_{12}+h_{21}h^*_{22} & |h_{12}|^2+|h_{22}|^2 \end{pmatrix} \tag{52}$$

which is of the form

$$\mathbf{h}^\dagger\mathbf{h} = \begin{pmatrix} \lambda_{11} & \lambda_{12}e^{-i\gamma} \\ \lambda_{12}e^{i\gamma} & \lambda_{22} \end{pmatrix} \tag{53}$$

with real λ_{ij} and γ depending on the elements of **h**. Now we can first pull out the phase γ and then diagonalize the real and symmetric matrix λ_{ij} with one rotation angle θ to get

$$\begin{aligned}\mathbf{h}^\dagger\mathbf{h} = &\begin{pmatrix} 1 & \\ & e^{i\gamma} \end{pmatrix}\begin{pmatrix} c & s \\ -s & c \end{pmatrix}\begin{pmatrix} e^{i\alpha} & \\ & e^{i\beta} \end{pmatrix} \\ &\times \begin{pmatrix} m_1^2 & \\ & m_2^2 \end{pmatrix} \\ &\times \begin{pmatrix} e^{-i\alpha} & \\ & e^{-i\beta} \end{pmatrix}\begin{pmatrix} c & -s \\ s & c \end{pmatrix}\begin{pmatrix} 1 & \\ & e^{-i\gamma} \end{pmatrix}\end{aligned} \tag{54}$$

where α and β are the arbitrary phases. Also, $m^2_{1,2} = \frac{\lambda_{11}+\lambda_{22}}{2} \mp \sqrt{(\frac{\lambda_{11}-\lambda_{22}}{2})^2+\lambda^2_{12}}$ and $\tan\theta = s/c = \frac{\lambda_{12}}{m_2^2-\lambda_{11}}$.

**The meaning of the phases α_i is exactly the familiar freedom of redefinition of the left-handed up and down quark fields and then absorbed in the right-handed fields thus disappearing from the theory.

Comparing this to (2) we see that **V** is equal to the first three matrices on the right-hand side and it is of the general form for a unitary 2×2 unitary matrix as derived in Section C

$$\mathbf{V} = \begin{pmatrix} 1 & \\ & e^{i\gamma} \end{pmatrix} \begin{pmatrix} c & s \\ -s & c \end{pmatrix} \begin{pmatrix} e^{i\alpha} & \\ & e^{i\beta} \end{pmatrix} \tag{55}$$

Similarly, one can construct **U**. It is crucial to remember that at this stage only the phases α and β are arbitrary, whereas γ and the angle θ are fixed by the original matrix **h**.

Now let us count the number of parameters. **h** has 4 real and 4 imaginary parameters. Compare this to the number of parameters in **U** and **V**. Each one has 1 angle and 3 phases, which with the two mass eigenvalues gives a total of 4 angles and 6 phases. Thus two combinations of phases in **U** and **V** will not appear in **h**. This can be seen as follows. From

$$\mathbf{h} = \mathbf{U}\mathbf{m}\mathbf{V}^\dagger \tag{56}$$

we see explicitly that of the 2 pairs or arbitrary phases in **U** and **V** only two combinations appear, that is $\alpha_\mathrm{U} - \alpha_\mathrm{V}$ and $\beta_\mathrm{U} - \beta_\mathrm{V}$. Two of the phases remain arbitrary and in the next section I will choose them to be α_V and β_V.

Now we have everything ready to construct the CKM matrix in the 2 generation cases, that we know consists of one angle only and no phases.

E. CKM matrix for the two generations

The 2×2 CKM matrix is given by

$$\mathbf{K} = \mathbf{V}^{u\dagger}\mathbf{V}^d = \\ \times \begin{pmatrix} e^{-i\alpha^u} & \\ & e^{-i\beta^u} \end{pmatrix} \begin{pmatrix} c^u & -s^u \\ s^u & c^u \end{pmatrix} \begin{pmatrix} 1 & \\ & e^{i\gamma} \end{pmatrix} \begin{pmatrix} c^d & s^d \\ -s^d & c^d \end{pmatrix} \begin{pmatrix} e^{i\alpha^d} & \\ & e^{i\beta^d} \end{pmatrix} \tag{57}$$

where $\gamma = \gamma^d - \gamma^u$.

At this point α^i and β^i are completely arbitrary (independent of the original Yukawa matrices) and I can use them to get rid of all the phases. For this purpose it is enough to notice that the three middle matrices on the rhs can be written as

$$\begin{pmatrix} c^u & -s^u \\ s^u & c^u \end{pmatrix} \begin{pmatrix} 1 & \\ & e^{i\gamma} \end{pmatrix} \begin{pmatrix} c^d & s^d \\ -s^d & c^d \end{pmatrix} = \begin{pmatrix} ce^{i\phi} & se^{i\xi} \\ -e^{i\gamma}se^{-i\xi} & e^{i\gamma}ce^{-i\phi} \end{pmatrix} = \begin{pmatrix} 1 & \\ & e^{i(\gamma-\phi-\xi)} \end{pmatrix} \begin{pmatrix} c & s \\ -s & c \end{pmatrix} \begin{pmatrix} e^{i\phi} & \\ & e^{i\xi} \end{pmatrix} \tag{58}$$

where

$$s \equiv |c^u s^d - s^u c^d e^{i\gamma}|\,,\; c \equiv |c^u c^d + s^u s^d e^{i\gamma}| \tag{59}$$

are real and satisfy $c^2 + s^2 = 1$, *i.e.* they describe one angle. Also $\xi \equiv \arg(c^u s^d - s^u c^d e^{i\gamma})$ and $\phi \equiv \arg(c^u c^d + s^u s^d e^{i\gamma})$. We can now choose for example $\alpha^u = 0$, $\beta^u = \gamma - \phi - \xi$, $\alpha^d = -\phi$, $\beta^u = -\xi$ so that indeed the CKM matrix in the 2 generation case is described by one angle only and no phases

$$\mathbf{K} = \begin{pmatrix} c & s \\ -s & c \end{pmatrix}. \tag{60}$$

It is interesting to note that although no CP violation is present in the CKM for the two generations, the rotation angle depends on the complex phases in the quark mass matrices.

Example. Let us assume that the mass matrix in the down sector is of the form [14]

$$\mathbf{h} = \begin{pmatrix} 0 & Be^{-i\gamma} \\ Be^{i\gamma} & A \end{pmatrix}, \tag{61}$$

and that there is a similar structure in the up sector. The mass eigenvalues and the rotation angle are

$$m_{d,s} = |A/2 \mp \sqrt{A^2/4 + B^2}|\,,\; \tan\theta^d = \frac{B}{m_s} = \sqrt{\frac{m_d}{m_s}} \tag{62}$$

and similarly for the up sector [††]. The CKM angle is given by [15]

$$\sin\theta = |c^u s^d - s^u c^d e^{i\gamma}| = |\sqrt{\frac{m_d}{m_s}} - \sqrt{\frac{m_u}{m_c}}e^{i\gamma}|\frac{1}{\sqrt{1+\frac{m_d}{m_s}}}\frac{1}{\sqrt{1+\frac{m_u}{m_c}}} \tag{63}$$

where $\gamma = \gamma^d - \gamma^u$. These results are *exact*.

F. Case of three generations

Let us now do the more nontrivial case of three generations. Let us again get the form for $\mathbf{V}$ in terms of elements of $\mathbf{h}$. $\mathbf{h}^\dagger\mathbf{h}$ is of the form

$$\mathbf{h}^\dagger\mathbf{h} = \begin{pmatrix} \lambda_{11} & \lambda_{12}e^{-i\gamma_{12}} & \lambda_{13}e^{-i\gamma_{13}} \\ \lambda_{12}e^{i\gamma_{12}} & \lambda_{22} & \lambda_{23}e^{-i\gamma_{23}} \\ \lambda_{13}e^{i\gamma_{13}} & \lambda_{23}e^{i\gamma_{23}} & \lambda_{33} \end{pmatrix} \tag{64}$$

with real λ_{ij} and γ_{ij} depending on the elements of $\mathbf{h}$. I will do the diagonalization by applying successive rotations and phase redefinitions. Various parametrizations of the unitary matrices that diagonalize $\mathbf{h}$ will depend on which order of phase redefinitions and rotations in the process of diagonalization we choose.

I will now continue with a specific choice but the steps can easily be repeated for any other representation. The choice is inspired by the often assumed hierarchical structure in the quark mass matrices with the biggest element at the 3-3 entry (and the other elements getting smaller and smaller as we move away from the 3-3 entry), so that one can easily follow the results below by assuming that the 1-3 rotations are small. However, needless to stress, the results below are exact and do not depend on any assumption.

In order to pull out all the phases I will need to do at least one rotation between phase redefinitions. First let me pull out the phase γ_{13} and do the rotation $\mathbf{R}'_{13}$ in the 1-3 sector to set the corresponding element to zero, and then pull out the remaining phases

$$\begin{aligned}\mathbf{h}^\dagger\mathbf{h} = &\begin{pmatrix} 1 & & \\ & 1 & \\ & & e^{i\gamma_{13}} \end{pmatrix}\begin{pmatrix} c'_{13} & & s'_{13} \\ & 1 & \\ -s'_{13} & & c_{13} \end{pmatrix}\begin{pmatrix} e^{i\gamma'_{12}} & & \\ & 1 & \\ & & e^{i\gamma'_{23}} \end{pmatrix} \\ \times &\begin{pmatrix} \lambda'_{11} & \lambda'_{12} & 0 \\ \lambda'_{12} & \lambda_{22} & \lambda'_{23} \\ 0 & \lambda'_{23} & \lambda'_{33} \end{pmatrix} \\ \times &\begin{pmatrix} e^{-i\gamma'_{12}} & & \\ & 1 & \\ & & e^{-i\gamma'_{23}} \end{pmatrix}\begin{pmatrix} c'_{13} & & -s'_{13} \\ & 1 & \\ s'_{13} & & c_{13} \end{pmatrix}\begin{pmatrix} 1 & & \\ & 1 & \\ & & e^{-i\gamma_{13}} \end{pmatrix}\end{aligned} \tag{65}$$

where $\lambda'_{11}, \lambda'_{33} = \frac{\lambda_{11}+\lambda_{33}}{2} \mp \sqrt{(\frac{\lambda_{11}-\lambda_{33}}{2})^2 + \lambda_{13}^2}$, $\tan\theta'_{13} = \frac{\lambda_{13}}{\lambda'_{33}-\lambda_{11}}$, $\lambda'_{23}e^{i\gamma'_{23}} = \lambda_{12}s'_{13}e^{-i\gamma_{12}} + \lambda_{23}c'_{13}e^{i\gamma_{23}-i\gamma_{13}}$ and $\lambda'_{12}e^{-i\gamma'_{12}} = \lambda_{12}c'_{13}e^{i\gamma_{12}} - \lambda_{23}s'_{13}e^{-i\gamma_{23}+i\gamma_{13}}$.

We are now left with diagonalizing the middle matrix, which I do in Appendix A. Contrary to a recent claim [9], a general real symmetric matrix with two off diagonal zeroes *cannot* be diagonalized with two rotations only. If one chooses, for example, the 2-3 rotation to get rid off λ'_{23} there will be a term $\lambda'_{12}s_{23}$ generated in the 1-3 position. Alternatively, one may think that by cleverly choosing two rotations one can simultaneously completely diagonalize the matrix. This however can easily be seen to fail unless we already start with the trivial case of, say, $\lambda'_{12} = 0$.

Thus, we need three rotations to diagonalize the middle matrix. There is no rule in which order the three rotations should be taken. I pick the first rotation to be a 1-3 rotation $\mathbf{R}''_{13}$ just so it is easy to combine later with $\mathbf{R}'_{13}$. My choice of the subsequent rotations being $\mathbf{R}_{23}$ and $\mathbf{R}_{12}$ is inspired again by the hierarchical structure. So $\mathbf{h}^\dagger\mathbf{h}$ becomes

$$\mathbf{h}^\dagger\mathbf{h} = \begin{pmatrix} 1 & & \\ & 1 & \\ & & e^{i\gamma_{13}} \end{pmatrix}\mathbf{R}'_{13}\begin{pmatrix} e^{i\gamma'_{12}} & & \\ & 1 & \\ & & e^{i\gamma'_{23}} \end{pmatrix}\mathbf{R}''_{13}\mathbf{R}_{23}\mathbf{R}_{12}\begin{pmatrix} e^{i\alpha_1} & & \\ & e^{i\alpha_2} & \\ & & e^{i\alpha_3} \end{pmatrix}$$

[††] The absolute values in the masses can be obtained by actually diagonalizing $\mathbf{h}^\dagger\mathbf{h}$.

$$\times \begin{pmatrix} m_1^2 & 0 & 0 \\ 0 & m_2^2 & 0 \\ 0 & 0 & m_3^2 \end{pmatrix}$$

$$\times \begin{pmatrix} e^{-i\alpha_1} & & \\ & e^{-i\alpha_2} & \\ & & e^{-i\alpha_3} \end{pmatrix} \mathbf{R}_{12}^T \mathbf{R}_{23}^T \mathbf{R}_{13}^{\prime\prime T} \begin{pmatrix} e^{-i\gamma_{12}'} & & \\ & 1 & \\ & & e^{-i\gamma_{23}'} \end{pmatrix} \mathbf{R}_{13}^{\prime T} \begin{pmatrix} 1 & & \\ & 1 & \\ & & e^{-i\gamma_{13}} \end{pmatrix} \tag{66}$$

in the obvious notation for the rotations. It is easy to obtain the relations between the rotation angles and the matrix elements and we leave the details to Appendix A. The phases α_i are the arbitrary phases. We can now identify the unitary matrix $\mathbf{V}$ as the first line on the rhs of the above equation

$$\mathbf{V} = \begin{pmatrix} e^{i\gamma_{12}'} & & \\ & 1 & \\ & & e^{i\gamma_{13}+i\gamma_{12}'} \end{pmatrix} \mathbf{R}_{13}' \begin{pmatrix} 1 & & \\ & 1 & \\ & & e^{i\gamma} \end{pmatrix} \mathbf{R}_{13}'' \mathbf{R}_{23} \mathbf{R}_{12} \begin{pmatrix} e^{i\alpha_1} & & \\ & e^{i\alpha_2} & \\ & & e^{i\alpha_3} \end{pmatrix} \tag{67}$$

where $\gamma = \gamma_{23}' - \gamma_{12}'$. We can bring the matrix into one of the forms for unitary matrices given in Table I. First, I combine the two 1-3 rotations, with the phase γ between, into one 1-3 rotation, between two phases, similarly to equation (58):

$$\mathbf{R}_{13}'(11\gamma)\mathbf{R}_{13}'' = (11, \gamma - \phi - \xi)\mathbf{R}_{13}(\phi 1 \xi) = (\xi 1, \gamma - \phi)\mathbf{R}_{13}(\phi - \xi, 11) \tag{68}$$

where $s_{13} = |c_{13}' s_{13}'' + s_{13}' c_{13}'' e^{i\gamma}|$, $\xi = \arg(c_{13}' s_{13}'' + s_{13}' c_{13}'' e^{i\gamma})$ and $\phi = \arg(c_{13}' c_{13}'' - s_{13}' s_{13}'' e^{i\gamma})$. After a further trivial phase redefinition we arrive at the form of a general arbitrary unitary matrix listed in Table I (matrix 3),

$$\mathbf{V} = (1\alpha_4\alpha_5)(-\delta 11)\mathbf{R}_{13}(\delta 11)\mathbf{R}_{23}\mathbf{R}_{12}(\alpha_1\alpha_2\alpha_3) =$$

$$\begin{pmatrix} 1 & & \\ & e^{i\alpha_4} & \\ & & e^{i\alpha_5} \end{pmatrix} \begin{pmatrix} c_{12}c_{23} + s_{12}s_{13}s_{23}e^{-i\delta} & s_{12}c_{13} - c_{12}s_{23}s_{13}e^{-i\delta} & s_{13}c_{23}e^{-i\delta} \\ -s_{12}c_{23} & c_{12}c_{23} & s_{23} \\ -c_{12}s_{13}e^{i\delta} + s_{12}s_{23}c_{13} & -s_{12}s_{13}e^{i\delta} - c_{12}s_{23}c_{13} & c_{13}c_{23} \end{pmatrix} \begin{pmatrix} e^{i\alpha_1} & & \\ & e^{i\alpha_2} & \\ & & e^{i\alpha_3} \end{pmatrix} \tag{69}$$

where $\delta = \phi - \xi$, $\alpha_4 = -\phi - \gamma_{12}'$, $\alpha_5 = \gamma_{13} + \gamma - 2\phi$.

As expected the unitary matrix $\mathbf{V}$ depends on 3 angles and 6 phases. Of the six phases only three of α_i are arbitrary $(\alpha_1, \alpha_2, \alpha_3)$, and others are fixed by the matrix $\mathbf{h}$. Notice of course the ambiguity in placing the αs on the left and right end, as well as the ambiguity in placing δ between the matrices.

The counting of parameters proceeds similarly to the 2×2 case. From $\mathbf{h} = \mathbf{U}\mathbf{m}\mathbf{V}^\dagger$ we see that the 9 real parameters on the lhs are matched on the rhs by the 3 angles in $\mathbf{U}$, 3 eigenvalues in $\mathbf{m}$ and 3 angles in $\mathbf{V}$. For the phases, $\mathbf{h}$ has 9 of them. On the other side, $\mathbf{U}$ and $\mathbf{V}$ have six phases each. However, three combinations of the six phases that were arbitrary (3 in $\mathbf{U}$ and 3 in $\mathbf{V}$) do not enter the rhs. I will again choose these arbitrary phases to be the $\alpha_1, \alpha_2, \alpha_3$ in $\mathbf{V}$ when we consider the CKM in the next Section.

G. CKM matrix for the three generations

I choose the parametrization derived in the previous section (but any other is equally good) for the two left-handed rotations

$$\mathbf{V}^i = (1\alpha_4^i\alpha_5^i)(-\delta^i 11)\mathbf{R}_{13}^i(\delta^i 11)\mathbf{R}_{23}^i\mathbf{R}_{12}^i(\alpha_1^i\alpha_2^i\alpha_3^i) \tag{70}$$

where $i = u, d$. Only the phases $\alpha_1^i, \alpha_2^i, \alpha_3^i$ do not depend on the original quark mass matrices. CKM is given by

$$\mathbf{K} = \mathbf{V}^{u\dagger}\mathbf{V}^d = (-\alpha_1^u - \alpha_2^u - \alpha_3^u)\mathbf{R}_{12}^{uT}\mathbf{R}_{23}^{uT}(-\delta_d, \alpha_4^d - \alpha_4^u, \delta^u - \delta^d)\mathbf{R}_{13}^{uT}(11\gamma)\mathbf{R}_{13}^d(\delta^d 11)\mathbf{R}_{23}^d\mathbf{R}_{12}^d(\alpha_1^d\alpha_2^d\alpha_3^d) \tag{71}$$

where $\gamma = \delta^d - \delta^u + \alpha_5^d - \alpha_5^u$.

This expression can be transformed into one of the parametrizations of the CKM with three rotations and one phase. We can arrive at any of the nine parametrizations given in Table I, but I choose the one requiring the least number of manipulations, the parametrization 4, and I derive it here. For this purpose I will use manipulations with rotations and as well the freedom in αs. However, since the manipulations of rotations involve phases between them, and keeping track becomes more cumbersome, I first derive the CKM with no phases. Then I consider the most general case of nonvanishing phases.

3×3 CKM with no phases

In this case the CKM from (71) is given by

$$\mathbf{K} = \mathbf{V}^{u\dagger}\mathbf{V}^{d} = \mathbf{R}_{12}^{uT}\mathbf{R}_{23}^{uT}\mathbf{R}_{13}^{uT}\mathbf{R}_{13}^{d}\mathbf{R}_{23}^{d}\mathbf{R}_{12}^{d} \tag{72}$$

First, notice that the product $\mathbf{R}_{13}^{uT}\mathbf{R}_{13}^{d}$ can be written as one 1-3 rotation $\mathbf{R}_{13}$ with angle $\theta_{13} = \theta_{13}^{d} - \theta_{13}^{u}$. Then, we can find rotations $\mathbf{R}_{12}''$, $\mathbf{R}_{23}$, $\mathbf{R}_{12}'''$, such that

$$\mathbf{R}_{23}^{uT}\mathbf{R}_{13}\mathbf{R}_{23}^{d} = \mathbf{R}_{12}''\mathbf{R}_{23}\mathbf{R}_{12}''' \tag{73}$$

where we read the relations between the angles from the (very useful!) Table I, comparing matrices 4 and 8, with all phases set to zero

$$c_{23} = c_{13}c_{23}^{u}c_{23}^{d} + s_{23}^{u}s_{23}^{d}\,,\; s_{12}''s_{23} = s_{13}c_{23}^{d}\,,\; s_{12}'''s_{23} = -s_{13}c_{23}^{u} \tag{74}$$

Finally, combining the two pairs of 1-2 rotations ($\mathbf{R}_{12}^{uT}$ and $\mathbf{R}_{12}''$, $\mathbf{R}_{12}^{d}$ and $\mathbf{R}_{12}'''$) into just two 1-2 rotations ($\mathbf{R}_{12}$ and $\mathbf{R}_{12}'$ respectively) with $\theta_{12} = \theta_{12}'' - \theta_{12}^{u}$ and $\theta_{12}' = \theta_{12}''' + \theta_{12}^{d}$ we get

$$\mathbf{K} = \mathbf{R}_{12}\mathbf{R}_{23}\mathbf{R}_{12}' = \begin{pmatrix} c_{12}c_{12}' - s_{12}'s_{12}c_{23} & c_{12}s_{12}' + s_{12}c_{23}c_{12}' & s_{12}s_{23} \\ -s_{12}c_{12}' - s_{12}'c_{12}c_{23} & -s_{12}s_{12}' + c_{12}c_{12}'c_{23} & c_{12}s_{23} \\ s_{12}'s_{23} & -c_{12}'s_{23} & c_{23} \end{pmatrix} \tag{75}$$

3×3 CKM with phases

Now let me consider the most general case in equation (71). The product of the two 1-3 rotations with the phase γ between can be combined into one 1-3 rotation $\mathbf{R}_{13}$ between two phase transformations, similar to equation (58) or (68). At this point we have

$$\mathbf{K} = (-\alpha_1^u - \alpha_2^u - \alpha_3^u)\mathbf{R}_{12}^{uT}(-\delta^d + \xi, \alpha_4^d - \alpha_4^u, \alpha_4^d - \alpha_4^u)\mathbf{R}_{23}^{uT}(11\rho)\mathbf{R}_{13}\mathbf{R}_{23}^{d}(\delta^d + \phi - \xi, 1, 1)\mathbf{R}_{12}^{d}(\alpha_1^d\alpha_2^d\alpha_3^d) \tag{76}$$

where $\rho = \alpha_5^d - \alpha_5^u - (\alpha_4^d - \alpha_4^u) - \phi$. Now we can recognize the product of the middle three rotations as one of the entries in Table I (matrix 8). We can immediately write it then as matrix 4, with appropriate phases on the left and right, since we relate two forms of the same unitary matrix. We call the phases β_i, $i = 1, ..., 5$ rather than α_i, to point out that they depend on the initial quark mass matrices

$$\mathbf{R}_{23}^{uT}(11\rho)\mathbf{R}_{13}\mathbf{R}_{23}^{d} = (1\beta_4\beta_5)\mathbf{R}_{12}''(\delta'11)\mathbf{R}_{23}\mathbf{R}_{12}'''(\beta_1\beta_2\beta_3) \tag{77}$$

Comparing in Table I we get

$$\begin{aligned} &c_{23} = |c_{13}c_{23}^{u}c_{23}^{d}e^{i\rho} + s_{23}^{u}s_{23}^{d}|\,,\; s_{12}''s_{23} = s_{13}c_{23}^{d}\,,\; s_{12}'''s_{23} = -s_{13}c_{23}^{u} \\ &\beta_3 = 0\,,\; \beta_5 = \arg(s_{23}^{u}s_{23}^{d} + c_{13}c_{23}^{u}c_{23}^{d}e^{i\rho})\,,\; \beta_1 = \rho - \beta_5 \\ &\beta_4 = \arg(c_{23}^{u}s_{23}^{d} - c_{13}c_{23}^{d}s_{23}^{u}e^{i\rho})\,,\; \beta_2 = -\beta_5 + \arg(s_{23}^{u}c_{23}^{d} - c_{13}s_{23}^{d}c_{23}^{u}e^{i\rho})\,,\; \delta' = \arg(c_{13}e^{i\beta_1} + s_{12}''s_{12}''')c_{23})\,, \end{aligned} \tag{78}$$

So we write

$$\mathbf{K} = (-\alpha_1^u - \alpha_2^u - \alpha_3^u)\mathbf{R}_{12}^{uT}(1\delta''1)\mathbf{R}_{12}''(\delta'11)\mathbf{R}_{23}\mathbf{R}_{12}'''(1\delta'''1)\mathbf{R}_{12}^{d}(\alpha_1^d\alpha_2^d\alpha_3^d) \tag{79}$$

where most of the phases were trivially absorbed into α_i and $\delta'' = \delta^d + \alpha_4^d - \alpha_4^u + \beta_4 - \xi$ and $\delta''' = -\delta^d + \beta_2 - \beta_1 - \phi + \xi$. What is left is to combine the two pairs of 1-2 rotations ($\mathbf{R}_{12}^{uT}$ and $\mathbf{R}_{12}''$, $\mathbf{R}_{12}^{d}$ and $\mathbf{R}_{12}'''$) with the phases between just two 1-2 rotations ($\mathbf{R}_{12}$ and $\mathbf{R}_{12}'$) between phases, similar to equation (58).

And finally I can choose α_i to rotate away all phases that depend on the original quark mass matrices, except one which we call δ. So I arrive at the form for the CKM

$$\mathbf{K} = (-\delta 11)\mathbf{R}_{12}(\delta 11)\mathbf{R}_{23}\mathbf{R}_{12}' = \begin{pmatrix} c_{12}c_{12}' - s_{12}'s_{12}c_{23}e^{-i\delta} & c_{12}s_{12}' + s_{12}c_{23}c_{12}'e^{-i\delta} & s_{12}s_{23}e^{-i\delta} \\ -s_{12}c_{12}'e^{i\delta} - s_{12}'c_{12}c_{23} & -s_{12}s_{12}'e^{i\delta} + c_{12}c_{12}'c_{23} & c_{12}s_{23} \\ s_{12}'s_{23} & -c_{12}'s_{23} & c_{23} \end{pmatrix} \tag{80}$$

which corresponds to matrix 4 in Table I.

As stressed before, there are essentially twelve different parametrizations of the CKM that are given in Table I, and we can arrive at any of them. It is just a matter of picking the order of rotations in the process of diagonalization, and manipulations in the CKM to get to any of the standard forms with three rotation angles and one phase.

Although it maybe cumbersome, the above results are exact. It is instructive to repeat the above excercise approximately for the often assumed case of hierarchical quark mass matrices, where the largest element is h_{33}, and the elements get smaller as we get farther away from this element. Then the angles in the CKM parametrization derived above (matrix 4 in Table 1) are in simple approximate relation with the elements of the Yukawa matrices. For this purpose let us first look at the derivation of the diagonalization matrices $\mathbf{V}^i$, i=u,d, in Section C. First, as advertised before, let me show that we can in first approximation neglect the 1-3 rotations. We see that because of the assumed hierarchy $\theta_{13}^{i'} \approx \lambda_{13}^i/\lambda_3^i \approx (h_{21}^i h_{23}^i + h_{31}^i h_{33}^i)/h_{33}^{i2}$ is very small. Also from Appendix A, $\theta_{13}^{i''} \approx \lambda_{23}^{i'}\lambda_{12}^{i'}/\lambda_3^{i2}$ is very small. Now, from Appendix A, $\theta_{23} \approx \lambda_{23}^i/\lambda_3^i \approx h_{32}^i/h_{33}^i$ and $\theta_{12}^i \approx \lambda_{12}^i/\lambda_2^i \approx (h_{21}^i h_{22}^i + h_{31}^i h_{32}^i)/m_2^{i2}$. Now, let us look at CKM. Since we neglect 1-3 rotations, $\mathbf{R}_{12}''$ and $\mathbf{R}_{12}'''$ can be neglected, and $\theta_{23} \approx |\theta_{23}^d - \theta_{23}^u e^{i\rho}|$, while $\theta_{12} \approx -\theta_{12}^u$ and $\theta_{12}' \approx \theta_{12}^d$.

Example. Let us assume that the up and down quark mass matrices are of the form [16,17,7]

$$\mathbf{h}^u = \begin{pmatrix} 0 & C & 0 \\ C & 0 & B \\ 0 & B & A \end{pmatrix}, \mathbf{h}^d = \begin{pmatrix} 0 & F & 0 \\ F & E & 0 \\ 0 & 0 & D \end{pmatrix} \tag{81}$$

with all the nonzero entries complex in principle. I can trivially pull out their phases

$$\mathbf{h}^u = \begin{pmatrix} e^{i\gamma_C - i\gamma_B} & & \\ & e^{i\gamma_B - i\gamma_A} & \\ & & 1 \end{pmatrix} \begin{pmatrix} 0 & C & 0 \\ C & 0 & B \\ 0 & B & A \end{pmatrix} \begin{pmatrix} e^{i\gamma_C - i\gamma_B + i\gamma_A} & & \\ & e^{i\gamma_B} & \\ & & e^{i\gamma_A} \end{pmatrix} \tag{82}$$

and similarly for $\mathbf{h}^d$

$$\mathbf{h}^d = \begin{pmatrix} e^{i\gamma_F - i\gamma_E} & & \\ & 1 & \\ & & 1 \end{pmatrix} \begin{pmatrix} 0 & F & 0 \\ F & E & 0 \\ 0 & 0 & D \end{pmatrix} \begin{pmatrix} e^{i\gamma_F} & & \\ & e^{i\gamma_E} & \\ & & e^{i\gamma_D} \end{pmatrix} \tag{83}$$

where A, B, C, D, E, F are now real and positive.

$\mathbf{h}^d$ is now trivially diagonalized with one rotation $\mathbf{R}_{12}^d$

$$\begin{aligned}\mathbf{h}^d = & \begin{pmatrix} e^{i\gamma_F - i\gamma_E} & & \\ & 1 & \\ & & 1 \end{pmatrix} \mathbf{R}_{12}^d \begin{pmatrix} e^{i\alpha_1} & & \\ & e^{i\alpha_2} & \\ & & e^{i\alpha_3} \end{pmatrix} \\ & \times \begin{pmatrix} m_d & 0 & 0 \\ 0 & m_s & 0 \\ 0 & 0 & m_b \end{pmatrix} \\ & \times \begin{pmatrix} e^{-i\beta_1} & & \\ & e^{-i\beta_2} & \\ & & e^{-i\beta_3} \end{pmatrix} \mathbf{R}_{12}^{dT} \begin{pmatrix} e^{i\gamma_F} & & \\ & e^{i\gamma_E} & \\ & & e^{i\gamma_D} \end{pmatrix}\end{aligned} \tag{84}$$

where the rotation angle is given by

$$\tan\theta_{12}^d = \frac{F}{m_s} = \sqrt{\frac{m_d}{m_s}}. \tag{85}$$

In order to get the CKM as easy as possible I choose to diagonalize the up quark matrix with the 1-2 2-3 1-2 rotations

$$\begin{pmatrix} 0 & C & 0 \\ C & 0 & B \\ 0 & B & A \end{pmatrix} = \mathbf{R}_{12}^{u'}\mathbf{R}_{23}^u\mathbf{R}_{12}^u \begin{pmatrix} e^{i\alpha_1} & & \\ & e^{i\alpha_2} & \\ & & e^{i\alpha_3} \end{pmatrix} \begin{pmatrix} m_u^2 & & \\ & m_c^2 & \\ & & m_t^2 \end{pmatrix} \begin{pmatrix} e^{-i\beta_1} & & \\ & e^{-i\beta_2} & \\ & & e^{-i\beta_3} \end{pmatrix} \mathbf{R}_{12}^{uT}\mathbf{R}_{23}^{uT}\mathbf{R}_{12}^{u'T} \tag{86}$$

where, using the matrix 4 in Table I with all phases set to zero,

$$t_{12}^{u'} = \frac{C}{m_t}, t_{23}^u = \frac{AB}{m_t^2 - B^2 - C^2}\frac{1}{c_{12}^{u'}}, t_{12}^u = -\frac{BC}{m_c^2 - C^2}\frac{s_{23}^u}{c_{12}^{u'}} - t_{12}^{u'}c_{23}^u \tag{87}$$

and I have used the shorthand $t \equiv s/c$. The mass squares $m_{u,c,t}^2$ are the solutions of the cubic equation $\det(\mathbf{h}^\dagger\mathbf{h} - m^2 1) = (C^2 - m^2)[(B^2 + C^2 - m^2)(A^2 + B^2 - m^2) - A^2B^2] - B^2C^2(B^2 + C^2 - m^2) = 0$. This can be written as $(B^2 + C^2 - m^2)^2 m^2 = (C^2 - m^2)^2 A^2$, reflecting the fact that one can also look for eigenvalues of $\mathbf{h}^u$ itself, but with all eigenvalues positive. Also, notice $m_u m_c m_t = C^2 A$.

Now, we can easily read the matrices $\mathbf{V}^{u,d}$ and compute the CKM

$$\mathbf{K} = \mathbf{V}^{u\dagger}\mathbf{V}^{d} = \begin{pmatrix} e^{-i\alpha_1^u} & & \\ & e^{-i\alpha_2^u} & \\ & & e^{-i\alpha_3^u} \end{pmatrix} \mathbf{R}_{12}^{uT}\mathbf{R}_{23}^{uT}\mathbf{R}_{12}^{u'T} \begin{pmatrix} 1 & & \\ & e^{i\gamma} & \\ & & 1 \end{pmatrix} \mathbf{R}_{12}^{d} \begin{pmatrix} e^{i\beta} & & \\ & e^{i\beta} & \\ & & 1 \end{pmatrix} \begin{pmatrix} e^{i\alpha_1^d} & & \\ & e^{i\alpha_2^d} & \\ & & e^{i\alpha_3^d} \end{pmatrix} \quad (88)$$

where $\beta = \gamma_C - \gamma_B + \gamma_F - \gamma_E$ and $\gamma = 2\gamma_B - \gamma_A - \gamma_C - \gamma_F + \gamma_E$, and αs are the arbitrary phases. Again, we can combine the two 1-2 rotations with the phase γ into one 1-2 rotation just as in equation (58), $\mathbf{R}_{12}^{u'T}(1\gamma 1)\mathbf{R}_{12}^{d} = (1, \gamma - \phi - \xi, 1)\mathbf{R}'_{12}(\phi\xi 1)$, where

$$s'_{12} = |c_{12}^{u'} s_{12}^{d} + c_{12}^{d} s_{12}^{u'} e^{i\gamma}| \quad (89)$$

The freedom in αs can be used to rotate away all phases except one. This brings the CKM to the form of matrix 4 in Table I

$$\mathbf{K} = (-\delta 11)\mathbf{R}_{12}(\delta 11)\mathbf{R}_{23}\mathbf{R}'_{12} \quad (90)$$

where $\delta = -\gamma + \phi + \xi$, $\theta_{12} = -\theta_{12}^{u}$ and $\theta_{23} = -\theta_{23}^{u}$. The rotation angles are given in (85), (87) and (89). This completes the *exact* solution of the example.

One can check the approximate results previously obtained. The cubic equation is solved approximately for $m_t \approx A$, $m_c \approx B^2/A$ and $m_u \approx C^2/m_c$. The mixing angles are $t_{12}^{u'} = \sqrt{m_u m_c}/m_t$ (exact!), $t_{23}^{u} \approx \sqrt{m_c/m_t}$, $t_{12} \approx C/m_c = \sqrt{\frac{m_u}{m_c}}$, and $t_{12}^{d} = \sqrt{\frac{m_d}{m_s}}$ (exact!). We see that t''_{12} is very small compared to t_{12}^{d} and for all practical purposes one can take $t'_{12} \approx t_{12}^{d}$. So we have to leading order for the CKM elements $K_{cb} \approx K_{ts} \approx s_{23} \approx \frac{B}{m_t} \approx \sqrt{\frac{m_c}{m_t}}$, $|\frac{K_{ub}}{K_{cb}}| = |t_{12}| \approx \sqrt{\frac{m_u}{m_c}}$ and $|\frac{K_{td}}{K_{ts}}| = |t'_{12}| \approx \sqrt{\frac{m_d}{m_s}}$.

H. Conclusions

In conclusion, CKM can be parametrized in terms of three rotation angles and one phase, but it is not clear which order of rotations and positioning of the phase is best to use in a given model of quark masses. The problem is that one must construct the CKM from the unitary rotations on the left-handed quark fields. Each of the left up and down diagonalization matrices consists of three angles and six phases and one must combine them in a nontrivial way to obtain the CKM in a parametrization with three angles and one phase. There are essentially 12 possible parametrizations (up to overall phase multiplications) and I list them in Table I.

Some forms may turn out to be more practical than others. For example, if the elements of the quark mass matrices exhibit a hierarchy, then the CKM parametrization as matrix 4 of Table I seems to be the one that is most easily obtained from the diagonalization matrices. In particular, this form has simple elements in the top or bottom entries and may prove convenient in the analysis of heavy quark processes [7–9].

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Appendix A. Diagonalization of a real symmetric matrix with two texture zeroes

We want to diagonalize the matrix

$$\mathbf{M}^2 = \begin{pmatrix} \lambda_{11} & \lambda_{12} & 0 \\ \lambda_{12} & \lambda_{22} & \lambda_{23} \\ 0 & \lambda_{23} & \lambda_{33} \end{pmatrix} \quad (91)$$

The angles of the necessary rotations are then determined in terms of the λ_{ij} and the eigenvalues λ_i, $i = 1,2,3$. The eigenvalues λ_i, are the solutions of the cubic equation $\det(\mathbf{M}^2 - \lambda\mathbf{1})$

$$(\lambda_{11} - \lambda)[(\lambda_{22} - \lambda)(\lambda_{33} - \lambda) - \lambda_{23}^2] - (\lambda_{33} - \lambda)\lambda_{12}^2 = 0 \quad (92)$$

and we order them so that λ_1 is smallest and λ_3 largest.

As stressed in the text, we need three rotations in order to diagonalize this matrix. We choose the following order (although any other is equally good)

$$\mathbf{M}^2 = \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix} = \mathbf{R}_{12}^T \mathbf{R}_{23}^T \mathbf{R}_{13}^T \begin{pmatrix} \lambda_{11} & \lambda_{12} & 0 \\ \lambda_{12} & \lambda_{22} & \lambda_{23} \\ 0 & \lambda_{23} & \lambda_{33} \end{pmatrix} \mathbf{R}_{13}\mathbf{R}_{23}\mathbf{R}_{12} \tag{93}$$

We can rewrite the above equation

$$\begin{pmatrix} \lambda_{11} & \lambda_{12} & 0 \\ \lambda_{12} & \lambda_{22} & \lambda_{23} \\ 0 & \lambda_{23} & \lambda_{33} \end{pmatrix} \mathbf{R}_{13}\mathbf{R}_{23}\mathbf{R}_{12} = \mathbf{R}_{13}\mathbf{R}_{23}\mathbf{R}_{12} \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix} \tag{94}$$

from which we can read off the rotation angles

$$\begin{aligned} \frac{s_{13}}{c_{13}} &= \frac{\lambda_{23}\lambda_{12}}{(\lambda_3 - \lambda_{22})(\lambda_3 - \lambda_{11}) - \lambda_{12}^2} \\ \frac{s_{23}}{c_{23}} &= \frac{\lambda_{23}c_{13} + \lambda_{12}s_{13}}{\lambda_3 - \lambda_{22}} \\ \frac{s_{12}}{c_{12}} &= \frac{\lambda_{12}c_{23} + s_{23}s_{13}(\lambda_2 - \lambda_{11})}{(\lambda_2 - \lambda_{11})c_{13}} . \end{aligned} \tag{95}$$

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