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Integration of Schwinger Equation for $(\phi^*\phi)^2_d$ Theory

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Abstract

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A general solution for the Schwinger equation for the generating functional of the complex scalar field theory with $(\phi^*\phi)^2_d$ interaction has been constructed. The method is based on the reduction of the order of this equation using the particular solution ~~from~~ ref. [1].

Аннотация

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Построено общее решение уравнения Швингера для производящего функционала теории комплексного скалярного поля с взаимодействием $(\phi^*\phi)^2_d$. Метод построения решения основан на понижении порядка уравнения Швингера с помощью указанного в [1] частного решения.

The Schwinger equations for the generating functional and related system of Dyson equations for the Green's functions (vacuum expectation values) are one of the ways of representing dynamical equations of quantum field theory (QFT). In the framework of perturbative QFT this technique is equivalent to other well known methods, such as operator formalism, functional integral technique, etc. At the same time, since the Schwinger equations are functional–differential ones, one can impose the problem of integrating these equations. Speaking about the integration of the Schwinger equation we will imply, by analogy with integration of ordinary differential equations, the reduction of these equations to some simplest equations in the functional derivatives. Such nonperturbative integration, may turn out to be useful in studying the problems going beyond the perturbation theory, e.g., spontaneous breakdown of symmetry.

Let us consider the theory of a complex scalar field in Euclidean space with the interaction Lagrangian $\mathcal{L}_{int} = -\frac{\lambda}{2}(\phi^*\phi)^2$. In this model the Schwinger equation has the form

$$\lambda \frac{\delta^2 G}{\delta\eta(xx)\delta\eta(yx)} = \delta(x-y)\cdot G + \int dx'\eta(xx')\frac{\delta G}{\delta\eta(yx')}\,. \qquad (1)$$

Here $G[\eta]$ is a generating functional of the Green's functions, $\eta(xy)$ is a bilocal source, $x, y \in E_d$. The source switching off corresponds to $\eta(xy) = \eta_0(xy) \equiv (m^2-\partial^2)\delta(x-y)$, where m^2 is the mass of a free field. The generating functional G is normalized by the condition $G[\eta_0] = 1$. At a switched off source the derivatives are the Green's functions (VEVs)

$$\left.\frac{\delta^n G}{\delta\eta(y_n x_n)...\delta\eta(y_1 x_1)}\right|_{\eta=\eta_0} = (-1)^n < 0|T\{\phi(x_1)...\phi(x_n)\phi^*(y_1)...\phi^*(y_n)\}|0 > \,.$$

At $\lambda = 0$ Eq. (1) has the solution $G_0 = det(\eta_0\eta^{-1})$ which is the generating functional of a free field. The G_0 based iteration scheme reproduces perturbation series over the powers of the coupling constant λ.

When $\lambda \neq 0$ the integration of equation (1) consists of two subsequent steps. The first one is the reduction of the order of equation (1) through a substitution, that uses a particular solution of this equation. The second step is to solve the resultant equation.

As is noted in ref.[1] Schwinger equation (1) has a particular solution, which is a quadratic exponential of the source η. In a more general form the functional

$$G_p[\eta] = \exp\left\{\frac{1}{2\lambda}\int dxdx'dydy' K(xx'|yy')\eta(yx)\eta(y'x')\right\}$$

is a particular solution of equation (1) if the kernel K satisfies the system of equations

$$\begin{cases} K(xx|yx) = \delta(x-y), \\ \\ K(xx'|yy')K(xx''|xy'') = \delta(x-y'')K(x'x''|yy'). \end{cases} \tag{2}$$

There exists at least one solution of system (2): $K(xx'|yy') = \delta(y-y')$.

Let us introduce the functional $W(xy|\eta)$, defined by the relation

$$\frac{\delta G}{\delta\eta(yx)} = \frac{1}{\lambda}\int dx'dy' K(xx'|yy')\eta(y'x')\cdot G - W(xy)\,. \tag{3}$$

Substituting (3) into (1) and with account of (2) we obtain the equations for W

$$\begin{cases} \lambda\frac{\delta W(xy)}{\delta\eta(xx)} = \int dx'\eta(xx')W(x'y) - \int dx'dy' K(xx'|yy')\eta(y'x')W(xx) \\ \\ \lambda\frac{\delta W(xx)}{\delta\eta(yx)} = \int dx'\eta(xx')W(x'y) - \int dx'dy' K(xx'|xy')\eta(y'x')W(xy) \end{cases} \tag{4}$$

Equations (4) are linear ones of the 1-st order w.r.t. the functional derivatives over the source η, which means that substitution (3) does reduce the order of the Schwinger equation (1). Two equations for W are a consequence of the commutability condition of the functional differentiation.

In order to integrate equations (4), i.e. to reduce them to the simplest equations in the functional derivatives, we shall make one more substitution

$$W(xy|\eta) = V(xy|\eta)\exp\{Z[\eta]\}, \tag{5}$$

where the functional V is the solution of equations

$$\frac{\delta V(xy)}{\delta\eta(xx)} = \frac{\delta V(xx)}{\delta\eta(yx)} = 0. \tag{6}$$

Then from equations (4) we obtain the equations for the functional Z

$$\begin{cases} \lambda V(xy)\frac{\delta Z}{\delta\eta(xx)} = \int dx'\eta(xx')V(x'y) - \int dx'dy'K(xx'|yy')\eta(y'x')V(xx), \\ \\ \lambda V(xx)\frac{\delta Z}{\delta\eta(yx)} = \int dx'\eta(xx')V(x'y) - \int dx'dy'K(xx'|xy')\eta(y'x')V(xy). \end{cases} \tag{7}$$

The functional Z can be normalized by the condition $Z[\eta_0] = 0$.

Formulae (2)–(7) solve the problem of integrating Schwinger equation (1) in the following sense: for any solution of system (6) and kernel K, satisfying the system of equations (2), we can calculate all the derivatives of Z using formulae (7), thus we shall determine the functional Z, and consequently the functional W using formula (5). Then formula (3) allows us to calculate the derivatives of G in the point η_0, which solves the imposed problem.

One can easily construct numerous solutions for the system of equations (6), both polynomial in the source η and non–polynomial (e.g., the functional $\frac{\delta}{\delta\eta}(\det\eta)$ belongs to the latter ones). Consequently, the general solution for the Schwinger equation contains functional arbitrariness, given by the general solution of equation (6). Therefore Schwinger equations in the nonperturbative region possess more freedom in choosing solutions, than the arbitrariness connected with boundary conditions (see refs. [1,2]).

As far as the system of equations (2) for the kernel K, one may point out at least one solution for the system. The author does not know whether there exist any other solutions for system (2).

In connection with this the relations, that do not contain the kernel K, are of interest. One of them can be obtained in the following way. Let us integrate the second equation in (2) over x', x'', y' and y'' with the weight $\eta(y'x')\eta(y''x'')$ and exclude K with the help of relation (3). As a result we get the relation

$$\begin{aligned} &\int dx'\eta(xx')\left[\frac{\delta G}{\delta\eta(yx')} + W(x'y)\right]\cdot G = \\ &= \lambda\left[\frac{\delta G}{\delta\eta(yx)} + W(xy)\right]\left[\frac{\delta G}{\delta\eta(xx)} + W(xx)\right]. \end{aligned} \tag{8}$$

The derivative of G is the propagator of the field ϕ, the source being switched off:

$$\Delta(x-y) \equiv \langle 0|T\{\phi(x)\phi^*(y)\}|0\rangle = -\left.\frac{\delta G}{\delta\eta(yx)}\right|_{\eta=\eta_0}.$$

We shall also denote the functional $W(xy)$ through $\bar{\Delta}(x-y)$, the source being switched off. From relation (8) at $\eta = \eta_0$ we obtain an equation for the difference $\Delta_\Phi \equiv \Delta - \bar{\Delta}$

$$(m^2 - \partial^2)\Delta_\Phi(x) = -\lambda\Delta_\Phi(0)\Delta_\Phi(x). \tag{9}$$

Let us introduce a "renormalized mass"

$$m_{\rm I}^2 \equiv m^2 + \lambda\Delta_\Phi(0). \tag{10}$$

Three cases are possible: 1) $m_{\rm I}^2 > 0$; 2) $m_{\rm I}^2 < 0$ and 3) $m_{\rm I}^2 = 0$.

Leaving apart two first cases, consider the case $m_{\rm I}^2 = 0$ in more detail. In this case the solution for equation (9) in the class of tempered distributions is a constant

$$\Delta_\Phi(x) = \Delta_\Phi(0) \equiv \Delta_0,$$

consequently, in agreement with (10)

$$\Delta_0 = -\frac{m^2}{\lambda}. \tag{11}$$

From relation (11) it follows that the case $m_{\rm I}^2 = 0$ corresponds to the Goldstone model (which is just the model under study at $m^2 < 0$). Indeed in the Goldstone model the following relation is valid

$$\Delta(x) = -\frac{m^2}{\lambda} + \frac{1}{2}\{\Delta_1(x) + \Delta_2(x)\}.$$

Here Δ is the propagator of the initial field ϕ , and Δ_1 and Δ_2 are propagators of "physical" fields of the Goldstone model ϕ_1 and ϕ_2, i.e., the fields,which are excitations of "true" vacuum

$$\phi(x) = \Phi_0 + \frac{1}{\sqrt{2}}\{\phi_1(x) + i\phi_2(x)\},$$

where $|\Phi_0|^2 \equiv -m^2/\lambda$. Consequently, if one identifies $\bar{\Delta}$ with a half sum of ϕ_1 and ϕ_2 field propagators, then relation (11) will exactly coincide with the classical result of the Goldstone model.

As for the choice of the solution for equation (6), one needs an additional physical principle. It may be quite possible that the requirement for the correspondence with the perturbation theory at $m^2 > 0$, $\lambda \to +0$ could be such a principle. However it is the presence of an essential singularity in λ at zero makes the application of this principle quite difficult. Really, at $\lambda \to 0$, Eq.(2) would imply $K = O(1)$ behaviour of the kernel K. Recognizing such behaviour of K we obtain (from (3) and (7)) that $W = O(1/\lambda)$ and $Z = O(1/\lambda)$. Consequently

$$V = O\Big(\frac{1}{\lambda}\exp\{O(1/\lambda)\}\Big), \tag{12}$$

and when choosing the solutions for equation (6), agreeing with the perturbation theory, from the very beginning one should take into consideration the essential singularity in the coupling constant of type (12).

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