



RU9810171

INSTITUTE OF THEORETICAL  
AND EXPERIMENTAL PHYSICS

45-94

ITEP

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THE QCD STRING WITH QUARKS. II.  
LIGHT CONE HAMILTONIAN

29 - 17

MOSCOW 1994

THE QCD STRING WITH QUARKS. 2. LIGHT CONE HAMILTONIAN: Preprint  
ITHEP 45-94/

A.Yu.Dabín, A.B.Kaidalov, Yu.A.Simonov - M., 1994 - 24p.

The light-cone Hamiltonian is derived from the general gauge - and Lorentz - invariant expression for the  $q\bar{q}$  Green's function, containing confinement via the area law for the Wilson loop. The resulting Hamiltonian contains in a nonadditive way contributions from quark and string degrees of freedom. Different limiting dynamical regimes of the system of quarks connected by the string are found. In the limit of no transverse motion one recovers the 't Hooft's 1+1 QCD Hamiltonian. The correspondence with the rest frame Hamiltonian spectrum is established. The important notion of the light-cone string, contributing significantly to the energy-momentum is discussed.

Ref. - 20.

## 1 Introduction

We have studied recently ([1], hereafter denoted as I) the quark-antiquark system in the confining vacuum. Starting from the QCD Lagrangian and assuming the minimal area law for the Wilson loop we have derived the dynamics of the system, where in addition to quark and antiquark also the QCD string appears at large distances,  $R > T_g$ , where  $T_g$  is the correlation length of the gluonic vacuum.

We have found in I the Hamiltonian of the system in the c.m. frame, which contains degrees of freedom of quarks and those of the string. The latter enters via an auxiliary function  $\nu$ , introduced in I, which physically measures the energy density of the string.

The Hamiltonian describes two different limiting regimes (for light quarks); for low orbital momentum  $n_r \gg L$  one obtains the relativistic linear potential Hamiltonian [2] describing radial excitations, while for high  $L \gg n_r$  one gets the regime of a rotating string.

It is remarkable that when correction terms are taken into account both regimes yield [3] almost the same standard slope  $\alpha' = (2\pi\sigma)^{-1}$  of Regge trajectories. This outcome also agrees with the result of numerical quantization in [4]. Our derivation in I essentially exploited the c.m. frame, and we have neglected in the path integral the backtracking paths of the quark and antiquark, arguing that those are suppressed on dynamical grounds.

The light-cone frame is advantageous from several points of view. First, the light-cone dynamics is believed to be simpler, especially for perturbative contributions [5]. Second, the back-

tracking paths do not contribute on the light cone for kinematical reasons. Last, but not the least, the light-cone formalism based on light-cone Hamiltonian, allows one to make contact with the parton picture in QCD [6] and to calculate formfactors and structure functions in terms of the eigenfunctions of the light-cone Hamiltonian.

Recently a promising formalism was proposed and shown to be effective at least for treatment of perturbative interaction – the light cone quantization [7]. Our light-cone Hamiltonian which will be obtained below in section 2 contains also the nonperturbative interaction between quarks – the string, and one can test on it methods of [7] or modification of those.

It is a purpose of the present paper to apply the method of I in the light-cone frame and to calculate the light-cone Hamiltonian for the system of quark and antiquark in the nonperturbative (confining) vacuum. As in I, we exploit the formalism of the Feynman-Schwinger representation for the  $q\bar{q}$  Green's function [8,9] together with the Hamiltonian formalism of [10,9]. We neglect for simplicity quark pair creation and perturbative gluon exchanges, concentrating only on the nonperturbative interaction, producing (initially in Euclidean space) the minimal area surface, bounded by paths of quark and antiquark. The Hamiltonian we are looking for, depends on the chosen evolution parameter  $\tau$  and is defined after continuation into Minkowski space on the 3-hypersurface (light cone plane) in  $4d$ .

The quantum Hamiltonian is in general both first and second quantized, i.e. is an operator in terms of field creation-annihilation operators and also an operator in space of coordinates of particles and antiparticles.

Since as in I we use the minimal area law asymptotics (which corresponds to neglect of string excitations) and neglect quark pair creation together with perturbative gluon exchanges there is a possibility to obtain only first-quantized Hamiltonian. The price paid in I was that backtracking paths had to be neglected too, since they correspond (after approximations being made) to pair creation processes from the vacuum. We find below the light-cone Hamiltonian, which is again only first-quantized, and additional neglect of no backtracking is not necessary since it is contained automatically in the light-cone frame.

Thus the light-cone dynamics also reduces to the relativistic quantum mechanics of quarks connected by the string, as in I, but without additional assumptions.

As well as in I we find two dynamical regimes of the "minimal" QCD string (corresponding to the minimal area law asymptotics of the averaged Wilson loop): for large orbital excitations  $L \approx |L_z| \gg n_r$ , the mass squared  $M^2$  and the total momentum  $P_+$  are determined by the rotating in transverse plane string contribution; for large radial quantum numbers  $n_r \gg L$  one gets the relativistic potential Hamiltonian so that the string contribution is "inert" in the leading order. We find the agreement between the calculations of the spectrum in the limit  $L \rightarrow \infty$  or  $n_r \rightarrow \infty$  in the rest and light-cone frames, postponing the detailed comparison of the Regge trajectories in I and II till the next publication.

We postpone the detailed discussion of relation of our Hamiltonian to the standard relativistic quantum mechanics formalism [11],[12] based on the Dirac paper [11] till the next publication. We note only that usual assumption, that the interaction

doesn't contribute to the total momentum  $P_+$  and to the orbital momentum is not valid for our system because the string carries the part of  $P_+$ .

The paper is organized as follows. In section 2 the light-cone Hamiltonian is derived from our general expressions in I. In Section 3 we study the spectrum of the obtained Hamiltonian and elaborate in detail the limiting cases such as the t'Hooft's 1+1 QCD at  $N_C \rightarrow \infty$ , the heavy quarkonia case and three limiting relativistic regimes. Section 4 is devoted to summary and conclusions. In Appendix we establish a connection between our approach and t'Hooft 1+1 QCD equation.

## 2 The light-cone Hamiltonian

Given a  $q\bar{q}$  Green's function in the coordinate space  $G(x\bar{x}; y\bar{y})$ , where  $x\bar{x}(y\bar{y})$  are final (initial) 4-coordinates of quark and anti-quark, one can define the Hamiltonian  $H$  through the equation (in the Euclidean space-time)

$$\frac{\partial G}{\partial T} = -HG \quad (2.1)$$

where  $T$  is an evolution parameter corresponding to some choice of a 3d hypersurface  $\Sigma$ . In the particular case of the c.m. Hamiltonian in I the role of  $T$  is played by the center-of-mass Euclidean time coordinate  $T = \frac{x_4 + \bar{x}_4}{2}$  and the hypersurface  $\Sigma$  is a hyperplane  $x_4 = \bar{x}_4 = \text{const}$ .

In general,  $H$  is a i) second - quantized and a ii) first - quantized operator, which means that i)  $H$  has matrix elements connecting different number of particles and ii)  $H$  is an operator acting on the coordinates of each particle.

In our case, we neglect all perturbative gluon exchanges (small  $\alpha_s$ , at all distances [13]) and quark-pair creation (small  $1/N_C$ ) and keep only nonperturbative confining interaction leading in Euclidean space to the area law of the Wilson loop. With all that, the first-quantized Hamiltonian can be defined only at large distances  $R \gg T_g$  [14] where  $T_g$  is the vacuum correlation length [15].

Thus the problem becomes that of relativistic quantum mechanics and in I we have found the relativistic Hamiltonian in the c.m. frame.

In this paper we aim at finding Hamiltonian in the light-cone frame, which can be done directly via (2.1) and the Feynman-Schwinger representation of the  $q\bar{q}$  Green's function [8,9]. This type of analysis in a more general case, when also perturbative gluons are taken into account, will be a subject of another publication, and here instead we shall follow the formalism of I, adjusting it to our specific frame.

With the notations for the vectors  $a_\mu, b_\mu$

$$ab = a_\mu b_\mu = a_+ b_+ - a_0 b_0 = a_\perp b_\perp + a_- b_- + a_+ b_+, \quad (2.2)$$

$$a_\pm = \frac{a_3 \pm a_0}{\sqrt{2}},$$

one can define the hypersurface  $\Sigma$  through the  $q\bar{q}$  coordinates  $z_\mu, \bar{z}_\mu$  as

$$z_+(\tau) = \bar{z}_+(\bar{\tau}) \quad (2.3)$$

and the kinetic part  $K + \bar{K}$  of the action  $A$ , (I, eqs(25-26)).

$$A = K + \bar{K} + \sigma S_{\min}, \quad (2.4)$$

has the form (after the continuation of eq.(22) into the Euclidean space)

$$\begin{aligned} K + \bar{K} &= \frac{1}{4} \int_0^s \dot{z}_\mu^2(\tau) d\tau + \frac{1}{4} \int_0^s \dot{\bar{z}}_\mu^2(\bar{\tau}) d\bar{\tau} + \int_0^s m_1^2 d\tau + \int_0^s m_2^2 d\tau \\ &= \int_0^T dz_+ \left[ \frac{\mu_1}{2} (\dot{z}_+^2 + 2\dot{z}_-) + \frac{\mu_2}{2} (\dot{\bar{z}}_+^2 + 2\dot{\bar{z}}_-) + \frac{m_1^2}{2\mu_1} + \frac{m_2^2}{2\mu_2} \right] \end{aligned} \quad (2.5)$$

where instead of proper time  $\tau$  we have used  $z_+$  and introduced new variables  $\mu_1, \mu_2$  via

$$2\mu_1(z_+) = \frac{dz_+}{d\tau} ; \quad 2\mu_2(z_+) = \frac{d\bar{z}_+}{d\tau} ; \quad z_+ = \bar{z}_+ \quad (2.6)$$

The condition of no backtracking  $\dot{z}_+, \dot{\bar{z}}_+ > 0$  discussed above implies  $\mu_1 > 0, \mu_2 > 0$ . Dots in (2.5) imply derivatives in  $z_+$ , while

$$T = \frac{x_+ + \bar{x}_+}{2} = x_+ \quad (2.7)$$

For the minimal area surface  $S_{min}$  we use the approximation that it is the worldsheet of the straight line connecting  $z_\mu(z_+)$  and  $\bar{z}_\mu(z_+)$  with the same value of the evolution parameter  $z_+$ , i.e.

$$S_{min} = \int_0^T dz_+ \int_0^1 d\beta [\dot{w}^2 w'^2 - (\dot{w}w')^2]^{1/2} \quad (2.8)$$

where

$$w_\mu(z_+; \beta) = z_\mu(z_+) \beta + \bar{z}_\mu(z_+) (1 - \beta) \quad (2.9)$$

and dot and prime denote derivatives in  $z_+$  and  $\beta$  respectively.

This approximation corresponds to the instantaneous formation of the string (flux tube) in accordance with the positions of quarks as in [10,1]. We now introduce "center-of-masses" and relative coordinates, (see [16] for a detailed discussion)

$$\dot{R}_\mu = x(\tau) \dot{z}_\mu + (1 - x(\tau)) \dot{\bar{z}}_\mu, \quad \dot{r}_\mu = \dot{z}_\mu - \dot{\bar{z}}_\mu \quad (2.10)$$



where function  $x(\tau)$  will be determined below and also auxiliary functions  $\eta(\beta, x_+), \nu(\beta, x_+)$  as in I will enter our action to get rid of the square root in (2.8).

Similarly to I we obtain

$$G(x\bar{x}, y\bar{y}) = \int D\mu_1(z_+) D\mu_2(z_+) D\nu \cdot D\eta DR_\mu D\tau_\mu e^{-A} \quad (2.11)$$

where

$$A = \frac{1}{2} \int_0^T dz_+ \{ a_1 \dot{R}_\perp^2 + 2a_1 \dot{R}_- - 2c_1 \dot{R}_\perp \tau_\perp + 2a_2 \dot{R}_\perp \dot{\tau}_\perp + (2.12) \\ + 2a_2 \dot{\tau}_- - 2c_2 \dot{\tau}_\perp \tau_\perp + a_3 \dot{\tau}_\perp^2 - 2c_1 \tau_- + a_4 \tau_\perp^2 + \frac{m_1^2}{\mu_1} + \frac{m_2^2}{\mu_2} \}$$

and we have used notations

$$a_1 = \mu_1 + \mu_2 + \int_0^1 \nu(z_+, \beta) d\beta \quad (2.13)$$

$$a_2 = \mu_1(1-x) - x\mu_2 + \int_0^1 d\beta \nu(1-x\beta - x(1-\beta)) \quad (2.14)$$

$$a_3 = \mu_1(1-x)^2 + \mu_2 x^2 + \int_0^1 \nu(z_+, \beta) (\beta-x)^2 d\beta \quad (2.15)$$

$$a_4 = \int_0^1 \frac{\sigma^2}{\nu} d\beta + \int_0^1 \eta^2 \nu d\beta \quad (2.16)$$

$$c_1 = \int_0^1 \eta \nu d\beta \quad (2.17)$$

$$c_2 = \int_0^1 \eta \nu (\beta-x) d\beta \quad (2.18)$$

One can determine now  $x(\tau)$  from the requirement that  $\dot{R}$  is to be decoupled from  $\dot{\tau}$  to provide the diagonalization [16] of the quadratic velocity part of the action (2.12). Formally it results in the condition  $a_2 = 0$  which gives

$$x = \frac{\mu_1 + \int \nu \beta d\beta}{a_1}; \quad 1-x = \frac{\mu_2 + \int \nu (1-\beta) d\beta}{a_1}; \quad (2.19)$$

As we shall see from Section 3 function  $x(\tau)$  plays the role of the Feinman light cone variable

Integration over  $\eta$  in (2.11) with  $A$  given by (2.12) can be done in the same way as in I and one gets a new effective action  $A'$  (here and in what follows we take into account only the exponent and not the preexponential factor, since we are interested in terms in the exponent linearly growing with  $T$ ). We have

$$A' = \frac{1}{2} \int dz_+ \left\{ \frac{m_1^2}{\mu_1} + \frac{m_2^2}{\mu_2} + a_1(\dot{R}_\perp^2 + 2\dot{R}_-) + a_3 \dot{r}_\perp^2 + \int \frac{\sigma^2}{\nu} d\beta \cdot r_\perp^2 \right. \quad (2.20)$$

$$\left. - \frac{(r_- + \dot{R}_\perp r_\perp + (\langle \beta \rangle - x) \dot{r}_\perp r_\perp)^2}{r_\perp^2 (\int \nu d\beta)^{-1}} - \frac{(\dot{r}_\perp r_\perp)^2 \int \nu (\beta - \langle \beta \rangle)^2 d\beta}{r_\perp^2} \right\},$$

where we define  $\langle \beta \rangle = \int \nu \beta d\beta / \int \nu d\beta$ .

We now proceed as in I to integrate over  $D\dot{R}_\mu$  instead of  $DR$  using relations

$$\int DR e^{-A'} = \int D\dot{R} \int d^3 \lambda e^{-A'} \cdot e^{i\lambda_\perp \int_0^T \frac{dR_\perp}{dt} dz_+ + i\lambda_+ \int \frac{dR_-}{dt} dz_+} \quad (2.21)$$

$$\cdot e^{-i\lambda_\perp (R_\perp(T) - R_\perp(0))} e^{-i\lambda_+ (R_-(T) - R_-(0))}$$

Fixing the frame of reference we finally have  $\lambda_\mu \rightarrow P_\mu$ , where  $P_\mu$  is total c.m. momentum so that one obtains for  $\vec{P}_\perp = 0$  the new effective action

$$A'' = \frac{1}{2} \int_0^T dz_+ \left\{ \frac{m_1^2}{\mu_1} + \frac{m_2^2}{\mu_2} + a_3 \dot{r}_\perp^2 + \int \frac{\sigma^2}{\nu} d\beta \cdot r_\perp^2 - \right. \quad (2.22)$$

$$\left. \nu_2 \frac{(\dot{r}_\perp r_\perp)^2}{r_\perp^2} - \frac{\nu_0 a_1 [r_- + (\langle \beta \rangle - x) \dot{r}_\perp r_\perp]^2}{r_\perp^2 (\mu_1 + \mu_2)} \right\}$$

where  $\nu_k = \int \nu(\beta, z_+) (\beta - \langle \beta \rangle)^k d\beta$ . Integration over  $D\dot{R}_-$  yields  $\delta(a_1 - P_+)$  with an important constraint

$$a_1 = P_+ \quad (2.23)$$

while integration over  $DR_+$  is trivial since  $A'$  does not depend on  $\dot{R}_+$ .

It is convenient to represent the constant (2.23) (existing for any point  $z_+$ , so that we have  $\pi_i \delta(a_1(z_+) - P_+)$ ) introducing the following representation of  $\delta$  - function

$$\delta(a_1(z_+) - P_+) = \frac{1}{2\pi} \int D\alpha(z_+) e^{i\alpha(z_+)(a_1(z_+) - P_+)} \quad (2.24)$$

Furthermore we go over into the Minkowski space, which means that

$$\begin{aligned} \mu_i &\rightarrow -i\mu_i^M, \quad \nu \rightarrow -i\nu^M \\ a_i &\rightarrow -ia_i^M, \quad A \rightarrow -iA^M \end{aligned} \quad (2.25)$$

For the Minkowski action we obtain (omitting from now on the superscript  $M$  everywhere)

$$\begin{aligned} A^M &= \frac{1}{2} \int dz_+ \left\{ -\frac{m_1^2}{\mu_1} - \frac{m_2^2}{\mu_2} + a_3 \dot{r}_\perp^2 - \int \frac{\sigma^2 d\beta}{\nu} r_\perp^2 - \right. \\ &\quad \left. - \nu_2 \frac{(\dot{r}_\perp r_\perp)^2}{r_\perp^2} - \frac{\nu_0 a_1}{(\mu_1 + \mu_2) r_\perp^2} [r_- + (\langle \beta \rangle - x) \dot{r}_\perp r_\perp]^2 + 2\alpha(a_1 - P_+) \right\} \end{aligned} \quad (2.26)$$

Let us now define the Hamiltonian, corresponding to the action  $A^M$

$$A^M = \int dz_+ L^M, \quad H = p_\perp \dot{r}_\perp - L^M \quad (2.27)$$

with  $p_\perp = \frac{\partial L^M}{\partial \dot{q}_\perp}$ . Simple calculations lead to the following expression

$$\begin{aligned} H &= \frac{1}{2} \left\{ \frac{m_1^2}{\mu_1} + \frac{m_2^2}{\mu_2} + \frac{p_\perp^2 - \frac{(p_\perp r_\perp)^2}{r_\perp^2}}{a_3} + \frac{(p_\perp r_\perp + \gamma r_-)^2}{\tilde{\mu} r_\perp^2} \right. \\ &\quad \left. + \int \frac{\sigma^2}{\nu} d\beta r_\perp^2 + \frac{\nu_0 a_1}{\mu_1 + \mu_2} \frac{r_-^2}{r_\perp^2} \right\} \end{aligned} \quad (2.28)$$

where we have used notations

$$\gamma = \nu_0 \langle \beta \rangle - \frac{\mu_1}{\mu_1 + \mu_2} = a_1 \left( x - \frac{\mu_1}{\mu_+} \right), \quad \bar{\mu} = \frac{\mu_1 \mu_2}{\mu_1 + \mu_2}$$

and the condition (2.23) is implied.

We note that additionally to canonically conjugated pairs  $\{x, (P_+ \tau_-)\}$  (see Section 3 for a short discussion),  $\{\vec{p}_\perp, \vec{r}_\perp\}$  Hamiltonian (2.28) contains the auxiliary field  $\nu(\tau, \beta)$  playing the role of the string energy density. Variables  $\mu_1, \mu_2, a_3$  are to be expressed with the help of eqs. (2.19), (2.23) in terms of  $x$  and  $\nu$ .

The integration over  $\nu(\tau, \beta)$  effectively amounts [1] to the substitution into eq. (2.28) its extremal value

$$\frac{\delta H}{\delta \nu(\tau, \beta)} \Big|_{\nu=\nu^{ext}} = 0 \quad (2.29)$$

Only after this substitution one is to construct the operator Hamiltonian acting on the wave functions (for more detailed discussion see forthcoming paper).

The Hamiltonian (2.28) is the central result of our paper. In the next section we shall study its properties.

### 3 Properties of the light-cone Hamiltonian

We consider in this section the case of heavy quarks and then different limiting relativistic regimes of the light-cone Hamiltonian.

### 3.1 Heavy quark limit of light-cone Hamiltonian

Assuming that quark masses are large,

$$m_1 \gg \sqrt{\sigma}, \quad m_2 \gg \sqrt{\sigma}, \quad (3.1)$$

one can prove that  $\nu$  does not depend on  $\beta$  in the leading order. Let us introduce the variable  $y = \frac{\nu}{P_+}$  corresponding to the part of the total momentum carried by the string. We shall demonstrate that the effective values of  $y$  satisfy the condition

$$y \ll 1 \quad (3.2)$$

First one is to expand kinetic part of (2.28) around the extremal value of  $x$

$$x_{ext} = \frac{m_1}{m_1 + m_2} \quad (3.3)$$

with the result

$$(m_1 + m_2)^2 + 2(m_1 + m_2) \frac{1}{2\bar{m}} (p_\perp^2 + p_z^2) + (m_1 + m_2)^2 y \quad (3.4)$$

where  $\bar{m} = m_1 m_2 / (m_1 + m_2)$  and we defined

$$p_z \equiv (m_1 + m_2)(x - m_1 / (m_1 + m_2)) \quad (3.5)$$

We stress here that kinetic part of eq. (2.28) actually contains not only the analog of the rest frame kinetic term  $\frac{p_\perp^2}{2\bar{m}}$ , but also the part of the potential  $(m_1 + m_2)y$ . Therefore the Hamiltonian (2.28) can not be represented in relativistic case as a sum of pure kinetic and pure potential term as it is assumed in some approaches [12].

Substituting (3.4) into (2.28) one arrives at the Hamiltonian in the form where auxiliary function  $y$  participates

$$H = \frac{1}{2P_+} \{ (m_1 + m_2)^2 + 2(m_1 + m_2) \left[ \frac{1}{2\bar{m}} (p_\perp^2 + p_z^2) + (3.6) \right. \}$$

$$+(m_1 + m_2) \frac{1}{2} \left( \frac{y}{r_\perp^2} \right) \left( r_\perp^2 + \left( \frac{P_+ r_-}{m_1 + m_2} \right)^2 + \frac{\sigma^2}{2(m_1 + m_2)} \left( \frac{r_\perp^2}{y} \right) \right) \Big\}$$

The integration over  $y$  in the path integral with the Hamiltonian (3.6) amounts [1] to the insertion of the extremal value of  $y$

$$y_{exp} = \frac{\sigma r_\perp^2}{m_1 + m_2} \cdot \left( r_\perp^2 + \left( \frac{P_+ r_-}{m_1 + m_2} \right)^2 \right)^{-1/2} \quad (3.7)$$

and the condition (3.2) is indeed satisfied.

Finally introducing  $r_z$ , canonically conjugated to  $p_z$  (3.16) via relation

$$r_z \equiv \frac{(P_+ r_-)}{m_1 + m_2} \quad (3.8)$$

one obtains the Hamiltonian of heavy quarkonia in the light-cone system:

$$H_{nonrel} = \frac{1}{2P_+} \left\{ (m_1 + m_2)^2 + 2(m_1 + m_2) \left( \frac{1}{2\bar{m}} \vec{p}^2 + \sigma |\vec{r}| \right) \right\} \quad (3.9)$$

or since  $H_{nonrel} = \frac{1}{2P_+} M_{nr}^2$ , one has in the leading order

$$M_{nr} \cong m_1 + m_2 + \frac{1}{2\bar{m}} \vec{p}^2 + \sigma |\vec{r}|, \quad (3.10)$$

in agreement with the usual nonrelativistic result in the c.m. system. We note also that variable  $(x - \frac{m_1}{m_1 + m_2})$  is canonically conjugated to  $(P_+ r_-)$ , which follows from eqs. (3.5), (3.8).

We emphasize that after elimination of the auxiliary function the  $O(3)$  rotation invariance of the Hamiltonian is recovered, which ensures that our straight-line ansatz on light-cone is compatible with the conservation of the angular momentum.

### 3.2 Relativistic dynamical regimes on light-cone

To consider relativistic regimes of the "minimal" QCD string with quarks in terms of light-cone variables, it is profitable to make correspondence with the dynamics of the system in the rest frame where as we proved in I there exist two limiting relativistic regimes. For the case of large orbital excitations (case 1 in what follows)  $L \gg n_r$  ( $n_r$  is the radial quantum number) the system behaves as the rotating string, which carries the main part of the orbital momentum and the energy. In the opposite case  $n_r \gg L$  (case 2) the string is nearly pure inert (constituting in the leading order the linear potential) and almost doesn't contribute into the kinetic part of the Hamiltonian. We have found in I that the spectrum is very close to the following asymptotic form

$$M_n^2 = 2\pi\sigma(2n_r + L + const) \quad (3.11)$$

When one goes over to the light-cone, there appears as we will demonstrate three different dynamical regimes, which can be connected to the two aforementioned of the rest frame ones by an infinite boost along  $z$ -axis. To establish this relation we are to recover first the counterpart of the rest frame  $r_z$  on the light cone. From the expression (3.8) for the heavy quarkonia limit of the Hamiltonian (2.28) one can anticipate, that it is the combination

$$(P_+ r_-)/M \quad (3.12)$$

where  $M$  is the total meson mass which corresponds to the rest frame  $r_z$  component. We note here that expression (3.12) obviously takes into account well known Lorentz squeezing of

longitudinal size for moving objects and determines the proper longitudinal scale of the light cone dynamics.

Due to the fact, that  $z$ -axis is distinguished from transverse ones the limiting dynamical regimes on the light cone are realized either for stretched configuration

$$\frac{|P_+ r_-|}{M} \gg |r_\perp| \quad (3.13)$$

or for squeezed into the perpendicular plane one

$$\frac{|P_+ r_-|}{M} \ll |r_\perp| \quad (3.14)$$

To form the corresponding wave packets in the rest frame one is to use for a given  $L \gg 1$  the set of adjoint Legendre polynomials  $P_L^{L_z}(\cos\theta)$ ,  $|L_z| \leq L$ . If  $\Delta\theta$  is angular smearing of the configuration with respect to  $z$ -axis, then the stretched wave packet,  $|r_z| \gg |r_\perp|$  is constructed from a set of  $n \sim 1/\Delta\theta$  polynomials  $P_L^{L_z}$  with  $|L_z| \ll L$ . On the other hand the squeezed configuration  $|r_z| \ll |r_\perp|$  is to be formed with use of  $n \sim 1/\Delta\theta$  different polynomials with  $(L - |L_z|) \ll L$ .

To consider these dynamical limits on light cone we first note that in the (almost) 2+1 squeezed case there are two independent possibilities to excite the system. The first one is to increase  $|L_z| \approx L$  keeping the radial quantum number to be constant. As we will derive below this regime on light cone corresponds to the transverse rotating string with the spectrum  $M^2 = 2P_+ H$

$$M^2 \rightarrow 2\pi\sigma \cdot L, \quad L \approx |L_z| \gg n_r \quad (3.15)$$

In the opposite case,  $n_r \gg L \approx |L_z|$  the transverse linear po-



tential describes the excitations of  $n_r$

$$H \rightarrow \frac{1}{2P_+} (2|\vec{p}_\perp| + \sigma|\vec{r}_\perp|)^2 \quad (3.16)$$

so that

$$M^2 \rightarrow 2\pi\sigma(2n_r), \quad n_r \gg L \approx |L_z| \gg 1 \quad (3.17)$$

We emphasize that these regimes can be interpreted as the transverse projections of the proper rest frame regimes. In contrast to that on the light cone there exists a quasi 1 + 1 dynamics of the stretched configuration peculiar for this frame and described, as we will prove, by the well known 1 + 1 QCD Hamiltonian of t'Hooft [17].

$$H \rightarrow \frac{1}{2P_+} (2\sigma|P_+r_-|), \quad n_r \gg L \gg |L_z| \quad (3.18)$$

Longitudinal excitations of this configuration obviously correspond in the rest frame to the increase of  $n_r$ . Taking into account the proper boundary conditions one obtains

$$M^2 \rightarrow 2\pi\sigma(2n_r), \quad n_r \gg L \gg |L_z| \quad (3.19)$$

We can conclude that at least asymptotically one recovers the rest frame form (3.11) of the spectrum, so that the regime (3.15) is the analog of the rest frame string regime while the regimes (3.16) and (3.18) are the counterparts of the rest frame potential dynamics.

Let us derive now the light cone asymptotics, discussed above.

We start with stretched configuration (3.13) and consider first 1 + 1 analog of our Hamiltonian (2.28). To go over to the two dimensional case one should omit everywhere transverse

degrees of freedom; some care is needed for the  $r_{\perp}^2$  terms in the denominators: one should keep in mind that  $\nu$  is an auxiliary function which can be redefined since it is to be found from the condition of the extremum (2.29). Correspondingly we introduce  $\bar{\nu}$  via

$$\bar{\nu} \equiv \nu/r_{\perp}^2 \quad (3.20)$$

which stays nonzero, while  $\nu$  and  $r_{\perp}^2$  tend to zero; putting in eq. (2.28)  $p_{\perp} = 0$  and  $r_{\perp} = 0$  one obtains the one-dimensional Hamiltonian

$$H_{long} = \frac{1}{2} \left\{ \frac{m_1^2}{\mu_1} + \frac{m_2^2}{\mu_2} + \int \frac{\sigma^2}{\bar{\nu}} d\beta + \frac{\bar{\nu}_0 a_1 r_-^2}{\mu_1 + \mu_2} \right\} \quad (3.21)$$

Here we used notation

$$\bar{\nu}_0 = \int_0^1 \bar{\nu} d\beta$$

It is clear from (3.21) that  $\bar{\nu}^{ext}$  does not depend on  $\beta$  (since  $\frac{\delta H}{\delta \nu(r, \beta)}$  does not depend on it), and we can rewrite

$$H_{long} = \frac{1}{2} \left\{ \frac{m_1^2}{\mu_1} + \frac{m_2^2}{\mu_2} + \frac{\sigma^2}{\bar{\nu}} + \frac{\bar{\nu} P_+ r_-^2}{\mu_1 + \mu_2} \right\} \quad (3.22)$$

Taking into account that  $\nu \rightarrow 0$  at  $r_{\perp}^2 \rightarrow 0$  one obtains

$$P_+ = a_1 = \mu_1 + \mu_2 + \nu_0 \rightarrow \mu_1 + \mu_2 \quad (3.23)$$

Finally the Hamiltonian after substitution of the extremum value of  $\bar{\nu}$  takes the form

$$H_{long} = \frac{m_1^2}{2\mu_1} + \frac{m_2^2}{2\mu_2} + \sigma |r_-| \quad (3.24)$$

Expressing  $\mu_1, \mu_2$  in terms of the familiar  $x$  parameter (2.19),  $0 \leq x \leq 1$

$$P_+ = p_{1+} + p_{2+} = \mu_1 + \mu_2; \quad \mu_1 = P_+ x, \quad \mu_2 = P_+(1-x) \quad (3.25)$$

we get

$$H = \frac{1}{2P_+} \left( \frac{m_1^2}{x} + \frac{m_2^2}{1-x} \right) + \sigma |r_-| \quad (3.26)$$

and comparing with  $H = \frac{M^2}{2P_+}$  one has finally

$$M^2 = \frac{m_1^2}{x} + \frac{m_2^2}{1-x} + 2\sigma |P_+ r_-| \quad (3.27)$$

Since  $x$  is canonically conjugated to  $P_+ r_-$  (see (3.5) and (3.8)) equation (3.27) is to be considered as an operator equation for the wave function  $\Psi(x)$  of a meson in the light-cone system

$$\left( \frac{m_1^2}{x} + \frac{m_2^2}{1-x} \right) \Psi(x) + \int K(x, y) \Psi(y) dy = M^2 \Psi(x) \quad (3.28)$$

The operator  $K$  is easily obtained from the Fourier transform of the last term in eq.(3.8)

$$K(x, y) \Psi(y) = -\frac{\sigma}{2\pi} \int \frac{dy \Psi(y)}{(x-y)^2} \quad (3.29)$$

Our equation (3.28) with the integral kernel (3.29) coincides with the t'Hooft integral equation, obtained in the framework of the 1+1 QCD for  $N_C \rightarrow \infty$  [17], when a proper identification of parameters is made (for details of this comparison see Appendix).

A. Let us consider the 3+1 limit (3.19) when  $n_r \gg L \gg |B_r|$  and the stretching condition (3.13) is satisfied. For this asymptotics one can keep only the following terms in the Hamiltonian

$$(2.28) \quad H = \frac{1}{2P_+} \left( \frac{(P_+ r_-) \int y d\beta}{r_\perp^2 (1 - \int y d\beta)} + \sigma^2 \int \frac{d\beta}{y} r_\perp^2 \right) \quad (3.30)$$

Substituting the extremal values of  $y(\tau, \beta)$

$$y(\tau, \beta) = \frac{\sigma r_\perp^2}{|P_+ r_-| + \sigma r_\perp^2} \rightarrow \frac{\sigma r_\perp^2}{|P_+ r_-|} \ll 1 \quad (3.31)$$

one obtains for the resulting Hamiltonian in the leading order

$$H = \frac{1}{2P_+} (2\sigma |P_+ r_-|) \quad (3.32)$$

This is again the well known t'Hooft Hamiltonian for 1+1 QCD. In the 3+1 QCD case it corresponds to the asymptotics (3.13) which leads to the following spectrum of eq. (2.28) (see Appendix for details).

$$M^2 = 2\pi\sigma(2n_r), \quad n_r \gg L \gg |L_z| \quad (3.33)$$

We have identified here the longitudinal excitations of the system with the radial quantum number excitations in agreement with consideration given above.

We note here that in this regime the string contribution is dominant, but corresponds to the light cone longitudinal linear potential. As a result  $y = \nu/P_+$  doesn't depend on parameter  $\beta$  along the string and we arrive at the dynamics which is the counterpart of the potential regime (case 2) of the rest frame.

B. Next we consider the case of configuration (3.14), when  $L \approx |L_z|$ , and the radial quantum number of the Hamiltonian (2.28)  $n_r$  is very large,

$$n_r \gg L \approx |L_z| \quad (3.34)$$

so that condition (3.14) is satisfied. In this case one can keep in (2.28) only the fourth and the fifth terms on the r.h.s. and write

$$H = \frac{1}{2P_+} \left\{ \frac{p_\perp^2}{x - \frac{1}{2} + \int \beta z d\beta} + \frac{p_\perp^2}{\int (1 - \beta) z d\beta - (x - \frac{1}{2})} + \sigma^2 r_\perp^2 \int \frac{d\beta}{1 - z} \right\} \quad (3.35)$$

where  $z(\tau, \beta) = 1 - y(\tau, \beta) = 1 - \frac{\nu(\tau, \beta)}{P_+}$ ;

Extremum in  $x$  and  $z$  is achieved for  $x = \frac{1}{2}$  and

$$z(\tau, \beta) = \frac{2|p_\perp|}{2|p_\perp| + \sigma|r_\perp|} \quad (3.36)$$

The leading term in (3.35) looks like

$$H \approx \frac{1}{2P_+} (2|p_\perp| + \sigma|r_\perp|)^2 \quad (3.37)$$

and yields the mass spectrum

$$M^2 = 2\pi\sigma(2n_r), \quad n_r = 1, 2, \dots \quad (3.38)$$

Here we have identified the transverse radial excitations with the excitations of radial quantum number  $n_r$ .

In this potential like regime the string and the quarks make comparable contribution to  $M^2$ . The string constitutes the transverse linear potential, so that one has string energy density  $\nu/P_+$  independent on  $\beta$  which corresponds to the case 2 of the rest frame.

C. Consider now the case of large  $L \approx |L_z|$ , which corresponds to the configuration (3.15), discussed at the beginning of this Section. As in the previous case, the motion is almost in the transverse plane, both  $p_\perp$  and  $r_\perp$  are large and roughly

perpendicular to each other so that condition (3.14) is satisfied. The dominant terms in (2.28) are

$$H \cong \frac{1}{2P_+} \left\{ \frac{L^2}{\tau_\perp^2 \bar{a}_3} + \sigma^2 \int \frac{d\beta}{y} \tau_\perp^2 \right\} \quad (3.39)$$

where  $\bar{a}_3(y) = a_3/P_+$ . We can find the extremal value of  $\tau_\perp^2$  from (3.39) and obtain the effective Hamiltonian of the transverse rotating string without quarks.

$$H_{eff} = \frac{1}{2P_+} 2|L|\sigma \left( \frac{1}{\bar{a}_3} \int \frac{d\beta}{y} \right)^{1/2} \quad (3.40)$$

with the conditions that  $a_3$  is given by (2.15) and  $P_- = a_1$ .

The minimization of  $H_{eff}$  with respect to  $y(\tau, \beta)$  under the constraint  $a_1 = P_+$  yields as in I for the limiting case (3.15).

$$y_{ext} = \frac{\nu_0(\beta)}{M_0} = \frac{1}{M_0} \left( \frac{8\sigma L}{\pi} \right)^{1/2} \frac{1}{\sqrt{1 - 4(\beta - \frac{1}{2})^2}} \quad (3.41)$$

and

$$H_{eff} = \frac{M_0^2}{2P_+}, \quad M_0^2 = 2\pi\sigma L \quad (3.42)$$

Thus we recover for the configuration 2, where  $|L_z| \approx L \gg n_r$ , again the correspondence with the rest frame spectrum (3.11).

In this string regime (case 1 of the rest frame) the main contribution comes from the rotating transverse string and as a consequence  $\nu/P_+$  does depend on  $\beta$  in accordance with Lorentz gamma factor (see I for details).

## 4 Conclusions

In this paper we have derived from the QCD Lagrangian the light-cone Hamiltonian, describing quarks connected by the

string and analyse different dynamical regimes of the system. The main assumption (supported by the cluster expansion arguments and lattice data [9]) is the area law for the Wilson loop.

The resulting Hamiltonian (2.28) is both reasonable and unexpected. Reasonable, since the spectrum (estimated here asymptotically) coincides with the c.m. spectrum found in I, and agrees well with experimental meson spectra [18].

The form of Hamiltonian is at the same time unexpected from point of view of the standard light-cone theory developed heretofore. The main new point in our formalism is the considerable contribution of the string into the total momentum  $P_+$  and orbital momentum  $L^2$ . Therefore the usual separation of the pure kinetic and pure potential part in the light-cone Hamiltonian

$$H = \frac{2\sigma}{2P_+}(M_0^2 + W) \quad (4.1)$$

where  $M_0^2$  is the free particle kinetic part, is not valid here. In our 3+1 case the string is not inert and interacts in the complicated way with quarks. In particular for the case (A) of  $n_r \gg L \gg |L_z|$  one obtains

$$H \approx \frac{2\sigma}{2P_+}|P_+ r_-| \quad (4.2)$$

so that  $P_+$  enters  $M^2$  in the drastic way, which is unusual for the standard formulation [11,12].

This fact is also supported by an independent argument - a comparison with the 1+1 QCD shows that the dominant term in the t'Hooft's equation coincides with (4.2), as we have demonstrated it in Section 3.2 and Appendix.

The presence of nonperturbative string term like (4.2) is unexpected also from the point of view of the light-cone quantization approach [7], where nonperturbative contributions are missing. In this context our Hamiltonian can be considered as a useful lesson how to incorporate nonperturbative effects into the formalism.

To complete the light-cone formalism for spinless quarks connected by the string one needs to construct 4 dimensional tensor  $M_{\mu\nu}$ , total angular momentum  $L^2$ , check Poincare group commutations between operators and compare c.m. and light-cone wave functions. It is also clear, that the wave function  $\psi(x, p_{\perp})$  is to be found by numerical computations with the Hamiltonian (2.28). This will allow numerous applications for formfactors, structure functions etc.

The work in this direction is now in progress.

The authors acknowledge financial support of the Russian Fund for Fundamental Research in the framework of the grant 93-02-14397.

We are grateful for useful discussions to A.S.Gorski, L.A.Kon-dratyuk, I.M.Narodetsky and K.A.Ter-Martirosyan.



## Appendix

**Comparison of eq.(3.28) with the t'Hooft's equation  
in the 1 + 1 QCD**

One can rewrite the potential term in (3.28) as

$$V(r_-)\Psi(r_-) = \int \tilde{V}(p-q)\tilde{\Psi}(q)\frac{dq}{2\pi} \quad (\text{A.1})$$

where the Fourier transform  $\tilde{V}(p)$  is defined via

$$\tilde{V}(p) = \int_{-\infty}^{\infty} V(r_-)e^{ipr_-} dr_-, \quad V(r_-) = 2P_+\sigma|r_-| \quad (\text{A.2})$$

For the integral in (A.2) to have sense, one has to introduce a cut-off  $e^{-\gamma|r_-|}$  and finally put  $\gamma \rightarrow 0$ . With that from (A.2) one obtains

$$\tilde{V}(p) = 2P_+\sigma\left[\frac{1}{(\gamma-ip)^2} + \frac{1}{(\gamma+ip)^2}\right] \quad (\text{A.3})$$

With the definition of the principal value integral as in [17] one has

$$\tilde{V}(p) = -4P_+\sigma P\left(\frac{1}{p^2}\right) \quad (\text{A.4})$$

where

$$P \int \frac{\varphi(k)dk}{k^2} = \frac{1}{2} \int \frac{\varphi(k+i\gamma)dk}{(k+i\gamma)^2} + \frac{1}{2} \int \frac{\varphi(k-i\gamma)dk}{(k-i\gamma)^2} \quad (\text{A.5})$$

Insertion of (A.4)-(A.5) into (A.1) with the definitions

$$p = P_+x; \quad q = P_+y \quad (\text{A.6})$$

allows finally to represent (A.1) as

$$V(r_-)\Psi(r_-) = -\frac{2\sigma}{\pi} P \int \frac{\tilde{\Psi}(y)iy}{(x-y)^2} \quad (\text{A.7})$$

Eq.(A.7) coincides with the t'Hooft's kernel [17].

A special attention should be given to the mass terms in the full t'Hooft's equation

$$M^2 \varphi(x) = \left( \frac{\bar{m}_1^2}{x} + \frac{\bar{m}_2^2}{1-x} \right) \varphi(x) - P(g^2/\pi) \int_0^1 \frac{\varphi(y) dy}{(y-x)^2} \quad (\text{A.8})$$

where

$$\bar{m}_i^2 = m_i^2 - \frac{g^2}{\pi} \quad (\text{A.9})$$

and  $m_i$  are the current quark masses.

In the  $1+1$  QCD at  $N_c \rightarrow \infty$  the mass shift  $(-g^2/\pi)$  is due to the infrared region contribution of the selfenergy loop graphs, and the negative sign is due to attractive Coulomb (confinement) interaction  $g^2|r_-|$ . The situation is different in the  $3+1$  QCD, where the same loop graph is only ultraviolet divergent and yields a usual renormalization of the quark propagator. Therefore there the current quark mass is defined at some normalization scale (usually at  $\mu = 1\text{GeV}$ ) and is ascribed the known value (e.g.  $m_u = 4\text{MeV}, m_d = 7\text{MeV}$  etc). However due to chiral symmetry breaking there appears a chiral mass of quark in the form of a nonlocal gauge-covariant operator  $\hat{M}(x, y)$  [19].

At large distances,  $R \gg \rho$  ( where  $\rho$  is the average instanton size,  $\rho \approx 0.2 \div 0.3\text{fm}$ ) one can approximate  $\hat{M}$  by a local gauge-invariant mass  $M_{ch}$ , which finally enters into the Hamiltonian (1.73) via  $\sqrt{p^2 + (M_{ch} + m)^2}$ . The spectrum of (A.8) is calculated in [17] and at large  $n$  the WKB approximation yields [17]

$$M_n^2 = g^2 \pi n + (\bar{m}_1^2 + \bar{m}_2^2) \ln n + C \quad (\text{A.10})$$

The constant is close to  $g^2 \frac{3}{4} \pi$ .

The lowest mass values for  $m_1 = m_2 = 0$ , are approximately

$$M_0^2 = 0; \quad M_1^2 \cong 0.6 \frac{g^2}{\pi}; \quad M_2^2 \cong 1.43 \frac{g^2}{\pi}$$

To compare with our eq.(3.28), one should replace  $g^2 \rightarrow 2\sigma$ . The spectrum (A.10) is to be compared with that of relativistic quark model [2], which asymptotically is

$$M_n^2 = 4\pi\sigma n + 2m^2 + 4m^2 \ln \frac{M_n}{m} + m_0^2 \quad (\text{A.11})$$

with  $m_0^2 = 3\pi\sigma$ . Taking into account that for 3+1 case only odd solutions  $\varphi(r_+) = -\varphi(r_-)$  (and even  $n$  in eq. (a.10)) are relevant one can see a close correspondence of (A.10) and (A.11) up to a constant  $2m^2$ .

The wave functions  $\varphi(x)$ , solutions of (A.8) behave at the boundary  $x = 0$  as  $\varphi(x) \sim x^{\beta_1}$ , with  $\pi\beta_1 \text{ctg}\pi\beta_1 + \alpha_1 = 0$  [17]. Boundary conditions are

$$\varphi_n(0) = \varphi_n(1) = 0, \quad n = 1, 2, \dots \quad (\text{A.12})$$

Approximately  $\varphi_n(x) \approx \text{sin}n\pi x$ ,  $n = 1, 2, \dots$ . For  $m_1 = m_2 = 0$  there appears a ground state  $n = 0$  with  $M_0 = 0$  with wave function  $\varphi(x) \equiv 1$ , which has no counterpart in 3 + 1.

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Струна КХД с кверками П.Гамльтониен на световом конусе.

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Подписано к печати 14.06.94. Формат 60x90 1/16. Офсетн. печ.  
Усл.-печ.л. 1,5. Уч.-изд.л. 1,1. Тираж 200 экз. Заказ 45.  
Индекс 3649

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Отпечатано в ИТЭФ, 117259, Москва, Б.Черемушкинская, 35

**ИНДЕКС 3649**

Препринт 45-94, ИТЭФ, 1994