A REVIEW OF PLASMA POLARIMETRY
(theory and techniques)

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SUMMARY

A review of plasma polarimetry is presented. First the theory is discussed in general, exact analytic solutions of the polarization evolution equation are presented and then approximate analytic solutions. Numerical integration of the evolution equation is also discussed. The design of experiments is then considered, with special attention to the techniques of polarization modulation (both progressive and alternating modulation). Different alternative configurations are described for progressive modulation which are of special interest because they can be realized in the far infrared and because they allow a measurement of phases rather than amplitudes. The effects of refraction are then considered. Finally the combination of polarimetry and interferometry on the same instrument is discussed, including the effects of polarization modulation.

(POLARIMETRY, PLASMA, FARADAY EFFECT, COTTON-MOUTON EFFECT)

RIASSUNTO

Viene presentata una rassegna della teoria e delle tecniche della polarimetria del plasma. Prima viene discussa la teoria in generale, vengono presentate soluzioni analitiche esatte dell’equazione per l’evoluzione della polarizzazione e poi soluzioni analitiche approssimate. Anche l’integrazione numerica dell’equazione di evoluzione viene discussa. Poi viene considerato il progetto degli esperimenti con particolare attenzione per le tecniche di modulazione della polarizzazione (modulazione sia progressiva che alternata). Vengono descritte diverse configurazioni per la modulazione progressiva che sono di speciale interesse perché possono essere realizzate anche nel lontano infrarosso e perché permettono misure di fase piuttosto che di ampiezza. Vengono quindi considerati gli effetti di rifrazione. Infine viene discussa la combinazione di polarimetria e interferometria sullo stesso strumento, includendo gli effetti della modulazione della polarizzazione.
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1. INTRODUCTION

The study of the change of the state of polarization of electromagnetic radiation, propagating across a magnetized plasma, is of interest since it provides an important plasma diagnostic technique which can be used to determine the magnetic field distribution and hence the current density in the plasma, as well as the particle density. However this diagnostic, up to now, has not been exploited in experiments to its full potential. The present overview contains theoretical results obtained over many years [1-16] and is intended as an aid towards the development and a full exploitation of the diagnostic.

For high-frequency radiation a plasma, in the presence of a magnetic field, is a birefringent and optically active medium. Indeed for propagation perpendicular to the magnetic field, a plasma is purely birefringent and presents the Cotton-Mouton effect [17,18], while for propagation parallel to the field it is optically active and presents the Faraday effect [17,18]. In the former effect the ellipticity of the polarization ellipse changes, while in the latter the ellipse (and, for linear polarization, the plane) rotates, with a constant ellipticity, and only its orientation changes. For propagation at an arbitrary angle to the magnetic field, in general the two effects combine in a rather complex way. A further complication is due to the fact that in most experiments the plasma is not uniform: the plasma density and the magnetic field vector change with position along the radiation path.

Up to now the conditions of plasma polarimetric measurements have been chosen so that, to good approximation, the change of polarization is a pure Faraday rotation and, if z is an arbitrary propagation direction, the rotation angle $\alpha$ is given by $\alpha = 26.2 \lambda^2 n B_z dz$, where the integral is taken over the radiation path and $\alpha$ is in radians, $\lambda$ is the wavelength (in mm), $n$ is the plasma density (in $10^{20}$ m$^{-3}$), $B_z$ is the z component of the magnetic field (in T) and $z$ is in m. In this case only the parameter $\alpha$ is measured and it is used to obtain informations on the distribution of magnetic field $B(r)$, when $n(r)$ is known. However the treatment can be
extended to cases where plasma effects are not small and the changes of polarization do not have simple analytic expressions, such as that given above for $\alpha$. Furthermore, as will be discussed in Sections 4 and 8, the conditions can also be chosen so that up to 3 independent parameters are measured on each beam of radiation (and actually one can measure up to 9 parameters, which are not independent). This represents a significant increase in the amount of information which can be obtained from each probing beam, information which can be used to provide additional constraints for the reconstruction of MHD equilibria.

In order to provide a background for this extension of the technique, in Sections 2 and 3 we recall the conventional methods for describing the state of polarization and the changes of polarization due to propagation across common optical components. In Section 4 we discuss the change of polarization due to propagation across a magnetized plasma. In Sections 5 and 6 we discuss exact and, respectively, approximate solutions of the equation for the evolution of the state of polarization, while Section 7 presents methods for the numerical integration of this equation.

In order to facilitate the handling of the increased information which polarimetry can provide, the design of experimental configurations is discussed in Section 8, including the important ingredient consisting of the modulation of the state of polarization. Possible plasma polarimetric measurements are analysed in Section 9. The effects of refraction, which at first are supposed negligible, are taken into consideration in Section 10. The combination of polarimetric and interferometric measurements is described in Section 11.

Finally the use of polarimetric measurements in plasma physics is discussed in Section 12.

It should be noted that the treatment of polarimetry presented here is quite general: it can be applied to any magnetic confinement configuration (tokamak, stellarator, RFP etc...) and no symmetry assumptions are required.

### 2. DESCRIPTION OF THE STATE OF POLARIZATION OF ELECTROMAGNETIC RADIATION

Let us consider fully polarized electromagnetic radiation, with frequency $\omega$, propagating in the z direction. There are various possible methods of describing the state of polarization. In general the tip of the electric field vector describes an ellipse, see Fig.1, and the electric field components can be written in the alternative equivalent forms

\[
\begin{align*}
E_x &= e_x \cos(\omega t + \delta_x) = A \cos \omega t - B \sin \omega t \\
E_y &= e_y \cos(\omega t + \delta_y) = C \cos \omega t - D \sin \omega t
\end{align*}
\]  

(2.1)

where $e_x$, $e_y$, $\delta_x$, $\delta_y$ and $A$, $B$, $C$, $D$ are constants. The shape of the ellipse and the sense of rotation are fully described by $\eta$ and $\phi$, where
Fig. 1 The polarization ellipse (the sense of rotation for $\chi > 0$ is shown)

$$\tan \eta = \frac{e_y}{e_x}$$ $$\phi = \delta_y - \delta_x$$ (2.2)

If $a$ and $b$ are the major and minor semi-axes of the ellipse, the state of polarization can also be fully described (see fig. 1) by the angle $\psi$ (with $0 \leq \psi \leq \pi$) between the major semi-axis and the $x$ direction and by the angle $\chi$ defined by $\tan \chi = \pm \frac{b}{a}$ (where $-\pi/4 \leq \chi \leq \pi/4$) with positive or negative sign for clockwise or anticlockwise rotation (looking towards the radiation source). It can be shown [19] that the following relationships hold

$$\tan 2\psi = \frac{2e_x e_y}{e_x^2 - e_y^2} \cos(\delta_y - \delta_x) = \tan 2\eta \cos \phi = \frac{2(AC + BD)}{A^2 + B^2 - C^2 - D^2}$$

$$\sin 2\psi = \frac{2e_x e_y}{e_x^2 + e_y^2} \sin(\delta_y - \delta_x) = \sin 2\eta \sin \phi = \frac{2(AD - BC)}{A^2 + B^2 + C^2 + D^2}$$ (2.3)

between the four sets of parameters ($\psi$, $\chi$, ($e_x$, $e_y$, $\delta_x$, $\delta_y$), ($\eta$, $\phi$) and (A, B, C, D).

It is also very useful to introduce the Stokes vector. We shall consider both a reduced Stokes vector, $s$, and a full Stokes vector, $S$. (Vectors will be indicated by bold roman symbols). The former has three components, $s_1 = \cos 2\chi \cos 2\psi$, $s_2 = \cos 2\chi \sin 2\psi$, $s_3 = \sin 2\chi$, or, using matrix notation,
\[
\mathbf{s} = \begin{bmatrix}
\cos 2\chi \cos 2\psi \\
\cos 2\chi \sin 2\psi \\
\sin 2\chi
\end{bmatrix}
\]  \hspace{1cm} (2.4)

so that

\[|\mathbf{s}|^2 = s_1^2 + s_2^2 + s_3^2 = 1. \hspace{1cm} (2.5)\]

The full Stokes vector, for radiation of intensity \(I\), has four components \(\mathbf{S} = (S_0, S_1, S_2, S_3)\) which are given by \(S_0 = I, S_1 = s_1 I, S_2 = s_2 I, S_3 = s_3 I\) and so \(S_1^2 + S_2^2 + S_3^2 = S_0^2 = I^2\). Using matrix notation this can be written

\[
\mathbf{S} = \begin{bmatrix} S_0 \\ S_1 \\ S_2 \\ S_3 \end{bmatrix} = I \times \begin{bmatrix} 1 \\ s_1 \\ s_2 \\ s_3 \end{bmatrix} \hspace{1cm} (2.6)
\]

Furthermore it can be shown [19] that the following relationships hold

\[
2S_0 = 2I = A^2 + B^2 + C^2 + D^2
\]
\[
2S_1 = 2s_1 I = A^2 + B^2 - C^2 - D^2
\]
\[
2S_2 = 2s_2 I = 2(AC + BD)
\]
\[
2S_3 = 2s_3 I = 2(AD - BC)
\]  \hspace{1cm} (2.7)

The full Stokes vector is necessary when the intensity of the radiation is not constant during propagation, such as, for instance, when polarizers are used, and when the degree of polarization is not constant. The reduced Stokes vector is sufficient when the intensity and the degree of ionization are constant during propagation, such as for plasma propagation (in the approximation which we will use). In general for partially polarized radiation one has \(S_0 = I, S_1 = ps_1 I, S_2 = ps_2 I, S_3 = ps_3 I\), with \(p\) the degree of polarization, so that \(S_1^2 + S_2^2 + S_3^2 = p^2 S_0^2 \leq S_0^2 = I^2\). However we will consider fully polarized radiation (\(p=1\)).
It is clear that $s=(1,0,0)$ represents linearly polarized radiation aligned with the x axis; $s=(0,\pm1,0)$ represents linearly polarized radiation oriented at ±45° to the x axis; $s=(0,0,\pm1)$ represents circular polarized radiation with either of the two senses of rotation.

Finally it is convenient to refer to the Poincaré sphere. This is a sphere of unit radius in $(s_1, s_2, s_3)$ space, see fig.2, where each possible state of polarization is uniquely represented by a point P on the sphere, having latitude $2\chi$ and longitude $2\psi$. Two orthogonal polarizations are represented by two diametrically opposite points on the sphere. It can be shown [20,21] that the evolution of the polarization during propagation in a uniform non-absorbing medium, having (slow and fast) characteristic refractive indexes $\mu_1$ and $\mu_2$ (with $\mu_1>\mu_2$), is represented on the Poincaré sphere by a rotation about an axis passing through the points representing the characteristic polarizations (which are orthogonal) and the angle of rotation is $\Delta\phi=L(\mu_1-\mu_2)\omega/c$ where L is the path length. Therefore the evolution of the polarization can be represented by the vector equation

$$\frac{ds(z)}{dz} = \Omega \times s(z)$$

(2.8)

where $|\Omega| = d\Delta\phi/dz = (\mu_1-\mu_2)\omega/c$ and the direction of $\Omega$ is opposite to that of the fast characteristic polarization vector $s_{c2}$, so that
\[ \Omega = -\frac{\omega}{c} (\mu_1 - \mu_2) s_{c2} \]  

(2.9)

It should be noted that this equation has the opposite sign from that appearing in refs. [1,3] because here we are using the opposite convention for the sign of \( \chi \), as is more common now, and for this reason also the signs appearing later in the expressions for the components of \( \Omega \), \( \Omega_1 \) and \( \Omega_2 \), of a plasma have opposite sign here to those in [1,3].

For a uniform medium \( \mu_1, \mu_2 \) and \( s_{c2} \) do not depend on \( z \), so that also \( \Omega \) is a constant and eq.2.8 can be integrated easily. As we shall see eq.2.8 is very useful because it can be extended to describe propagation in a non-uniform medium simply by considering \( \mu_1, \mu_2, s_{c2} \), and hence also \( \Omega \) in eq.2.8, as functions of \( z \). This corresponds to considering the medium as approximately uniform over each infinitesimal slab of thickness \( dz \).

For completeness, we recall that the state of fully polarized radiation can also be described by the Jones vector method [18]. Here the electric field of the wave (see eq. 2.1) is represented by the real part of the complex, two-dimensional Jones vector

\[ E = \exp(i\omega t) \begin{bmatrix} E_x \\ E_y \end{bmatrix} = \exp(i\omega t) \begin{bmatrix} e_x \exp(i\delta_x) \\ e_y \exp(i\delta_y) \end{bmatrix} \]  

(2.10)

The relationship between the Jones vector and the (\( \psi, \chi \)) description is given by eqs.2.3 together with the following identity (see [19])

\[ \tan(\delta_x - \delta_y) = \frac{\Re(E_x)\Re(E_y) - \Re(E_y)\Re(E_x)}{\Re(E_x)\Re(E_y) + \Re(E_x)\Im(E_y)} \]  

(2.11)

where \( \Re \) and \( \Im \) indicate the real and imaginary parts respectively. Finally also the relationship with the reduced Stokes vector is obtained by recalling that \( \tan 2\psi = s_2/s_1 \) and \( \sin 2\chi = s_3 \), together with eq. 2.5.

3. CHANGE OF THE STATE OF POLARIZATION DUE TO OPTICAL COMPONENTS

For the design and analysis of polarimetric measurements it is often necessary to describe the effect on the state of polarization due to optical components, in particular polarizers and retarders. These effects are described most simply by means of the 4x4 matrix \( M \) (called the Mueller matrix [22]), characteristic of each optical component, which gives the output Stokes vector in terms of the input Stokes vector by...
\[ S_{\text{out}} = M \cdot S_{\text{in}} \quad (3.1) \]

which, as usual, is equivalent to \((S_{\text{out}})_j = M_{jk} (S_{\text{in}})_k\) for \(j=1, 2, 3\) and \(4\), where a repeated index indicates summation with respect to the index and \(M_{jk}\) is an element of \(M\). (Square matrices will always be indicated by bold italic symbols). Of course, if a system consists of more than one component, say \(M_1, M_2\) and \(M_3\), then the matrix of the entire system is the product of the matrices, ordered right to left, \(M = M_3 \cdot M_2 \cdot M_1\).

Here we will recall the expressions for the Mueller matrices of ideal polarizers and retarders. More information and in particular the expressions for non-ideal components can be found in [22].

The Mueller matrix \(M_p(\theta)\) for a polarizer having its axis at an angle \(\theta\) to the \(x\) direction is

\[
M_p(\theta) = \frac{1}{2} \begin{bmatrix}
1 & \cos 2\theta & \sin 2\theta & 0 \\
\cos 2\theta & \cos^2 2\theta & \sin 2\theta \cos 2\theta & 0 \\
\sin 2\theta & \sin 2\theta \cos 2\theta & \sin^2 2\theta & 0 \\
0 & 0 & 0 & 0
\end{bmatrix} \quad (3.2)
\]

It can be verified that a polarizer in general changes the intensity of the radiation. The matrix \(M_R(\rho, \gamma)\) for a retarder, having a relative retardation \(\rho\) and its principal axis at an angle \(\gamma\) to the \(x\) direction is

\[
M_R(\rho, \gamma) = \begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & \cos^2 2\gamma + \cos \rho \sin^2 2\gamma & (1-\cos \rho) \sin 2\gamma \cos 2\gamma & -\sin \rho \sin 2\gamma \\
0 & (1-\cos \rho) \sin 2\gamma \cos 2\gamma & \sin^2 2\gamma + \cos \rho \cos^2 2\gamma & \sin \rho \cos 2\gamma \\
0 & \sin \rho \sin 2\gamma & -\sin \rho \cos 2\gamma & \cos \rho
\end{bmatrix} \quad (3.3)
\]

where \(G=(1+\cos \rho)/2\) and \(H=(1-\cos \rho)/2\).

It is clear that, since all the elements of the first row are zero except the first one, the intensity of the radiation is unchanged here and therefore one can also use the \(3\times3\) matrix,
\[
M_r(\rho, \gamma) = \begin{bmatrix}
\cos^2 \gamma + \cos \psi \sin^2 \gamma & (1 - \cos \rho) \sin^2 \gamma & - \sin \psi \sin 2\gamma \\
(1 - \cos \rho) \sin^2 \gamma & \sin^2 \gamma + \cos \psi \cos^2 \gamma & \sin \psi \cos 2\gamma \\
\sin \psi \sin 2\gamma & - \sin \psi \cos 2\gamma & \cos \rho
\end{bmatrix}
\]

\[
G + H \cos 4\gamma \\
H \sin 4\gamma \\
\sin \psi \sin 2\gamma - \sin \psi \cos 2\gamma \\
\cos \rho
\]

(3.4)

to be applied to the reduced Stokes vector \( s \), instead of \( S \). For simplicity, in the following comments we shall consider the 3\times3 matrix.

In particular, for a quarter-wave retarder \( (\rho = \pi/2) \) we have

\[
M_r(\pi/2, \gamma) = \begin{bmatrix}
\cos^2 \gamma & \sin^2 \gamma & - \sin 2\gamma \\
\sin^2 \gamma & \cos^2 \gamma & \cos 2\gamma \\
\sin 2\gamma & - \cos 2\gamma & 0
\end{bmatrix}
\]

(3.5)

and so, for \( \gamma = 0 \),

\[
M_r(\pi/2, 0) = \begin{bmatrix}
1 & 0 & 0 \\
0 & 0 & 1 \\
0 & -1 & 0
\end{bmatrix}
\]

(3.6)

and, for \( \gamma = \pi/4 \),

\[
M_r(\pi/2, \pi/4) = \begin{bmatrix}
0 & 0 & -1 \\
0 & 1 & 0 \\
1 & 0 & 0
\end{bmatrix}
\]

(3.7)

It can be seen that a quarter-wave retarder with \( \gamma = 0 \) (or \( \pi/2, \pi \) or \( 3\pi/2 \)) exchanges \( s_2 \) and \( s_3 \), the second and third components of the reduced Stokes vector, (with a change of a sign), while for \( \gamma = \pi/4 \) (or \( 3\pi/4, 5\pi/4 \) or \( 7\pi/4 \)) it exchanges \( s_1 \) and \( s_3 \), the first and third components (again with a change of a sign).

For a half-wave retarder \( (\rho = \pi) \) we have
\[ M_t(\pi, \gamma) = \begin{bmatrix} \cos 4\gamma & \sin 4\gamma & 0 \\ \sin 4\gamma & -\cos 4\gamma & 0 \\ 0 & 0 & -1 \end{bmatrix} \]  

(3.8)

which shows that a half-wave retarder changes the input $\Psi$ into $-\Psi + 2\gamma$ and the input $\chi$ into $-\chi$.

Again for completeness we recall that the electric field of the wave can also be described by the (two-dimensional) Jones vector (see Section 2). In this case each optical component is characterized by a $2 \times 2$ matrix, its so-called Jones matrix. The expressions for the Jones matrices of common optical components can be found for instance in ref.[18].

4. CHANGE OF THE STATE OF POLARIZATION DUE TO PROPAGATION IN PLASMA

4.1 Change of output polarization for a specific input polarization

In order to analyse the evolution of the polarization of radiation propagating across a magnetized plasma, we shall assume that the frequency $\omega$ of the radiation is much greater than the particle collision frequency and larger than the plasma frequency, $\omega_p = (4\pi ne^2/m)^{1/2}$ where $n$ is the plasma density, so that the cold plasma approximation is adequate and absorption of the radiation by the plasma is negligible. Then, if we indicate by $\theta$ the angle between $\mathbf{B}$ and $\mathbf{z}$, the propagation direction, and by $\beta$ the angle between $\mathbf{z} \times (\mathbf{B} \times \mathbf{z})$ and $\mathbf{y}$ (see fig.3), the refractive indexes $\mu_1$ and $\mu_2$ of the slow and fast characteristic waves are given (see eg [23]) by

![Fig. 3 Magnetic field vector diagram](image)
\[(\mu_{1,2})^2 = 1 - \frac{\omega_p^2}{\omega^2} \left[ 1 + \frac{\omega_c^2}{\omega^2} \frac{\sin^2 \theta}{2(1-\omega_p^2/\omega^2)} \left[ -1 \pm (1+F^2)^{1/2} \right] \right]^{-1} \] (4.1)

where \(\omega_c\) is the electron cyclotron frequency, \(\omega_c = eB/mc\). Furthermore the polarization parameters \(\psi_{c2}\) and \(\chi_{c2}\) describing the state of polarization, \(s_{c2}\), of the fast characteristic wave are given by

\[\psi_{c2} = -\beta \quad \text{and} \quad \tan \chi_{c2} = -[(1+F^2)^{1/2} - 1]/F\] (4.2)

where

\[F = \frac{2\omega}{\omega_c} \left[ 1 - \frac{\omega_p^2}{\omega^2} \right] \cos \theta \frac{\cos \theta}{\sin^2 \theta} \] (4.3)

Thus, from eqs. 2.5 and 4.2, we have

\[s_{c2} = \frac{1}{(1+F^2)^{1/2}} \times \begin{bmatrix} \cos 2\beta \\ -\sin 2\beta \\ -F \end{bmatrix} \] (4.4)

If \(B_x, B_y, B_z\) are the components of \(B\) and if we call

\[N = (\omega_p/\omega)^2\] (4.5)

\[D = 1 - \left[ \frac{e}{\omega mc} \right]^2 \left\{ \frac{B_x^2 + B_y^2}{1 - N} + B_z^2 \right\} \] (4.6)

\[G = \frac{\omega_p^2}{2\omega^4} \left[ \frac{e}{\omega mc} \right]^2 \left\{ \frac{B_x^2 + B_y^2}{1 - N} \right\} \frac{1}{D} \] (4.7)

then we can put eq. 4.1 in the form

\[(\mu_{1,2})^2 = 1 - (N/D) + G \pm G(1+F^2)^{1/2}\] (4.8)
\[ \mu_1 - \mu_2 = \frac{\mu_1^2 - \mu_2^2}{\mu_1 + \mu_2} = \frac{2G(1+F^2)^{1/2}}{\mu_1 + \mu_2} \]  

We can now use these expressions to evaluate for a magnetized plasma the vector \( \mathbf{\Omega} \), introduced in Section 2 and defined by eq.2.9. One obtains the following expressions

\[ \mathbf{\Omega} = \frac{\omega_p^2}{(\mu_1 + \mu_2) c \omega^3 D} \times \left[ \left( \frac{e}{mc} \right)^2 \left( \frac{B_x^2 - B_y^2}{1 - N} \right) \right. \]

\[ \left. \left( \frac{e}{mc} \right)^2 \left( \frac{2B_x B_y}{1 - N} \right) \right] - \frac{2 \omega e}{mc} B_z \]  

Again the differences in sign appearing in eqs. 4.1, 4.2 and 4.10, compared with the corresponding expressions reported in refs.[1,3,11], are due to the use of the opposite convention for the sign of \( \chi \).

When, as we shall assume for simplicity, \( \omega_p^2 < \omega^2 \) and \( \mu_1^2 + \mu_2^2 = 2 \), one has

\[ \mathbf{\Omega} = \frac{\omega_p^2}{2c \omega^3 (1 - \omega_c^2 / \omega^2)} \times \left[ \left( \frac{e}{mc} \right)^2 (B_x^2 - B_y^2) \right. \]

\[ \left. \left( \frac{e}{mc} \right)^2 2B_x B_y \right] - \frac{2 \omega e}{mc} B_z \]  

In general \( \mathbf{\Omega} \) depends on \( z \) through the plasma parameters \( B_x, B_y, B_z \) and \( \omega_p \). For a uniform plasma \( \mathbf{\Omega} \) is a constant vector and the evolution of the state of polarization is described by eq.2.8. For the case, which is most interesting for plasma diagnostics, where the plasma is non-uniform and \( \mathbf{\Omega} = \mathbf{\Omega}(z) \), we shall make two assumptions. The first one, which will later be dropped, is that gradients of plasma parameters transverse to the propagation direction are small so that refraction of the radiation beam is negligible. The second one is that parallel gradients are also small and that the characteristic length of the non-uniformity is large compared with the radiation wavelength. Then we can use a WKB approximation where the plasma is considered locally approximately uniform and in each infinitesimal slab the plasma parameters are constant but they change from slab to slab. This implies simply using eq.2.8 with \( \mathbf{\Omega} \) a function of \( z \), to describe the evolution of polarization, namely,
\[ \frac{ds(z)}{dz} = \Omega(z) \times s(z) \]  

(4.12)

It is convenient also to write this equation in dimensionless form. If \( a \) is a characteristic length of the plasma geometry (for instance the minor radius of a tokamak) and we call \( Z = z/a \), the normalized coordinate, let us define the dimensionless vector \( \mathbf{T}(Z) \) by

\[ \mathbf{T} = a \Omega = \frac{a \omega_p^2}{(\mu_1 + \mu_2) c \omega D} \times \begin{bmatrix} \left( \frac{e}{mc} \right)^2 \frac{B_x^2 - B_y^2}{1 - N} \\ \left( \frac{e}{mc} \right)^2 \frac{2B_x B_y}{1 - N} \\ 2 \omega \frac{e}{mc} B_z \end{bmatrix} \]  

(4.13)

so that, when \( \omega_p^2 \gg \omega^2 \), one has approximately

\[
\mathbf{T} = \frac{a \omega_p^2}{2c \omega^3(1 - \omega_c^2/\omega^2)} \times \begin{bmatrix} \left( \frac{e}{mc} \right)^2 (B_x^2 - B_y^2) \\ \left( \frac{e}{mc} \right)^2 2B_x B_y \\ 2 \omega \frac{e}{mc} B_z \end{bmatrix}
\]

\[
= \frac{a \omega_p^2}{2c \omega^3(1 - \omega_c^2/\omega^2)} \times \begin{bmatrix} -\omega_c^2 \sin^2 \theta \cos 2\beta \\ \omega_c^2 \sin^2 \theta \sin 2\beta \\ 2 \omega \omega_c \cos \theta \end{bmatrix}
\]  

(4.14)

Then the evolution equation, eq.4.12, becomes

\[ \frac{ds(Z)}{dZ} = \mathbf{T}(Z) \times s(Z) \]  

(4.15)

This vector equation (as also eq.4.12) represents a system of three coupled scalar equations for \( s \). For a given plasma configuration, i.e. for given \( \mathbf{T}(Z) \) or \( \Omega(z) \) and for a given initial (input) state of polarization \( s_0 \), the system can be integrated across the plasma, from the input coordinate \( Z_0 \) to the output \( Z_1 \), and so it can provide the final (output) state of polarization, \( s_1 \). This is essential for the design and analysis of measurements. For a special distribution of plasma parameters, i.e. a special form of \( \mathbf{T}(Z) \) or \( \Omega(z) \), there is an exact analytic solution to
eqs.4.15 or 4.12 which will be discussed in Section 5. In general the integration can only be performed numerically (see Section 7), however when the changes of polarization are small an approximate analytic solution exists (see Section 6).

With the previous analysis we have obtained a formalism which, for a given input $s_0$, provides the output $s_1$. This is fully adequate when $s_0$ is constant in time. However, as will be discussed later, there are powerful experimental techniques which require that $s_0$ changes with time (time modulation of the polarization). In this case the previous formalism can become very cumbersome since it requires numerical integrations for many values of $s_0$. We will now extend the analysis to obtain a formalism which directly provides the output $s_1$, for any possible input $s_0$.

4.2 Change of output polarization for all possible input polarizations.

Let us first define the 3×3 matrix $A$ by

\[
A(Z) = \begin{bmatrix}
A_{11} & A_{12} & A_{13} \\
A_{21} & A_{22} & A_{23} \\
A_{31} & A_{32} & A_{33}
\end{bmatrix} = \begin{bmatrix}
0 & -T_3 & T_2 \\
T_3 & 0 & -T_1 \\
-T_2 & T_1 & 0
\end{bmatrix}
\] (4.16)

where $T_1$, $T_2$, $T_3$ are the components of $T$ (given by eq.4.13, or eq.4.14 when $\omega_p^2 << \omega^2$).

Then the vector equation, eq.4.15, describing the evolution of polarization, can also be written, using matrix notation, as

\[
\frac{ds}{dZ} = A(Z) \cdot s
\] (4.17)

Here the product $A \cdot s$ is the vector whose components are $(A \cdot s)_i = A_{ij} s_j$ for $i=1,2,3$ and a repeated index will denote summation according to the index. It follows from the properties of linear first-order differential equations with variable coefficients [24, 25] that the solutions of the system of Eqs.(4.17), corresponding to the initial value $s = s_0$, can be put in the form

\[
s(Z) = M(Z) \cdot s_0
\] (4.18)

and so, using eqs 4.17 and 4.18, it is readily verified that the matrix $M(Z)$ satisfies the differential equation

\[
\frac{dM}{dZ} = A(Z) \cdot M(Z)
\] (4.19)
with the initial determination $M(Z_0) = 1$, the unit matrix, and the matrix product $A \cdot M$, as usual, is defined as the matrix whose elements are $(A \cdot M)_{jk} = A_{jm}M_{mk}$, for $j,k=1,2,3$. The matrix equation, Eq.(4.19), corresponds to a system of nine scalar differential equations for the elements of $M$. Only three of these nine elements are independent since the condition $|s(Z)|=1$, which follows from Eq.(4.15), implies that $M_{jm}M_{jn} = 1$ for $m=n$ and $M_{jm}M_{jn} = 0$ for $m \neq n$, with $m,n=1,2,3$. From these conditions it can be shown that also $M_{mj}M_{nj} = \delta_{mn}$ and so we have

$$M_{j1}^2 + M_{j2}^2 + M_{j3}^2 = M_{1j}^2 + M_{2j}^2 + M_{3j}^2 = 1,$$  \hspace{1cm} \text{(4.20)}$$

for $j=1,2,3$. In matrix theory such a matrix is called an orthogonal matrix and has other interesting properties (see eg [26]).

When $T(Z)$ is known, also $A(Z)$ is known from Eq.(4.16) and then by integrating the system of Eqs.(4.19) starting from $Z=Z_0$, where $s=s_0$ and $M=1$, it is possible to determine $M(Z)$. In particular we have

$$s_1 = M(Z_1) \cdot s_0$$ \hspace{1cm} \text{(4.21)}$$

which shows that the plasma transition matrix $M(Z_1)$ allows us to obtain immediately $s_1$ for any $s_0$. This matrix therefore describes completely the change of polarization for radiation crossing the plasma. A $3 \times 3$ matrix operating on the reduced Stokes vector is sufficient here because the plasma is considered lossless and there is no attenuation of the radiation and no change in the degree of polarization (see Section 2).

Conversely using eq.4.21 it is possible, as will be shown later, to determine experimentally the transition matrix $M(Z_1)$ (i.e. its nine elements) by measuring the output $s_1$ for various forms of the input $s_0$. It should be noted that the experimentally determined elements of $M(Z_1)$ are nine independent quantities, in contrast with the theoretical values derived for known plasma parameters, i.e. for given $T(Z)$. There is therefore the possibility of some redundancy of the measurements with respect to the measured quantities.

Of course we can also use the dimensional variable, $z$, and define $M(z)$ so that

$$s(z) = M(z) \cdot s_0$$ \hspace{1cm} \text{(4.22)}$$

and

$$\frac{dM}{dz} = A(z) \cdot M(z)$$ \hspace{1cm} \text{(4.23)}$$

where now
\[
A(z) = \begin{bmatrix}
0 & -\Omega_3 & \Omega_2 \\
\Omega_3 & 0 & -\Omega_1 \\
-\Omega_2 & \Omega_1 & 0
\end{bmatrix}
\] (4.24)

and the plasma transition matrix is now \(M(z_1)\).

5. SPECIAL CASES WITH EXACT ANALYTIC SOLUTION OF THE EVOLUTION EQUATION

We will now discuss particular distributions of plasma parameters, i.e. particular forms of \(\Omega(z)\) or \(T(Z)\), which allow an exact analytic solution of eq. 4.12, or eq. 4.15, the equation for the evolution of the state of polarization during propagation.

5.1 Uniformly sheared plasma

Let us first consider the case of uniform magnetic shear, where the plasma density and hence the plasma frequency are constant, \(B_z\) is constant and \(B_x = B_\perp \cos \alpha\) and \(B_y = B_\perp \sin \alpha\) (see fig.3), with the transverse field \(B_\perp\) constant, \(\alpha = \pi/2 - \beta = g\zeta + \alpha_0\) and \(g\) is also a constant giving the rate of shear. The constant \(\alpha_0\) can always be reduced to zero by a convenient choice of the orientation of the \(x, y\) axes, and so in the following we shall put \(\alpha_0 = 0\). Hence, for this case of uniform shear, \(\Omega_3\) is constant and also \(\Omega_\perp\) is constant where

\[
\Omega_\perp = \frac{\omega_p^2}{2\cos^3(\zeta + \Omega_\perp^2)} \left[ \frac{e}{m c} \right]^2 \frac{\Omega_\perp}{B_\perp^2}
\] (5.1)

For simplicity we use eq.4.14 valid for \(\omega_p^2 = \omega_\perp^2\), while the extension to the general case of eqs.4.13 is straightforward. From eq.4.14 the first two components of \(\Omega\) can be written as \(\Omega_1 = -\Omega_\perp \cos 2\alpha\) and \(\Omega_2 = -\Omega_\perp \sin 2\alpha\) and so the three components of eq.4.12 are

\[
\frac{ds_1}{dz} = -\Omega_\perp s_3 \sin 2\alpha - \Omega_3 s_2
\] (5.2)

\[
\frac{ds_2}{dz} = \Omega_3 s_1 + \Omega_\perp s_3 \cos 2\alpha
\] (5.3)

\[
\frac{ds_3}{dz} = -\Omega_\perp s_2 \cos 2\alpha + \Omega_\perp s_1 \sin 2\alpha
\] (5.4)
By introducing the auxiliary functions \( p(z) \) and \( q(z) \) defined as

\[
p = s_1 \sin 2\alpha - s_2 \cos 2\alpha \quad \text{and} \quad q = s_1 \cos 2\alpha + s_2 \sin 2\alpha \quad (5.5)
\]

so that

\[
s_1 = psin2\alpha + qcos2\alpha \quad \text{and} \quad s_2 = -pcos2\alpha + qsin2\alpha \quad (5.6)
\]

with \( \alpha = g z \), the system of equations 5.2-5.4 can be transformed into the following system for the functions \( p, q \) and \( s_3 \)

\[
\frac{d^3 s_3}{dz^3} = -\Omega_0^2 \frac{ds_3}{dz} \quad (5.7)
\]

\[
\Omega_\perp p = \frac{ds_3}{dz} \quad (5.8)
\]

\[
\frac{dq}{dz} = \frac{(\Omega_3 - 2g)}{\Omega_\perp} \frac{ds_3}{dz} \quad (5.9)
\]

where \( \Omega_0^2 = \Omega_\perp^2 + (\Omega_3 - 2g)^2 \). These equations can be integrated directly to give

\[
s_3 = \text{Acos}\Omega_0 z + \text{Bsin}\Omega_0 z + C \quad (5.10)
\]

\[
p = -\frac{\Omega_0}{\Omega_\perp} (\text{Asin}\Omega_0 z - \text{Bcos}\Omega_0 z) \quad (5.11)
\]

\[
q = \frac{(\Omega_3 - 2g)}{\Omega_\perp} (\text{Acos}\Omega_0 z + \text{Bsin}\Omega_0 z) + D \quad (5.12)
\]

where \( A, B, C \) and \( D \) are integration constants and, from eq.5.3 evaluated at \( z = 0 \), it follows that

\[
D = -\frac{\Omega_1 C}{(\Omega_3 - 2g)}. \quad \text{Using eqs.} \ 5.11, \ 5.12 \ \text{in the eqs.} \ 5.6 \ \text{the explicit expressions for} \ s_1(z) \ \text{and} \ s_2(z) \ \text{are found.}
The integration constants can be written in terms of the initial (or boundary values) \( s_{01}, s_{02} \) and \( s_{03} \), where \( s_{0} = s(0) \), namely

\[
A = [\Omega_1 (\Omega_3 - 2g)/\Omega_0^2]s_{01} + (\Omega_1/\Omega_0)^2 s_{03}
\]

\[
B = - (\Omega_1/\Omega_0)s_{02}
\]

\[
C = s_{03} - A
\]

It can be seen that \( \cos 2gz \) and \( \sin 2gz \) only appear in \( s_1(z) \) and \( s_2(z) \), but not in \( s_3(z) \).

Writing the solution in terms of \( s_{01}, s_{02} \) and \( s_{03} \) rather than \( A, B \) and \( C \), it is clear that \( s_1(z), s_2(z), s_3(z) \) are linear combinations of \( s_{01}, s_{02}, s_{03} \) and it is indeed possible to cast the solution \( s(z) \) in the form \( s(z) = M(z)s_0 \) (or \( s_i(z) = M_{ij}(z)s_0 \), for \( i=1,2,3 \)), as discussed in Section 4a. If we put \( u = \Omega_1/\Omega_0 \) and \( v = (\Omega_3 - 2g)/\Omega_0 \), so that \( u^2 + v^2 = 1 \), and if we denote \( \Omega_0z \) by \( \gamma \), then the elements \( M_{ij}(z) \) of the transfer matrix \( M(z) \) are given by

\[
M_{11}(z) = v^2 \cos 2\alpha \cos \gamma + u^2 \cos 2\alpha - v \sin 2\alpha \sin \gamma
\]

\[
M_{12}(z) = - v \cos 2\alpha \sin \gamma - \sin 2\alpha \cos \gamma
\]

\[
M_{13}(z) = - uv \cos 2\alpha (1 - \cos \gamma) - u \sin 2\alpha \sin \gamma
\]

\[
M_{21}(z) = v^2 \sin 2\alpha \cos \gamma + u^2 \sin 2\alpha + v \cos 2\alpha \sin \gamma
\]

\[
M_{22}(z) = - v \sin 2\alpha \sin \gamma + \cos 2\alpha \cos \gamma
\]

\[
M_{23}(z) = - uv \sin 2\alpha (1 - \cos \gamma) + u \cos 2\alpha \sin \gamma
\]

\[
M_{31}(z) = - uv (1 - \cos \gamma)
\]

\[
M_{32}(z) = - u \sin \gamma
\]

\[
M_{33}(z) = 1 - u^2 (1 - \cos \gamma)
\]
Thus the uniformly sheared plasma is a non-trivial example where the polarimetry transfer matrix, which in general is obtained only numerically (see Section 7), can be given an analytic expression. The latter is useful also in the general case, where the magnetic field and the plasma density have a more complicated behaviour, for an approximate evaluation of the various elements of the transfer matrix.

From the solution given above it is possible to determine the parameters of the so-called "reference waves" or characteristic waves [5,6], which in a uniformly sheared plasma propagate with a constant ellipticity, i.e. constant values of $\chi$ and $s_3$, and linearly varying $\psi$. Indeed for $s_3$ constant, we must have $A=B=0$, or $(\Omega_3-2g)s_0+\Omega_1s_3=0$ and $s_0=0$, and so $s_0=\pm(\Omega_3-2g)/\Omega_0$. For these parameters, the two possible waves are described by

\[ s_1=\pm(\Omega_1/\Omega_0)\cos 2gz; \quad s_2=\pm(\Omega_1/\Omega_0)\sin 2gz; \quad -s_3=\pm(\Omega_3-2g)/\Omega_0 \]  

which correspond to $\psi=gz$, $\chi=-(1/2)\arcsin[(\Omega_3-2g)/\Omega_0]$ and respectively $\psi=gz+\pi/2$, $\chi=(1/2)\arcsin[(\Omega_3-2g)/\Omega_0]$. The two reference waves have the same constant absolute ellipticity and opposite sense of rotation of the electric field vector, while the orientation of the polarization ellipse follows that of the transverse magnetic field. When $\Omega_3=2g$ the reference waves are linearly polarized.

The structure of the reference waves can also be obtained directly, by making use of the Poincaré sphere. Since for a uniformly sheared plasma the instantaneous rotation axis precesses around the $s_3$ direction with a constant modulus, it is found [6] that there are two polarization states, those of the reference waves, which also simply precess around the $s_3$ axis and so have a constant ellipticity but rotating orientation.

5.2 A more general case

An exact analytic solution for the evolution equation can also be found for more general forms of $\mathbf{\Omega}(z)$ than the uniformly sheared case. In order to show this let us write the three components of the vector $\mathbf{\Omega}(z)$ in the form

\[ \Omega_1=\Omega\cos \eta \cos \phi; \quad \Omega_2=\Omega\cos \eta \sin \phi; \quad \Omega_3=\Omega \sin \eta \]  

where $\Omega$, $\eta$ and $\phi$ are in general all functions of $z$ (and, comparing with eq.4.14, $\phi=2\beta$). With this form the three scalar equations corresponding to the evolution equation, eq.4.12,

\[ \begin{align*}
\frac{ds_1}{dz} &= \Omega_2s_3 - \Omega_3s_2; \\
\frac{ds_2}{dz} &= \Omega_3s_1 - \Omega_1s_3; \\
\frac{ds_3}{dz} &= \Omega_1s_2 - \Omega_2s_1
\end{align*} \]  

can be transformed by simple algebra into
\[
\frac{1}{\Omega} \frac{d(s_2+is_1)}{dz} = -i \sin \eta (s_2+is_1) - \cos \eta \exp(-i\phi) s_3
\] (5.18)

\[
\frac{1}{\Omega} \frac{d(s_2-is_1)}{dz} = i \sin \eta (s_2-is_1) - \cos \eta \exp(i\phi) s_3
\] (5.19)

\[
\frac{1}{\Omega} \frac{ds_3}{dz} = \cos \eta \left[ \frac{\exp(i\phi)(s_2+is_1)+\exp(-i\phi)(s_2-is_1)}{2} \right]
\] (5.20)

where \( i \) is the imaginary unit \((i^2=-1)\). If we introduce the new variables

\[
U(z) = \exp(i\phi)(s_2+is_1); \quad V(z) = \exp(-i\phi)(s_2-is_1)
\] (5.21)

and change the independent variable from \( z \) to \( w \) by means of

\[
w = \int \cos \eta \Omega \, dz
\] (5.22)

then eqs. 5.20 become

\[
\frac{dU}{dw} = \imath hU - s_3; \quad \frac{dV}{dw} = -\imath hU - s_3; \quad 2 \frac{ds_3}{dw} = U + V
\] (5.23)

where we have put

\[
h(z) = \frac{1}{\cos \eta} \left[ \frac{1}{\Omega} \frac{d\phi}{dz} - \sin \eta \right]
\] (5.24)

Finally from the first two of eqs. 5.23 we have

\[
\frac{d(U+V)}{dw} = \imath h(U-V) - 2s_3
\] (5.25)

\[
\frac{d(U-V)}{dw} = \imath h(U+V)
\] (5.26)
When \( h(z) \) is a constant, i.e. when

\[
\frac{1}{\cos \eta} \left[ \frac{1}{\Omega} \frac{d\phi}{dz} - \sin \eta \right] = \text{constant} \tag{5.27}
\]

from eqs. 5.25, 5.26 and the last of eqs. 5.23 we have, putting \( k^2 = h^2 + 1 \),

\[
\frac{d^2(U+_V)}{dw^2} = -k^2 (U+_V) \tag{5.28}
\]

which can be integrated to give

\[
U + V = Q_1 \exp(ikw) + Q_2 \exp(-ikw) \tag{5.29}
\]

with \( Q_1 \) and \( Q_2 \) integration constants. By integrating now eq. 5.26

\[
U - V = \frac{h}{k} \left[ Q_1 \exp(ikw) - Q_2 \exp(-ikw) \right] + Q_3 \tag{5.30}
\]

Then eq. 5.25 gives \( s_3 \) in terms of \( (U+_V) \) and \( (U-_V) \) which now are known, while \( s_1 \) and \( s_2 \) are obtained from eqs. 5.21, 5.29 and 5.30.

Thus we have shown that an analytic solution exists when the condition of eq. 5.27 holds. It is obvious that the uniformly sheared plasma is a special case where the condition is satisfied because \( \eta, \Omega \) and \( d\phi/dz = 2g \) are all constants. It can be seen that in general \( kw \) (in eqs. 5.29 and 5.30) corresponds to \( \Omega Q \) of the uniformly sheared case (eqs. 5.10–5.12). The condition of eq. 5.27 can also be written in the form

\[
\frac{1}{\Omega(z)} \left[ 2 \frac{d\alpha}{dz} - Q_3(z) \right] = \text{constant} \tag{5.31}
\]

The exact analytic solutions of the evolution equation which have been discussed in this Section constitute a convenient model for the more general case where a numerical solution is necessary (see Section 7).
6. APPROXIMATE ANALYTIC SOLUTIONS OF THE EVOLUTION EQUATION

In this Section we will discuss some approximate analytic solutions of the evolution equation, eq.4.12 or eq.4.19. These are valid either when the entire change of polarization due to plasma propagation is small or when two of the components of the vector $\Omega$ are much smaller than the third component.

6.1 The case where $W \ll 1$.

Let us define $W(z)$ by

$$W(z) = \int_{Z_0}^{Z} dZ' \|Q(Z')\| \equiv \int_{Z_0}^{Z} dZ' \|T(Z')\|$$

so that $dW/dz = \|\Omega\| = \Omega$ and, as in Section 4, we can use either $Q(z)$ or $T(Z)$. Then it is easy to verify that the solution of eq.4.12 can be written in the form of a series as follows

$$s(z) = s_0 + a_0(z) + \sum_{n=1}^{\infty} a_n(z)$$

(6.2)

where

$$a_0(z) = \int_{Z_0}^{Z} dz' \Omega(z') \times s_0 ; \quad a_n(z) = \int_{Z_0}^{Z} dz' \Omega(z') \times a_{n-1}(z')$$

(6.3)

Since it can be shown [2] that $|a_n| \leq W^{(n+1)/(n+1)!}$, when $W<1$ the series is convergent and an approximate analytic solution is found by truncating the series at a certain point. The error thereby incurred is evaluated in [2]. When $W \ll 1$, to lowest order in $W$, we have

$$s(z) \approx s_0 + \int_{Z_0}^{Z} dz' \Omega(z') \times s_0$$

(6.4)

Let us define the dimensionless parameters $W_1$, $W_2$ and $W_3$ by
\[ W_j = \int_{z_0}^{z_1} dz \Omega_j(z) \quad \text{(for } j=1,2,3) \] (6.5)

Then from eq.4.11 we have

\[ W_1 = C_1 \int_{z_0}^{z_1} n(z)(B_x^2 - B_y^2) dz \] (6.6)

\[ W_2 = 2C_1 \int_{z_0}^{z_1} n(z)B_x(z)B_y(z) dz \] (6.7)

\[ W_3 = C_3 \int_{z_0}^{z_1} n(z)B_z(z) dz \] (6.8)

where \( C_1 = 2.42 \times 10^{-20} \lambda^3 \text{ (m}^2/\text{T}^2) \), \( C_3 = 5.23 \times 10^{-19} \lambda^2 \text{ (m}^2/\text{T}) \) and \( \lambda \) is in mm. It should be noted that if \( n(z) \) and \( B_y(z) \) are even functions of \( z \) while \( B_x(z) \) is odd, as can occur in a tokamak for vertical propagation, then \( W_2 = 0 \).

If we choose \( s_0 = (1,0,0) \) then from eq.6.4 we have \( s_1 = s(z_1) = (1, W_3, -W_2) \) which for \( |W_2|, |W_3| \ll 1 \) represents a polarization which has rotated by an angle \( \delta \psi = W_3/2 \) and has an ellipticity given by \( -W_2/2 \) (see eq.2.4). If we choose \( s_0 = (0,1,0) \), i.e. linearly polarized radiation oriented at \( 45^\circ \) to \( x \) and \( y \), then from eq.6.4 we have \( s_1 = s(z_1) = (W_3, 1, W_1) \) which for \( |W_1|, |W_3| \ll 1 \) represents a polarization which has rotated by an angle \( \delta \psi = -W_3/2 \) and has an ellipticity given by \( W_1/2 \). Therefore it is clear that \( W_1 \) and \( W_2 \) represent the effect of plasma birefringence (Cotton-Mouton effect), while \( W_3 \) represents the Faraday effect.

Using eqs.6.6 and 6.8, for \( \lambda = 0.1 - 0.2 \text{ mm} \) and for example vertical propagation in a tokamak with \( n = 1 \times 10^{20} \text{ m}^{-3} \), \( B_T = 5 \text{T} \), \( I = 1 \text{ MA} \), \( a = 0.3 \text{ m} \), one finds that \( W_1 \) and \( W_3 \) are in the range 0.1 to 0.9 and so it can be seen that the condition \( |W_1|, |W_3| \ll 1 \) is not always satisfied in practice.

In general the output polarization can be characterized either by \( s_1 \) or equivalently by \( \psi_1 \) and \( \chi_1 \) according to eq. 2.4. However, as we shall see, the former is often more convenient since the detector signals are directly related to the three components of \( s_1 \).

For any given input polarization \( s_0 \), the output \( s_1 \) can be evaluated from eq.6.2 to any desired order in \( W \) by truncating the series at the appropriate term. However it is convenient to use the
corresponding plasma transition matrix $M(Z_1)$ which allows us to obtain $s_1$ for any $s_0$ by means of eq.4.22. It can be verified (see also [11]) that, when $W \ll 1$, the transition matrix can be written as the Neumann series [24],

$$M(Z_1) = 1 + \int \limits_{Z_0}^{Z_1} dZ A(Z) + \int \limits_{Z_0}^{Z_1} dZ A(Z) \int \limits_{Z_0}^{Z_1} dZ' A(Z') + \int \limits_{Z_0}^{Z_1} dZ A(Z) \int \limits_{Z_0}^{Z_1} dZ' A(Z') \int \limits_{Z_0}^{Z_1} dZ'' A(Z'') + ...$$

which satisfies eq.4.19. This expression, which is equivalent to eq.6.2, is useful since it can provide an approximate analytical expression for $M(Z_1)$, valid when plasma effects are small.

We can write

$$M(Z_1) = 1 + M_1 + M_2 + M_3 + ...$$

where the components of the matrices $M_1$, $M_2$, $M_3$ ... are given by

$$(M_1)_{jk} = \int \limits_{Z_0}^{Z_1} dZ A_{jk}(Z)$$

$$(M_2)_{jk} = \int \limits_{Z_0}^{Z_1} dZ A_{jm}(Z) \int \limits_{Z_0}^{Z_1} dZ' A_{mk}(Z')$$

$$(M_3)_{jk} = \int \limits_{Z_0}^{Z_1} dZ A_{jm}(Z) \int \limits_{Z_0}^{Z_1} dZ' A_{mn}(Z') \int \limits_{Z_0}^{Z_1} dZ'' A_{nk}(Z'')$$

... and so on ....,

where again repeated indices imply summation and the elements of the matrix $A$ are defined by eq.4.16. Therefore, if we put (for $j,k=1,2,3$)

$$W_j = \int \limits_{Z_0}^{Z_1} dZ T_{j}(Z) ; \quad W_{jk} = \int \limits_{Z_0}^{Z_1} dZ T_{j}(Z) \int \limits_{Z_0}^{Z_1} dZ' T_{k}(Z')$$

(6.14)
where the first equation coincides with eq.6.5, we have

\[
M_1 = \begin{bmatrix}
0 & -W_3 & W_2 \\
W_3 & 0 & -W_1 \\
-W_2 & W_1 & 0
\end{bmatrix}
\] (6.15)

\[
M_2 = \begin{bmatrix}
-(W_{22}+W_{33}) & W_{21} & W_{31} \\
W_{12} & -(W_{33}+W_{11}) & W_{32} \\
W_{13} & W_{23} & -(W_{11}+W_{22})
\end{bmatrix}
\] (6.16)

and it can be verified that in general \( W_{jk} + W_{kj} = W_{j} W_{k} \) so that \( W_{jj} = \frac{W_{j}}{2} \).

When plasma effects are small (i.e. when \( W_j^2 \ll 1 \) for \( j=1,2,3 \)), by taking the Neumann expansion (eq.6.10) up to second order, we have the approximate relation

\[
M(Z_1) = \begin{bmatrix}
1-(W_2^2 + W_3^2)/2 & -(W_3 - W_{21}) & (W_2 + W_{31}) \\
(W_3 + W_{12}) & 1-(W_1^2 + W_3^2)/2 & -(W_1 - W_{32}) \\
-(W_2 - W_{13}) & (W_1 + W_{23}) & 1-(W_1^2 + W_2^2)/2
\end{bmatrix}
\] (6.17)

It should be noted that all the expressions given above are valid for arbitrary propagation directions and arbitrary forms of the plasma density profile \( n(r) \), appearing in \( \omega_p \) of eq.4.11, and of the magnetic field profile \( B(r) \). In specific plasma configurations and for particular directions of propagation, there may be important simplifications.

For a tokamak having up/down symmetry with respect to the equatorial plane, if we consider propagation in the vertical direction and take \( x \) in the direction of the major radius of the torus, then \( T_1(Z) \) and \( T_3(Z) \) are even functions of \( Z \) while \( T_2(Z) \) is odd, so that \( W_2 = 0 ; W_{13} = W_{31} = W_1 W_3/2 ; W_{12} = -W_{21} ; W_{23} = -W_{32} \). (The same is also true for propagation in the equatorial plane of an axisymmetric tokamak, taking \( x \) in the vertical direction). The transition matrix then becomes approximately, for \( W_1^2 \ll 1, W_3^2 \ll 1 \),

\[
M(Z_1) \approx \begin{bmatrix}
1 & -(W_3 - W_{21}) & W_1 W_3/2 \\
(W_3 - W_{21}) & 1 & -(W_1 - W_{32}) \\
W_1 W_3/2 & (W_1 - W_{32}) & 1
\end{bmatrix}
\] (6.18)

so that eqs.4.21 become
(s_1)_1 = (s_0)_1 - (W_3 - W_21)(s_0)_2 + (W_1 W_3/2)(s_0)_3 \quad (6.19)

(s_1)_2 = (W_3 - W_21)(s_0)_1 + (s_0)_2 - (W_1 - W_32)(s_0)_3 \quad (6.20)

(s_1)_3 = (W_1 W_3/2)(s_0)_1 + (W_1 - W_32)(s_0)_2 + (s_0)_3 \quad (6.21)

The approximate expressions for \( \mathbf{M}(Z_i) \) corresponding to the first few terms of the Neumann series (eqs.4.18 and 4.19) are also useful when plasma effects are not small: they can be used for an evaluation of the relative importance of the elements of \( \mathbf{M}(Z_i) \), without going through the numerical integration of the system of eqs.4.19. It should be noted that the quantity \( W_2 \) provides a measure of the deviation from symmetry.

6.2 The case where \( W = W_j; W_{j+1} \ll 1; W_{j+2} \ll 1 \)

Let us now suppose that the condition \( W \ll 1 \) is not necessarily satisfied, but one of the components of \( \mathbf{Q} \) (say \( Q_3 \)) is much larger than the others and so two of the \( W_j \)'s are small (say \( W_1, W_2 \ll 1 \)).

When, for instance, \( Q_3 \gg Q_j, Q_2 \), namely the case when the Faraday effect dominates, and we only assume \( W_1, W_2 \ll 1 \), we can write \( \mathbf{M}(z) \) in the form

\[
\mathbf{M}(z) = \mathbf{M}^{(0)} + \mathbf{M}^{(1)} + \mathbf{M}^{(2)} + \ldots
\]

where \( \mathbf{M}^{(0)}(z) \) is the solution of the system of eqs.4.23 when \( \Omega_1 \) and \( \Omega_2 \) are neglected with respect to \( \Omega_3 \) (and it represents the order zero), \( \mathbf{M}^{(1)}(z) \) corresponds to retaining terms of first order in \( \Omega_1 \) and \( \Omega_2 \) in the system, \( \mathbf{M}^{(2)}(z) \) to retaining terms of second order in \( \Omega_1 \) and \( \Omega_2 \), and so on. Thus \( \mathbf{M}^{(0)}(z) \) is given by the solution of the system of scalar equations

\[
\frac{d\mathbf{M}^{(0)}}{dz} = \mathbf{A}_0(z) \cdot \mathbf{M}^{(0)}(z)
\]

with

\[
\mathbf{A}_0(z) = \begin{bmatrix}
0 & -\Omega_3 & 0 \\
\Omega_3 & 0 & 0 \\
0 & 0 & 0
\end{bmatrix}
\]
and with the initial condition: \( M^{(0)} = I \) for \( z = 0 \). By integrating, one obtains directly

\[
M^{(0)}(z) = \begin{bmatrix}
\cos W_3 & -\sin W_3 & 0 \\
\sin W_3 & \cos W_3 & 0 \\
0 & 0 & 1
\end{bmatrix}
\] (6.25)

Furthermore, it can be seen that \( M^{(n+1)}(z) \) (for \( n = 0, 1, 2, \ldots \)) is given by the solution of the following system of equations (with \( h, j, k = 1, 2, 3 \)) for its nine elements

\[
\frac{d}{dz} M_{1h}^{(n+1)} = -\Omega_3 M_{2h}^{(n+1)} + \Omega_2 M_{3h}^{(n)}
\] (6.26)

\[
\frac{d}{dz} M_{2j}^{(n+1)} = \Omega_3 M_{1j}^{(n+1)} - \Omega_1 M_{3j}^{(n)}
\] (6.27)

\[
\frac{d}{dz} M_{3k}^{(n+1)} = -\Omega_2 M_{1k}^{(n)} + \Omega_1 M_{2k}^{(n)}
\] (6.28)

with the initial condition: \( M^{(n+1)} = 0 \) for \( z = 0 \). This system can be integrated considering \( M^{(n)}(z) \) to be known. Thus, from eqs. 6.26 and 6.27 we have (for \( h = 1, 2, 3 \))

\[
\frac{d}{dz} [M_{1h}^{(n+1)} + iM_{2h}^{(n+1)}] = i\Omega_3 [M_{1h}^{(n+1)} + iM_{2h}^{(n+1)}] + (\Omega_2 - i\Omega_1) M_{3h}^{(n)}
\]

where \( i \) is the imaginary unit, and after integration we obtain (taking into consideration the initial condition)

\[
M_{1h}^{(n+1)} + iM_{2h}^{(n+1)} = \exp(iW_3) \int_0^z dz' (\Omega_2' - i\Omega_1') M_{3h}^{(n)} \exp(-iW_3')
\]

where the prime indicates functions of \( z' \). By separating the real and imaginary parts, we have

\[
M_{1h}^{(n+1)} = \int_0^z dz' \left[ \Omega_2' \cos(W_3 - W_3') + \Omega_1' \sin(W_3 - W_3') \right] M_{3h}^{(n)}
\]
\[ M_{2h}^{(n+1)} = \int_0^z [\Omega_2 ' \sin(W_3 - W_3') - \Omega_1 ' \cos(W_3 - W_3')] M_{3h}^{(n)} \]

which can also be written in the more convenient form

\[ M_{1h}^{(n+1)} = P_h^{(n)} \cos W_3 + Q_h^{(n)} \sin W_3 \quad (6.29) \]

\[ M_{2h}^{(n+1)} = P_h^{(n)} \sin W_3 - Q_h^{(n)} \cos W_3 \quad (6.30) \]

where

\[ P_h^{(n)} = \int_0^z [\Omega_2 ' \cos W_3' - \Omega_1 ' \sin W_3'] M_{3h}^{(n)} \quad (6.31) \]

\[ Q_h^{(n)} = \int_0^z [\Omega_2 ' \sin W_3' + \Omega_1 ' \cos W_3'] M_{3h}^{(n)} \quad (6.32) \]

From eqs.6.28 we have directly (again for \( h = 1, 2, 3 \))

\[ M_{3h}^{(n+1)} = \int_0^z [\Omega_1 ' M_{2h}^{(n)} - \Omega_2 ' M_{1h}^{(n)}] \]

Thus eqs.6.29, 6.30 and 6.33 provide \( M^{(n+1)} \) when \( M^{(n)} \) is known. Since \( M^{(0)} \) is known, all higher order matrices can be constructed by recursively applying these equations for each successive order. In this way we find

\[ M^{(1)} = \begin{bmatrix} 0 & 0 & (P_3^{(0)} \cos W_3 + Q_3^{(0)} \sin W_3) \\ 0 & 0 & (P_3^{(0)} \sin W_3 - Q_3^{(0)} \cos W_3) \\ -P_3^{(0)} & Q_3^{(0)} & 0 \end{bmatrix} \quad (6.34) \]

where
\[ P_3^{(0)} = \int_0^z [\Omega_2' \cos W_3' - \Omega_1' \sin W_3'] \, dz \]

\[ Q_3^{(0)} = \int_0^z [\Omega_2' \sin W_3' + \Omega_1' \cos W_3'] \, dz \]

Also it can be verified that, in general,

\[ M_{13}^{(2m)} = M_{23}^{(2m)} = M_{31}^{(2m)} = M_{32}^{(2m)} = 0 \]

and

\[ M_{11}^{(2m+1)} = M_{21}^{(2m+1)} = M_{12}^{(2m+1)} = M_{22}^{(2m+1)} = M_{33}^{(2m+1)} = 0, \]

(for \( m=0,1,2,... \)). Namely, for even values of \( n \), \( M^{(n)} \) has the same distribution of zeros as \( M^{(0)} \) and, for odd values of \( n \), the same as \( M^{(1)} \).

When a component different from \( \Omega_3 \) is dominant, it is sufficient to change the identification of the indices and the same analysis holds. This can therefore be applied both when the Faraday effect and when the Cotton-Mouton effect is dominant.

When \( B_x^2 > B_y^2 \) the condition \( \Omega_3 > \Omega_1 \), from eqs. 4.11, becomes \( \omega B_z (e/mc) B_x^2 \), (or \( \lambda B_x^2 / B_z \approx 21.6 \) mmT, where \( \lambda \) is the radiation wavelength). This condition is satisfied for a typical large tokamak (ITER) having average plasma density \( 1 \times 10^{20} \) m\(^{-3} \), toroidal magnetic field 6T, plasma current 24MA, minor radius 3m, major radius 8m, elongation 1.6, where, for equatorial propagation with \( \lambda = 0.1 \) mm, one has \( W_1 = 0.3, \, W_3 = 19 \) and \( \omega = \Omega_3 / \Omega_1 = 63 \).

Finally when, for example \( \Omega_1 = \Omega_2 = 0 \), then all the higher order matrices are identically zero and \( M = M^{(0)} \); the solution simply involves \( W_3 \) and again this particular solution does not require \( |W| < 1 \) for its validity. This was recognized by O'Rourke [27] for the case when only the Faraday effect is present, but the results given above show that it is also true when only the pure Cotton-Mouton effect is present, with \( \Omega_2 = \Omega_3 = 0 \) (and then \( W_1 \) appears in eq.6.25 instead of \( W_3 \)).
7. NUMERICAL INTEGRATION OF THE EVOLUTION EQUATION

When plasma effects are large (and especially when \(W > 1\)) the approximate analytic solutions of the propagation equation, discussed in Section 6, can no longer be used and it becomes necessary to integrate this equation numerically. This can be done directly by integrating numerically the system of scalar equations given by eqs.4.19 or 4.23. However this does not ensure that \(\mathbf{M}\) remains a unitary matrix, as is required by Eqs.4.23 and 4.24, and so significant numerical errors can accumulate in the integration, especially when the Cotton-Mouton and Faraday effects are large. We shall therefore discuss two alternative schemes (see [15]), for the numerical integration of the system of nine coupled scalar equations given by Eq.4.23, which leave \(\mathbf{M}\) exactly unitary. Scheme I is simpler but less accurate and is reported here for completeness, while Scheme II is a little more complex but can tolerate larger steps in the numerical integration. These schemes are especially useful when the off-axis elements of \(\mathbf{M}\) are large.

**Scheme I.** For this scheme, a slab model is taken and all plasma quantities are assumed constant within each space step, of width \(\Delta z\), so that we can use the analytic solution of the propagation equation for a uniform medium. This solution can be found easily by introducing the three vectors \(\mathbf{M}_j\) (\(j = 1, 2, 3\)) defined by \(\mathbf{M}_j = (M_{1j}, M_{2j}, M_{3j})\) where \(M_{ij}\) (\(i, j = 1, 2, 3\)) are the elements of \(\mathbf{M}\). Then Eq.4 can be written as

\[
\frac{d\mathbf{M}_j(z)}{dz} = \mathbf{\Omega} \times \mathbf{M}_j \tag{7.1}
\]

showing that the nine coupled scalar equations of Eq.4 for the nine \(M_{ij}\)'s actually consist of 3 sets of 3 coupled equations, each set containing only 3 of the \(M_{ij}\)'s. The solution of Eq.6, when \(\mathbf{\Omega}\) is constant, is

\[
\mathbf{M}_j = a_j \sin \Omega z + b_j \cos \Omega z + c_j \tag{7.2}
\]

where \(\Omega = (\Omega_1^2 + \Omega_2^2 + \Omega_3^2)^{1/2}\),

\[
\begin{align*}
  a_j &= \mathbf{\Omega} \times \mathbf{M}_{j0}/\Omega \\
  b_j &= \mathbf{M}_{j0} - (\mathbf{\Omega} \cdot \mathbf{M}_{j0})\mathbf{\Omega}/\Omega^2 \\
  c_j &= (\mathbf{\Omega} \cdot \mathbf{M}_{j0})\Omega/\Omega^2
\end{align*} \tag{7.3}
\]

and \(\mathbf{M}_{j0}\) is the initial value of \(\mathbf{M}_j\). This solution can also be written in matrix notation as

\[
\mathbf{M} = a \sin \Omega z + b \cos \Omega z + c \tag{7.4}
\]
where the matrices $a$, $b$ and $c$ are defined by

$$\begin{align*}
a &= A \cdot M(0)/\Omega \\
b &= M(0) - \Omega \cdot M(0)/\Omega^2 \\
c &= \Omega \cdot M(0)/\Omega^2
\end{align*}$$

(7.5)

and $\Omega = \Omega \Omega$, i.e. $\Omega_{ij} = \Omega_j \Omega_i$, and $M(0)$ is the initial value of $M$. From eqs. 7.2 or 7.4, the final value of $M$ after crossing the slab of thickness $\Delta z$ can also be written in a more compact form as

$$M(\Delta z) = \left\{ I + \frac{\sin \Omega \Delta z}{\Omega} A + \frac{1}{2} \left[ \frac{\sin(\Omega \Delta z/2)}{\Omega^2} \right] A^2 \right\} \cdot M(0)$$

(7.6)

and $I$ is the unit matrix.

The complete solution, for a step-wise constant plasma, is obtained by recursively applying eq.7.6, or 7.4, or 7.2, and by using the output of each slab as the input to the successive one and by putting $M = I$ as the input to the first slab. In this way $M$ remains unitary exactly and the output $M(z_1)$ is a good approximation when $\Delta z$ is sufficiently small compared to the scalelength $L$ of variation of plasma quantities: indeed the relative error is of the order of $\Delta z/L$.

It should be noted that an exact analytic solution of eq.4.23 also exists for $\Omega(z)$ within the class of functions which satisfy eq.5.27 (see Section 5) and a numerical integration scheme could also be based on this kind of solution, by using, for each space step, an approximating $\Omega(z)$ within the special class of functions.

Scheme II. This scheme for numerical integration is based on a quasi-analytical method that exploits the (so-called) symmetric split decomposition \[28, 29\]. We first note that, by using the Magnus type of integration \[30, 31\], the solution to eq.4.23 can be written as

$$M(z) = \exp \left\{ \int_0^z A(z')dz' + \frac{1}{2} \int_0^z dz_1 \int_0^{z_1} dz_2 [A(z_2);A(z_1)] + \ldots \right\} \cdot M$$

(7.7)

where the commutation $[P;Q] = P \cdot Q - Q \cdot P$. The second term in the exponent and the successive contributions, containing higher order commutations, provide the "time-ordering" corrections. If again we limit ourselves to small $z$-intervals, $\Delta z$, we can infer that the neglected time-ordering corrections are small, of order $(\Delta z/L)^3$ or smaller (see \[30,31\]). We now put
\[ A_1 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix}, \quad A_3 = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \]

so that

\[ A(z) = \sum_{j=1}^{3} \Omega_j(z) A_j \quad (7.8) \]

and use the symmetric-split decomposition [28, 29], namely for arbitrary matrices \( S \) and \( T \)

\[
\exp[(S+T)\delta t] = \exp(S \delta t/2) \exp(T \delta t) \exp(S \delta t/2) \\
+ [(S+2T) : [S, T]] (\delta t)^3/24, \quad (7.9)
\]

neglecting the last term in this expression. From eq.7.7 we obtain

\[
M(Az) = R_1(Az) \cdot R_2(Az) \cdot M(0) \quad (7.10)
\]

where

\[
R_1(Az) = \exp((W_3/2)A_3) \exp((W_2/2)A_2) \exp((W_1/2)A_1) \quad (7.11)
\]

\[
R_2(Az) = \exp((W_1/2)A_1) \exp((W_2/2)A_2) \exp((W_3/2)A_3) \quad (7.12)
\]

and (see eq.6.5)

\[
W_j(\Delta z) = \int_0^{\Delta z} \Omega_j(z)dz \quad (7.13)
\]

It is easy to realise that \( R_1 \) and \( R_2 \) are rotation matrices and that, putting \( w_j = W_j/2 \),

\[
R_2(w_1,w_2,w_3) = \{ R_1(-w_1,-w_2,-w_3) \}^+ \quad (7.14)
\]
with

\[ R_1(w_1, w_2, w_3) = \begin{bmatrix} \cos w_3 \cos w_2 & -\sin w_3 \cos w_1 + \cos w_3 \sin w_2 \sin w_1 & \sin w_3 \sin w_1 + \cos w_3 \sin w_2 \cos w_1 \\ \sin w_3 \cos w_2 & \cos w_3 \cos w_1 + \sin w_3 \sin w_2 \sin w_1 & -\cos w_3 \sin w_1 + \sin w_3 \sin w_2 \cos w_1 \\ -\sin w_2 & \cos w_2 \sin w_1 & \cos w_2 \cos w_1 \end{bmatrix} \]

and \((P)^+\) indicates the transpose matrix of \(P\). If we divide the integration interval into \(N\) sub-intervals \(\Delta z\), we can write the output matrix \(M\) in the form

\[ M \equiv \prod_{n=1}^{N} R_{1,n} \cdot R_{2,n} \cdot 1 \quad \text{(7.15)} \]

where

\[ R_{1,n} = R_1(w_{1,n}, w_{2,n}, w_{3,n}), \quad w_{j,n} = \int_{z(n-1)}^{z(n)} dz' \Omega_j(z')/2 \]

and \(z(n)-z(n-1)=\Delta z\). For this scheme, due to the truncations used in eqs.7.7 and 7.9, the relative error is of the order of \((\Delta z/L)^3\) and again \(M\) remains unitary.

We have thus presented two alternative schemes for the evaluation of the polarimetric transition matrix \(M\) of a magnetized plasma when the plasma parameters are assigned, i.e. when the vector \(\Omega(z)\) is known. Scheme I is simpler and is subject to an error of the order of \(\Delta z/L\), while for Scheme II this error is of order \((\Delta z/L)^3\). For both schemes it is ensured that \(M\) remains unitary exactly.

The schemes for numerical integration discussed above can be used in the analysis of experiments for the reconstruction of MHD equilibria. In this kind of analysis the space dependence of plasma parameters is determined by a best fit procedure, minimizing the differences between measured quantities and quantities computed assuming parametrized forms for plasma quantities. The present schemes can be used when the elements of \(M\) are included among the measured quantities. The schemes are particularly useful when the Cotton-Mouton and Faraday effects are not small, so that the results of Section 6 cannot be applied.

8. DESIGN OF EXPERIMENTS

For a beam of radiation crossing a plasma the output polarization \(s\) is related to the input \(s_0\) by eq.4.21 (or eq.4.22). Therefore the plasma is fully characterized by the transition matrix \(M\) and so the maximum amount of information which can be extracted from polarimetric measurements
consists in the determination of the nine elements of $M$. These nine quantities can indeed be measured independently (although, as shown in Section 4, only three are functionally independent) and they can then be used to determine the plasma parameters $n(r)$ and $B(r)$. As shown in Section 6, when plasma effects are small, the measurements can at most determine $W_1$, $W_2$ and $W_3$ which are related to plasma parameters by eqs. 6.6 to 6.8. Furthermore from these equations it can be seen that, for typical plasma parameters (say $n=1\times 10^{20} m^{-3}$, $B_T=5T$, $B_p=0.3T$, $L=0.5m$), in order to have $W_1 \geq 0.01$ or $W_3 \geq 0.01$ it is necessary to have $\lambda \geq 0.1mm$ or respectively $\lambda \geq 0.03 mm$. Thus the wavelength of the radiation used must be in the far infrared (FIR) range and this places some strong limitations on the components which can be used in plasma polarimetry.

In view of the design of systems for plasma polarimetric measurements, in this Section we shall first discuss some components of a polarimetric system, namely the possible analyser configurations and the kinds of modulation of the state of polarization which are possible. Then we shall present alternative schemes for the measurement of the elements of $M$, or, when plasma effects are small, of $W_1$, $W_2$ and $W_3$. The various schemes have different advantages and disadvantages.

### 8.1 Analyser configurations

Let us consider radiation having a Stokes vector $S=(S_0, S_1, S_2, S_3)$ or, using the components of the reduced Stokes vector, $S=S_0(1, s_1, s_2,s_3)$. In order to measure its state of polarization, one can use an analyser (see fig.4) consisting in general of a quarter-wave retarder, whose axis is at an angle $\gamma$ to the reference direction (the $x$ direction), followed by a polarizer, with its axis at an angle $\theta$ to the $x$ direction, followed finally by a detector. Using the expressions given in Section 3 for the Mueller matrices, $M_R$, for a retarder (eq.3.3 or 3.5), and $M_p$, for a polarizer (eq.3.2), one can find the Stokes vector $S_D$ of the radiation after the polarizer as $S_D=M_pM_R&S$. The first component of this vector, $(S_D)_0$, is the intensity of the radiation falling on the detector and so, apart from a calibration constant, it represents the detector signal.

By following this procedure it is easily shown that, if the retarder is absent, the intensity on the detector, $I_D$, is given by

$$I_D(\theta) = \frac{1}{2} (S_0 + \cos 2\theta S_1 + \sin 2\theta S_2) = \frac{1}{2} I_0 (1 + \cos 2\theta s_1 + \sin 2\theta s_2) \quad (8.1)$$

where $I_0$ is the intensity ($I_0=S_0$) entering the analyser. In particular for $\theta=0$ and $\theta=\pi/4$ we find $I_D(0)=I_0(1+s_1)/2$ and $I_D(\pi/4)=I_0(1+s_2)/2$.

If the retarder is present, the intensity on the detector is
Fig. 4 The analyser configuration. A quarter-wave retarder is denoted by $R(\pi/2)$ and a polarizer by $P$

\[ I_D(\gamma, \theta) = \frac{1}{2} I_0 \left\{ 1 + \cos(2\theta - 2\gamma)(\cos 2\gamma s_1 + \sin 2\gamma s_2) + \sin(2\theta - 2\gamma) s_3 \right\} \] (8.2)

and in particular, if $\theta = \gamma + \pi/4$, then $I_D(\gamma, \gamma + \pi/4) = I_0 (1 + s_3)/2$, independent of $\gamma$.

For $\theta = \gamma$, we have $I_D(\gamma, \theta) = I_D(\theta)$.

Thus, with a convenient choice of analyser configuration, it is possible to determine $s_1$, $s_2$ and $s_3$ directly from measurements of $I_D/I_0$. (We recall that $s_1^2 + s_2^2 + s_3^2 = 1$). It can be seen that, as mentioned previously, the quantities $s_1$, $s_2$ and $s_3$ are related directly to measured quantities, while the polarimetric parameters $\psi$ and $\chi$ (see eq. 2.5) are related only indirectly. This is a reason for the importance of the Stokes vector.

In general, if we put

\[ A(\gamma, \theta) = \cos 2\gamma \cos(2\theta - 2\gamma) \equiv \frac{1}{2} \cos 2\theta + \frac{1}{2} \cos(2\theta - 4\gamma) \]
\[
B(\gamma, \theta) = \sin 2\gamma \cos (2\theta - 2\gamma) = \frac{1}{2} \sin 2\theta - \frac{1}{2} \sin (2\theta - 4\gamma)
\]  
(8.3)

\[
C(\gamma, \theta) = \sin (2\theta - 2\gamma)
\]

we have

\[
2I_D(\gamma, \theta)/I_0 = 1 + A_s + B_s + C_s
\]  
(8.4)

Let us suppose that the analyser is placed immediately after the plasma, so that if, s and s_o are the Stokes vectors of the radiation leaving and, respectively, entering the plasma, one has

\[s = M s_o\]

(see eq.4.21). Then we can write

\[
2I_D(\gamma, \theta)/I_0 = 1 + A_{M_{ij}}(s_o)_j + B_{M_{ij}}(s_o)_j + C_{M_{ij}}(s_o)_j
\]

\[
= 1 + J(s_o)_1 + K(s_o)_2 + L(s_o)_3
\]  
(8.5)

where repeated indexes denote summation, \((s_o)_j\) for \(j=1,2,3\) are the components of \(s_o\) and we have put

\[
J = A_{M_{11}} + B_{M_{21}} + C_{M_{31}}
\]

\[
K = A_{M_{12}} + B_{M_{22}} + C_{M_{32}}
\]

\[
L = A_{M_{13}} + B_{M_{23}} + C_{M_{33}}
\]  
(8.6)

Thus it is clear that in general, if \(I_D/I_0\) is measured for three known values of \(s_o\), one can determine \(J, K\) and \(L\) for a particular set of values of \(\theta\) and \(\gamma\). If this is done for three independent sets, all the elements of the plasma transition matrix are determined. It should be recalled that these elements are not all independent since \(M_{jm} M_{jn} = \delta_{mn}\), for \(m,n=1,2,3\) (see Section 4). In practice, as we shall see, important information on \(M\) can be obtained with fixed \(\theta\) and \(\gamma\) by using a time-varying (modulated) \(s_o\).

8.2 Modulation of the state of polarization

It will be shown that for polarimetric measurements it is very useful to modulate the state of polarization of the radiation. Such a modulation can be applied to the input radiation entering the plasma or to the output radiation leaving the plasma. The former case is more convenient for multichannel systems since a single modulator can operate for many beams. The modulation consists in applying a time variation to some polarimetric parameter, say \(\delta\). We shall consider
two kinds of modulation, the *progressive modulation* where $\delta = \omega_m t$ and the *alternating modulation* where $\delta = \delta_0 \cos \omega_m t$, and the modulation frequency $\omega_m$ is a constant (much smaller than the radiation frequency).

The relative merits of the two kinds of modulation will be discussed later: they are also connected with the availability of modulating devices at the radiation wavelength of interest (which is usually in the far infrared). In this Section we will begin by considering *progressive modulation*.

### 8.2.1 progressive modulation

We shall consider two methods to produce a progressive modulation of the state of polarization.

**Method I.** The first method is to include in the optical path a retarder rotating at a constant frequency $\omega_0/2\pi$. The retarder can be inserted in the optical path before crossing the plasma (input modulation) or after (output modulation). If $s=(s_1, s_2, s_3)$ is the reduced Stokes vector of the radiation entering the retarder, and $s'$ the Stokes vector after the retarder, then we have

$$s' = M_r(\rho, \omega_0 t) \cdot s$$

(8.7)

where $M_r(\rho, \omega_0 t)$, the Mueller matrix for a retarder having retardation $\rho$ and orientation $\gamma = \omega_0 t$, is given by eq. 3.4. Of special interest are the cases $\rho = \pi/2$ and $\rho = \pi$. For $\rho = \pi/2$, we have

$$M_r(\pi/2, \omega_0 t) = \begin{bmatrix}
(1+\cos 4\omega_0 t)/2 & (\sin 4\omega_0 t)/2 & -\sin 2\omega_0 t \\
(\sin 4\omega_0 t)/2 & (1-\cos 4\omega_0 t)/2 & \cos 2\omega_0 t \\
\sin 2\omega_0 t & -\cos 2\omega_0 t & 0
\end{bmatrix}$$

(8.8)

and, if $s=(s_1, 0, s_3)$, then

$$s' = \frac{1}{2} \begin{bmatrix}
s_1 - 2s_3 \sin 2\omega_0 t + s_1 \cos 4\omega_0 t \\
2s_3 \cos 2\omega_0 t + s_1 \sin 4\omega_0 t \\
2s_1 \sin 2\omega_0 t
\end{bmatrix}$$

(8.9)

which has a double modulation at the frequencies $2\omega_0$ and $4\omega_0$.

For $\rho = \pi$, we have
\[
M_r(\pi, \omega_0t) = \begin{bmatrix}
\cos 4\omega_0t & \sin 4\omega_0t & 0 \\
\sin 4\omega_0t & -\cos 4\omega_0t & 0 \\
0 & 0 & -1
\end{bmatrix}
\] (8.10)

It can be verified that in this case the input polarization is rotated by an angle \(2\gamma = 4\omega_0t\), with no change of ellipticity (in absolute value) and there is modulation at the single frequency \(4\omega_0\) (see also eq.3.8).

**Method II.** For the second method to obtain a progressive modulation, the radiation source produces the input modulation by combining (see fig.5) linearly polarized radiation at a frequency \(\omega\) (having an amplitude \(a\) and orientation at an angle \(\mu\) to the x direction) together with linearly polarized radiation at a frequency \((\omega + \delta\omega)\), having an amplitude \(b\) and orientation at an angle \(\nu\) to the x direction. Using eqs.2.7, one finds that the full Stokes vector of the combined radiation, \(S_h\), is given by

\[
S_h = \begin{bmatrix}
1 + h_1 \cos \delta\omega t \\
g_2 + h_2 \cos \delta\omega t \\
g_3 + h_3 \cos \delta\omega t \\
h_4 \sin \delta\omega t
\end{bmatrix}
\] (8.11)

where \(h_1 = h \cos (\nu - \mu)\), \(h_2 = h \cos (\nu + \mu)\), \(h_3 = h \sin (\nu + \mu)\), \(h_4 = h \sin (\nu - \mu)\), \(g_2 = (a^2 \cos 2\mu + b^2 \cos 2\nu)/(a^2 + b^2)\) and \(g_3 = (a^2 \sin 2\mu + b^2 \sin 2\nu)/(a^2 + b^2)\) with \(h = 2ab/(a^2 + b^2)\). Thus \(S_h\) is modulated at the frequency \(\delta\omega\) and in general also the intensity, \((S_h)_0\), is modulated.

Of special interest is the case \(\mu = 0, \nu = \pi/2\) for which \(h_1 = h_2 = g_3 = 0, h_3 = h_4 = h, g_2 = g = (a^2 - b^2)/(a^2 + b^2)\) so that now the intensity is constant (and the reduced Stokes vector can be used).

Obviously, with method I, one can modulate both the input and the output states of polarization, whereas, with method II, only the input can be modulated. With method I frequencies \(\omega_0/2\pi = 1\,\text{kHz}\) have been obtained [32-34], also in the far infrared. For method II, the frequency

---

**Fig.5 Electric field diagram for high-frequency modulated source**
shift $\delta \omega$ can be obtained, according to the wavelength of interest, by various methods which have been used for heterodyne measurements. The Doppler shift due to diffraction from a rotating grating-wheel has been used to produce $\delta \omega/2\pi$ up to 3MHz on FIR, far-infrared radiation [35-38]. Optoacoustic modulators (Bragg cells) can produce $\delta \omega/2\pi$ of tens of MHz [39, 40] for shorter wavelengths (infrared or less). A single FIR laser can produce radiation at two wavelengths (with $\delta \omega=1$MHz) on two orthogonal linearly polarized modes [41]. Alternatively twin lasers [42, 43] can be used to provide two beams having a precise frequency difference (up to tens of MHz) at various discrete wavelengths also in the far infrared. Finally, for operation with millimetric microwaves, it is possible [44] to modulate the frequency of a single generator (eg a backward-wave tube) with a saw-tooth signal, so that the frequency periodically increases linearly with time. If one splits the generator output into two beams and then delays one of the two, the resulting two beams have a constant frequency difference, which can be adjusted by changing the delay, and the two beams can be used for polarimetry [44].

Thus it is clear that much higher frequencies can be obtained by method II of modulation than by method I. However method I produces a two-frequency modulation and so, as we shall see, it allows the determination of two polarimetric parameters from the two phase measurements on the two frequency components. It is possible to obtain this feature also at very high frequency by combining the two methods.

Method IIa. Indeed let us consider the source of method II, followed by a retarder of retardation $\rho$, rotating at a frequency $\omega_r/2\pi$ (see fig.6). We suppose for simplicity that $\mu=0$ and $\nu=\pi/2$ so that the polarization of radiation leaving the source can be described by the reduced Stokes vector $s_s=(g, h\cos\delta\omega t, h\sin\delta\omega t)$. Then the polarization $s_d$ after the rotating retarder is given by

$$s_d = M_r(\rho, \omega_r t) \cdot s_s$$

$$\begin{bmatrix} G+H\cos4\omega_r t & H\sin4\omega_r t & -\sin\rho\cos2\omega_r t \\ H\sin4\omega_r t & G-H\cos4\omega_r t & \sin\rho\sin2\omega_r t \\ -\sin\rho\sin2\omega_r t & -\sin\rho\cos2\omega_r t & \cos\rho \end{bmatrix} \cdot \begin{bmatrix} g \\ h\cos\delta\omega t \\ h\sin\delta\omega t \end{bmatrix}$$

(8.12)

where $G=(1+\cos\rho)/2$ and $H=(1-\cos\rho)/2$ (see Section 3). It can be seen that, besides a dc component, $s_d$ has components at various frequencies, $2\omega_r$, $4\omega_r$, $\delta\omega$, $\delta\omega\pm2\omega_r$, $\delta\omega\pm4\omega_r$. If we consider only the two components at the highest frequencies ($\delta\omega+2\omega_r$) and ($\delta\omega+4\omega_r$), for the purpose of computing the contributions to the output intensity at these frequencies, we can use the corresponding partial Stokes vector $s_{dp}$, given by:

$$s_{dp} = \frac{h}{2} \begin{bmatrix} \sin\rho\cos(\delta\omega+2\omega_r)t + H\sin(\delta\omega+4\omega_r)t \\ \sin\rho\sin(\delta\omega+2\omega_r)t - H\cos(\delta\omega+4\omega_r)t \\ -\sin\rho\cos(\delta\omega+2\omega_r)t \end{bmatrix}$$

(8.13)
which is obtained from eq.8.12 by taking only terms at the two highest frequencies, provided that \( \sin \rho \neq 0 \). Thus, by combining methods I and II, we have high-frequency modulation simultaneously at more than one frequency (eq.8.12) and in particular at two frequencies (eq.8.13), which as we shall see is very useful.

By comparing eqs.8.9 and 8.13 it can be seen that, if we neglect the DC part in \( s' \), put \( \sin \rho = 1 \) and make the correspondences

\[
\begin{align*}
    s_1 &= s_3 = \frac{h}{2}, & (\delta \omega + 2 \omega t) &= 2 \omega_0 t + \pi/2, & (\delta \omega + 4 \omega t) &= 4 \omega_0 t + \pi/2, \\
\end{align*}
\]

(8.14)

then \( s' \) and \( s_{dp} \) become identical. This implies that the treatment, which we will give later for the case where \( s' \) represents the input polarization (method I), can also be applied to the case where \( s_{dp} \) (or rather \( s_d \)) is the input polarization (method IIa), by using these correspondences.

Method IIb. An alternative possibility is to use a half-wave retarder, \( \rho = \pi \), so that from eq.8.12 we have

\[

s_d(\pi) = \begin{bmatrix}
    g \cos 4 \omega_0 t + h \sin 4 \omega_0 t \cos \delta t \\
    g \sin 4 \omega_0 t - h \cos 4 \omega_0 t \cos \delta t \\
    -h \sin \delta \omega_0 t
\end{bmatrix}
\]

(8.15)

and this, when \( \delta \omega \gg 4 \omega_0 \), can be considered as modulated at the high frequency \( \delta \omega \), with an amplitude which changes slowly at the frequency \( 4 \omega_0 \).

It must be stressed that progressive modulation (with both methods I and II) can be used readily also at far infrared wavelengths, which are required for plasma polarimetry.

### 8.2.2 alternating modulation

Alternating modulation is produced by special optical components (such as photoelastic modulators or electro-optical modulators) which introduce an alternating time behaviour on
some polarimetric parameter. Up to now such optical components have been developed only for the visible and near infrared wavelengths, and they are widely used for ellipsometry of materials [45] and in astronomy [46,47]. The interest for plasma polarimetry is mainly in the far infrared, however combined interferometer/polarimeter measurements have also been performed in the near infrared [48] and future measurements have been proposed at this wavelength [49]. Furthermore it may be possible (see [50]) to obtain polarization modulation from a FIR cavity pumped by a CO₂ laser having active modulation at 10.6 μm. Therefore we will briefly discuss also the principles of alternating modulation.

The modulators used up to now are based on induced optical activity (e.g., Faraday effect) or on induced birefringence. In order to be specific we shall discuss the latter and in particular the use of the photoelastic modulator, PEM. This is a rectangular bar of a suitable transparent material (e.g., fused silica) which is made to oscillate (at 20 to 100 kHz) by a piezoelectric transducer. The elastic standing wave in the bar makes it become birefringent, with an oscillating relative retardation. Thus the PEM has a Mueller matrix, $M_r(\rho, \gamma)$, given by eq. 3.4 where the retardation $\rho$ is time dependent according to $\rho = \rho_0 \sin \omega_m t$.

For example, let us consider an analyzer (see fig. 7) consisting of such a PEM, with its reference axis parallel to the x direction, followed by a polarizer with axis at $45^\circ$ to the x direction, and then the detector. Using the expressions of Section 3, one finds that, for radiation of intensity $I_0$ having Stokes vector $s = (s_1, s_2, s_3)$, the intensity $I$ on the detector is

$$I = \frac{1}{2} I_0 \left[ 1 + s_2 \cos \rho + s_3 \sin \rho \right]$$  \hspace{1cm} (8.16)$$

We can now recall the expressions

$$\cos(\rho_0 \sin \omega_m t) = J_0(\rho_0) + 2 \sum_{n=1}^{\infty} J_{2n}(\rho_0) \cos[2n\omega_m t]$$

$$\sin(\rho_0 \sin \omega_m t) = 2 \sum_{n=1}^{\infty} J_{2n-1}(\rho_0) \sin[(2n-1)\omega_m t]$$

where $J_n$ is the Bessel function of order $n$. Therefore, if we measure the amplitudes of the DC component of $I$, $I_{\text{DC}}$, and of the $\omega_m$ and $2\omega_m$ components, $I_\omega$ and $I_{2\omega}$, we have

$$\frac{I_\omega}{I_{\text{DC}}} = \frac{2s_3 J_1(\rho_0)}{1 + s_2 J_0(\rho_0)}; \hspace{1cm} \frac{I_{2\omega}}{I_{\text{DC}}} = \frac{2s_2 J_2(\rho_0)}{1 + s_2 J_0(\rho_0)}$$  \hspace{1cm} (8.17)
Since $\rho_0$ and hence $J_0(\rho_0)$, $J_1(\rho_0)$, and $J_2(\rho_0)$ can be determined by calibration measurements, the Stokes components $S_2$ and $S_3$ (and hence $\psi$ and $\chi$) are determined from eq.8.17. These expressions become simpler if the modulation amplitude $\rho_0$ is adjusted ($\rho_0=2.405$) so that $J_0=0$, $J_1=0.519$, $J_2=0.432$.

This method has been used to measure very small fractional polarizations ($<3\times10^{-7}$) in the radiation from the Sun [47]. Unfortunately modulators for the far infrared (which is of greatest interest for plasma diagnostics) do not yet exist. However there are, for instance, plans for measuring in the ITER tokamak the plasma Faraday rotation for tangential chords, using CO\textsubscript{2} laser radiation at 10.6 $\mu$m [49].

9. PLASMA POLARIMETRIC MEASUREMENTS

As we have shown in Section 4, the polarimetric behaviour of a plasma, for each chord, is fully described by the plasma transition matrix $M(z_1)$, and we shall henceforth omit the $z_1$ dependence. Therefore plasma measurements can, at most, determine the nine elements of $M$. When plasma effects are small, as we have found in Section 6, one has

$$M = \begin{bmatrix}
M_{11} & M_{12} & M_{13} \\
M_{21} & M_{22} & M_{23} \\
M_{31} & M_{32} & M_{33}
\end{bmatrix}$$

$$\approx \begin{bmatrix}
1-(W_2^2+W_3^2)/2 & -(W_3-W_21) & (W_2+W_31) \\
(W_3+W_12) & 1-(W_1^2+W_3^2)/2 & -(W_1-W_32) \\
-(W_2-W_13) & (W_1+W_23) & 1-(W_1^2+W_2^2)/2
\end{bmatrix}$$
\[
M = \begin{bmatrix} 1 & -(W_3-W_{21}) & W_1 W_3/2 \\
(W_3-W_{21}) & 1 & -(W_1-W_{32}) \\
W_1 W_3/2 & (W_1-W_{32}) & 1 
\end{bmatrix}
\]  
(9.1)

where \(W_j\) and \(W_{jk}\) are given by eqs.6.5 to 6.8 and 6.14, and the last expression holds when, due to symmetry, \(W_2=0\) and \(W_{13}=W_{31}=W_1 W_3/2\). In general, to lowest order, we have approximately

\[
M = \begin{bmatrix} 1 & -W_3 & W_2 \\
W_3 & 1 & -W_1 \\
-W_2 & W_1 & 1 
\end{bmatrix}
\]  
(9.2)

This expression is very useful for a rough estimate of the elements of \(M\) and also it shows the roles of \(W_1\) and \(W_2\), which describe the Cotton-Mouton effect, and of \(W_3\), which describes the Faraday effect (see Section 6).

As we have seen in Section 4, the elements of \(M\) are not independent and in general we have, from eq.4.20,

\[
M_{11} = \pm \sqrt{1-M_{21}^2-M_{31}^2} = \pm \sqrt{1-M_{12}^2-M_{13}^2}
\]

\[
M_{22} = \pm \sqrt{1-M_{12}^2-M_{32}^2} = \pm \sqrt{1-M_{21}^2-M_{23}^2}
\]  
(9.3)

\[
M_{33} = \pm \sqrt{1-M_{13}^2-M_{23}^2} = \pm \sqrt{1-M_{31}^2-M_{32}^2}
\]

and, when plasma effects are not too large, the + sign must be taken in all these expressions (as can be seen by continuity with the case with no plasma, for which \(M_{11}=M_{22}=M_{33}=1\)).

9.1 Measurements without modulation of the state of polarization

In this Section we will consider plasma measurements where no modulation is applied to the state of polarization, but we will also discuss the possible modulation of the polarimetric signal (see Section 9.1.1).

In order to illustrate the measurements without modulation of the polarization, let us first consider some particular cases of special interest. If \(s_0=(1, 0, 0)\) is the constant Stokes vector of the radiation entering the plasma and we use the analyser configuration of Section 8.1, then from eq.8.5 we have for \(R=2I_D(\gamma, \theta)/I_0\) the following
Fig. 8 Measurement configuration for $s_0=(1,0,0)$ and a) $\theta=\pi/2$, b) $\theta=\pi/4$, c) $\theta=-\pi/4$. $E$ denotes the electric field of the input radiation and $P$ a polarizer.

$$R = 2I_D(\gamma, \theta)/I_0 = 1 + J = 1 + AM_{11} + BM_{21} + CM_{31}$$

(9.4)

If we now choose $\gamma=\theta=\pi/2$, namely no retarder and the polarizer crossed with the input polarization (see fig.8a), we have from eqs.8.3 $B=C=0$ and $A=-1$ so that

$$R = 1 - M_{11} \approx (W_2^2 + W_3^2)/2.$$  \hfill (9.5)
Here, and in the following, the sign of approximate equality refers to the conditions, discussed in Section 6, where plasma effects are small and eqs.9.1 or 9.2 apply. We see that a measurement of $R$ provides $M_{11}$ and, for small plasma effects, the quantity $(W_2^2 + W_3^2)$ which reduces to $W_3^2$ when by symmetry $W_2 = 0$. Therefore one measures the Faraday effect, quadratically. If we choose instead $\gamma = \theta = \pm \pi/4$, namely again no retarder and the polarizer at $\pm 45^\circ$ to the input polarization (see figs.8b and 8c), we have $A = C = 0$ and $B = \pm 1$ so that

$$R_{\pm} = 1 \pm M_{21} = 1 \pm W_3.$$  \hspace{1cm} (9.6)

If one arranges to measure both $R_+$ and $R_-$ (and this can be done, see fig.9, by using a beam splitter and two analysers [7]) one has $(R_+ - R_-) = 2M_{21} = 2W_3$ and so one measures the Faraday effect, linearly.

If the input Stokes vector $s_0 = (0, 1, 0)$ is taken, then we have from eq.8.5
Fig. 10 Measurement configuration for $s_0=(0,1,0)$ and a) $\theta=0$, b) $\theta=\pi/2$, c) $\gamma=0$, $\theta=\pi/4$, d) $\gamma=0$, $\theta=-\pi/4$. E denotes the electric field of the input radiation, R denotes a quarter-wave retarder and P a polarizer

\begin{equation}
R = 1 + K = 1 + AM_{12} + BM_{22} + CM_{32}
\end{equation}

and, if we choose $\gamma=\theta=0$ or $\pi/2$ (see figs.10a and 10b), then $B=C=0$ and $A=\pm1$ so that
\[ R_\pm = 1 \pm M_{12} = 1 \mp W_3 \]  \hspace{1cm} (9.8)

and again the Faraday effect is measured. If instead we choose \( \theta - \gamma = \pm \pi/4 \) (the case \( \gamma = 0, \theta = \pm \pi/4 \) is shown in figs.10c and 10d), then \( A = B = 0 \) and \( C = \pm 1 \) so that

\[ R_\pm = 1 \pm M_{32} = 1 \pm W_1 \]  \hspace{1cm} (9.9)

and now the Cotton-Mouton effect is measured. The quantity \( W_1 \), which is relevant when plasma effects are small, is given by eq.6.6 and, as we shall see, it is important in tokamak polarimetry because it provides a measurement of the electron line density.

From the examples given above it is clear that, by a convenient choice of the input polarization \( s_0 \) and of the analyser parameters \( \gamma \) and \( \theta \), it is possible to measure any one of the elements of the plasma transition matrix \( M \). Indeed, from eqs.8.5 and 8.6, by choosing \( s_0 \) conveniently one can determine \( J \) or \( K \) or \( L \) and, from eqs.8.3, one can have \([A=1, B=C=0]\) or \([B=1, A=C=0]\) or \([C=1, A=B=0]\) so that (see eqs 8.6) any chosen element of \( M \) is determined. In general these elements are related to plasma quantities through a complex integration (of eqs.4.19 or 4.23) however when plasma effects are small the elements of \( M \) have the simple analytic expressions recalled in eqs.9.1 and 9.2. More than one element can be measured simultaneously, by using beam splitters and more than one analyser (see fig.9 and [7]). The measurements discussed here have the disadvantages that they rely on measurements of amplitudes, that two signals are required for each element to be measured (one being possibly a monitor of the input intensity) and that measurements are made in the frequency band of the time evolution of plasma quantities. All these features imply significant contributions to the noise. As we shall see in Section 9.1.1 the last feature can be improved without applying modulation to the state of polarization. All the features are improved by modulation of the polarization (see Sections 9.2 and 9.3).

9.1.1 Modulation of the polarimetric signal without modulation of the input polarization

Even without modulating the state of polarization of the input radiation, it is possible to have a high-frequency modulation on the signal to be measured. This can be obtained by having, besides the probing beam at a frequency \( \omega \), also a reference beam at a frequency \( \omega + \delta \omega \), and mixing the two beams on the detector. Since this is quadratic, a harmonic component at a frequency \( \delta \omega \) appears on the signal, whose amplitude is proportional to the square root of the product of the intensities of probing and reference beams. This method is very convenient (and has been used extensively [51,52]) when the polarimeter is combined with an interferometer, which in any case requires a reference beam at a shifted frequency.

It should be noted however that, if the input state of polarization is not modulated, the polarimetric information appears in the form of an amplitude, and not of a phase. A more detailed discussion is given in Section 11.
9.2 Measurements with progressive modulation of the state of polarization

We will now discuss measurements with progressive modulation and we will put particular stress on the possibility, in this case, of performing measurements of phase rather than amplitude, in contrast to the case for alternating modulation.

9.2.1 Modulation of the input

We will begin by discussing progressive modulation applied to the input radiation before crossing the plasma and first we will consider the general case where we may be interested in any of the components of $M$. Later we will consider a particular case of special treatment.

9.2.1.1 General treatment

We shall consider three kinds of progressive modulation of the input polarization. The first kind corresponds to modulation methods I or IIa of Section 8 and we will consider the input configurations U1, U2 and U3 which are of this kind. The second kind corresponds to modulation method IIb of Section 8 and again we will consider configurations V1, V2 and V3 of this second kind. Finally the third kind corresponds to modulation method II (without rotating retarder) and here we will consider the configurations W1, W2 and W3.

Configurations U1, U2 and U3. These configurations can be produced either by modulation method I of Section 8.2.1 (low frequency) or by modulation method IIa. Let us first consider the configurations produced by method I, which are illustrated in fig.11. The configuration U1 is produced by sending linearly polarized monochromatic radiation (whose electric field is oriented at an angle $\theta$ to the x axis) first through a fixed quarter-wave retardation plate ($\rho = \pi/2$) having retardation axis aligned with the x axis and then through a second rotating quarter-wave plate whose retardation axis makes an angle $\alpha = \omega_0 t$ with the x axis. The Stokes vector of the radiation entering the rotating retarder is $s = (\cos \theta, 0, -\sin \theta)$ and we are in the conditions corresponding to the modulation method I of Section 8.2.1. Therefore, if we put $p = s_1 = \cos \theta$, $q = s_3 = -\sin \theta$, from eq.8.9 we have, for the Stokes vector $s_{ou1}$ produced by configuration U1,

$$s_{ou1} = \frac{1}{2} \begin{bmatrix} p - 2q\sin 2\alpha + p\cos 4\alpha \\ 2q\cos 2\alpha + psin 4\alpha \\ 2p\sin 2\alpha \end{bmatrix} \quad (9.10)$$

The configurations U2 and U3 are produced by sending radiation U1 through a third quarter-wave plate, respectively aligned with the x axis and at 45° to the x axis. Therefore if we put (see eqs.3.6 and 3.7)
Fig. 11 Configurations U1, U2 and U3, using Method I for low-frequency modulation of the input polarization.

\[ R_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}; \quad R_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix}; \quad R_3 = \begin{bmatrix} 0 & 0 & -1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \tag{9.11} \]

we can write

\[ \mathbf{soul}_1 = R_1 \cdot \mathbf{soul}_1, \quad \mathbf{soul}_2 = R_2 \cdot \mathbf{soul}_1, \quad \mathbf{soul}_3 = R_3 \cdot \mathbf{soul}_1 \tag{9.12} \]

and so we have
\[ \mathbf{sou}_2 = \frac{1}{2} \begin{bmatrix} p - 2q\sin^2\alpha + p\cos^4\alpha \\ 2p\sin^2\alpha \\ - 2q\cos^2\alpha - p\sin^4\alpha \end{bmatrix} \] (9.13)

\[ \mathbf{sou}_3 = \frac{1}{2} \begin{bmatrix} - 2p\sin^2\alpha \\ 2q\cos^2\alpha + p\sin^4\alpha \\ p - 2q\sin^2\alpha + p\cos^4\alpha \end{bmatrix} \] (9.14)

If we now define the vectors \( \mathbf{f}_2 \) and \( \mathbf{f}_4 \) and the matrices \( \mathbf{D}_2 \) and \( \mathbf{D}_4 \) as

\[ \mathbf{f}_2 = (0, \cos^2\alpha, \sin^2\alpha), \quad \mathbf{f}_4 = (1, \cos^4\alpha, \sin^4\alpha) \] (9.15)

and

\[ \mathbf{D}_2 = q \begin{bmatrix} 0 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & p/q \end{bmatrix}, \quad \mathbf{D}_4 = p/2 \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \] (9.16)

we can write

\[ \mathbf{sou}_1 = \mathbf{D}_2 \cdot \mathbf{f}_2 + \mathbf{D}_4 \cdot \mathbf{f}_4 \] (9.17)

and so from eq.9.12, for \( j=1, 2 \) and \( 3 \), we have

\[ \mathbf{sou}_j = R_j \cdot \mathbf{sou}_1 = (R_j \cdot \mathbf{D}_2) \cdot \mathbf{f}_2 + (R_j \cdot \mathbf{D}_4) \cdot \mathbf{f}_4 \] . (9.18)

where \( \mathbf{f}_2 \) contains the modulation at \( 2\omega_0 \) and \( \mathbf{f}_4 \) the modulation at \( 4\omega_0 \).

If the radiation of configurations U1, U2 or U3 is sent through a plasma with transition matrix \( \mathbf{M} \), the corresponding output Stokes vectors, \( \mathbf{sou}_j \) (for \( j=1,2,3 \)), are

\[ \mathbf{sou}_j = \mathbf{M} \cdot \mathbf{sou}_j \]

\[ = \mathbf{M} \cdot [R_j \cdot (\mathbf{D}_2 \cdot \mathbf{f}_2)] + \mathbf{M} \cdot [R_j \cdot (\mathbf{D}_4 \cdot \mathbf{f}_4)] \]

\[ = (M \cdot R_j) \cdot (\mathbf{D}_2 \cdot \mathbf{f}_2) + (M \cdot R_j) \cdot (\mathbf{D}_4 \cdot \mathbf{f}_4) \]
\[ M_j = M \cdot R_j, \quad G_{2j} = M_j \cdot D_2, \quad G_{4j} = M_j \cdot D_4 \] (9.20)

It can be seen that \( M_1 \equiv M \), while \( M_2 \) is obtained from \( M \) by exchanging columns 2 and 3 and then changing sign of column 2, \( M_3 \) is obtained from \( M \) by exchanging columns 1 and 2 and then changing sign of column 3, so that \( M_1, M_2 \) and \( M_3 \) contain the same elements, in absolute value, as \( M \), but in changed positions.

From eqs.9.15 and 9.19, the k-th component of \( s_{uj} \) is given by

\[ (s_{uj})_k = (G_{4j})_{k1} + (G_{2j})_{k2} \cos^2 \alpha + (G_{2j})_{k3} \sin^2 \alpha + (G_{4j})_{k2} \cos^4 \alpha + (G_{4j})_{k3} \sin^4 \alpha \]

\[ = P_{ujk} + Q_{uj2k} \sin(2\omega_0 t + \Phi_{uj2k}) + Q_{uj4k} \sin(4\omega_0 t + \Phi_{uj4k}) \] (9.21)

where we have put

\[ \tan \Phi_{uj2k} = \frac{(G_{2j})_{k2}}{(G_{2j})_{k3}} = \frac{(M_j)_{km} (D_2)_{m2}}{(M_j)_{km} (D_2)_{m3}} = \frac{(M_j)_{k2}}{(p/q)(M_j)_{k3} - (M_j)_{k1}} \] (9.22)

\[ \tan \Phi_{uj4k} = \frac{(G_{4j})_{k2}}{(G_{4j})_{k3}} = \frac{(M_j)_{km} (D_4)_{m2}}{(M_j)_{km} (D_4)_{m3}} = \frac{(M_j)_{k1}}{(M_j)_{k2}} \] (9.23)

Here, and in the following, a repeated index on matrix elements indicates summation over this index. In these expressions, \( (M_j)_{k1}, (M_j)_{k2} \) and \( (M_j)_{k3} \) are not independent; they are related (see eq.4.20) by

\[ (M_j)_{k1}^2 + (M_j)_{k2}^2 + (M_j)_{k3}^2 = 1. \] (9.24)

The expressions for \( P_{ujk}, Q_{uj2k} \) and \( Q_{uj4k} \) are easily found.

Using a very simple output analyser, as discussed in Section 8.1, it is possible to obtain a signal proportional to \( (1 + s_1), (1 + s_2) \) or \( (1 + s_3) \), i.e. in the present case to \( 1 + (s_{uj})_k \) for \( k = 1, 2 \) or 3. By a convenient filtering of such a signal (see eq.9.21), one can measure both the phase
$\Phi_{uj2k}$ of the $2\omega_0$ component and the phase $\Phi_{uj4k}$ of the $4\omega_0$ component. Therefore from eqs. 9.22 to 9.24, it is possible to determine $(M_j)_{k1}$, $(M_j)_{k2}$ and $(M_j)_{k3}$ and if this is repeated for $k=1, 2, 3$ (i.e. the analyser configuration is changed) the entire matrix $M_j$, and hence $M$, is determined. This procedure can be repeated for $j=1, 2,$ and $3$, i.e. for input configurations U1, U2 and U3, and so, by selecting one of these, a transposed matrix $M_j$ is selected and the roles of the elements of $M$ in the expressions for $\Phi_{uj2k}$ and $\Phi_{uj4k}$ can be exchanged. The value of $\omega_0$ should always be sufficiently large that $\pi/2\omega_0$ is small compared to the characteristic time of variation of $M$.

In this discussion of configurations U1, U2 and U3 we have referred to method I where the modulation is made by a rotating retarder. However, as described in Section 8.2.1, the same analysis can be used for method IIb (which produces high-frequency modulation) by using the correspondence of eq. 8.14. The high-frequency version of configurations U1, U2 and U3 is illustrated in fig. 12.

\[ \text{Fig. 12 Configurations U1, U2 and U3, using Method IIa for high-frequency modulation of the input polarization} \]
Fig. 13 Configurations V1, V2 and V3 for modulation of the input polarization

It is clear that configurations U1, U2, U3, produced by method I, constitute a special case where the retarder has a retardation $\delta = \pi/2$. Also for other values of $\delta$ the two frequencies $2\omega_0$ and $4\omega_0$ are present, with other amplitudes. However, for $\delta = \pi$, only the frequency $4\omega_0$ occurs and this value of $\delta$ can be used (i.e. a half-wave retarder) when only one polarimetric phase measurement is required (see e.g. [33]). In this case we have essentially the configurations W1, W2, W3, to be discussed later.

Configurations V1, V2 and V3. These configurations are illustrated in fig.13. The configuration V1 is produced by combining linearly polarized radiation at a frequency $\omega$, having the electric field parallel to the x axis, $E_x = a \cos \omega t$, together with coherent linearly polarized radiation at a frequency $(\omega + \delta \omega)$, having the electric field parallel to the y axis, $E_y = b \cos(\omega + \delta \omega) t$. The radiation then passes through a rotating half-wave retardation plate ($\pi$ relative delay) whose retardation axis makes an angle $\beta = \omega t$ with the x axis. This is the configuration we considered for modulation method IIb in Section 8.2.1 and so, from eq.8.15, the Stokes vector $s_{OV1}$ produced by configuration V1 is given by
where the constants $g$ and $h$ are $g=\frac{a^2-b^2}{a^2+b^2}$ and $h=\frac{2ab}{a^2+b^2}$.

The configurations $V_2$ and $V_3$ are produced by sending radiation $V_1$ through a third quarter-wave plate, respectively aligned with the $x$ axis and at 45° to the $x$ axis. Therefore the Stokes vectors of configurations $V_2$ and $V_3$ are given respectively (see eqs. 3.6, 3.7) by

$$S_{ov2} = g \cos 4\beta + h \sin 4\beta \cos \delta \omega t$$

$$-g \sin 4\beta + h \cos 4\beta \cos \delta \omega t$$

$$S_{ov3} = h \sin 5\delta \omega t$$

$$g \sin 4\beta - h \cos 4\beta \cos \delta \omega t$$

$$g \cos 4\beta + h \sin 4\beta \cos \delta \omega t$$

All the configurations $V_1$, $V_2$ and $V_3$ provide an input polarization having a double time-modulation and we will be interested in the case where $4\omega_r \ll \delta \omega$, so that $\beta$ represents a slow modulation and $\delta \omega t$ a fast one, but the expressions given in this Section are valid for arbitrary values.

It can be verified that, for each of $V_1$, $V_2$ and $V_3$, as $\beta$ increases, the polarization alternates successively (for $\beta=\pi/4$ and $\beta=(2n+1)\pi/8$ with $n=0, 1, 2, \ldots$) between two of the three configurations $W_1$, $W_2$ and $W_3$ which will be introduced later (see eqs. 9.35 to 9.37). Specifically, apart from an inessential phase, $V_1$ alternates between configurations $W_1$ and $W_2$; $V_2$ alternates between $W_1$ and $W_3$; $V_3$ alternates between $W_2$ and $W_3$.

The frequency shift $\delta \omega$ can be generated by one of the various methods we have discussed in Section 8.2.1.

If we now define the matrix $D$ and the vector $f$ by
\[ D = \begin{bmatrix} g \cos 4\beta & h \sin 4\beta & 0 \\ g \sin 4\beta & -h \cos 4\beta & 0 \\ 0 & 0 & -h \end{bmatrix}, \quad f = \begin{bmatrix} 1 \\ \cos \delta \omega t \\ \sin \delta \omega t \end{bmatrix} \] (9.28)

we can write

\[ s_{ov1} = D \cdot f \] (9.29)

and so, for \( j = 1, 2 \) and \( 3 \), we have

\[ s_{ovj} = R_j \cdot s_{ov1} = (R_j \cdot D) \cdot f \] (9.30)

If the radiation of configurations \( V_1, V_2 \) or \( V_3 \) is sent through a plasma with transition matrix \( M \), the corresponding output Stokes vectors, \( s_{vj} \) (for \( j = 1, 2, 3 \)), are

\[ s_{vj} = M \cdot s_{ovj} = M \cdot [R_j \cdot (D) \cdot f] = (M \cdot R_j \cdot (D) \cdot f) \]
\[ = ([M \cdot R_j] \cdot D) \cdot f = [M_j \cdot D] \cdot f = G_j \cdot f \] (9.31)

where we have put

\[ G_j = M_j \cdot D, \] (9.32)

and \( M_j \) is given by eq. 9.20. Then, using eqs. 13, the \( k \)-th component of \( s_{vj} \) is given by

\[ (s_{vj})_k = (G_j)_k1 + (G_j)_k2 \cos \delta \omega t + (G_j)_k3 \sin \delta \omega t = P_{vj} + Q_{vj} \sin(\delta \omega t + \Phi_{vj}) \] (9.33)

where we have put

\[ \tan \Phi_{vj} = \frac{(G_j)_k2}{(G_j)_k3} = \frac{(M_j)_km(D)m2}{(M_j)_km(D)m3} = \frac{(M_j)_k2}{(M_j)_k3} \cos 4\omega rt - \frac{(M_j)_k1}{(M_j)_k3} \sin 4\omega rt \] (9.34)

The expressions for \( P_{vj} \) and \( Q_{vj} \) are easily found. The quantities \( P_{vj}, Q_{vj} \) and \( \Phi_{vj} \), in eq. 9.33, are all functions of time since they contain the elements of \( D \), however, if \( 4\omega r \ll \delta \omega \), they change very slowly compared to the \( \delta \omega \) modulation and, for short times, they are approximately constant.
Using a very simple output analyser, as discussed in Section 8.1, it is possible to obtain a
signal proportional to \((1+s_1), (1+s_2)\) or \((1+s_3)\), i.e. to \(1+(svj)k\) for \(k=1,2,3\) in the present
case. On such a signal it is possible to measure the (slowly varying) phase \(\Phi_{vjk}(t)\), which tends
to be independent of any amplitude changes due to fluctuations of the input intensity or to
refraction. From the measurement of \(\Phi_{vjk}\), using eqs.9.34 and 9.24, it is possible to determine
\((Mj)k1, (Mj)k2\) and \((Mj)k3\) and if this is repeated for \(k=1,2,3\) (ie the analyser configuration is
changed) the entire matrix \(M_j\), and hence \(M\), is determined. This procedure can be repeated for
\(j=1,2,3\), i.e. for input configurations \(V1, V2\) and \(V3\), and again, selecting one of these, a
transposed matrix \(M^T_j\) is selected and the roles of the elements of \(M\) in the expression for \(\Phi_{vjk}\)
again can be exchanged.

The condition \(\delta\omega \gg \omega_r\) can be satisfied readily (see Section 8.2.1). However \(\omega_r\) should be
sufficiently large that \(\pi/2\omega_r\) is small compared to the characteristic time of variation of \(M\).

Configurations \(W1, W2\) and \(W3\). These configurations can be produced either in a
high-frequency version or in a low-frequency one. The high-frequency version is illustrated in
fig.14. The configuration \(W1\) is produced by combining linearly polarized radiation at a
frequency \(\omega\), having the electric field parallel to the x axis, \(E_x=a \cos \omega t\), together with coherent
linearly polarized radiation at a frequency \((\omega+\delta\omega)\), having the electric field parallel to the y axis,
\(E_y=b \cos(\omega+\delta\omega) t\). This is is the configuration we considered for modulation method II in
Section 8.2.1, with \(\mu=0\) and \(\nu=\pi/2\), and so (see eq.8.11) the Stokes vector \(s_{W1}\) produced by
configuration \(W1\) is given by

\[
s_{W1} = \begin{bmatrix} g \\ h \cos \delta \omega t \\ h \sin \delta \omega t \end{bmatrix} \tag{9.35}
\]

The configuration \(W2\) is produced by sending radiation \(W1\) through a half-wave retarder \((\rho=\pi)\)
whose axis is at an angle \(\pi/8\) to the x direction and so its Stokes vector is

\[
s = \begin{bmatrix} h \cos \delta \omega t \\ g \\ -h \sin \delta \omega t \end{bmatrix} \tag{9.36}
\]

The configuration \(W3\) is produced by sending radiation \(W2\) through a quarter-wave retarder
\((\rho=\pi/2)\) whose axis is aligned with the x direction, and so its Stokes vector is
A low-frequency version of configurations W1, W2 and W3 can also be produced by using a source at frequency \( \omega \), together with a rotating half-wave retarder. Indeed, if we send linearly polarized monochromatic radiation (whose electric field is oriented at an angle \( \theta \) to the x axis) first through a fixed quarter-wave retarder having retardation axis aligned with the x axis and then through a second rotating half-wave retarder whose retardation axis makes an angle \( \gamma = \omega_1 t \) with the x axis (see fig.15), the resulting radiation has \( s = (h \cos \omega_1 t, h \sin \omega_1 t, -g) \) where \( g = -\sin 2\theta \) and \( h = \cos 2\theta \), as can be verified using eqs.3.6 and 3.8. This is the Stokes vector of
configuration W3, namely $s_{ow3}$, with $-4\omega_1$ instead of $\delta\omega$ (see eq.9.37). By adding a fixed quarter-wave retarder at 0 or 45° to the x axis, configurations W2 or W1 respectively are produced. In all three cases if it is required to have $g=0$ and $h=1$, one can simply omit the first quarter-wave retarder and then $\theta$ only introduces a constant phase.

It can be verified (by comparing figs. 11 and 15) that the low-frequency versions of configurations W1, W2 and W3 coincide with the configurations U3, U2 and U1 produced by method I, when the rotating quarter-wave retarder is replaced by a rotating half-wave retarder.

From eqs. 9.35-9.37 it can be verified that, if we call

$$
N_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}; \quad N_2 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix}; \quad N_3 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & -1 \\ -1 & 0 & 0 \end{bmatrix}
$$

(9.38)
then we can write the three Stokes vectors (for \( j=1,2,3 \)) as

\[ s_{owj} = N_j \cdot s_{ow1} \]  

(9.39)

Obviously this expression (and all the following) are valid both for the high-frequency and for the low-frequency versions of \( W_1, W_2, W_3 \).

If the radiation of configurations \( W_1, W_2 \) or \( W_3 \) is sent through a plasma with transition matrix \( M \), the corresponding output Stokes vectors, \( s_{wj} \) (for \( j=1,2,3 \)), are

\[ s_{wj} = M \cdot s_{owj} = M \cdot N_j \cdot s_{ow1} = H_j \cdot s_{ow1} \]  

(9.40)

where we have called

\[ H_j = M \cdot N_j \]  

(9.41)

Again it can be seen that \( H_1 = M \), while \( H_2 \) and \( H_3 \) are obtained from \( M \) by exchanging some columns and changing some signs, so that \( H_1, H_2 \) and \( H_3 \) contain the same elements, in absolute value, as \( M \) but possibly in changed positions and, from eq.4.20,

\[ (H_j)_{k1}^2 + (H_j)_{k2}^2 + (H_j)_{k3}^2 = 1. \]  

(9.42)

Then, using eqs.9.35, 9.39 and 9.40, the k-th component of \( s_{wj} \) is given by

\[ (s_{wj})_k = (H_j)_{kn} (s_{ow1})_n \]

\[ = g(H_j)_{k1} + h(H_j)_{k2} \cos \delta \omega t + h(H_j)_{k3} \sin \delta \omega t \]

\[ = g(H_j)_{k1} + hQ_{wj} \sin(\delta \omega t + \Phi_{wj} ) \]  

(9.43)

where we have put

\[ \tan \Phi_{wj} = \frac{(H_j)_{k2}}{(H_j)_{k3}} \]  

(9.44)

\[ Q_{wj}^2 = (H_j)_{k2}^2 + (H_j)_{k3}^2 \]  

(9.45)
By using the output analysers discussed in Section 8.1, again it is possible to measure
\(1+(s w j)k\) and to determine the phase \(\Phi_{wjk}\) and the amplitude ratio, \(hQ_{wjk}/[1+g(H_j)k]\), of the 
\(\delta\omega\) and DC components. Therefore, from eqs.9.42, 9.44, 9.45, it is possible to determine 
\((H_j)_k\), \((H_j)_k^2\) and \((H_j)_k^3\), and if this is repeated for \(k=1,2,3\) (ie the analyser configuration is 
changed) the entire matrix \(H_j\), and hence \(M\), is determined.

This procedure can be repeated for \(j=1, 2\) and 3, i.e. for input configurations \(W_1, W_2\) and \(W_3\),
and so, by selecting one of these, a transposed matrix \(H_j\) is selected and the roles of the 
elements of \(M\) in the expressions for \(\Phi_{wjk}\) and \(Q_{wjk}\) can be exchanged.

9.2.1.2 Cases of special interest

We shall now consider some cases of special interest, namely the input configurations \(U_1, V_2, \ W_1, W_2\) and \(W_3\) (see Section 9.2.1.1), together with an output analyser which selects \(s_2\) (see 
Section 8.1) so that the corresponding analyser signals are \([1+(s u_1)_2]\), \([1+(s v_2)_2]\) and 
\([1+(s w j)_2]\), with \(j=1,2,3\). It should be recalled that configuration \(U_1\) can be produced either at 
low frequency as in fig.11 or at high frequency as in fig.12. Thus all the configurations of this 
Section (and indeed all those of Section 9.2.1) can be produced at high frequency.

For configuration \(U_1\), from eqs.9.21 to 9.23 we have

\[(s u_1)_2 = P u_{12} - Q u_{122} \cos(2\omega t + \Phi_2) + Q u_{142} \sin(4\omega t + \Phi_4)\]  (9.46)

with

\[\tan \Phi_2 = \frac{1}{\tan \Phi u_{122}} = \frac{q M_{23}}{p M_{22}^2} \frac{M_{21}}{M_{22}}; \quad \tan \Phi_4 = \tan \Phi u_{142} = \frac{M_{21}}{M_{22}}\]

\[P u_{12} = p M_{21}/2; \quad Q u_{122} = [(p M_{23} - q M_{21})^2 + q^2 M_{22}^2]^{1/2}; \quad Q u_{142} = [M_{21}^2 + M_{22}^2]^{1/2} p/2\]  (9.47)

For configuration \(V_2\), from eqs.9.33 and 9.34 we have

\[(s v_2)_2 = P v_{22} + Q v_{22} \sin(\delta \omega t + \Phi)\]  (9.48)

with
\[
\tan \Phi = \tan \Phi_{22} = -\frac{M_{23}}{M_{22}} \cos 4\omega t - \frac{M_{21}}{M_{22}} \sin 4\omega t
\]

\[
P_{V22} = g \left( M_{21} \cos 4\omega t - M_{23} \sin 4\omega t \right)
\]

\[
Q_{v22} = -h \left[ M_{22}^2 + (M_{21} \sin 4\omega t + M_{23} \cos 4\omega t)^2 \right]^{1/2}
\]

For configurations \(W_1, W_2\) and \(W_3\), from eqs. 9.43 to 9.45 we have

\[
(s_{wj})^2 = g(H_j)^2 + hQ_{wj2} \sin (\delta \omega t + \Phi_{wj2})
\]

where, for \(j=1, 2\) and 3 respectively,

\[
(H_1)^{21} = M_{21}; \quad Q_{w12} = (M_{22}^2 + M_{23}^2)^{1/2}; \quad \tan \Phi_{w12} = \frac{M_{22}}{M_{23}}
\]

\[
(H_2)^{21} = M_{22}; \quad Q_{w22} = (M_{21}^2 + M_{23}^2)^{1/2}; \quad \tan \Phi_{w22} = \frac{M_{21}}{M_{23}}
\]

\[
(H_3)^{21} = -M_{23}; \quad Q_{w32} = -(M_{21}^2 + M_{22}^2)^{1/2}; \quad \tan \Phi_{w32} = \frac{M_{21}}{M_{22}}
\]

The expressions given in this Section, which are valid for plasma effects of arbitrary size, are particularly significant. This can be seen by by considering them for the case of small plasma effects, when \(M_{21} = W_3, M_{23} = -W_1, M_{22} = 1 - (W_1^2 + W_3^2)/2 \approx 1\) (see eq. 6.17). In this case, for configuration \(U_1\), eqs.9.47 become

\[
\tan \Phi_2 = \frac{p}{q} W_1 + W_3; \quad \tan \Phi_4 = W_3; \quad P_{u12} = p W_3/2;
\]

\[
Q_{u122} = [(pW_1 + qW_3)^2 + q^2]^{1/2}; \quad Q_{u142} = [1 + W_3^2]^{1/2} p/2.
\]

For configuration \(V_2\), the eqs.9.49 become
\[ \tan \Phi(t) = W_1 \cos 4\omega t - W_3 \sin 4\omega t \]

\[ P_{V22} = g(W_3 \cos 4\omega t + W_1 \sin 4\omega t) \quad (9.55) \]

\[ Q_{V22} = -h [1 + (W_3 \sin 4\omega t - W_1 \cos 4\omega t)^2]^{1/2} \]

For configurations \( W_1, W_2 \) and \( W_3 \), the eqs. 9.51 to 9.53 become, for \( j = 1, 2 \) and 3 respectively,

\[ (H_1)_{21} = W_3; \quad Q_{W12} = (1 + W_1^2)^{1/2} = 1; \quad \tan \Phi_{W12} = -\frac{1}{W_1} \quad (9.56) \]

\[ (H_2)_{21} = 1; \quad Q_{W22} = (W_3^2 + W_1^2)^{1/2}; \quad \tan \Phi_{W22} = \frac{W_3}{W_1} \quad (9.57) \]

\[ (H_3)_{21} = W_1; \quad Q_{W32} = -(1 + W_3^2)^{1/2} = -1; \quad \tan \Phi_{W32} = -W_3 \quad (9.58) \]

We conclude therefore that one can determine both \( W_3 \) and \( W_1 \), i.e. both the Faraday effect and the Cotton-Mouton effect (see eqs. 6.6 and 6.8), either by using the input configuration \( U_1 \) and measuring the phases \( \Phi_2 \) and \( \Phi_4 \) or by using the input \( V_2 \) and measuring \( \Phi(t) \) or by using one of the input configurations \( W_1, W_2 \) or \( W_3 \) and measuring phases and amplitudes, as discussed in ref [12]. We also see the significance of the particular configurations considered in this Section.

We recall that configurations \( W_1, W_2 \) and \( W_3 \) can be realized either in the high-frequency version or in the low-frequency one. The low-frequency version of \( W_3 \) was used in refs. [33, 34].

### 9.2.2 Modulation of the output

It is also possible to obtain phase measurements of the elements of \( M \) by using a constant input state of polarization and modulating the output polarization [12]. In order to illustrate this, let us consider the constant input Stokes vector \( \mathbf{s}_0 = (0, 1, 0) \), corresponding to linearly polarized radiation oriented at 45° to the x axis. Then, after crossing the plasma, we have \( \mathbf{s}_1 = M \mathbf{s}_0 = (M_{12}, M_{22}, M_{32}) \). Let us consider an analyser (see fig.16) consisting of a rotating quarter-wave retarder (whose retardation axis makes an angle \( \alpha = \omega_0 t \) with the x axis), followed by a fixed quarter-wave retarder (having retardation axis aligned with the x axis), followed in turn by a fixed linear polarizer at an angle \( \theta \) to the x axis. Then, if the input radiation intensity is \( I_0 \), it can be shown that the intensity \( I_1 \) after the polarizer is given by
Fig. 16 Analyser for modulation of the output polarization

\[
\frac{4I_1}{I_0} = 2 + pM_{12} + p(M_{22} \sin 4\omega_0 t + M_{12} \cos 4\omega_0 t)
- 2q[M_{22} \cos 2\omega_0 t - (M_{12} + pM_{32}/q) \sin 2\omega_0 t]
\] (9.59)

where \(p = \cos 2\theta\), \(q = \sin 2\theta\), and \(\theta\) can be chosen so as to optimise the measurement. (The expression for \(I_1\) is similar to eq.9.46 with eq.9.47, where \(M_{21}\) is replaced by \(M_{12}\) and \(M_{23}\) by \(-M_{32}\)). We see that, by measuring this intensity, from the phases of the components at \(4\omega_0\) and \(2\omega_0\), one can determine the quantities \(M_{12}/M_{22}\) and \((M_{12} + pM_{32}/q)/M_{22}\), i.e. \(M_{12}/M_{22}\) and \(M_{32}/M_{22}\). For small plasma effects, \(M_{12} \approx -W_3\), \(M_{22} \approx 1\), \(M_{32} = W_1\) (see eq.6.17) and so again one determines \(W_3\) and \(W_1\).

By using as input \(s_0 = (1, 0, 0)\) or \(s_0 = (0, 0, 1)\) the other elements of \(M\) are determined.

When modulation of the output polarization is used, one needs as many modulators as the measurement lines operating in parallel. Therefore for a multichannel system the instrument becomes very complex.

9.2.3 Modulation of both input and output

It is also possible to use at the same time input and output modulation of the state of polarization. In this case the signal on a single detector contains many harmonic components (at least 12). It can be shown (see [14]) that, if we call \(m_{jk} = M_{jk}/M_{11}\), then by measuring the phases of 8 of these components it is possible to determine all the \(m_{jk}\) and by measuring an amplitude also \(M_{11}\) is determined. Thus the full plasma transition matrix \(M\) is determined essentially by phase measurements on a single detector signal. We have recalled this briefly for completeness and the details can be found in ref.[14] (where a different convention is used for the sign of retardation \(\rho\), which is opposite to that used in this text).

9.3 Measurements with alternating modulation of the state of polarization

In order to illustrate measurements using alternating modulation, described in Section 8.2.2, let us consider a constant input Stokes vector \(s_0 = (0, 1, 0)\), corresponding to linearly polarized
radiation oriented at 45° to the x axis. Then, after crossing the plasma, we have \( s_1 = M s_0 = (M_{12}, M_{22}, M_{32}) \). Let us now suppose that after the plasma we have a photoelastic modulator (PEM), whose axis is oriented at 45° to the x axis, followed by a polarizer with axis aligned with the x axis and then by a detector. If \( \rho = \rho_0 \sin \omega t \) is the retardation of the PEM, its Mueller matrix (see eq. 3.4) is given by

\[
M_r(\rho, \pi/4) = \begin{bmatrix}
\cos \rho & 0 & -\sin \rho \\
0 & 1 & 0 \\
\sin \rho & 0 & \cos \rho
\end{bmatrix}
\]

and then it is found that the intensity on the detector is

\[
I + (s_1)_1 = M_{12} \cos \rho - M_{32} \sin \rho
\]

Therefore from eqs. 8.17 we find for amplitude ratios of the \( \omega_m \), the \( 2\omega_m \) and the DC components of the signal

\[
\frac{I_{\omega}}{I_{DC}} = \frac{2M_{32}J_1(\rho_0)}{1 + M_{12}J_0(\rho_0)} ; \quad \frac{I_{2\omega}}{I_{DC}} = \frac{2M_{12}J_2(\rho_0)}{1 + M_{12}J_0(\rho_0)}
\]

and so from a measurement of these amplitudes one can determine \( M_{12} \) and \( M_{32} \). Again these expressions become simpler if the modulation amplitude \( \rho_0 \) is adjusted (\( \rho_0 = 2.405 \)) so that \( J_0 = 0, J_1 = 0.519, J_2 = 0.432 \). If plasma effects are small (see eq. 6.17), \( M_{12} = -W_3, M_{32} = W_1 \) and so again one determines \( W_3 \) and \( W_1 \).

As discussed in Section 8.2.2, this technique can provide very accurate measurements, however there is the problem that today the required modulators exist only for wavelengths at the near infrared (about 10 \( \mu \)m) or shorter, and at these wavelengths the Faraday and Cotton-Mouton effects are often extremely small for usual laboratory plasmas.

10. EFFECTS OF REFRACTION

In our discussion up to this point we have neglected the effects of transverse gradients of the refractive index on wave propagation (see Section 4) and so we have neglected refraction. We will now take it into account and consider the ray path, in general, a three-dimensional curve (see fig. 17) given by \( r(w) = (x(w), y(w), z(w)) \), where \( w \) is an abscissa along the ray. Then the unit vector, \( t \), tangent to the ray is given by

\[
\frac{dr}{dw} = t
\]
In the WKB approximation, where the wavelength is much smaller than the characteristic length of gradients of the refractive index, $\mu(x, y, z)$ of the medium, the equation for the evolution of $t$ (see [53, 54]) is given by

$$\frac{dt}{dw} = \frac{1}{\mu} (\nabla \parallel \mu) = \frac{1}{\mu} [\nabla \mu - (\nabla \mu \cdot t) t]$$

(10.2)

where $\nabla \mu = (\partial \mu / \partial x, \partial \mu / \partial y, \partial \mu / \partial z)$ or equivalently by $d(\mu t)/dw = \nabla \mu$. The eqs. 10.1 and 10.2 constitute the ray-tracing equations and, when $\mu(x, y, z)$ is given, they can be integrated, starting from an initial position $r_0$ with an initial direction $t_0$, so as to provide the ray trajectory $r(w)$, and in particular the output position and direction after traversing the medium.

For a plasma the refractive indexes of the slow and fast characteristic waves, $\mu_1$ and $\mu_2$, are given by eq. 4.1 and we will consider the case where $\omega_c^2 \ll \omega^2$, so that for the purpose of ray-tracing $\mu_1 = \mu_2 = \mu = (1 - \omega_p^2/\omega^2)^{1/2}$. In this case the rays of both characteristic waves coincide and we can continue to use the analysis of polarization evolution given above, with the difference that now all the integrations must be carried out along $w$ with a changing direction $t$ instead of along an unchanging propagation direction. The ray trajectory in general is obtained by integrating the eqs. 10.1 and 10.2 with $\mu^2 = 1 - \omega_p^2/\omega^2$. When also $\omega_p^2 \ll \omega^2$, then

$$\mu(x, y, z) \approx 1 - \omega_p^2/2\omega^2 = 1 - 2\pi e^2 n(x, y, z)/m\omega^2$$

(10.3)

and $\mu$ becomes a linear function of the plasma density $n$. 

*Fig. 17 Diagram of ray trajectory*
Let us now consider the case where refraction is small and the deflection of the ray trajectory is small. Then to lowest order the ray is just a straight line, the $w$ axis, having the direction of $t_0$ and starting at $w_0=w(x_0, y_0, z_0)$. Let us define $u$ and $v$ as two orthogonal directions, both perpendicular to $t$. From eqs.10.1 and 10.2 we obtain the following approximate expressions for the transverse displacements (from a straight line), $\Delta u$ and $\Delta v$, of the ray, when it leaves the plasma at $w=w_1$ and for the transverse components of $t$ at $w=w_1$:

$$
\Delta u = \frac{w_1}{w_0} \int \frac{\partial \ln u}{\partial w'} dw'; \quad \Delta v = \frac{w_1}{w_0} \int \frac{\partial \ln v}{\partial w'} dw'.
$$

(10.5)

Of course these expressions are valid only if $|u|, |v| \ll 1$. They are very useful for a rapid estimate of the effects of refraction, in particular on the coupling of the beam to the analyser and to the detector.

For ray propagation in an axisymmetric configuration, such as a tokamak, it is convenient to define the two orthogonal directions, $u$ and $v$, perpendicular to $t$ as follows. If $z$ is the unit vector in the axial direction of the cylindrical coordinates $(R, \phi, z)$ and $\phi_0$ is the unit vector in the toroidal direction at the input position, then we put

$$
u = z \quad \text{if} \quad \phi_0 \times t_0 = 0
$$

$$
u = \frac{\phi_0 \times t_0}{|\phi_0 \times t_0|} \quad \text{if} \quad \phi_0 \times t_0 \neq 0
$$

(10.6)

In this way $u$ is perpendicular to $\phi_0$ and also, if we consider propagation in the meridian plane or on planes $z=constant$, we have that $u$ remains perpendicular to the toroidal direction everywhere along the ray.
11. THE COMBINATION OF POLARIMETRIC AND INTERFEROMETRIC MEASUREMENTS

Any polarimetric measurement can be combined together with an interferometric measurement on the same chord across the plasma.

Indeed let us consider a polarimetric set-up such as any one of those considered up to here (see fig.18a), which provides an output intensity falling on the detector \( I = 2I_0 f(t) \), where the polarimetric information is contained in \( f(t) \). Then the electric field \( E_p \) at the output of the analyser is \( E_p = \sqrt{4I_0 f(t)} \cos(\omega t + \phi) \), if \( \omega \) is the radiation frequency and \( \phi \) is the interferometric phase shift produced by the plasma. Let us now (see fig.18b) add a reference beam (having a frequency \( \omega + \delta \omega_1 \) and intensity \( I_r \), corresponding to an electric field \( E_r = \sqrt{2I_r} \cos(\omega + \delta \omega_1) t \), which does not cross the plasma, and we combine this beam with the reference beam on a quadratic detector \( D_2 \). Then the beat signal \( S_p \) produced by \( D_2 \) is

\[
S_p = \sqrt{8I_0 I_r} f(t) \cos(\delta \omega_1 t - \phi)
\]  

(11.1)

Similarly the beat signal \( S_r \) produced by the detector \( D_1 \) of fig.18b is

\[
S_r = \sqrt{8I_0 I_r} f_0(t) \cos(\delta \omega_1 t)
\]  

(11.2)

where \( f_0(t) \) corresponds to \( I/2I_0 \) when plasma effects are zero. Now, from a comparison of \( S_p \) and \( S_r \), one can obtain both the polarimetric information, contained in the difference between \( f(t) \) and \( f_0(t) \), and the interferometric information, \( \phi \).

Such a combination of polarimetry and interferometry has been used mainly in cases [51, 52] where the input polarization is not modulated, so that \( f_0 \) is constant and \( f(t) \) is only due to (slow) changes of plasma parameters. However the combination has also been used with input modulation produced by a rotating half-wave retarder [33, 34].

It is important to note that, if the input polarization is not modulated, the polarimetric information appears only in the form of an amplitude, whereas, if the input polarization is modulated the polarimetric information can be obtained from the phases.

In order to illustrate the combination of polarimetry and interferometry, let us consider a specific example: namely the case where the input modulation is produced by the configuration \( U_1 \) (discussed in Section 9.2.1.2). Then, from eq.9.46, we have

\[
f(t) = P + Q_2 \cos(2\omega_0 t + \Phi_2) + Q_4 \cos(4\omega_0 t + \Phi_4)
\]  

(11.3)

where we have put \( P = 1 + Pu_{12} \), \( Q_2 = -Qu_{122} \), \( Q_4 = Qu_{142} \), and the phases \( \Phi_2 \) and \( \Phi_4 \) are related to the elements of the plasma transition matrix according to eq.9.47 or, when plasma effects are small, to eq.9.54. Also we have
\[ f_0(t) = P_0 + Q_{20} \cos(2\omega_0 t) + Q_{40} \cos(4\omega_0 t) \]  

(11.4)

where the subscript \( o \) indicates quantities evaluated for zero plasma effects. Thus we find for the signals in this case

\[ S_p/\sqrt{8I_0 r} = [P + Q_2 \cos(2\omega_0 t + \Phi_2) + Q_4 \cos(4\omega_0 t + \Phi_4)]^{1/2} \cos(\delta \omega t - \phi) \]

(11.5)

\[ S_r/\sqrt{8I_0 r} = [P_0 + Q_{20} \cos(2\omega_0 t) + Q_{40} \cos(4\omega_0 t)]^{1/2} \cos(\delta \omega t) \]
and indeed the polarimetric information is contained in the phases $\Phi_2$ and $\Phi_4$ (see eqs. 9.47 or 9.54). From a measurement of $S_p$ and $S_r$ we can determine all the three quantities $\Phi_2$, $\Phi_4$ and $\phi$. This is especially simple when $\delta \omega_1 \approx \omega_0$ (see also [33]).

It should be noted that $f(t)$ can also contain high frequencies, such as for example $\delta \omega$ of configurations $W_1$, $W_2$ and $W_3$ (see section 9.2.1.2). In this case it may be preferable to have $\delta \omega_1$ sufficiently different from $\delta \omega$.

We have thus found that there are no difficulties for the polarimetric measurements in combining interferometry and polarimetry together on the same instrument and using polarization modulation. However there can be important difficulties for the interferometric measurements. Indeed, in the absence of modulation the interferometric phase $\phi$ is given as usual by $\phi = \Delta = \int (\omega/c)(1 - \omega_p^2/\omega^2)^{1/2} dz$ but it can be shown [55, 16] that in the presence of polarization modulation one has $\phi = \Delta + \varphi_m(t)$. It is found [16] that for a plasma with finite birefringence (even if it is very small) the time dependence of $\varphi_m$ is always important. The undesired contribution of $\varphi_m$ to the interferometric phase can be eliminated by a convenient filtering of the interferometer signals but this implies a significant loss in the time resolution of the interferometer signals. It should be noted that for most plasma experiments, and in particular for tokamaks, plasma birefringence is always finite and so the question must be considered when polarization modulation is used.

12. THE USE OF POLARIMETRY IN PLASMA PHYSICS

In plasma physics and nuclear fusion research polarimetry can be used to determine $B(r)$ and $n(r)$ the space distribution of magnetic field and plasma density. When the former is known also the distribution of current density $j(r)$ can be obtained from the Maxwell equations.

Up to now all the measurements were made in conditions where the plasma effects are small, i.e. the wavelength was chosen so that $W \ll 1$. In this case, as shown in Section 6, the entire polarimetric information consists in the parameters $W_1$, $W_2$ and $W_3$, where $W_1 = \int n(B_x^2 - B_y^2) dz$, $W_2 = \int nB_xB_y dz$, $W_3 = \int nB_z dz$ (see eqs. 6.6-6.8). Furthermore in conditions of symmetry one has $W_2 = 0$. Most experiments have concentrated on measuring $W_3$, the Faraday effect (while birefringence was considered a disturbance). To our knowledge, only two experiments [56, 44] have measured $W_1$, the Cotton-Mouton effect. A special interest in this measurement is due to the fact that, in a tokamak for vertical propagation, $W_1$ is essentially proportional to the line integral of the plasma density and so polarimetry is an alternative to interferometry [7, 12]. Polarimetry has the advantage over interferometry that it does not require a reference beam and the corresponding large rigid structure (the C-frame). Therefore it is intrinsically less sensitive to vibrations and simpler to realise.

When the geometry of the plasma is known (or assumed) and plasma effects are small, it is possible measure $W_1$ and $W_3$ along many chords and apply one of the various inversion techniques [34, 52, 57, 58] to obtain $B(r)$ and $n(r)$. Actually, in all the applications up to now,
n(r) was measured by interferometry and polarimetry was used to measure only W_3 and so determine B(r).

Often however the geometry is not a priori known, as for example in elongated divertor tokamaks, and also there is an advantage to operate with large plasma effects in order to improve the accuracy of the measurements (and then the polarimetric information is no longer given simply by W_1, W_2 and W_3). When either one of these conditions occur, it is no longer possible to use the usual inversion techniques. However, as we have seen, in general the polarimetric measurements determine the elements of the plasma transition matrix \( \mathbf{M} \) and these can be used for the reconstruction of the MHD equilibrium. Indeed, first one can obtain a first order determination of the equilibrium [59-61] by neglecting the polarimetric measurements and using the other available measurements. Then, in successive iterations, one can use also the polarimetric measurements as constraints, by minimizing the difference between the measured elements of \( \mathbf{M} \) and those computed (using the techniques discussed in Section 7) with B(r) and n(r) obtained in the previous step of the iteration.

A task for the future remains the study of the convergence of the procedure and of the improvement which can result by including polarimetric measurements but in any case there is the possibility of increasing the amount of information which can be obtained from each probing polarimetric beam.

We have shown that, when modulation of the input polarization is used, the polarimetric information can be derived from measurements of phases rather than of amplitudes and this is advantageous since it allows a reduction of the signal to noise ratio. The polarimeter signal can be arranged so as to have more than one harmonic component (see Section 9.2) and so with a single detector it is possible to measure more than one polarimetric parameter. Furthermore it should be noted that the polarimetric phases (namely \( \Phi_{uj2k}, \Phi_{uj4k}, \Phi_{vjk}, \) and \( \Phi_{wjk} \) appearing in eqs. 9.22, 9.23, 9.34, 9.44 and 9.47 to 9.48) all occur as a tangent. Thus, even for very large plasma effects, the polarimetric phase is always included in the range \(-\pi/2 \) to \( \pi/2 \), which may be experimentally advantageous.

Section 9.2.1.2 contains the description of some polarimeter configurations of particular interest, especially when plasma effects are not large.

The treatment of polarimetry presented here is quite general: it can be applied to any magnetic plasma configuration (tokamak, stellarator, RFP etc...) and no particular symmetry assumptions are required. Polarimetric measurements up to now have been made mainly in tokamaks because these have provided the highest plasma densities and magnetic fields and so the largest polarimetric effects.

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REFERENCES


