Non-leptonic Decays of $K$-mesons within the Chiral Quark Model

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Last but not least I will thank my wife, Monika, for her encouragement, patience, love and care during the work with this thesis.
Preface

This thesis has been submitted for the degree of Doctor Scientiarum and is mainly based upon four papers

• A. E. Bergan and J. O. Eeg,
  $K - \bar{K}$ Mixing in a Low Energy Effective QCD,
  Zeitschrift für Physik C 61, 511-516 (1994)

• A. E. Bergan and J. O. Eeg,
  Momentum Dependence of the Penguin Interaction,
  Accepted for publication in Zeitschrift für Physik C

• A. E. Bergan,
  Non-factorizable Contribution to $B_K$ of order $(G^3)$,

• A. E. Bergan and J. O. Eeg
  The Self-penguin Contribution to $K \rightarrow 2\pi$
  Accepted for publication in Physics Letters B

In the papers made together with my supervisor, I have worked through all the calculations. In addition I have done some calculations with a fermion propagator in an external field. This yielded some momentum dependent functions that has been used in the calculation of gluon condensate corrections. Some of the results in table 5.3 have not been seen elsewhere. However, they have not been published. I have also done some unpublished calculations about the effects of a condensed penguin.
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Chapter 1

Introduction

1.1 Motivation

The aim of elementary particle physics is to discover new fundamental laws of nature and to classify and develop a profound theoretical framework that arranges particles in simple patterns and describes their interactions in a unified way. The Standard Model of elementary particle physics fits this prescription and it has withstood all experimental tests so far. Quantum chromodynamics, QCD, is a part of the Standard Model which describes interactions among quarks. It contains roughly two parts which are essentially different. The high energy part which can be described by a perturbative framework and in principle does not represent any problem while in practice is very tedious to calculate. The second part concerns low energy effects which cannot be calculated in Standard Model QCD due to a strong coupling constant. However, phenomenological models or ansatizes based on appropriate symmetries can be used to mimic the effects of low energy QCD. One of the frontiers of today's particle physics is attached to effective models of QCD.

Even though there has not been observed any deviation from the Standard Model there are some sectors that still need to be investigated more deeply due to seemingly improper descriptions which probably are due to their complexity or maybe more interesting, new physics. One of the most interesting areas in this concern is non-leptonic decays which describes interactions containing only hadronic states. An example of such processes, comprising merely mesons, are $K \to 2\pi$ decays. These processes have puzzled physicists in many years. They exhibit unexpected ratios between different decays. This is referred to as the $\Delta I = 1/2$ rule which refers to a change of isospin in the initial and final states. Another phenomenon which occurs in $K \to 2\pi$ decays is CP-violation. This is indicated by a non-zero ratio of $\epsilon'/\epsilon$. It has to do with the matter/antimatter asymmetry in the universe and there are several experiments measuring its value. Another important process in non-leptonic decays is $K^0 - \bar{K}^0$ mixing. Especially, there are some diverging points of view related to the long distance effects and they should therefore be investigated further. The study of the $\Delta I = 1/2$ rule, $\epsilon'/\epsilon$ and $K^0 - \bar{K}^0$ mixing is the main motivation for this thesis.
The purpose is to calculate effects within the Chiral Quark Model which is a low energy model of QCD.

The thesis is organized as follows. After this loose motivation the Standard Model is introduced with a focus on low energy QCD in addition to a more thorough and quantitative description of the motivating $\Delta I = 1/2$ rule. The last part of this introductory chapter is devoted to symmetries of QCD and the introduction of Goldstone bosons which are relics of a seemingly broken symmetry. The second chapter contains standard approaches to non-leptonic decays including some historical remarks about the $\Delta I = 1/2$ rule. The third chapter describes perturbative corrections to non-leptonic decays yielding effective Lagrangians at quark level. This also includes a short description of the renormalization procedure. In the next chapter Chiral Perturbation Theory $\chi PT$ is introduced as an effective Lagrangian of QCD at very low energies while in the fifth chapter we present $\chi QM$. This model also contains soft gluons which can be used to describe gluon condensate contributions. This part contains results which have not been presented elsewhere. The main results of applying the $\chi QM$ in $K$-meson decays are presented in chapter six together with some ideas about the condensed penguin diagram. Chapter seven contains a short résumé while some details concerning Clebsch-Gordan coefficients and technical details appearing in $K \to \pi$ analysis are left for the appendix.

1.2 Introduction to the Standard Model

The Standard Model consists of quarks and leptons which constitute all matter. In addition, there are mediators of the different forces, among them the electromagnetic field (photon or $\gamma$) is most familiar due to its presence in our daily life. In 1984 there was a major breakthrough in particle physics due to the discoveries of the $W^\pm$ and $Z$ bosons. They were necessary ingredients in the unified theory of $SU(2)_L \otimes U(1)$ which was introduced already in the 1960’s. The unification refers to electromagnetic and weak forces and this is now a part of the Standard Model. The $SU(2)_L \otimes U(1)$ refers to the symmetry of the Lagrangian which describes the interactions. A gauge field acts as a mediator of the different forces and it was originally included in order to have a symmetry of local phase transformations. This meant that experimentally measured quantities should not depend on wherever the experiment took place or what kind of measuring devices one used. Choosing a particular gauge corresponds in a way to choose a measuring unit. The fermionic fields in the Standard Model are two-component left-handed fields transforming as doublets under $SU(2)$ and right-handed singlets.

The complete Standard Model also exhibits $SU(3)_c$ symmetry, where the index $c$ denotes colour. This symmetry is thought to be exact in the quark sector. It gives rise to eight gluons which mediate the strong force.

The matter fields in the Standard Model are 6 leptons and 6 quarks. The charges in units of the elementary charge in the leptonic sector are $(-1)$ for the electron, the muon and the tau while the neutrinos are neutral. In the quark sector the charges
for $u, c, t$ are $2/3$ while for $d, s, b$ are $(-1/3)$. In addition, there are antiparticle states with opposite sign on the charges. The particles are divided into right-handed singlets and left-handed doublets. The right-handed quark and lepton fields are singlets in weak isospin:

$$ (u)_R, (d)_R, (s)_R, (c)_R, (b)_R, (t)_R, (\nu_e)_R, (\nu_\mu)_R, (\nu_\tau)_R, (\nu_\tau)_R. \quad (1.1) $$

The left-handed lepton doublets are:

$$ \left( \begin{array}{c} \nu_e \\ \nu_\mu \\ \nu_\tau \end{array} \right)_L, \left( \begin{array}{c} \nu_e' \\ \nu_\mu' \\ \nu_\tau' \end{array} \right)_L, $$

and the left-handed quark doublets are:

$$ \left( \begin{array}{c} u \\ d' \\ c \\ s' \end{array} \right)_L, \left( \begin{array}{c} u' \\ d'' \\ c' \\ s'' \end{array} \right)_L, \left( \begin{array}{c} t \\ b' \end{array} \right)_L, \left( \begin{array}{c} t' \\ b'' \end{array} \right)_L, $$

where the primed fields are

$$ \left( \begin{array}{c} d' \\ s' \\ b' \end{array} \right) = V \left( \begin{array}{c} d \\ s \\ b \end{array} \right), $$

and $V$ is the Cabbibo-Kobayashi-Maskawa ($CKM$) matrix which connects the weak eigenstates (primed) with the mass eigenstates (unprimed). One can write the unitary matrix $V$ as

$$ V = \begin{pmatrix} V_{ud} & V_{us} & V_{ub} \\ V_{cd} & V_{cs} & V_{cb} \\ V_{td} & V_{ts} & V_{tb} \end{pmatrix}. $$

The different quantities can be measured in experiments and the numbers are given in ref. [1]. The $CKM$ matrix can be parametrized by three angles and one phase. They can be determined experimentally by actual processes. A non-zero magnitude of the phase would indicate CP-violation. One should note, however, that this is not the only condition; in addition there are constraints on the mixing angles and Jarlskog has shown that the quark masses within a given charge sector must not exhibit degeneracies (e.g. $m_u \neq m_c$). This is contained in the Jarlskog determinant [2].

The expected top quark has recently been discovered at Fermilab by two collaborations. Its mass is found to be around $180 \text{GeV}$ which is considerably larger than the other quark masses. Even though there is no formal hindrance in extending the theory to more doublets, the four collaborations at LEP/CERN have shown that there are no more than three light neutrinos and therefore probably no more than three lepton families. The word "light" in this circumstance usually refers to masses smaller than about $M_\pi/2$. In astrophysics there are also indications of three lepton families. It has been determined through a study of light elements in the universe [3].

The last ingredient is the Higgs field. Without it, all other particles are massless in order to have a gauge invariant theory. By acquiring a non-zero vacuum expectation value of the Higgs field gives rise to masses of other particles through their
1.2 Introduction to the Standard Model

interactions. This is called the Higgs mechanism. The long awaited Higgs boson is
the only particle which is still missing. Unfortunately, its mass is not predicted by the
Standard Model but the energy range can be limited by looking at higher order cor-
rections and compare with experiment. This reflects the importance of high accuracy
experiments. The sensitivity to higher energy scales gives a unique opportunity to
do research at energy ranges which is not actually reachable with todays accelerator
devices. The Higgs particle have been searched for in various experiments and decay
channels but has not been found. However, this last path in the maze will hopefully
be found at the forthcoming Large Hadron Collider (LHC) at CERN. Even if it is
seemingly a dead end concerning Higgs, a lot of other theories like for instance su-
persymmetry will prosper and new ideas concerning masses on fermions and bosons
will pop up.

Formally, the basic interaction in the weak sector that we will be concerned with
can be written

$$\mathcal{L}_W = -\frac{g_W}{2} J^\mu W^\dagger_\mu + h.c.$$  (1.2)

where $g_W$ is the weak coupling constant. The $W$ boson is coupled to the charged
current which is

$$J_\mu = (\bar{u} \tau^a L) V \gamma_\mu \gamma^5 \begin{pmatrix} d \\ s \\ b \end{pmatrix},$$  (1.3)

where $L = (1 - \gamma_5)/2$ is a left-handed projection operator or chirality operator. The
separation of the fundamental particle contents in left- and right-handed doublets and
singlets reflects that nature can distinguish between left and right. This is a peculiar
thing about weak interactions and the $W$ fields only couple to the left-handed part
of the fermion fields. This is called $V - A$ theory, referring to the Dirac structure of
the interaction.

The strong sector of the Standard Model is written

$$\mathcal{L}_{QCD} = \bar{q} [i \gamma^\mu D_\mu - \mathcal{M}_q] q - \frac{1}{4} G^a_{\mu\nu} G^{a,\mu\nu}. \quad (1.4)$$

The covariant derivative and the gluon tensor are defined as

$$D_\mu = \partial_\mu - ig_s t^a A^a_\mu,$$  (1.5)

and

$$G^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + g_s f^{abc} A^b_\mu A^c_\nu,$$  (1.6)

respectively, where the $t^a$'s are the generators of the $SU(3)_c$ group and $A^a_\mu$'s are the
 gluon fields.

Due to the $SU(3)_c$ symmetry of the Standard Model each quark can appear in
three different states with definite quantum numbers. They can be denoted red,
blue and green. These states are sometimes called colour-charge, but have in fact
nothing to do with neither colour nor charge as we know from daily life. They should
merely be looked upon as three degrees of freedom per quark. It is postulated by
Greenberg that physical particles must be colourless, i.e. have a singlet structure in colour. From group theory one can construct singlets by combining $qqq$ or $qar{q}$ states which we identify as baryons and mesons, respectively. Four quark states $(qqqq)$ or hybrids $(qqgq)$ or glueballs $(gg)$, $(ggg)$ could also in principle exist, but should be rare compared to states with few quarks.

At high energies the coupling constant of strong interaction is small enough to allow a perturbative description. However, the word "constant" should be put in quotation marks since its magnitude varies with energy. The coupling depends on the energy-momentum scale $Q$ and including only one loop corrections it may be written

$$\alpha_s(Q^2) = \frac{4\pi}{\beta_0 \ln(Q^2/\Lambda_{QCD}^2)}$$

for $Q^2 \gg \Lambda_{QCD}^2$ and where $\beta_0$ is related to the number of quark flavours $N_f$ as $\beta_0 = 11 - \frac{2}{3} N_f$ and the constant $\Lambda_{QCD}$ is a characteristic scale of QCD and is of the order of a few hundred $MeV$. The functional dependence on $Q^2$ means that at large $Q^2$ the coupling constant becomes small and the theory is asymptotically free. Hence, a perturbative description in terms of weakly interacting quarks and gluons makes sense. A recipe for treating QCD radiative corrections in weak interactions is to make use of the operator product expansion and renormalization group methods [4]. The operator product expansion characterizes the product of non-local field operators at high energies with an appropriate basis of local operators and their coefficients. The coefficients are calculated with renormalization group methods. The long distance effects are accounted for by matrix elements of the local operators and must be treated by non-perturbative methods. This technique is standard with respect to effective non-leptonic Lagrangians due to the approximate point interaction in weak processes at low energies. In these interactions, two hadronic charged weak currents are coupled by the exchange of $W^\pm$ bosons. The hadronic current involves the $CKM$ matrix as seen from eq. 1.3. If the $CKM$ matrix $V$ was equal to unity, there would be only transitions between unprimed particles in the same multiplet and hence no non-leptonic decay. For interactions where the participating particles have small momenta compared to the $W^\pm$ mass scale, the two current process can be viewed as an effective four-fermion interaction with a strength of the order of the Fermi coupling constant. This is called Fermi theory. The effective Lagrangian can be written

$$\mathcal{L} = -\frac{G_F}{\sqrt{2}} \lambda_u \sum_i C_i Q_i$$

where $\lambda_u = V_u V_u^*$ and the $C_i$'s and $Q_i$'s are the short-distance Wilson coefficients and four-quark operators, respectively. The Fermi coupling constant is defined by

$$\frac{G_F}{\sqrt{2}} = \frac{g_W^2}{4M_W^2}$$

It is an example of an effective field theory and it is not renormalizable due to the dimensionful coupling $G_F$. We will see later that this is often the case in effective field
1.3 Hadronic Interactions and the $\Delta I = 1/2$ Rule

The behaviour of the coupling constant in the last section is characteristic for non-abelian Yang-Mills theories due to the gluon self-coupling. This is opposite to the abelian Quantum Electro-Dynamics (QED) where the coupling constant grows slightly with increasing energy. One can understand this behaviour from quantum fluctuations. The bare charge polarizes the vacuum and therefore the bare charge is screened. A low momentum probe will therefore see a smaller effective charge than a high momentum probe would do. For QCD the contrary is true. One says that QCD antiscreens because of nonlinear interactions among gluons. QCD effects have also analogies with a superconductor where the magnetic field is expelled from the superconductors interior (The Meissner effect). The QCD vacuum similarly expels the colour fields into the mesonic $q\bar{q}$ state. QCD vacuum is called a colour dielectric. This property is related to the gluon condensate

$$\langle 0 | \frac{\alpha_s}{\pi} G^2 | 0 \rangle = \frac{2\alpha_s}{\pi} \langle 0 | \sum_a [(B^a)^2 - (E^a)^2] | 0 \rangle. \quad (1.10)$$

The condensate is model dependent. Its value is of the order $(400 MeV)^4$.

Even though the underlying theory of QCD is $SU(3)_L \otimes SU(3)_R$ invariant the vacuum does not possess this invariance (hidden symmetry). This is due to the condensation of a $q\bar{q}$ pair which breaks chirality. For the condensed $u, d$ and approximately the $s$-quarks, the condensate is

$$\langle 0 | q\bar{q} | 0 \rangle = (-235 MeV)^3 \quad (1.11)$$

at 1 GeV. The seemingly broken symmetry is casted into a new form through the existence of massless pseudoscalar bosons corresponding to the broken generators of $SU(3)_L \otimes SU(3)_R$. The effect of a quark moving in a medium with condensed $q\bar{q}$ pairs is analogous to a spinning particle in a ferromagnet which feels the effect of a magnetic field created by all the aligned spins. The effect of this interaction is to create an energy gap between the zero-momentum quark and the vacuum. The energy gap is present even if the current quark mass vanishes and can thus be interpreted as the constituent quark mass.

Concerning the quark masses, half of them lies above this scale ($c, b, t$) and the other half lies below ($u, d, s$). The three heavy quarks can be treated in a heavy quark effective field theory while the light ones can be treated in low energy field
### 1.3 Hadronic Interactions and the $\Delta I = 1/2$ Rule

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<tr>
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<th>$u$</th>
<th>$d$</th>
<th>$s$</th>
<th>$c$</th>
<th>$b$</th>
<th>$t$</th>
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<td>Current mass [MeV]</td>
<td>4</td>
<td>8</td>
<td>150</td>
<td>1400</td>
<td>4900</td>
<td>180 000</td>
</tr>
<tr>
<td>Constituent mass [MeV]</td>
<td>200-300</td>
<td>200-300</td>
<td>400-500</td>
<td>1400</td>
<td>4900</td>
<td>180 000</td>
</tr>
</tbody>
</table>

Table 1.1: Table of current and constituent quark masses.

Theories like the Chiral Quark Model ($\chi QM$) and Chiral Perturbation Theory ($\chi PT$). When calculating short distance effects one uses current quark masses, but below the hadronic scale one has to use constituent quark masses. The current quark mass is the mass term in the QCD Lagrangian which explicitly breaks chiral invariance while the constituent quark mass stems from a non-trivial QCD vacuum. For the heavy quarks, the current and constituent quark masses are approximately equal, but for the light ones the masses are different. They are listed in tab.1.1.

The physical hadronic particles besides protons and neutrons are particles like $\pi, K, \Sigma, \Xi$ etc. The two first have the same quantum number as two-quark states and are called mesons while the last two, in addition to protons and neutrons, have quantum numbers corresponding to three quark states called baryons.

The classical example of a weak decay is the nuclear $\beta$-decay where a neutron decays into a proton, an electron and an antineutrino. The reason for calling the interaction "weak" is not primarily because of a small weak coupling constant $g_w$, but rather the heavy mass of the gauge bosons $W^\pm$ and $Z$ which mediates the weak force. Thus, they operate at short ranges. There exist also other types of weak interactions. They are classified in three categories:

- Leptonic (e.g. $\pi \rightarrow \mu \nu$)
- Semileptonic (e.g. $\beta$-decay)
- Non-leptonic (e.g. $\Sigma \rightarrow n\pi$, $K \rightarrow 2\pi$).

In this thesis we will be concerned with the last class of processes and essentially $K \rightarrow 2\pi$ decays and $K^0 - \bar{K}^0$ mixing.

One of the motivations for this thesis is to make a proper analysis of some of the Standard Model ingredients that could possibly give us a better explanation of the $\Delta I = 1/2$ rule. This selection rule emerges from the following: Kaons and pions belong to doublets and triplets under isospin rotations, respectively. If we only look at isospin, two pions can form 3 different isospin states. One can make an isoscalar $\pi \cdot \pi = \sum \pi^i \pi^i$, an isovector $\pi \times \pi = \pi^i \pi^j \epsilon^{ijk}$ or a tensor $\pi^i \pi^j + \pi^j \pi^i - \pi^k \pi^k \delta^{ij}/3$. The first and third is symmetric under exchange of the two particles while the second is antisymmetric. Due to the factorization of the overall wavefunction in space, flavour, spin and colour in addition to the generalized Bose principle which says that the overall wavefunction must be symmetric under the exchange of two bosons, the final state pions can only end up in a state with isospin 0 or 2 if there is no relative angular
momentum. Therefore, the decay of $K^0 \to \pi^+\pi^-$ can be submitted to a change of isospin by $3/2$ or $1/2$ unit. The experimental values can be found in ref. [1]

$$A(K^0 \to 2\pi)_{1/2} = 3.33 \cdot 10^{-7} GeV$$  \hspace{1cm} (1.12)

and

$$A(K^0 \to 2\pi)_{3/2} = 1.5 \cdot 10^{-8} GeV$$  \hspace{1cm} (1.13)

for $\Delta I = 1/2$ and $\Delta I = 3/2$, respectively. The former amplitude is suppressed by a factor of 22.2. The decay $K^- \to \pi^-\pi^0$ can only proceed through the $\Delta I = 3/2$ channel due to $I_z = -1$ in the final state. Thus, one can extract information of the $A(\Delta I = 3/2)$ amplitude. Even though strong interaction enhances the $\Delta I = 1/2$ part of the Lagrangian and reduces the $\Delta I = 3/2$ part, the amplitudes are of comparable size when non-perturbative interactions are not included. It has therefore been a challenge for theoretical physicists to describe the rather large difference between the amplitudes. In the baryon sector one also has the same enhancement (e.g. $\Sigma^+ \to n\pi^+$) and it seems that this phenomenon is a universal feature of non-leptonic decays [5].

In order to take into account non-perturbative effects there is need for low energy effective models. There have been attempts to describe the $\Delta I = 1/2$ rule in light of the large $N_c$ expansion. However, this procedure has failed and the rule must lie beyond leading order in $N_c$. Lattice calculations seem to be a good opportunity to look at long distance effects and some attempts seem to show a dominance of the so called "eight-graphs" and "eye-graphs" which indicates an explanation of the $\Delta I = 1/2$ rule. However, computer capacity and time consumption puts limits on how small the discretization and thus the accuracy can be. The $\Delta I = 1/2$ rule is also intimately related to the $\epsilon'$ parameter as a non-zero value depends on relations between $A(\Delta I = 3/2)$ and $A(\Delta I = 1/2)$. The theoretical value can be calculated by looking at the imaginary part of the amplitude and thus one has to include the top-quark.

1.4 External, Internal and Hidden Symmetries

In a theoretical framework the description of elementary particle physics must be indifferent with respect to where and when the experiment is performed (translations in space and time), whether the coordinate system is rotated or not and to uniformly moving coordinate systems (boosts). This is called Poincaré invariance and the transformations constitute the Poincaré group while rotations and boosts generate the proper Lorentz group $L^\dagger$. The arrow means that the 00-component of the transformation is greater or equal to +1. Invariance of the theory under a continuous transformation implies a conserved current according to Noethers theorem. For rotations, translations in space and time, the associated conserved quantities are angular momentum, momentum and energy, respectively. The scalar product $x_\alpha x^\alpha$ is invariant under proper Lorentz transformations, but also under the discrete parity ($P$) and time reversal ($T$) transformations. The full Lorentz group is given as the proper Lorentz group plus the three combinations $P$, $T$ and $PT$ acting on $L^\dagger$. 
1.4 External, Internal and Hidden Symmetries

The basic objects in particle physics are the fields and not the coordinates as in classical mechanics. This means that the fields have to transform according to some representation of the proper Lorentz group. If it is more or less obvious that nature should be invariant with respect to proper Lorentz transformations, there is no a priori reason why nature should exhibit \(P\) or \(T\) invariance. Therefore, one must acquire information through conservation tests. \(P\) and \(T\) can each only assume two different values since each of them is a two element group with the unit operator. In terms of the Dirac gamma matrices the discrete transformations on fermionic fields are

\[
P\psi(x)P^{-1} = \eta_P \gamma^0 \psi(x_P) \quad (1.14)
\]
\[
T\psi(x)T^{-1} = i\eta_T \gamma^1 \gamma^3 \psi^*(x_T) \quad (1.15)
\]

where the \(\eta\)'s are arbitrary phase factors and \(x_P\) and \(x_T\) means that there is a minus sign in front of the space and time component of the four vector \(x^a\), respectively. The time reversal operator can be shown to be antiunitary in order to have a lowest state in the particle spectrum [6].

An internal discrete symmetry is charge conjugation \(C\). Its operation exchanges particles and antiparticles. In terms of Dirac matrices, it transforms a field \(\psi\) as

\[
C\psi(x)C^{-1} = i\gamma^2 \gamma^0 \psi^T(x) \quad (1.16)
\]

where \(\eta\) is an arbitrary phase and \(\psi^T(x) = \psi_\alpha^\dagger \gamma_\alpha\beta\). The space time components are unaffected by charge conjugation.

The response of the normal-ordered Dirac operators

\[
S(x) = :\overline{\psi}(x)\psi(x): \quad (1.17)
\]
\[
P(x) = :\overline{\psi}(x)\gamma_5 \psi(x): \quad (1.18)
\]
\[
J^\mu(x) = :\overline{\psi}(x)\gamma^\mu \psi(x): \quad (1.19)
\]
\[
J_5^\mu(x) = :\overline{\psi}(x)\gamma^\mu \gamma_5 \psi(x): \quad (1.20)
\]
\[
T^{\mu\nu}(x) = :\overline{\psi}(x)\sigma^{\mu\nu} \psi(x): \quad (1.21)
\]

under the discrete transformations are listed in table 1.2. Note the location of indices [5].

The Standard Model QCD Lagrangian incorporates also continuous internal symmetries. For each flavour \(A\), one can perform a global \(U(1)\) phase transformation

\[
\psi_A \rightarrow \psi'_A = e^{i\delta} \psi_A \quad (1.22)
\]
\[
\overline{\psi}_A \rightarrow \overline{\psi}'_A = e^{-i\delta} \overline{\psi}_A \quad (1.23)
\]

and leave the rest unchanged. We suppress the space-time dependence of the quantum fields. According to Noether's theorem this gives the conserved current. The corresponding conserved charge is the quark number associated with the particular flavour in question

\[
N_A = \int d^3x \psi^\dagger_A \psi_A. \quad (1.24)
\]
Table 1.2: Transformation properties of Dirac bilinears under C, P and T.

This leads to baryon number conservation if we transform all quark fields with the same phase

$$B = \frac{1}{3} \sum_{A=1}^{N_f} N_A. \quad (1.25)$$

Another case of $U(1)^{N_f}$ is the conservation of electric charge

$$Q = \sum_{A=1}^{N_f} q_A N_A \quad (1.26)$$

If there are degenerate mass states, they will give rise to higher symmetries which will be reflected in multiplets of particles. The conserved current will then contain generators ($T^a$) of the corresponding group with the multiplet $\Psi$ of particles

$$V^a_\mu = \bar{\Psi} \gamma_\mu T^a \Psi \quad (1.27)$$

One can also have multiplets of particles even they are not degenerate in mass as for the lepton doublets mentioned in the first part of the introduction. An even higher symmetry is obtained if the quarks are massless. The resulting symmetry is then called chiral symmetry. This yields the conserved axial current

$$A^a_\mu = \bar{\Psi} \gamma_\mu \gamma_5 T^a \Psi \quad (1.28)$$

Due to the weird nature of weak interaction, it is fruitful to define left-handed and right-handed currents

$$L^a_\mu = V^a_\mu - A^a_\mu \quad (1.29)$$
$$R^a_\mu = V^a_\mu + A^a_\mu \quad (1.30)$$

and their corresponding charges

$$Q_L^a = Q^a - Q_5^a \quad (1.31)$$
$$Q_R^a = Q^a + Q_5^a \quad (1.32)$$
1.4 External, Internal and Hidden Symmetries

<table>
<thead>
<tr>
<th>Quantity</th>
<th>( C )</th>
<th>( P )</th>
<th>( T )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Coordinates</td>
<td>( \vec{r} \rightarrow \vec{r} )</td>
<td>( \vec{r} \rightarrow -\vec{r} )</td>
<td>( \vec{r} \rightarrow \vec{r} )</td>
</tr>
<tr>
<td>Time</td>
<td>( t \rightarrow t )</td>
<td>( t \rightarrow t )</td>
<td>( t \rightarrow -t )</td>
</tr>
<tr>
<td>Momentum</td>
<td>( \vec{p} \rightarrow \vec{p} )</td>
<td>( \vec{p} \rightarrow -\vec{p} )</td>
<td>( \vec{p} \rightarrow -\vec{p} )</td>
</tr>
<tr>
<td>Energy</td>
<td>( E \rightarrow E )</td>
<td>( E \rightarrow E )</td>
<td>( E \rightarrow E )</td>
</tr>
<tr>
<td>Angular momentum</td>
<td>( \vec{J} \rightarrow \vec{J} )</td>
<td>( \vec{J} \rightarrow \vec{J} )</td>
<td>( \vec{J} \rightarrow -\vec{J} )</td>
</tr>
<tr>
<td>Vector potential</td>
<td>( A \rightarrow -A )</td>
<td>( A \rightarrow -A )</td>
<td>( A \rightarrow -A )</td>
</tr>
<tr>
<td>Scalar potential</td>
<td>( \phi \rightarrow -\phi )</td>
<td>( \phi \rightarrow \phi )</td>
<td>( \phi \rightarrow \phi )</td>
</tr>
<tr>
<td>Electric field</td>
<td>( E \rightarrow -E )</td>
<td>( E \rightarrow -E )</td>
<td>( E \rightarrow E )</td>
</tr>
<tr>
<td>Magnetic field</td>
<td>( H \rightarrow -H )</td>
<td>( H \rightarrow H )</td>
<td>( H \rightarrow -H )</td>
</tr>
</tbody>
</table>

Table 1.3: Transformation properties of different quantities under \( C \), \( P \) and \( T \) transformations.

where

\[
Q^a = \int d^3 x V_0(x) \tag{1.33}
\]

\[
Q_0^a = \int d^3 x A_0(x). \tag{1.34}
\]

If the function \( \beta \) in 1.23 is space-time dependent, one talks about local symmetry transformations. In order to have a local symmetry, one has to introduce the gauge fields \( A_\mu^a \) (not to be confused by the axial current defined above). They naturally describe the interaction between fermions mediated by a vector field. The QCD Lagrangian in eq. 1.4 is then locally invariant if the current quark mass is zero and the gauge field transforms appropriately.

A fundamental theorem in particle physics is the CPT theorem which ensures the important property that a particle and its antiparticle have equal masses and lifetimes. It involves the combination of charge conjugation \( C \), parity \( P \) and time reversal \( T \) transformations and it is formally presented as [7]:

"Any theory described by a local Lagrangian field theory and invariant under proper ortochronous Lorentz transformations is also invariant under strong reflection, i.e. under the product of the operations of space inversion \( P \), time reversal \( T \) and charge conjugation \( C \) taken in any order, provided the usual spin-statistics connection holds".

This theorem has the amazing property that it relates the internal structure of particles to space and time. It should be emphasized that the CPT-theorem does not imply that a CP-transformation is equivalent to a \( T \)-transformation but that the \( T \)-symmetry is violated to the same extent as the CP-symmetry. We have shown in table 1.3 how different quantities transform under \( C \), \( P \) and \( T \) transformations. We can write the CPT invariance as

\[
(CPT)^{-1} \mathcal{L}(x)CPT = \mathcal{L}(-x). \tag{1.35}
\]
which means that the action $\int d^4x \mathcal{L}(x)$ is invariant under CPT.

In the mid-fifties one discovered that parity was broken in weak interactions due to the discrimination between left and right handed currents. The reigning belief was that the combined CP transformation was conserved. This was contradicted by the famous experiment of Christenson et al. in 1964 where kaon states with CP quantum number $-1$ were measured to decay into two pion states with CP quantum number $+1$. Due to the CPT theorem this implied that time reversal was broken. The weakly charged current is CP invariant and hence $T$ invariant when the $CKM$ matrix is real. In order to describe CP-violation the phase $\delta$ in the $CKM$ matrix must be non-vanishing. This can be seen by performing a $T$ transformation on the weak current. Since $T$ includes complex conjugation, this means that the transformed current would not be equal to the adjoint current due to the phase $\delta$. Still, there are no other experiments which have reported CP violation.

Another type of symmetry is called hidden symmetry. The phenomenon occurs when the symmetry of the ground state does not exhibit the full symmetry of the Lagrangian. It is intimately connected to the Goldstone theorem [8], [9]: Suppose it exists a field $\xi(x)$ with non-vanishing vacuum expectation value

$$\langle 0 | \xi | 0 \rangle \neq 0$$

and whose conserved charge $Q = \int d^3x j^0$ for which $\xi$ can be written as a commutator [10]

$$\xi = i [Q, \phi]$$

for some $\phi(x)$. If $Q|0\rangle = 0$ then $\langle 0 | \xi | 0 \rangle = 0$ while a non-vanishing $\langle 0 | \xi | 0 \rangle$ gives

$$\langle 0 | \xi | 0 \rangle = i \int d^3y \sum_n \langle 0 | j^0(y) | N \rangle \langle N | \phi(x) | 0 \rangle - \langle 0 | \phi(x) | N \rangle \langle N | j^0(y) | 0 \rangle$$

This implies the existence of a Goldstone state $|G\rangle$ such that

$$\langle 0 | j^0(y) | G \rangle \neq 0$$

Since $j^0(y)$ is invariant with respect to rotations, the Goldstone boson must be spinless. If the energy and momentum of the Goldstone state is $E$ and $\vec{p}$, respectively, we have that

$$\langle 0 | j^0(\vec{x}, t) | G \rangle = \langle 0 | e^{-i \vec{p} \cdot \vec{r} + iE^t} j^0(0, 0) e^{-i \vec{p} \cdot \vec{r} + iE^t} | G \rangle$$

The continuity equation gives

$$\langle 0 | j^0(0, 0) | G \rangle E - \langle 0 | j^0(0, 0) | G \rangle \vec{p} = 0$$

As $\vec{p}$ approaches zero the energy $E(\vec{p})$ also approaches zero. This means that the Goldstone boson is gapless. In a relativistic description, this means that the mass is zero.
Chapter 2

Non-leptonic Decays

2.1 Introduction

In non-leptonic decays a charged current is coupled to a $W^\pm$ gauge boson. The interaction term is written in eq. 1.2

$$\mathcal{L}_{\text{eff}} = -\frac{G_F}{\sqrt{2}} J^\dagger J + \text{h.c.}$$ (2.1)

where the Fermi coupling constant is defined in eq. 1.9. Such interactions are generally more difficult to analyze theoretically than leptonic and semi-leptonic decays due to two hadronic currents. If one inserts a complete set of intermediate states, one has to take into account all possible intermediate states up to $M_W$. Thus, we need theoretical control over low, intermediate and high energy scales in order to make reliable predictions. Processes described by such interactions can be both baryonic and mesonic such as $\Sigma \to n\pi$, $\Lambda \to p\pi$, $K \to 2\pi$, $K^0 \to K^0$. We look at products of currents that have $\Delta S = 1$. They will have the form

$$\bar{d} \gamma^\nu L u \bar{u} \gamma_\nu L s, \quad \bar{d} \gamma^\nu L c \bar{c} \gamma_\nu L s, \quad \bar{d} \gamma^\nu L t \bar{t} \gamma_\nu L s$$

All these products may contribute but if we consider kaons and pions naively expect that the first product is the most dominant due to the $u, d, s$ content of kaons and pions. The last two contribute through loop effects. The basic diagram for $s \to d$ transition is shown in fig. 2.1. The same figure also shows the approximated weak vertex as a point interaction. This is valid in low energy QCD due to a heavy $W$ boson.

Despite the difficulties in treating non-leptonic decays, one can regard the symmetries of the products of currents. The first product is constructed from two $SU(3)$ flavour octets and they have to combine into a symmetric product since they belong to the same octet. They will be a part of a 27-plet and an octet. The singlet part is
2.2 K Decays in Light of CPT Invariance

We outline here some consequences of the CPT theorem for K decays:

\[ a^{*}(I) = \langle f^\text{out}_{I} | \mathcal{L}_{\text{eff}}^{S=1} | K^{0} \rangle^* \]
\[ = \langle f^\text{out}_{I} | T^{-1} P^{-1} C^{-1} \mathcal{L}_{\text{eff}}^{S=1} \text{CPT} | K^{0} \rangle^* \]  

The index \( I \) refers to isospin. Since the time-reversal operator is an anti-unitary operator means that all momenta and spins of the corresponding states are reversed. This is denoted by the `symbol.

\[ T|K^0\rangle = \langle \tilde{K}^0 | \] 
\[ \langle f^\text{out}_{I} | T^{-1} = | f^\text{n}_{I} | \]
2.2 K Decays in Light of CPT Invariance

The $K^0$ has spin zero and if we are in its rest frame we can drop the -$\bar{\text{v}}$-symbol. This yields

$$a^*(I) = \langle K^0 | P^{-1} C^{-1} \mathcal{L}_{\text{eff}}^{\Delta S = 1} CP | f_{\text{in}}^\dagger \rangle^*$$

$$= \langle f_{\text{in}}^\dagger | P^{-1} C^{-1} \mathcal{L}_{\text{eff}}^{\Delta S = 1} CP | K^0 \rangle$$

From Watsons theorem the S-matrix relation between in- and out-states can be written

$$\langle f_{\text{in}}^\dagger | = \langle f_{\text{out}}^\dagger | e^{-2i\delta_I}$$

One can choose the CP phases such that

$$CP|K^0\rangle = -|\bar{K}^0\rangle$$

$$\langle f_{\text{in}}^\dagger | P^{-1} C^{-1} = \langle f_{\text{in}}^\dagger |$$

Hereby

$$a^*(I) = -e^{-2i\delta_I} \langle f_{\text{in}}^\dagger | \mathcal{L}_{\text{eff}}^{\Delta S = 1} | K^0 \rangle$$

$$a^*(I) = -e^{-2i\delta_I} \bar{a}(I)$$

This means that we have relations between the $K$ decay amplitude and its corresponding antiparticle decay.
2.3 Relations between $K \rightarrow 2\pi$ Amplitudes

In $K \rightarrow 2\pi$ transitions we can parametrize the amplitudes in such a way that symmetry factors are extracted from the amplitude expressions by use of the Wigner-Eckart theorem, leaving a reduced matrix element to be evaluated [7]. Due to the isospin structure of the pions, the normalized 2-pion states are given by

$$|\pi^+\pi^0\rangle = |I = 2, I_z = 1\rangle$$
$$|\pi^+\pi^-\rangle = \sqrt{\frac{2}{3}} |I = 0, I_z = 0\rangle \pm \sqrt{\frac{1}{3}} |I = 2, I_z = 0\rangle$$
$$|\pi^0\pi^0\rangle = \sqrt{\frac{1}{3}} |I = 0, I_z = 0\rangle - \sqrt{\frac{2}{3}} |I = 2, I_z = 0\rangle.$$  

The transitions $K \rightarrow 2\pi (I = n)$ where $n = 0, 2$ can be written

$$\langle I = n | L_{eff}^{\Delta S=1} | K^0 \rangle = a_n e^{i\delta_n}$$
$$\langle I = n | L_{eff}^{\Delta S=1} | \bar{K}^0 \rangle = -a_n e^{i\delta_n},$$

where the last line follow from CPT-invariance (eq.2.13) and $\delta_n$ is the $\pi-\pi$ phase shift. We thus have the following decay amplitudes:

$$A(K^0 \rightarrow \pi^+\pi^-) = \sqrt{\frac{2}{3}} a_0 e^{i\delta_0} + \sqrt{\frac{1}{3}} a_2 e^{i\delta_2}$$
$$A(K^0 \rightarrow \pi^+\pi^-) = -\sqrt{\frac{2}{3}} a_0 e^{i\delta_0} - \sqrt{\frac{1}{3}} a_2 e^{i\delta_2}$$
$$A(K^0 \rightarrow \pi^0\pi^0) = \sqrt{\frac{1}{3}} a_0 e^{i\delta_0} - \sqrt{\frac{2}{3}} a_2 e^{i\delta_2}$$
$$A(\bar{K}^0 \rightarrow \pi^0\pi^0) = -\sqrt{\frac{1}{3}} a_0 e^{i\delta_0} + \sqrt{\frac{2}{3}} a_2 e^{i\delta_2}$$
$$A(K^+ \rightarrow \pi^+\pi^-) = b_2 e^{i\delta_2}$$
$$A(K^- \rightarrow \pi^-\pi^-) = -b_2 e^{i\delta_2}.$$

We now split the total weak Lagrangian in terms that belong to different multiplets

$$L_{eff}^{\Delta S=1} = L\left(\frac{1}{2}\right) + L\left(\frac{3}{2}\right) + L\left(\frac{5}{2}\right).$$

The $L\left(\frac{3}{2}\right)$ part is of electromagnetic origin. If the penguin gluon is replaced by a photon or a Z boson, the amplitude will affect the theoretical value for $\epsilon'/\epsilon$. We will keep the $\Delta I = 5/2$ part in order to see its formal contribution but no effects from this term will be calculated in this thesis. We have

$$a_0 e^{i\delta_0} = \langle I = 0, I_z = 0 | L_{eff}^{\Delta S=1} | I = \frac{1}{2}, I_z = -\frac{1}{2}\rangle$$
$$= \langle I = 0, I_z = 0 | L\left(\frac{1}{2}\right) | I = \frac{1}{2}, I_z = -\frac{1}{2}\rangle.$$
2.3 Relations between $K \to 2\pi$ Amplitudes

We will now apply the Wigner-Eckart theorem. It reads

$$
\langle I', \ell_\pi' | \mathcal{L}(k_1, k_2) | I, I_\pi \rangle = \langle k, I, I_\pi | \mathcal{L}(k_1) | I' \rangle \langle I' | | I \rangle. \quad (2.29)
$$

The first and second term on the right-hand side is the Clebsch-Gordan coefficient and the reduced matrix element, respectively. Thus, we have to find the actual Clebsch-Gordan coefficients. They are given in the appendix. We are now left with the reduced matrix elements which does not depend on the $z$-component of the isospin.

$$
a_0 e^{i\delta_0} = \frac{1}{\sqrt{2}} \langle I = 0 | | \mathcal{L}(1/2) | I = 1/2 \rangle \quad (2.30)
$$

This yields

$$
a_0 = \frac{1}{\sqrt{2}} A_{1/2}. \quad (2.31)
$$

Similarly we get

$$
a_2 = \frac{1}{\sqrt{2}} (A_{3/2} + A_{5/2}) \quad (2.32)
$$

where the $A$'s denote reduced matrix elements

$$
A_{3/2} = \langle I = 2 | | \mathcal{L}(3/2) | I = 1/2 \rangle \quad (2.33)
$$

$$
A_{5/2} = \langle I = 2 | | \mathcal{L}(5/2) | I = 1/2 \rangle. \quad (2.34)
$$

Finally one gets

$$
A(K^0 \to \pi^+\pi^-) = \sqrt{1/3} A_{1/2} e^{i\delta_0} + \sqrt{1/6} (A_{3/2} + A_{5/2}) e^{i\delta_2} \quad (2.35)
$$

$$
A(\bar{K}^0 \to \pi^+\pi^-) = -\sqrt{1/3} A_{1/2} e^{i\delta_0} - \sqrt{1/6} (A_{3/2} + A_{5/2}) e^{i\delta_2} \quad (2.36)
$$

$$
A(K^0 \to \pi^0\pi^0) = \sqrt{1/6} A_{1/2} e^{i\delta_0} - \sqrt{1/3} (A_{3/2} + A_{5/2}) e^{i\delta_2} \quad (2.37)
$$

$$
A(\bar{K}^0 \to \pi^0\pi^0) = -\sqrt{1/6} A_{1/2} e^{i\delta_0} + \sqrt{1/3} (A_{3/2} + A_{5/2}) e^{i\delta_2} \quad (2.38)
$$

$$
A(K^+ \to \pi^+\pi^0) = (\sqrt{3}/2) A_{1/2} - \sqrt{1/3} A_{5/2} e^{i\delta_2} \quad (2.39)
$$

$$
A(K^- \to \pi^-\pi^0) = -\sqrt{3}/2 A_{1/2} + \sqrt{1/3} A_{5/2} e^{i\delta_2}. \quad (2.40)
$$
If we neglect electromagnetic effects ($A_{\frac{1}{2}} = 0$), we can write the amplitudes in the following form:

$$A(K^0 \rightarrow \pi^+\pi^-) = \sqrt{\frac{1}{3}} \left[ A_{\frac{1}{2}} e^{i\delta_0} + \sqrt{\frac{1}{2}} A_{\frac{3}{2}} e^{i\delta_2} \right]$$ (2.43)

$$A(K^0 \rightarrow \pi^+\pi^-) = -\sqrt{\frac{1}{3}} \left[ A_{\frac{1}{2}} e^{i\delta_0} + \sqrt{\frac{1}{2}} A_{\frac{3}{2}} e^{i\delta_2} \right]$$ (2.44)

$$A(K^0 \rightarrow \pi^0\pi^0) = \sqrt{\frac{1}{6}} \left[ A_{\frac{1}{2}} e^{i\delta_0} - \sqrt{2} A_{\frac{3}{2}} e^{i\delta_2} \right]$$ (2.45)

$$A(K^0 \rightarrow \pi^0\pi^0) = -\sqrt{\frac{1}{6}} \left[ A_{\frac{1}{2}} e^{i\delta_0} - \sqrt{2} A_{\frac{3}{2}} e^{i\delta_2} \right]$$ (2.46)

$$A(K^+ \rightarrow \pi^+\pi^0) = \frac{\sqrt{3}}{2} A_{\frac{3}{2}} e^{i\delta_2}$$ (2.47)

$$A(K^- \rightarrow \pi^-\pi^0) = -\frac{\sqrt{3}}{2} A_{\frac{3}{2}} e^{i\delta_2}.$$ (2.48)

In the literature the factors outside the square brackets are sometimes included in the amplitude. Note that in the amplitude expressions for $\pi^0\pi^0$ final states there is a factor $1/\sqrt{2}$ included. This factor takes into account that the final state pions are indistinguishable. In the chiral Lagrangians these factors shows up automatically.

### 2.4 Historical Aspects of the $\Delta I = 1/2$ rule

The $\Delta I = 1/2$ rule has for the last 50 years been subject to a lot of debate. Nature prefers a much larger $A_0$ amplitude than $A_2$ as for instance soft pion models naively would give without loop corrections. The selection rule is not only experimentally manifest for the decay of kaons, but for $s$-quark baryons (hyperons) as well [5]. The basic process for $s \rightarrow d$ dressed with gluons gives rise to a new operator. Similar short distance corrections give also birth to the penguin operator. This yields six operators with different coefficients. The earliest work by for instance Wilson [4], Gaillard-Lee [11] and Vainshtein-Zakharov-Shifman [12] focused on the short distance effects or the determination of the coefficients in front of the operators. These calculations gave a pull in the right direction, but were still far from the experimental result. This made people think that long distance effects could give some vital contributions. The works of Cohen-Manohar [13] and Bardeen-Buras-Gerard [14] were aimed at meson-loop calculations in a chiral quark model. In the last years one has realized that the long-distance effects are probably the most important. The gluon condensates have large effects on $A_0$ and $A_2$ in a favourable way. This is shown by for instance by Pich and de Rafael [15]. The last ingredient seems to be the renormalization of meson loops [16], [17]. In a paper by Antonelli et al. [18] they show that there is a strong enhancement for $A_0$ but only cosmetic changes for $A_2$. Some of these authors give an updated survey of the $\Delta I = 1/2$ rule [19]. There are however still some intriguing
2.5 Measurable CP-violating Quantities

The CP-violating parameters frequently referred to are $\epsilon$ and $\epsilon'$. The $\epsilon$ parameter refers to indirect CP-violation and involves the mixing between $K^0$ and $\bar{K}^0$ while $\epsilon'$ refers to direct CP-violation (e.g. a pure CP-odd $K^0$ state of $K^0$ decaying into two pions). At present, there is dissension as to whether a non-zero value of $\epsilon'$ has been found. In the absence of weak interaction, the kaon states $|K^0\rangle$ and $|\bar{K}^0\rangle$ are stable eigenstates of strangeness with eigenvalues $\pm 1$. Linear combinations of these states are the long-lived and short-lived kaon states

$$|K_S\rangle = \frac{1}{\sqrt{2(1 + |\tilde{\epsilon}|^2)}} [(1 + \tilde{\epsilon})|K^0\rangle - (1 - \tilde{\epsilon})|\bar{K}^0\rangle]$$

$$|K_L\rangle = \frac{1}{\sqrt{2(1 + |\tilde{\epsilon}|^2)}} [(1 + \tilde{\epsilon})|K^0\rangle + (1 - \tilde{\epsilon})|\bar{K}^0\rangle]$$

where $\tilde{\epsilon}$ is related to the indirect CP-violating parameter $\epsilon$. A vanishing $\tilde{\epsilon}$ would mean that the states $|K_S\rangle$ and $|K_L\rangle$ would be eigenstates of the CP-operator with eigenvalues $+1$ and $-1$, respectively. They are often called $K^0_l$ and $K^0_s$. By turning on the weak interaction, the states $|K_S\rangle$ and $|K_L\rangle$ are no longer stable and will decay after a while $t$. They obey an exponential decay law

$$|K_S\rangle \rightarrow e^{-\imath M_S t}|K_S\rangle$$

$$|K_L\rangle \rightarrow e^{-\imath M_L t}|K_L\rangle$$

where $t$ is the time in the kaons frame of reference, i.e. the proper time. The complex numbers $M_S$ and $M_L$ are defined as follows

$$M_{S,L} = m_{S,L} - \frac{i}{2} \Gamma_{S,L}$$

where $m_{S,L}$ and $\Gamma_{S,L}$ are masses and lifetimes for the kaon states, respectively. Inverting the expressions in eq.2.49 and eq.2.50 one finds that the $K^0$ and $\bar{K}^0$ evolves into each other:

$$K^0 \rightarrow \frac{1}{2} \left[ e^{-\imath M_L t} + e^{-\imath M_S t} \right] |K^0\rangle + \frac{1}{2} \frac{(1 - \tilde{\epsilon})}{(1 + \tilde{\epsilon})} \left[ e^{-\imath M_L t} - e^{-\imath M_S t} \right] |\bar{K}^0\rangle$$

points left. In the baryon sector a natural but naive estimate of the constituent quark mass could be an average of the nucleon and the $\Delta$ resonances. This would give a mass of order $350MeV$. However, in the mesonic sector a similar naive mass estimate could be obtained by averaging the $\pi$ and $\rho$ masses which could give a constituent quark mass of approximately $230MeV$. In order to fit experimental data the authors of [19] use a mass of $180MeV$ which seems to be very small. In another paper by the same authors [20] the quark mass must be around $220MeV$ in order to have a scheme independent estimate of $\epsilon'/\epsilon$. This value seems to be in accordance with a naive estimate also.
2.5 Measurable CP-violating Quantities

\[ \overline{K}^0 \rightarrow \frac{1}{2} [e^{-iM_L t} + e^{-iM_S t}] |\overline{K}^0\rangle + \frac{1}{2} \frac{(1 + \hat{\epsilon})}{(1 - \hat{\epsilon})} [e^{-iM_L t} - e^{-iM_S t}] |K^0\rangle \]  

(2.55)

During a short time interval \( \Delta t \) one finds

\[
\begin{pmatrix}
|K^0\rangle \\
|\overline{K}^0\rangle
\end{pmatrix} = \begin{pmatrix}
|K^0\rangle \\
|\overline{K}^0\rangle
\end{pmatrix} - i \Delta t \mathcal{M} \begin{pmatrix}
|K^0\rangle \\
|\overline{K}^0\rangle
\end{pmatrix}
\]

where \( \mathcal{M} \) is the complex mass-matrix for the \( K^0 - \overline{K}^0 \) system:

\[
\mathcal{M} = \frac{1}{2} \begin{bmatrix}
M_L + M_S & \frac{(1 - \hat{\epsilon})}{(1 + \hat{\epsilon})} (M_L - M_S) \\
\frac{(1 + \hat{\epsilon})}{(1 - \hat{\epsilon})} (M_L - M_S) & M_L + M_S
\end{bmatrix}
\]

where \( M_L, S \) are the complex numbers defined in eq.2.53. From the discussion of CPT in an earlier chapter, we know that the particles and antiparticles have equal masses and lifetimes. Thus, the equal diagonal elements of the matrix \( \mathcal{M} \) is a consequence of the CPT theorem.

The CP-violating parameter \( \hat{\epsilon} \) is defined as follows:

\[
\hat{\epsilon} = \frac{A(K_L \rightarrow (\pi\pi)_{I=0})}{A(K_S \rightarrow (\pi\pi)_{I=0})} \]  

(2.56)

\[
\hat{\epsilon} = \frac{\langle (\pi\pi)_0 | S | K^0 \rangle + \langle (\pi\pi)_0 | S | \overline{K}^0 \rangle + \hat{\epsilon} \langle (\pi\pi)_0 | S | K^0 \rangle - \langle (\pi\pi)_0 | S | \overline{K}^0 \rangle}{\langle (\pi\pi)_0 | S | K^0 \rangle - \langle (\pi\pi)_0 | S | \overline{K}^0 \rangle + \hat{\epsilon} \langle (\pi\pi)_0 | S | K^0 \rangle + \langle (\pi\pi)_0 | S | \overline{K}^0 \rangle} \]  

(2.57)

We utilize eqs.2.18 and 2.19 and find

\[
\hat{\epsilon} = \frac{(1 + \hat{\epsilon})A_0 - (1 - \hat{\epsilon})A_0^*}{(1 + \hat{\epsilon})A_0 + (1 - \hat{\epsilon})A_0^*} \]  

(2.58)

\[
\hat{\epsilon} = \frac{\hat{\epsilon} + \frac{\text{Im}A_0}{\text{Re}A_0}}{1 + i \frac{\text{Im}A_0}{\text{Re}A_0}} \]  

(2.59)

Hence, \( \hat{\epsilon} \) and \( \epsilon \) are related.

The parameter \( \epsilon' \) is a measure of direct CP-violation and is conventionally defined as

\[
\epsilon' = \frac{1}{\sqrt{2}} [\omega' - \epsilon \omega] \]  

(2.60)

where \( \omega \) is defined as

\[
\omega = \frac{A(K_S \rightarrow (\pi\pi)_{I=0})}{A(K_S \rightarrow (\pi\pi)_{I=0})} \]  

(2.61)

\[
\omega = \frac{(1 + \hat{\epsilon})A_2 + (1 - \hat{\epsilon})A_2^* e^{i(\delta_2 - \delta_0)}}{(1 + \hat{\epsilon})A_0 + (1 - \hat{\epsilon})A_0^* e^{i(\delta_2 - \delta_0)}} \]  

(2.62)

\[
\omega = \frac{\text{Re}A_2 + i \text{Im}A_2 e^{i(\delta_2 - \delta_0)}}{1 + i \frac{\text{Im}A_0}{\text{Re}A_0}} \]  

(2.63)
2.5 Measurable CP-violating Quantities

while

\[\omega' = \frac{A(K_L \rightarrow (\pi \pi)_{I=2})}{A(K_S \rightarrow (\pi \pi)_{I=0})} \]

\[= \frac{(1 + \xi)A_2 - (1 - \xi)A_0^*}{(1 + \xi)A_0 + (1 - \xi)A_0^*} e^{i(\xi_2 - \xi_0)} \]

\[= \frac{i Im A_2 + \xi Re A_2}{Re A_2} + \xi Re A_2 e^{i(\xi_2 - \xi_0)}. \]

(2.64)

(2.65)

(2.66)

Combining the last 3 expressions yields an exact expression for \(\epsilon'\).

\[\epsilon' = i(1 - \xi^2)e^{i(\xi_2 - \xi_0)} \frac{(Re A_0 Im A_2 - Re A_2 Im A_0)}{(Re A_0 + i\xi Im A_0)^2} \]

(2.67)

In order to make connection with parameters accessible to experiment one defines

\[\eta_{+-} = \frac{\langle \pi^+\pi^- | S | K_L \rangle}{\langle \pi^+\pi^- | S | K_S \rangle} \]

(2.68)

\[\eta_{00} = \frac{\langle \pi^0\pi^0 | S | K_L \rangle}{\langle \pi^0\pi^0 | S | K_S \rangle} \]

(2.69)

A decomposition of the pions into specified isotopic spin states yields

\[\eta_{+-} = \frac{\sqrt{2}/3 \times \langle (\pi^+\pi^-)_{I=0} | S | K_L \rangle + 1/3 \times \langle (\pi^+\pi^-)_{I=2} | S | K_L \rangle}{\sqrt{2}/3 \times \langle (\pi^+\pi^-)_{I=0} | S | K_S \rangle + 1/3 \times \langle (\pi^+\pi^-)_{I=2} | S | K_S \rangle} \]

(2.70)

and

\[\eta_{00} = \frac{\sqrt{2}/3 \times \langle (\pi^0\pi^0)_{I=2} | S | K_L \rangle - 1/3 \times \langle (\pi^0\pi^0)_{I=0} | S | K_L \rangle}{\sqrt{2}/3 \times \langle (\pi^0\pi^0)_{I=2} | S | K_S \rangle - 1/3 \times \langle (\pi^0\pi^0)_{I=0} | S | K_S \rangle} \]

(2.71)

By taking advantage of the above defined expressions for \(\epsilon, \epsilon', \omega\) and \(\omega'\) one finds that

\[\eta_{+-} = \epsilon + \frac{\epsilon'}{1 + \frac{\omega}{\sqrt{2}}} \]

(2.72)

\[\eta_{00} = \epsilon - \frac{2\epsilon'}{1 - \sqrt{2}\omega}. \]

(2.73)

These equations make contact between theoretical parameters and experimentally measurable quantities concerning CP-violation. The NA31 experiment at CERN and E731 at Fermilab are measuring \(\eta_{+-}\) and \(\eta_{00}\) and hence the indirect and the direct CP-violating parameters \(\epsilon\) and \(\epsilon'\). Note that the expression for \(\epsilon'\) is doubly suppressed through the \(\Delta I = 1/2\) rule. Remember that the experimental ratio \(\omega = Re A_2/Re A_0 \approx 1/22.2\).
Note that in the expressions used so far in this section no approximations have been used. However, certain approaches may be used.

This will enable us to relate the CP-violating parameter \( \epsilon \) to for instance the \( K^0 - \overline{K}^0 \) mass difference. It can be written [5]

\[
\epsilon \approx \frac{1}{\sqrt{2}} e^{i \pi / 4} \left[ \frac{\text{Im} M_{12}}{\Delta m_K} + \frac{\text{Im} A_0}{\text{Re} A_0} \right]. \tag{2.74}
\]

An expression for the ratio \( \epsilon'/\epsilon \) in terms of the quark operators \( Q_i \), which will be introduced in a later section, can be written in the following form

\[
\frac{\epsilon'}{\epsilon} = \frac{G_F}{2} \frac{\omega}{|| A_0 ||} \text{Im} \lambda \sum_{i=1}^{8} y_i \left\{ \langle Q_i \rangle_0 - \frac{1}{\omega} \langle Q_i \rangle_2 \right\} \tag{2.75}
\]

It has also been suggested that the \( K^0, \overline{K}^0 \) system can be exploited to test the CPT theorem. If violations of the CPT-theorem are found the two phases \( \phi_{\pm} \) and \( \phi_{00} \) would be different. They correspond to complex phases of \( \eta_{\pm} \) and \( \eta_{00} \), respectively.
Chapter 3

Effective Lagrangian at Quark Level

3.1 The Standard Approach to $K \rightarrow 2\pi$ Transitions

The pure electroweak non-leptonic Lagrangian is a local product of quark currents

$$\mathcal{L}_b = -4\frac{G_F}{\sqrt{2}} \lambda_u (\bar{d} \gamma^a L u)(\bar{u} \gamma_\alpha L s) + h.c. \quad (3.1)$$

where $\lambda_u = V_{us} V_{ud}^*$. The Lagrangian is valid in the low energy limit where the quark interactions with the $W$-boson can be approximated to be point like. The lowest order diagram is shown in fig. 2.1.

If we take into account QCD corrections of the order $\alpha_s$ to the lowest order diagram, $\lambda$ matrices will be inserted in the currents in eq. 3.1. This induces a new operator $(\bar{d} \gamma^a L s)(\bar{u} \gamma_\alpha L u)$. It can be seen by applying the completeness relation for the SU(3) generators

$$\lambda^a_{ij} \lambda^a_{kl} = 2(\delta_{ij} \delta_{jk} - \frac{1}{N_c} \delta_{ij} \delta_{kl}) \quad (3.2)$$

followed by a Fierz transformation for anticommuting fermion fields

$$\bar{\psi}_A \gamma^a L \psi_B \bar{\psi}_C \gamma_\alpha L \psi_D = \bar{\psi}_A \gamma^a L \psi_D \bar{\psi}_C \gamma_\alpha L \psi_B. \quad (3.3)$$

We see that the fields $\psi_B$ and $\psi_D$ have changed places. By moving a fermionic field through another fermionic field introduces a minus sign. It can also be shown that another minus sign shows up by exchanging the actual indices on the $\gamma$-matrices. This yields altogether the plus sign. Hence, when we take into account the short distance strong interaction effects, we get an effective weak interaction with a modification of the coefficient of the bare Lagrangian. Additionally, one gets coefficients and operator terms that is not contained originally. This is usually referred to as operator mixing under renormalization. Another $\alpha_s$ contribution to the effective Lagrangian is the penguin interaction which is represented in fig.3.1. This interaction will have an
3.1 The Standard Approach to $K \to 2\pi$ Transitions

Figure 3.1: The penguin interaction.

effective vertex $\propto j^p_\mu(s \to d)\mu(DG)_{\nu}^{\mu}$. If we use the equation of motion for the gluon field, $D_{\nu}^{ab}\pi_{\nu} = -g_2j^2_{\nu}$, it is equivalent to attach a quark current to the free end of the penguin interaction in fig.3.1. Hence, we write the penguin operator as

\[ L_p = -\frac{G_F}{\sqrt{2}} \lambda_u \frac{C_{\rho}}{2} (\bar{d} \gamma^\mu \lambda^a L s) \sum_{q=u,d,s} (\bar{q} \gamma_{\mu} \lambda^a q) + h.c. \tag{3.4} \]

Utilizing the completeness relation and the attending Fierz transformation

\[ \bar{\psi}_A \gamma^\alpha L \psi_B \bar{\psi}_C \gamma_\alpha R \psi_D = -2 \bar{\psi}_A R \psi_D \bar{\psi}_C L \psi_B \tag{3.6} \]

the penguin operator can be divided into four parts:

\[ L_p = -\frac{G_F}{\sqrt{2}} \lambda_u \frac{C_{\rho}}{2} \left[ \bar{d} \gamma^\mu L s \right] \sum_{q=u,d,s} (\bar{q} \gamma_{\mu} L q) + 2(\bar{d}^i \gamma^\mu L s^i) \sum_{q=u,d,s} (\bar{q}^i \gamma_{\mu} L q^i) \]

\[ -\frac{2}{N_c} \bar{d} \gamma^\mu L s \sum_{q=u,d,s} (\bar{q} \gamma_{\mu} R q) + 2(\bar{d}^i \gamma^\mu L s^i) \sum_{q=u,d,s} (\bar{q}^i \gamma_{\mu} R q^i) \right] + h.c. \tag{3.7} \]

\[ L_p = -\frac{G_F}{\sqrt{2}} \lambda_u \frac{C_{\rho}}{4} \left[ Q_4 + Q_6 - \frac{1}{N_c} (Q_3 + Q_5) \right] + h.c. \tag{3.8} \]

In the last step the Fierz transformation has been used and the $Q_i$'s are defined below. A popular approach is the large $N_c$ limit. However, although the number of colours is not at all large the approximation seems to work reasonably well. In the large $N_c$ limit, one can neglect $Q_3$ and $Q_5$ due to the factor $1/N_c$ in front.

The total effective Lagrangian reads \[ 15 \]

\[ L_{eff}^{Q-S} = -\frac{G_F}{\sqrt{2}} \lambda_u \frac{C_{+}}{2} Q_+ + \frac{C_{-}}{2} Q_- + C_3 Q_3 + C_4 Q_4 + C_5 Q_5 + C_6 Q_6 + h.c. \tag{3.9} \]

This is called an operator product expansion due to Wilson \[ 4 \]. When calculating amplitudes, the high energy and low energy part is governed by the Wilson coefficients
3.1 The Standard Approach to $K \to 2\pi$ Transitions

$(C's)$ and the hadronic matrix elements of the quark operators $(Q's)$, respectively. The $Q's$ are defined as

\begin{align*}
Q_+ & = Q_2 + Q_1 \\
Q_- & = Q_2 - Q_1 \\
Q_1 & = 4(\bar{d}\gamma^\alpha Ls)(\bar{u}\gamma_\alpha Lu) \\
Q_2 & = 4(\bar{d}\gamma^\alpha Lu)(\bar{u}\gamma_\alpha Ls) \\
Q_3 & = 4(\bar{d}\gamma^\alpha Ls) \sum_{q=u,d,s}(\bar{q}\gamma_\alpha Lq) \\
Q_4 & = 4 \sum_{q=u,d,s}(\bar{d}\gamma^\alpha Lq)(\bar{q}\gamma_\alpha Ls) \\
Q_5 & = 4(\bar{d}\gamma^\alpha Ls) \sum_{q=u,d,s}(\bar{q}\gamma_\alpha Rq) \\
Q_6 & = -8 \sum_{q=u,d,s}(\bar{d}\gamma^\alpha Rq)(\bar{q}\gamma_\alpha Ld). 
\end{align*}

The last four operators descend from the penguin interaction. There are also some additional operators due to the electroweak penguin interaction, i.e. where the gluon is replaced by a photon or $Z$ boson. They are denoted by $Q_{7-10}$:

\begin{align*}
Q_7 & = 6(\bar{d}Ls) \sum_{q=u,d,s} e_q(qRq) \\
Q_8 & = -12 \sum_{q=u,d,s} e_q(\bar{q}Ls)(\bar{d}Rq) \\
Q_9 & = 6(\bar{d}Ls) \sum_{q=u,d,s} e_q(\bar{q}Lq) \\
Q_{10} & = 6 \sum_{q=u,d} e_q(\bar{q}Ls)(\bar{d}Lq)
\end{align*}

The first six operators $Q_{1-6}$ contribute to CP-conserving processes while operators $Q_{3-10}$ contribute to CP-violating processes.

Of the first six $Q$-operators only five of them are independent since we have

\[ Q_2 + Q_3 = Q_1 + Q_4. \]

Strictly, this is only valid in the four dimensional limit. This fact stems from the Fierz transformation which receives additional contributions when $D \neq 4$.

\[ \bar{u}_L\gamma_\mu s_L = \bar{u}_R\gamma_\mu Ls \]

and for arbitrary dimension $D$ we write

\[ \gamma_\mu \to \tilde{\gamma}_\mu + \tilde{\gamma}_\mu \]
3.1 The Standard Approach to $K \rightarrow 2\pi$ Transitions

The first term on the right hand side denotes the first 4 dimensions and the second term takes care of the remaining $D - 4$ dimensions. In the 't Hooft/Veltman regularization scheme, which we will introduce later, the relation in eq. 3.23 still holds. This can be seen from the second term in eq. 3.1 which gives a vanishing contribution when $\gamma_5$ and $\gamma_\mu$ commutes. We will not use an extended version of eq. 3.23 even though we sometimes will work in the naive dimensional regularization scheme. The main reason for this is calculational convenience and that the error is of subleading nature.

One can form different linear combinations of the operators such that each combination is contained in a multiplet with certain quantum numbers. In Clebsch-Gordan tables one can find the coefficients [1], [21]. We can now associate the different states with the pseudoscalar particles. More details appear in the appendix.

The pseudoscalar particles can now be written in terms of the corresponding quark flavour contents:

\[ K^- = \bar{u}s \]
\[ \bar{K}^0 = \bar{d}s \]
\[ \pi^+ = \bar{d}u \]
\[ \pi^0 = \frac{1}{\sqrt{2}}(\bar{u}u - \bar{d}d) \]
\[ \eta_8 = \frac{1}{\sqrt{6}}(\bar{u}u + \bar{d}d - 2\bar{s}s) \]

By suppressing the Dirac structure this yields

\[ |27; Y = -1, I = 1/2, I_z = +1/2\rangle = \frac{2}{\sqrt{30}} O_3 \]

This expression for $O_3$ is the same as in reference [22] as far as the flavour content is concerned. Similar expressions for operators with definite quantum numbers can be found for other linear combinations of $Q$-operators as well.

\[ |27; Y = -1, I = 3/2, I_z = +1/2\rangle = \frac{2}{\sqrt{6}} [(\bar{u}s)(\bar{d}u) + (\bar{d}s)(\bar{u}u) - (\bar{d}s)(\bar{d}d)] O_4 \]
3.1 The Standard Approach to $K \rightarrow 2\pi$ Transitions

\[ |\bar{s}_1; Y = -1, I = 1/2, I_z = +1/2 \rangle - \frac{2}{\sqrt{30}} |\bar{s} \otimes 1\rangle \]

\[ = \frac{2}{\sqrt{30}} [3(\bar{u}s)(\bar{d}u) - 2(\bar{d}s)(\bar{u}u) + (\bar{d}s)(\bar{d}d) + (\bar{d}s)(\bar{s}s) - \sum_{q=u,d,s} (\bar{d}s)(\bar{q}q)] \]

\[ = -\frac{6}{\sqrt{30}} [(\bar{d}s)(\bar{u}u) - (\bar{u}s)(\bar{d}u)] \]

\[ = \frac{6}{\sqrt{30}} \mathcal{O}_1 \]

\[ |\bar{s}_1; Y = -1, I = 1/2, I_z = +1/2 \rangle + 5\frac{2}{\sqrt{30}} |\bar{s} \otimes 1\rangle \]

\[ = \frac{2}{\sqrt{30}} [3(\bar{u}s)(\bar{d}u) - 2(\bar{d}s)(\bar{u}u) + (\bar{d}s)(\bar{d}d) + (\bar{d}s)(\bar{s}s) + 5 \sum_{q=u,d,s} (\bar{d}s)(\bar{q}q)] \]

\[ = \frac{6}{\sqrt{30}} [(\bar{u}s)(\bar{d}u) + (\bar{d}s)(\bar{u}u) + 2(\bar{d}s)(\bar{d}d) + 2(\bar{d}s)(\bar{s}s)] \]

\[ = \frac{6}{\sqrt{30}} \mathcal{O}_2 \]

The four $\mathcal{O}_1$-states are seen to be orthogonal. We can now relate the states of definite quantum numbers to linear combinations of $Q$-operators that transform in the same way as the states above. We suppress the Dirac and colour parts and consider only the flavour part.

\[ \frac{1}{4}(Q_1 + \frac{2}{3}Q_2 - \frac{1}{3}Q_3) \]

\[ = \frac{1}{3} [2(\bar{d}u)(\bar{u}s) + 2(\bar{d}s)(\bar{u}u) - (\bar{d}s)(\bar{d}d) - (\bar{d}s)(\bar{s}s)] \]

\[ = \frac{1}{9} \frac{\sqrt{30}}{2} |27; Y = -1, I = 1/2, I_z = +1/2 \rangle + \frac{5}{9} \frac{\sqrt{6}}{2} |27; Y = -1, I = 3/2, I_z = +1/2 \rangle \]

\[ \frac{1}{4}(Q_1 - Q_2 + 2Q_3) \]

\[ = 3(\bar{u}s)(\bar{d}u) - 2(\bar{d}s)(\bar{u}u) + (\bar{d}s)(\bar{d}d) + (\bar{d}s)(\bar{s}s) \]

\[ = \frac{\sqrt{30}}{6} |\bar{s}_1; Y = -1, I = 1/2, I_z = +1/2 \rangle \]

The operators $Q_5$ and $Q_6$ come from the penguin interaction and are clearly members of an octet because of the octet times singlet structure in flavour (see eq.3.4). We have seen that operators producing $\Delta I = 3/2$ transitions contain only left-handed fermion fields while a $\Delta I = 1/2$ transition also includes right-handed fermion fields such that one has to evaluate the matrix element of scalar and pseudoscalar densities. Theories with explicit massive terms also deal with left- and right-handed field densities. This could be an indication that the $\Delta I = 1/2$ rule has its origin in chiral symmetry breaking.
For the operators in the effective Lagrangian we have pointed out the different linear combinations which belong to a certain multiplet. These combinations are defined by the operators $O_i$. Their coefficients can be determined through the original coefficients. The four operators with their respective coefficients yield the following Lagrangian:

$$
\mathcal{L} = -\frac{G_F}{\sqrt{2}} \lambda_a \left\{ \left[ \frac{1}{2} C_+ + \frac{1}{2} C_4 - \frac{1}{2} C_3 \right] O_1 + \left[ \frac{1}{10} C_+ + \frac{1}{2} C_4 + \frac{1}{2} C_3 \right] O_2 + \frac{1}{15} C_+ O_3 + \frac{1}{3} C_4 O_4 \right\}
$$

(3.51)

where $O_1$ and $O_2$ are pure $\Delta I = 1/2$ operators while $O_3$ and $O_4$ are pure $\Delta I = 3/2$ operators. In addition, the operators $O_{5-6}$ contribute only to $\Delta I = 1/2$. The numerical values of the different coefficients varies with the renormalization point and have been extensively studied throughout the literature. See for instance [15], [14].

### 3.2 The $K^0 - \bar{K}^0$ Transition

There is only one process measured which exhibits $\Delta S = 2$ transition and that is $K^0 - \bar{K}^0$ mixing. This process has been extensively studied throughout the literature and gives raise to the $K_L - K_S$ mass difference and CP-violating parameter $\epsilon$. The Standard Model prediction is in relatively good accordance with experimental results but the two main contributing effects, short distance and long distance, yields a precise calculation difficult. The standard contributing box diagrams were shown in fig.2.2.

The calculation of an effective $\Delta S = 2$ Hamiltonian from the Standard Model was done by Gaillard and Lee more than twenty years ago [23]. In the same paper they were also able to estimate the mass of the charm quark prior to its experimental discovery due to the experimentally small mass difference between the long-lived and short-lived kaons. They estimated the c-quark mass to be less than $2\text{GeV}$ which fully agrees with experiment. Prior to this, Glashow, Iliopoulos and Maiani (GIM) had predicted its existence in 1970 [24]. They argued that the c-quark had to exist in order to avoid a too large decay amplitude for certain processes.

Fortunately, contrary to the $K \to 2\pi$ interaction, perturbative QCD corrections does not generate new operators and one gets an effective Lagrangian of the form

$$
\mathcal{L}_{\text{box}} = -\frac{G_F}{4\pi^2} M_W^2 b(\mu) \mathcal{O}(\Delta S = 2) \left[ \lambda_1^2 \eta_1 S(x_e) + \lambda_2^2 \eta_2 S(x_t) + 2\lambda_c \lambda_i \eta_3 S(x_c, x_t) \right],
$$

(3.52)

where the mass ratio $x_q = m_q^2/M_W^2$ and the $S$ functions are the Inami-Lim functions [25]

$$
S(x) = x \left[ \frac{1}{4} + \frac{9}{4} \frac{1}{(1-x)} - \frac{3}{2} \frac{1}{(1-x)^2} \right] - \frac{3}{2} \left[ \frac{x}{1-x} \right]^3 \ln x,
$$

(3.53)

$$
S(x_c, x_t) = -x_c \ln x_c + x_c \left[ \frac{x_t^2 - 8x_t + 4}{4(1-x_t)^2} \ln x_t + \frac{3}{4} \frac{x_t}{x_t - 1} \right]
$$

(3.54)
which depend on the masses of the charm and top quarks. The operator is written as

$$O(\Delta S = 2) = (\bar{d}_\mu Ls)(\bar{d}_\mu Ls).$$

(3.55)

The factors $b(\mu)$ and the $\eta'_i$'s descend from short-distance QCD corrections. To leading order

$$b(\mu) = \alpha_s(\mu)^{-2/9}.\quad (3.56)$$

Note that, due to the unitarity of the $CKM$ matrix we have that $\lambda_u + \lambda_c + \lambda_t = 0$. Numerically we also have $|\lambda_u|^2 \gg |\lambda_t|^2$. The real part of the sharp brackets in eq. 3.52 is proportional to $\Delta m_K = m_{K_L} - m_{K_S}$ and one gets contributions from all three terms while the imaginary part is connected to $\epsilon$ and only the last two terms may contribute.

The first term in $L_{box}$ is $CKM$ favoured and gives short distance contribution to $\Delta m_K$. Significant long distance contributions to $\Delta m_K$ are obtained if the quark pair $\bar{q}q$ in the loop in fig. 2.2 is interpreted as bound states of pions. Due to the large top quark mass, the second term dominates the theoretical value of $\epsilon$ even though this term is doubly $CKM$ suppressed. The third term is an interference between a $c$-quark and a $t$-quark running in the "box-loop". It was previously thought to be the most dominating part of the $\epsilon$ parameter due to an assumed low top quark mass.

The measured mass difference between the short lived and long lived kaons is experimentally known to a very high accuracy [1]

$$\Delta m_K = m_{K_L} - m_{K_S} = (3.522 \pm 0.016) \cdot 10^{-12} \text{MeV}.\quad (3.57)$$

If this energy difference had been of electromagnetic origin, it would have corresponded to a photon of wavelength 35 cm. This can be compared with the energy difference in atomic Hydrogen where the splitting between the total spin zero and spin one state corresponds to a wavelength of 21 cm. Thus, the energy difference between $K_L$ and $K_S$ is smaller than the Hydrogen splitting. The essential theoretical uncertainties can mainly be addressed to the $CKM$ parameters and the hadronic matrix element of the relevant quark operators.

The non-diagonal $K^0 - \bar{K}^0$ matrix element of the $\Delta S = 2$ operator is parametrized by the $B_K$ parameter and is conventionally defined by

$$\langle K^0 | O(\Delta S = 2) | \bar{K}^0 \rangle = \frac{4}{3} f_K^2 m_K^2 B_K \quad (3.58)$$

$B_K$ is in fact dependent on the renormalization scale $\mu$. However, the combination

$$\hat{B}_K = b(\mu) B_K(\mu) \quad (3.59)$$

must in principle be renormalization scale independent.

In order to evaluate the hadronic matrix elements one has to deal with four quark operators. They can be dealt with in different models or approximations. One of these approaches is the vacuum saturation approximation (VSA) where one assumes that the main contribution from four quark operators come from the part where one
3.3 Renormalization

has split the four quark operator in two and inserted vacuum states in between. This yields the following factorization of the $\mathcal{O}(\Delta S = 2)$ operator:

$$\langle f|\mathcal{O}(\Delta S = 2)|i\rangle = \langle f|\bar{d}\gamma_{\mu}Ls|0\rangle\langle 0|\bar{d}\gamma^{\mu}Ls|i\rangle$$ \hspace{1cm} (3.60)

and its Fierz-transformed version

$$\langle f|\mathcal{O}(\Delta S = 2)^{F}|i\rangle = \langle f|\bar{d}^{j}\gamma_{\mu}Ls^{k}|0\rangle\langle 0|\bar{d}^{k}\gamma^{\mu}Ls^{j}|i\rangle$$ \hspace{1cm} (3.61)

A typical matrix element is defined as

$$\langle 0|\bar{d}\gamma^{\mu}Ls|K^{0}\rangle = -i\frac{\sqrt{2}}{2}f_{K}k^{\mu}$$ \hspace{1cm} (3.62)

and

$$\langle 0|\bar{d}^{k}\gamma^{\mu}Ls^{j}|K^{0}\rangle = -i\frac{\delta^{jk}\sqrt{2}}{N_{c}}f_{K}k^{\mu}$$ \hspace{1cm} (3.63)

The vacuum saturation approach accommodates only a factorizable contribution which yields $B_{K} = 3/4$ in the large $N_{c}$ limit. By adding the factorizable next to leading order contribution $1/N_{c}$ increases $B_{K}$ to 1. Lattice calculations seem to indicate that the physical $B_{K}$ is numerically around 0.7 while Sum Rules give a value around 0.55. Other models may also give values around 0.3. Generally, $B_{K}$ is determined to lie between 0.3 and 1.2.

3.3 Renormalization

In dimensional regularization the scale $\mu$ is a priori completely arbitrary since it emerges from regularized loop integrals. It appears through the claim that the coupling constant must be a dimensionless quantity. Since there are no restrictions on $\mu$, we can choose $\pi\mu$ or $4\pi\mu$ as our renormalization point. These choices would lead to different vertex functions and Wilson coefficients but they will be related to one another through finite renormalization constants. The group structure can be indicated by choosing three different renormalization schemes $S$, $T$ and $U$ where $U$ can be related to $S$ via $T$ or directly to $S$:

$$(S \rightarrow U) = (S \rightarrow T)(T \rightarrow U)$$ \hspace{1cm} (3.64)

Under these transformations the physical content will remain invariant. These transformations are therefore elements of an abstract symmetry group which is called the Renormalization Group.

A crucial ingredient in the renormalization group analysis is the $\beta$ function. It is defined by the equation

$$\mu \frac{d}{d\mu}g(\mu) = \beta[g(\mu)]$$ \hspace{1cm} (3.65)

Suppose that the $\beta$ function vanishes for a specific argument of the coupling constant
3.3 Renormalization

Figure 3.2: Two typical cases of the $\beta$-function. Figure a) shows an ultraviolet stable fixed point $g_0$ as $s \to \infty$. Figure b) shows an infrared stable fixed point $g_0$ as $s \to 0$.

$g_0$. See fig. 3.2. This is denoted as a fixed point. Suppose further that $\beta(g_0) < 0$ for values of $g$ slightly bigger than $g_0$. As the scaling of momenta $s \to \infty$, $g$ will fall in towards $g = g_0$ if $g$ is larger than $g_0$ at some value of $s$. Similarly if $\beta(g) > 0$ for $g$ less than $g_0$, $g$ will increase towards $g_0$ as $s \to \infty$. This type of fixed point is called ultraviolet stable. The infrared stable fixed point is indicated by considering the mirror function around the vertical line through the ultraviolet fixed point. The third type of fixed point is called unstable. This is the case if the $\beta$ function has a similar shape of a parabola and a minimum at $\beta(g_0) = 0$.

In the $\overline{MS}$ scheme the renormalized quark masses $m_i(\mu)$ are determined by the differential equation

$$\mu \frac{d}{d\mu} m_i(\mu) = -\gamma(g(\mu)) m_i(\mu).$$

(3.66)

The $\beta$- and $\gamma$-functions are determined from loop diagrams to be

$$\beta(g) = -\beta_0 \frac{g^3}{16\pi^2} - \beta_1 \frac{g^5}{(16\pi^2)^2} + \mathcal{O}(g^7)$$

(3.67)

$$\gamma(g) = \gamma_0 \frac{g^2}{4\pi^2} + \gamma_1 \frac{g^4}{(4\pi^2)^2} + \mathcal{O}(g^5)$$

(3.68)

where the constants are defined as follows

$$\beta_0 = 11 - \frac{2}{3} N_f$$

(3.69)
The solutions to the differential equations in eqs. 3.65 and 3.66 are [26]

\[
\begin{align*}
\beta_1 &= 102 - \frac{38}{3} N_f \\
\gamma_0 &= 2 \\
\gamma_1 &= \frac{101}{12} \frac{5}{18} N_f
\end{align*}
\]  

(3.70) (3.71) (3.72)

We will mostly be confronted with the low energy region and use the decoupling theorem [27] to put \( N_j = 3 \) in the equations above since the heavy degrees of freedom only manifest themselves as couplings in the low energy region.

The scale dependent quark condensate in terms of the running masses of the \( u \)-quark and \( d \)-quark is

\[
\langle \bar{q}q \rangle(\mu) = -\frac{f^2 m_2^2}{m_u(\mu) + m_d(\mu)}.
\]  

(3.75)

If we take the renormalization group invariant masses

\[
\bar{m}_u + \bar{m}_d = 12 \text{MeV}
\]  

(3.76)

at \( \mu = 1 \text{GeV} \), we find \( \langle \bar{q}q \rangle(\mu = 1 \text{GeV}) = (-235 \text{MeV})^3 \).

We will now take a closer look at the coefficients \( C_i \) in a former section. Physical results will not depend on the renormalization scale so the product of the Wilson coefficients and the hadronic matrix element of the quark operators must be \( \mu \)-independent i.e.

\[
\frac{d}{d\mu} C_i(\mu) \langle Q_i \rangle(\mu) = 0.
\]  

(3.77)

This is usually a difficult task to show, because the coefficients and matrix elements are calculated at different energy scales. For the density operator \( Q_6 \) the independence can be shown exactly in the leading \( 1/N_c \) limit, but for operators containing \( \gamma_\mu \)'s no one has been able to explicitly show the independence.

The anomalous dimension matrix for all operators is defined by the mixing of the operators \( Q_i \) under renormalization

\[
\mu \frac{d^2}{d\mu^2} \langle Q_i \rangle = -\frac{1}{2} \gamma_{ij} \langle Q_j \rangle
\]  

(3.78)
3.3 Renormalization

In ref.[28] they find to one-loop ($n_f$ is the number of massless quark flavours)

$$\gamma_{ij} = \frac{\alpha_s N_c}{\pi} \times \frac{1}{2} \begin{pmatrix}
\frac{3}{N_c^2} & \frac{3}{N_c^2} & 0 & 0 & 0 \\
\frac{1}{N_c} & \frac{1}{N_c} & \frac{1}{2N_c} & \frac{1}{2N_c} & \frac{1}{2N_c} \\
0 & 0 & \frac{1}{3N_c} & \frac{1}{3N_c} & \frac{1}{3N_c} \\
0 & 0 & \frac{1}{3N_c} & \frac{1}{3N_c} & \frac{1}{3N_c} \\
0 & 0 & \frac{1}{3N_c} & \frac{1}{3N_c} & \frac{1}{3N_c} \\
\end{pmatrix}
$$

The matrix element $(Q_\pm)$ takes care of the low energy part of a certain process, while the $C_i$'s govern the high energy region. They satisfy the following renormalization group equation:

$$\left[ \mu \frac{\partial}{\partial \mu} + \beta(\alpha_s) \frac{\partial}{\partial \alpha_s} - \gamma^T(\alpha_s, \alpha) \right] C_i\left( \frac{M_w^2}{\mu^2}, \alpha_s, \alpha \right) = 0 \quad (3.79)$$

where $\gamma^T$ means the transposed anomalous dimension matrix and $\alpha$ is the weak coupling constant. The general solution to eq.3.79 is

$$C_i\left( \frac{M_w^2}{\mu^2}, \alpha_s, \alpha \right) = C_i(1, \alpha_s(M_w), \alpha) T_{\alpha_s} \left\{ \exp \left[ \int_{\alpha_s(M_w)}^{\alpha_s(\mu)} \frac{\gamma^T(z, \alpha)}{z \beta(z)} \, dz \right] \right\} \quad (3.80)$$

Since $\gamma^T(z, \alpha)$ is a matrix function one must order the matrices when expanding the exponential function. This is formally written $T_{\alpha_s}$, which denotes ordering in the QCD coupling constant such that it is decreasing from left to right. This is similar to the time-ordered product in the Dyson expansion of the S-matrix in perturbation theory. In the basis of $Q_+$ and $Q_-$, the $C_\pm$ coefficients are given by

$$C_\pm = \left[ \frac{\alpha_s(\mu^2)}{\alpha_s(M_w^2)} \right]^{\gamma_\pm/\beta_0} \quad (3.81)$$

$$= \left[ 1 + \beta_0 \frac{\alpha_s(\mu^2)}{4\pi} \cdot \ln\left( \frac{M_w^2}{\mu^2} \right) \right]^{\gamma_\pm/\beta_0} \quad (3.82)$$

The net effect of the higher order corrections is an enhancement of the $Q_-$ operator while the $Q_+$ operator is suppressed. The exponent $\gamma_\pm$ is the anomalous dimension of the operator $Q_\pm$. Numerically, at one loop they are -2 and 4 respectively, which yields values of $C_+$ and $C_-$ approximately 0.7 and 1.9 at $\mu = 1\, GeV$. Thus, we see that the anomalous dimension pulls in the right direction concerning the $\Delta I = 1/2$ rule, but the enhancement is still not strong enough, so there must also be other mechanisms.
Chapter 4

Chiral Perturbation Theory ($\chi PT$)

4.1 Effective Low Energy Field Theories

In order to do calculations in QCD at low energies one needs effective models since the Standard Model does not describe mesonic decays but merely the interactions at quark level. The root of the problem is to get a better description of the confinement effect in QCD. This is due to the growth of the strong coupling constant when approaching lower and lower energies. Hence, we can not treat the interactions perturbatively at low energies. However, $\chi PT$ is an effective theory for low energy QCD while the intermediate energy region is covered by models like a Nambu-Jona-Lasinio model or $\chi QM$.

In this chapter we will comment on effective field theories in general and especially look at $\chi PT$. The $\chi QM$ is left for the next chapter. We have already been in contact with an effective theory in the former section. However, that was at the quark level, purely. An effective low energy theory of QCD will also include mesons.

Our description of physical phenomena relies very much on symmetry. Fields can be represented in various ways through different redefinitions, but the physics remains the same as long as the symmetry requirements are the same. An important theorem due to Haag [29] states the following: If the S-matrix is going to yield the same physics independent of field redefinitions, then the transformed field has to start with a linear term in the original field in order to keep the same particle singularities. Except for this criterion the two fields can be related nonlinearly. Put in another way, the redefined field $\phi'$ is related to $\phi$ by:

$$\phi' = \phi F(\phi) \quad (4.1)$$

and

$$F(0) = 1. \quad (4.2)$$

The field $\phi'$ is the transformed field, non-linearly related to $\phi$, and $F(\phi)$ is an arbitrary function of $\phi$ which fulfills the constraint above. This is called representation independence. To illustrate Haag's theorem, ref. [5] applies it on elastic $\pi^+\pi^0$ elastic scattering at tree level. It is shown that three different Lagrangians give the same
leading order result for the amplitude. Another application is the 'Partial Conserva-
tion of the Axial Current (or PCAC)' theorem which states that one may either use
the pion field $\pi$ or the divergence of the axial vector current $\partial_\mu A^\mu$ as the pseudoscalar
state.

Another useful theorem due to Appelquist and Carazzone [27] can be introduced
by asking the question: If one has a renormalizable theory at a high energy scale
with certain degrees of freedom, how does the theory behave at much lower ener-
gies? The theorem states that the actual degrees of freedom at the lower energy
scale have non-renormalizable corrections due to interactions at high energies. These
corrections are of order $(1/M_H)^N$ where $M_H$ is a typical mass at high energies and
$N$ could be $1, 2, \ldots \infty$. These corrections are non-renormalizable due to a dimensionful
coupling constant. Even though there are infinitely many correction terms, the series
has usually a fast convergence due to a heavy mass in the denominator. Thus, the
series can in practice be truncated after one or two terms. Fermi theory is a typical
application of the Appelquist/Carazzone theorem because correction terms to the
free theory contains four-quark operators of the three light quarks ($u, d, s$) times the
Fermi coupling constant $G_F$. The typical mass parameter in this case is then the
$W$-boson mass. Effects of the three heavy quarks ($c, b, t$) can be seen in the Wilson
coefficients. This means that, even at low energies, one can see relics of high energy
physics through the coupling constants. This could be a very important window to
search for physics beyond the Standard Model. Fine tuning of the experimental pa-
rameters can give us vital information about energies which we will never be able
to reach with today's technique of building accelerator devices. If it is possible to
measure different parameters with high enough accuracy, we could for instance tell
at what scale the suggested supersymmetric particles (SUSY) would come into play.

The difference between a so called fundamental and an effective theory is rather
vague. In fact, all theories are in a way effective. This is so because they can only
be applied at certain energy scales. We will describe the different energy regions in
three stages (see fig. 4.1). At high energies (above $\Lambda = 1\text{GeV}$) one can perturbatively
apply the Standard Model QCD Lagrangian while at energies below $200\text{MeV}$, say,
we have an effective meson theory governed by $\chi PT$. Previously, there were no good
models to calculate the coefficients in the expansion of $\chi PT$. This energy gap has
been somehow bridged by the $\chi QM$ for instance. It is therefore possible to relate
different parameters in the models. Hence, the energy range which the $\chi QM$
can be applied is then between $200\text{MeV}$ and $1000\text{MeV}$ which can be called intermediate
energies. It consists of the Standard Model QCD Lagrangian in addition to a term
that takes into account spontaneous symmetry breaking occurring at the hadronic
scale of order $\Lambda$. This term introduces the pseudoscalar meson octet. It is important
to note that the $\chi QM$ is a model ansatz of QCD in such a way that $\chi PT$ can be
regarded as an effective theory of both the Standard Model QCD Lagrangian and of
the $\chi QM$.

Maybe, a better model at intermediate energies is the extended Nambu-Jona-
Lasinio model (ENJL) [30]. In the mean-field approximation of the ENJL model the
$\chi QM$ is justified. One should note that the different energy regions are not sharply
Figure 4.1: The three stage model.
This three stage model has analogies with a bag model with a shell which is smeared out. The inner part of the bag \((R < R_1)\) is associated with the high energy region where we have quarks and gluons. Then \((R_1 < R < R_2)\), there is an intermediate region consisting of quarks and mesons. In the outer part, we have the pure mesonic region for \(R > R_2\).

In the case of \(\chi PT\) the low energy expansion relies on the expansion in energy which in turn is associated with factors of momenta. The standard references to \(\chi PT\) is the works done by Gasser and Leutwyler [16], [31]. The expansion is valid since the energy is small. To a given order in \(p/M\) one can truncate the series at this order. For instance, to order \(\mathcal{O}(p^4)\) in the expansion one has to use the \(\mathcal{O}(p^4)\) Lagrangian at tree level in addition to \(\mathcal{O}(p^2)\) used at both tree level and as loop diagrams. This is described by Weinberg’s power counting theorem [32]. In order to make physical predictions, the coefficients have to be properly renormalized. The number of possible terms is strongly increasing for higher orders. This complexity puts a practical limit to how many loops one may calculate. Such a theory is non-renormalizable, which means that one has to add infinitely many counterterms in order to get rid of the infinities. However, when calculating amplitudes to a given order one can add a finite number of counterterms.

\section{\(\chi PT\)}

\(\chi PT\) is in principle an effective theory for QCD although it can not be derived from QCD. \(\chi PT\) represents a tool for calculating hadronic matrix elements at very low energies. It is an expansion in momentum and also in current quark masses around the chiral limit. The fields occurring in \(\chi PT\) are only meson fields. To a certain power in momenta in \(\chi PT\), one has to include all chiral \(SU(3)_L \otimes SU(3)_R\) invariant and Lorentz invariant terms that one may construct. The chiral symmetry can only be broken by the current quark mass matrix and not by terms which are by construction non-invariant. In \(\chi PT\) the number of terms is infinite and thereby contains an infinite number of coupling constants. Chiral symmetry does not relate the different couplings, so each one has to be determined experimentally. However, in certain limits they can be related. At low energies higher order terms in momenta are suppressed and only the first terms need to be maintained. To a given order the couplings can be determined experimentally. The coupling constants should in principle also be obtained from QCD. However, this is not possible in practice but can be done with the \(\chi QM\).

In the construction of \(\chi PT\) the current quark masses counts as \(\mathcal{O}(p^2)\). In terms of the basic meson fields contained in the exponential \(\Sigma\) and the external field sources \(s, p, r, l\), the lowest order terms in \(\chi PT\) for strong interactions reads

\[\mathcal{L}^{\chi PT}_{2} = f^2 \frac{g^2}{4} Tr(D_\mu \Sigma D^\mu \Sigma^\dagger) + f^2 \frac{g^2}{4} Tr(\chi^\dagger \Sigma + \Sigma^\dagger \chi) \] (4.3)
where the sigma field,
\[ \Sigma = e^{2\pi f/s}, \]
contains the Goldstone fields. The constant \( f_\pi = 93 \text{MeV} \) has been extracted from leptonic \( \pi \)-decay data and the pion matrix is
\[
\Pi = \sum_a t^a \pi^a = \frac{1}{\sqrt{2}} \begin{pmatrix} \frac{\pi^0}{\sqrt{2}} + \frac{\pi^\pm}{\sqrt{6}} & \pi^\mp \sqrt{2} + \frac{\pi^0}{\sqrt{6}} & \pi^+ \\ \pi^- & \frac{\pi^0}{\sqrt{2}} - \frac{\pi^\pm}{\sqrt{6}} & K^- \\ K^+ & K^0 & \frac{2\pi^0}{\sqrt{6}} \end{pmatrix}.
\]
The factor \( \chi \) is
\[ \chi = 2B_0(s + ip) \]
where \( B_0 \) is given as the ratio of a meson mass squared and the sum of the corresponding current quark masses
\[ B_0 = \frac{m^2}{m_u + m_d} = \frac{m_K^2}{m_u + m_s} = \frac{m_{K^0}^2}{m_d + m_s} = \frac{3m^2}{m_u + m_d + 4m_s}. \]

It can be read off directly by expanding \( \Sigma \) to second order in the meson fields and setting the external scalar source equal to the quark mass matrix \( M \). The \( B_0 \) parameter can also be related to the quark condensate
\[ \langle \bar{q}q \rangle = -B_0 f^2(1 + \mathcal{O}(M)). \]

The \( \Sigma \) field transforms the following way:
\[ \Sigma \rightarrow \Sigma' = U_L \Sigma U_R^\dagger, \]

The covariant derivative acting on the meson fields is
\[ D_\mu \Sigma = \partial \Sigma - i\gamma_5 \Sigma + i\gamma_\mu. \]

The number of a priori unknown constants are strongly increasing to higher order. To order \( \mathcal{O}(p^4) \) there are 12 constants to be determined by experiment. They are conventionally denoted \( L_i \):
\[
L_{4,PT} = L_1 Tr(D_\mu \Sigma^\dagger D^\mu \Sigma)^2 + L_2 Tr(D_\mu \Sigma^\dagger D_\nu \Sigma) Tr(D^\mu \Sigma^\dagger D^\nu \Sigma) + L_3 Tr(D_\mu \Sigma^\dagger D^\mu \Sigma D_\nu \Sigma^\dagger D^\nu \Sigma) + L_4 Tr(D_\mu \Sigma^\dagger D^\nu \Sigma) Tr(\chi^\dagger \Sigma + \Sigma^\dagger \chi) + L_5 Tr(D_\mu \Sigma^\dagger D^\mu \Sigma(\chi^\dagger \Sigma + \Sigma^\dagger \chi)) + L_6 Tr(\chi^\dagger \Sigma + \Sigma^\dagger \chi)^2 + L_7 Tr(\chi^\dagger \Sigma - \Sigma^\dagger \chi)^2 + L_8 Tr(\chi^\dagger \Sigma \chi^\dagger \Sigma + \Sigma^\dagger \chi \Sigma^\dagger \chi) + L_9 Tr(\chi^\dagger \Sigma \chi^\dagger \Sigma + \Sigma^\dagger \chi \Sigma^\dagger \chi) + L_{10} Tr(\Sigma^\dagger F_{\mu\nu} \Sigma F_{\lambda\delta}) + L_{11} Tr(F_{\mu\nu} F_{\lambda\delta} F_{\rho\eta} F_{\chi\gamma}) + L_{12} Tr(\chi^\dagger \chi)
\]
where \( F_{\mu\nu} \) and \( F^\dagger_{\mu\nu} \) are the field strength tensors associated with the external fields \( r_\mu \) and \( t_\mu \). In addition to Lorentz invariance and local chiral symmetry transformation,
Table 4.1: The renormalized coupling constants $L_i(M_p)$. $L_{11}$ and $L_{12}$ are not directly accessible to experiment.

<table>
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<tr>
<th>$i$</th>
<th>$L_i(M_p) \times 10^3$</th>
<th>source</th>
<th>$\Gamma_{i1}$</th>
<th>$\Gamma_{i2}$</th>
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<tr>
<td>1</td>
<td>$0.4 \pm 0.3$</td>
<td>$K^{\pm} \to \pi \pi$</td>
<td>$3/32$</td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>$1.35 \pm 0.3$</td>
<td>$K^{\pm} \to \pi \pi$</td>
<td>$3/16$</td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>$-3.5 \pm 1.1$</td>
<td>$K^{\pm} \to \pi \pi$</td>
<td>$0$</td>
<td></td>
</tr>
<tr>
<td>4</td>
<td>$-0.3 \pm 0.5$</td>
<td>Zweig rule</td>
<td>$1/8$</td>
<td></td>
</tr>
<tr>
<td>5</td>
<td>$1.4 \pm 0.5$</td>
<td>$F_K : F_\pi$</td>
<td>$3/8$</td>
<td></td>
</tr>
<tr>
<td>6</td>
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<td>$11/144$</td>
<td></td>
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<tr>
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<td>12</td>
<td></td>
<td></td>
<td>$5/24$</td>
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</table>

Table 4.1: The renormalized coupling constants $L_i(M_p)$. $L_{11}$ and $L_{12}$ are not directly accessible to experiment.

the strong effective Lagrangian is also invariant under parity and charge conjugation. In order to calculate amplitudes of order $O(p^2)$, one must calculate one-loop diagrams with all vertices from $\mathcal{L}_2$ and tree diagrams with a single vertex from $\mathcal{L}_4$ given above together with any number of vertices from $\mathcal{L}_2$. In order to take into account for the chiral anomaly one must also include the Wess-Zumino-Witten functional [33] and [34]. To one-loop the renormalized $L_i$ constants are determined by [35] in table 4.1. The scale dependence of the $L_i$'s can be written

$$ L_i(\mu_2) = L_i(\mu_1) + \frac{\Gamma_i}{(4\pi)^2} \ln \left( \frac{\mu_1}{\mu_2} \right). $$

Analogous to the $\beta$-function in QCD, the coefficients $\Gamma_i$ serves as the $\beta$-functions of the $L_i$'s.

The effective chiral Lagrangian also contains a weak sector. Due to non-conservation of parity and charge conjugation in weak interactions, the effective low energy Lagrangian must also include non-conservation of the respective transformations.

From the lowest order term in strong $\chi PT$, one can describe the corresponding weak sector of $\chi PT$ by introducing a projection operator $\lambda_+ = \lambda_6 + i\lambda_7$ into the lowest order term in chiral perturbation theory where $\lambda_6$ and $\lambda_7$ are Gell-Mann matrices. This means that the original $P$-even effective Lagrangian from the strong sector, which is realized by $\chi PT$, becomes $P$-odd and thus makes it possible to handle transitions as $K \to 2\pi, 4\pi, \cdots$ etc. where an odd number of mesons participate. To lowest order in the chiral expansion, the weak sector is described by

$$ \mathcal{L}_W^{(2)} = \text{Tr} [\lambda_+ \partial_\mu \Sigma \partial^\mu \Sigma^\dagger]. $$(4.12)
There are other terms as well. They will be outlined from the $\chi QM$ in section 5.3 and to order $O(p^2)$ the result can be written

$$L_{xW}^{\Delta S=1} = G^{(0)}(Q_{7,8}) Tr(\lambda^3_7 \Sigma^1 \lambda^1_7 \Sigma) + G_{g}(Q_{3-10}) Tr(\lambda^2_3 D_{\mu} \Sigma^1 D_{\mu} \Sigma) + G_{sL}(Q_{1,2,9,10}) Tr(\lambda^2_s D_{\mu} \Sigma) Tr(\lambda^1_s D_{\mu} \Sigma^1) + G_{sL}(Q_{7,8}) Tr(\lambda^2_s D_{\mu} \Sigma) Tr(\lambda^1_s D_{\mu} \Sigma) + G_{sL}(Q_{1,2,9,10}) Tr(\lambda^2_s D_{\mu} \Sigma) Tr(\lambda^1_s D_{\mu} \Sigma^1) + G_{sL}(Q_{7,8}) Tr(\lambda^2_s D_{\mu} \Sigma) Tr(\lambda^1_s D_{\mu} \Sigma^1) + Tr(\lambda^1_3 D_{\mu} \Sigma D_{\mu} \Sigma^1) Tr(\lambda^1_7 \Sigma^1). \quad (4.13)$$

The Gell-Mann $SU(3)$ flavour matrices are defined by $(\lambda^j_3)_{ik} = \delta_{ij} \delta_{jk}$. The notation is to remind the reader about which four-quark operators that corresponds to which bosonized term. The $G_{g}$ represents the pure octet part from the gluonic penguin interaction with which we will be concerned. The terms proportional to $G_{sL}^{a,b,c}$ are admixtures of the octet and 27-plet of the interaction. The $G^{(0)}$ term is the constant part arising in the isospin violating and $G_s$ electroweak components where $G_{sL}^{a,b,c}$ represents the corresponding $O(p^2)$ momentum corrections.

The $\Sigma$ field can be expanded in terms of the pion fields to read off the coefficients. Due to the incorporated symmetries in the effective Lagrangian, $K \rightarrow n\pi$ processes can be interrelated. We will look at the off-shell $K \rightarrow \pi$ transition to order $O(p^2)$ and later on draw conclusions at the same order in momenta for especially $K \rightarrow 2\pi$ processes. We find the ratio between $K^- \rightarrow \pi^-$ and $K^0 \rightarrow \pi^+\pi^-$ for the octet amplitudes to be

$$A(K^0 \rightarrow \pi^+\pi^-)_B = \frac{i(m^2_K - m^2_{\pi})}{\sqrt{2} f(p_K \cdot p_\pi)} A(K^- \rightarrow \pi^-)_B. \quad (4.14)$$

The effective realization of the quark currents in terms of pseudoscalar fields is obtained by comparing the standard QCD Lagrangian with $\chi PT$. By differentiating with respect to the external sources (as for instance $W^\mu$ and $M_\mu$) in both models one can compare the results and obtain an effective bosonization of the quark currents.

We will especially emphasize the penguin $Q_6$ operator. The results are given in [15] and read

$$(\bar{q}_i \gamma^\mu L_{qi}) = -\frac{\partial L}{\partial l^\mu_{ji}} = -\frac{if^2}{2}(\Sigma D_{\mu} \Sigma^1)_{ij} + O(p^3) \quad (4.15)$$

$$(\bar{q}_i \gamma^\mu R_{qi}) = -\frac{\partial L}{\partial r^\mu_{ji}} = -\frac{if^2}{2}(\Sigma^1 D_{\mu} \Sigma)_{ij} + O(p^3) \quad (4.16)$$

$$(\bar{q}_i L_{qi}) = -\frac{\partial L}{\partial M_{ij}} = (-1) \left[ 2B_0 \frac{f^2}{4} \Sigma^1_{ji} + 2B_0 L_5 (D_{\mu} \Sigma^1 D_{\mu} \Sigma^1)_{ji} + \cdots \right] \quad (4.17)$$
(\bar{q}, R q) = -\frac{\partial \mathcal{L}}{\partial \mathcal{M}_{ij}} = (-1) \left[ 2B_0 \frac{f_\pi^2}{4} \Sigma_{ji} + 2B_0 L_5 (\Sigma^\dagger D^\mu \Sigma^D_u \Sigma)_{ji} + \cdots \right] \quad (4.18)

In order to find an effective realization of the \( Q_6 \) operator we see that we have to go to \( \mathcal{O}(p^4) \) in \( \chi PT \) in order to find a contribution since the lowest order result only gives us \( \Sigma \Sigma^\dagger = 1 \). Beyond the first trivial order this yields

\[
(\bar{q}, R q) (\bar{q} L s) = 2B_0^2 f_\pi^2 L_5 \left[ D^\mu \Sigma^D_u \Sigma \right]_{32}
\]

(4.19)

In fact, when calculating the matrix element of the \( Q_6 \)-operator will give an estimate of the \( L_5 \)-coefficient. The \( K^- \rightarrow \pi^- \) transition is related to the \( L_5 \) coefficient by

\[
\langle \pi^- | Q_6 | K^- \rangle = \frac{-32 \langle \bar{q} q \rangle_0^2 (p_K \cdot p_\pi)}{f_\pi^4} L_5.
\]

(4.20)

Following [15] we define the coupling constant \( g_8^{(1/2)} \) of the octet chiral Lagrangian. In term of this, the virtual \( K \rightarrow \pi \) amplitude can be written:

\[
\mathcal{M}(K^- \rightarrow \pi^-)_8 = -\sqrt{2} G_F \lambda_u k^2 g_8^{(1/2)} f_K f_\pi,
\]

(4.21)

where \( k \) is the momentum of the virtual meson transition. The standard contribution to \( g_8^{(1/2)} \) for the penguin contribution, mainly due to \( Q_6 \), can be written as

\[
g(Q_6)_s \equiv g_8^{(1/2)}(Q_6)_s = -16 \operatorname{Re} C_6(\mu) \left( \frac{\langle \bar{q} q \rangle_0}{f_\pi^3} \right)^2 L_5,
\]

(4.22)

where \( L_5 \) is the coupling constant of the relevant term in the strong chiral Lagrangian of \( \mathcal{O}(p^4) \) as defined in eq. 4.10.

The largest contribution to \( g_8^{(1/2)} \) is coming from the four quark operator \( Q_- \) of left-left type, which gives the amplitude[15]

\[
g_8^{(1/2)}(Q_-) = \frac{1}{2} C_- [1 - \frac{1}{N_c} (1 - \delta)]
\]

(4.23)

where \( C_- \) is the Wilson coefficient of the operator \( Q_- \), which is \( C_- \simeq 2 \) at \( \mu \simeq 0.8 GeV \), and the quantity \( \delta \) represents non-factorizable gluon condensate corrections which will be introduced in the next section:

\[
\delta \equiv \frac{N_c < \frac{\alpha_s}{\pi} G^2 >}{32 \pi^2 f_\pi^4}.
\]

(4.24)

Writing \( < \frac{\alpha_s}{\pi} G^2 > = \eta^4 \), the quantity \( \eta \) is of the order 400 MeV, which gives a value of \( \delta \) around 3 (\( \delta \simeq 2.6, 3.0, 3.7 \) for \( \eta = 376, 390, 410 \) MeV, respectively.) Using \( \eta = 376 MeV \) and \( \mu \simeq 0.8 GeV \), one obtains \( g_8^{(1/2)}(Q_-) \simeq 1.6 \), while for \( \eta = 390 MeV \) and \( \mu \simeq 320 MeV \) one obtains [15] \( g_8^{(1/2)}(Q_-) \simeq 2.6 \). Anyway, the prediction will be below the experimental value for the total \( \Delta I = 1/2 \) amplitude [36, 15]

\[
g^{(1/2)}_{exp} = 5.1.
\]

(4.25)
4.2 An Example of the FKW Theorem in $\chi^PT$

The lowest order terms in $\chi^PT$ is shown in eq.4.3. In the weak sector of $\chi^PT$ there is a term which is normally not cited

$$\mathcal{L}_8 = \overline{g}_8 \text{Tr}(\lambda_8 \chi \Sigma^I + \Sigma \chi^I \lambda_8^I). \quad (4.26)$$

We will show how this term gives a non-vanishing contribution in the $K^0 \rightarrow \pi^+\pi^-$ decay. It is important to realize that $\mathcal{L}_8$ gives rise to tadpole diagrams. The two contributions are shown in fig. 4.2. If we expand $\mathcal{L}_8$, the actual terms become

$$\mathcal{L}_8 = 2B_0 (m_s - m_d) \overline{g}_8 [\frac{-\sqrt{2} K^0}{f} + \frac{\sqrt{2}}{3f^3} K^0 \pi^+ \pi^-] \quad (4.27)$$

This gives the Feynman rules for the weak vertices. In the strong sector one finds

$$\frac{f^2}{4} \text{Tr}(\partial_\mu \Sigma \partial^\mu \Sigma^I) = \frac{1}{2} \partial_\mu \pi^a \partial^\mu \pi^a + \frac{1}{6f^2} [((\pi^a \partial_\mu \pi^a)^2 - \pi^a \pi^a (\partial_\mu \partial^\mu \pi^b)] + \cdots \quad (4.28)$$

where we have made use of $\text{Tr}(\lambda^a \lambda^b) = 2\delta^{ab}$ and $\text{Tr}(\lambda^a \lambda^b \lambda^c \lambda^d) = 2(\delta^{ab} \delta^{cd} - \delta^{ac} \delta^{bd} + \delta^{ad} \delta^{bc})$. Taking also into account the terms proportional to the current quark masses in the strong sector one finds for the actual terms

$$\mathcal{L}_s(\pi^4) = \frac{1}{3f^2} [((\pi^+ \partial_\mu \pi^-)(K^0 \partial^\mu K^0) + (\pi^+ \partial_\mu \pi^-)(K^0 \partial^\mu K^0)] \quad (4.29)$$
By using $\partial \rightarrow ip_u$ for incoming and $\partial \rightarrow -ip_u$ for outgoing particles, one finds that the momentum dependent terms give a factor $2(m_u^2 - m_d^2)$. Combining all the terms and using $i/(-m_K^2)$ for the tadpole propagator give the total amplitude as

$$
\mathcal{M} = \frac{2\sqrt{2}B_0g_8}{f^3}(m_s - m_d) \left[ \frac{B_0(m_u + 2m_d + m_s)}{6m_K^2} + \frac{1}{6} \left( 1 - \frac{m_u^2}{m_K^2} \right) - \frac{1}{3} \right] (4.33)
$$

$$
= \frac{2\sqrt{2}B_0g_8}{f^3}(m_s - m_d) \left[ \left( \frac{m_u^2}{6m_K^2} + \frac{1}{6} \right) + \frac{1}{6} \left( 1 - \frac{m_u^2}{6m_K^2} \right) - \frac{1}{3} \right] (4.34)
$$

$$
= 0 (4.35)
$$

This is an example on the Feinberg-Kabir-Weinberg theorem where $\mathcal{L}_8$ does not contribute to any physical processes.

Another way of looking at the weak extra term, $\mathcal{L}_8$, is the following: because $\chi$ is proportional to the current quark mass matrix one can redefine the masses. The new mass matrix can then be diagonalized by a proper transformation. Hence, the $\mathcal{L}_8$ term does not contribute to any physical process as shown explicitly by the diagrammatic calculation.
Chapter 5

The Chiral Quark Model (\(\chi QM\))

5.1 \(\chi QM\)

As a bridge between high and low energy QCD, the \(\chi QM\) has been vindicated as an effective model of QCD at intermediate energies [17], [30]. This is the mixed phase of the three stage model. The Lagrangian can be written in the following way:

\[
\mathcal{L} = \mathcal{L}_{QCD} + \Delta \mathcal{L}_\chi,
\]

(5.1)

where

\[
\Delta \mathcal{L}_\chi = -M(q_L^c q_R^c + q_R^c \Sigma^l q_L^c)
\]

(5.2)

Even though the QCD Lagrangian has the same form as in eq. 1.4, we have now only three quarks. They form a triplet under \(SU(3)\) transformations

\[
q = \begin{pmatrix} u \\ d \\ s \end{pmatrix}
\]

and the \(c, b, t\) quarks and the heavy gauge bosons are integrated out. The coupling constant \(M\) has dimension mass and acts as an infrared regulator.

In the \(\chi QM\) the pions and quarks are treated as distinguished fundamental particles. This is contrary to short distance effects which are calculated by assuming the valence quark approximation. The pions are considered as an external field and in the version of the \(\chi QM\) that we have used there are no propagating pion fields at tree level. They occur only after integrating out the quarks. One should therefore not have the picture in mind that the pions consist of quark/antiquark pairs. This has been called the European version of \(\chi QM\). In the American version of the model they add the kinetic pion term \(Tr(\partial_\mu \Sigma \partial^\mu \Sigma^l)\) at the tree level. This could be justified because all terms obeying certain symmetry requirements \(SU(3)_L \otimes SU(3)_R\) should be present in the Lagrangian. We will, however, use the European approach.

One may think that in the effective theory with \(\bar{q}q\) states and pions as fundamental particles would give a double counting in number of states. However, in reference [37]
they argue that this is not the case since otherwise the $\bar{q}q$ state would have been pushed far beyond the $\Lambda_X$ energy scale and hence it would not have been a part of the effective theory.

The sigma field and its transformation properties where introduced in eqs.4.4,4.8, and the left- and right-handed quark fields transform as

$$q_L \rightarrow q'_L = U_L q_L$$ (5.3)

$$q_R \rightarrow q'_R = U_R q_R.$$ (5.4)

In the chiral limit, where $M_q \rightarrow 0$, the Lagrangian is seen to be $SU(3)_L \otimes SU(3)_R$ invariant and it is triggered by the chiral symmetry breaking. This way of parametrizing chiral symmetry breaking is built in by hand and the $\chi QM$ is therefore an ansatz or a phenomenological model. It can not obtained from a fundamental theory by an integration of a certain field in a path integral formalism. One can however look at the model as having integrated out the heavy degrees of freedom leaving only the three light quarks $u, d, s$ and soft gluons. It is important to note that the gluons are soft, i.e. having low energies. We will later come back to how they condense and give rise to gluon condensate corrections. Effects of having integrated out the heavy degrees of freedom are assembled in the mass parameter $M$. One should note that the terms in the Lagrangian connected to the constituent quark mass $M$ and the current quark mass $M_q$ have different transformation properties. We will go to the chiral limit and ignore the current quark mass.

The constituent quark mass $M$ can be interpreted in different ways. It can for instance be regarded as an effective mass due to a surrounding pion "cloud" or, in terms of Feynman path integral formalism, as the accumulated effect of integrating out the heavy degrees of freedom.

Due to the term $\Delta L_X$ which couple quarks to mesons, one can calculate hadronic matrix elements as loop diagrams. In this way one incorporates off-shell effects and non-local effects as momentum variation of the effective interactions can be integrated out.

The $\chi QM$ is thought to apply for momenta of the order and below the scale of chiral-symmetry breaking, which we define to be

$$\Lambda_X = 2\pi f_\pi \sqrt{6/N_c},$$ (5.5)

where $N_c = 3$ is the number of colours (-numerically $\Lambda_X = 0.83 \text{ GeV}$). The model can be used down to a scale of about $200 \text{ MeV}$.

The model has appealing features such that from this model one can for instance integrate out the meson modes to get the Nambu-Jona-Lasinio model. Another feature is that we can obtain an effective meson theory $\chi PT$ by integrating out the quark degrees of freedom in the $\chi QM$ [17]. The coefficients in $\chi PT$ can therefore be determined with the $\chi QM$.

Nature does not reveal $SU(3)_L \otimes SU(3)_R$ symmetry due to the lack of a parity doublet partner of the proton. Since a continuous symmetry is broken, Goldstones
Table 5.1: These are the exceptions where \( a_1 \) is not equal to unity. See also text for explanation.

Theorem can be applied \[8\]. It states that to each generator of a continuous symmetry group that is spontaneously broken there exists a spinless and massless particle. They are zero energy excitation modes due to the degenerate vacuum structure. Since eight generators are broken, there should exist eight massless particles in nature. We do not know about massless octets but there is a meson octet \( J^P = 0^- \) which is massless in the limit where the current quark masses are vanishing. Thus, the inclusion of the pseudoscalar meson octet in accordance with the \( SU(3)_L \otimes SU(3)_R \) symmetry is a vital ingredient in the \( \chi Q M \). The spontaneous symmetry breaking is parametrized through the existence of approximate Goldstone bosons. Spontaneous symmetry breaking in QCD is demonstrated through the non-zero vacuum expectation values of \( \langle 0 | \bar{u} u | 0 \rangle \), \( \langle 0 | \bar{d} d | 0 \rangle \) and \( \langle 0 | \bar{s} s | 0 \rangle \). The pion masses are related to the vacuum condensates by the current algebra Ward identity from eq. 4.7:

\[
(m_u + m_d)\langle 0 | \bar{u} u + \bar{d} d | 0 \rangle = -2f_\pi^2 m_\pi^2 (1 + O(M)). \tag{5.6}
\]

We see that in the chiral limit the pion becomes massless.

In order to obtain Feynman rules the Lagrangian \( \Delta L_x \) can be expanded in powers of \( 1/f_\pi \)

\[
\Delta L_x = -M\bar{q}q - 2iM\frac{\bar{q}\gamma_5 q}{f_\pi} + 2M\frac{\bar{q}\Pi^2 q}{f_\pi} + \cdots \tag{5.7}
\]

and this yields the Feynman rules. Each meson-quark-quark interaction gives a factor

\[
a_1 \frac{\sqrt{2} M}{f_\pi} \gamma_5, \tag{5.8}
\]

where the constant \( a_1 \) stem from \( SU(3) \) symmetric factors from eq.5.7 and is equal to one except for the cases where \( \pi^0 \) or \( \eta_8 \) is involved. The exceptions are listed in table 5.1. The vertex factor corresponds to the interaction in fig. 5.1a. For meson-meson-quark-quark vertices one gets a factor

\[
a_2 \frac{iM}{f_\pi^2}, \tag{5.9}
\]
Figure 5.1: Typical diagrams generated by the $\Delta \mathcal{L}_\chi$ Lagrangian. There exists also vertices with three, four etc. mesons interacting with the quarks.

<table>
<thead>
<tr>
<th>Vertex</th>
<th>$a_2$</th>
<th>Vertex</th>
<th>$a_2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\bar{u}u\pi^0\pi^0$</td>
<td>$\frac{1}{2}$</td>
<td>$dd\pi^0\pi^0$</td>
<td>$\frac{1}{2}$</td>
</tr>
<tr>
<td>$\bar{u}u\eta_8\eta_8$</td>
<td>$\frac{1}{2}$</td>
<td>$dd\eta_8\eta_8$</td>
<td>$\frac{1}{2}$</td>
</tr>
<tr>
<td>$du\pi^-\eta_8$</td>
<td>$\frac{1}{2}$</td>
<td>$\bar{u}d\pi^+\eta_8$</td>
<td>$\frac{1}{2}$</td>
</tr>
<tr>
<td>$\bar{s}uK^-\pi^0$</td>
<td>$\frac{1}{2}$</td>
<td>$\bar{u}sK^+\pi^0$</td>
<td>$\frac{1}{2}$</td>
</tr>
<tr>
<td>$\bar{s}uK^-\eta_8$</td>
<td>$\frac{1}{2}$</td>
<td>$\bar{u}sK^+\eta_8$</td>
<td>$\frac{1}{2}$</td>
</tr>
<tr>
<td>$\bar{s}dK^0\pi^0$</td>
<td>$-\frac{1}{\sqrt{2}}$</td>
<td>$\bar{d}sK^0\pi^0$</td>
<td>$-\frac{1}{\sqrt{2}}$</td>
</tr>
<tr>
<td>$\bar{s}dK^0\eta_8$</td>
<td>$-\frac{1}{\sqrt{2}}$</td>
<td>$\bar{d}sK^0\eta_8$</td>
<td>$-\frac{1}{\sqrt{2}}$</td>
</tr>
<tr>
<td>$\bar{s}s\eta_8\eta_8$</td>
<td>$\frac{1}{3}$</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Table 5.2: These are the exceptions where $a_2$ is not equal to unity. See also text for explanation.
where $a_2$ is equal to unity except for the cases shown in table 5.2. The vertex factor corresponds to the interaction in fig. 5.1b.

When calculating amplitudes in the $\chi QM$, one can be faced with terms from individual diagrams which seems to break chiral symmetry. However, this is only preliminary. By summing all possible contributions to a given order such terms will cancel and the result will be proportional to momentum squared in accordance with chiral symmetry. This means that in the chiral limit the amplitudes vanish. This is the FKW theorem due to Feinberg, Kabir and Weinberg [38]. See also [39]. An exception is the $Q_{11}$ operator which breaks chiral symmetry.

There is also a rotated picture for $\chi QM$. The $Q_{L,R}$ are the rotated quark fields

$$q_L = \xi^\dagger Q_L, \quad \bar{q}_L = \overline{Q}_L \xi, \quad q_R = \xi Q_R, \quad \bar{q}_R = \overline{Q}_R \xi^\dagger \quad (5.10)$$

where $\xi^2 = \Sigma$ and $(\xi^\dagger)^2 = \Sigma^\dagger$. This rotation corresponds to transform the $\chi QM$ Lagrangian into a pure mass term

$$\mathcal{L}_\chi \rightarrow -M(\overline{Q}_R Q_L + \overline{Q}_L Q_R) \quad (5.11)$$

while the quark coupling to the mesons is transfered to the kinetic term in the Lagrangian

$$\overline{Q}(\gamma^\mu \gamma_\mu + \gamma^\mu \gamma_5 \gamma_5 A_\mu) Q \quad (5.12)$$

where

$$\gamma_\mu = (R_\mu + L_\mu)/2 \quad (5.13)$$

$$A_\mu = (R_\mu - L_\mu)/2 \quad (5.14)$$

and

$$R_\mu = \xi^\dagger i \partial_\mu \xi + \xi^\dagger r_\mu \xi \quad (5.15)$$

$$L_\mu = \xi i \partial_\mu \xi^\dagger + \xi l_\mu \xi^\dagger \quad (5.16)$$

In this rotated picture the quark-meson couplings will also contain $\gamma_\mu$ matrices. However, the physical picture is still the same. The axial vector can be written

$$A_\mu = -\frac{i}{2} \xi (D_\mu \Sigma^\dagger) \xi = \frac{i}{2} \xi^\dagger (D_\mu \Sigma) \xi^\dagger \quad (5.17)$$

and one can see that it gives rise to the lowest order term in the strong Lagrangian, i.e. $Tr(D_\mu \Sigma D^\mu \Sigma^\dagger) \propto Tr(A_\mu A^\mu)$.

### 5.1.1 Relations between $f_\pi$, $M$, $\Lambda$ and $\langle \bar{q}q \rangle$

In $\chi QM$ one can calculate the pion decay constant. In eq.4.4 we have used $f_\pi$ instead of $f$ in the exponential. This is correct to lowest order but in our version of the $\chi QM$ it is also given by a loop diagram due to the meson-quark coupling in $\Delta \mathcal{L}_\chi$. This factor with corrections is called $f_\pi^{(0)}$. Fig. 5.2 shows the lowest order loop with gluon
5.1 Relations between $f_\pi$, $M$, $\Lambda$ and $\langle \bar{q}q \rangle$

condensate corrections. We refer to section 5.4 for details concerning the correction terms. The result reads

$$f_\pi^{(0)} = \frac{N_c M^2}{4 \pi^2 f} \left[ f_\pi + \frac{\pi^2}{6 N_c M^4} \left( \frac{\alpha_s G^2}{\pi} \right) + \cdots \right]$$  \hfill (5.18)

This is contrary to the American version where the pion loop is a correction to the lowest order result. The size of the higher order condensates depends on the ratio between the gluon condensates and the constituent quark masses but is generally thought to be relatively small. The physical pion decay constant has been experimentally determined to be $f = 93.3 \text{MeV}$ and the 'bare' decay constant $f_\pi^{(0)}$ is usually approximated by $f$. In practice one puts $f_\pi^{(0)} = f = f_\pi$. However, at intermediate stages in the calculations it can be useful to distinguish them. The quantity $f_\pi$ is dependent on the regularization scheme. In a sharp cut-off or Pauli-Villars regularization the leading behaviour is $\ln(\Lambda^2)$ where $\Lambda$ is a phenomenological ultraviolet cutoff and the logarithmic term is absorbed in the physical pion decay constant. In dimensional regularization the singularity turns out to be $f_\pi = \Gamma(\epsilon)$. Thus, a naive logarithmic divergence turns out to be proportional to $1/\epsilon$ in the dimensional analysis. This means that the renormalized pion decay constant has embedded the $\epsilon$-singularity. The condensate corrections are rather large and influence the size of the cut-off significantly in order to keep $f_\pi$ constant.

We have a similar expression for the quark condensate. The lowest order term is quadratically divergent and the correction terms are proportional to $1/M$. If we include first order corrections the quark condensate is

$$\langle \bar{q}q \rangle^{(0)} = -\langle \bar{u} \bar{d} \rangle f_\pi^{(0)} \int \frac{d^D p}{(2\pi)^D} \text{Tr}[iS(p)]$$

$$= \frac{N_c M}{4 \pi^2} C_q - \frac{1}{12 M} \langle \frac{\alpha_s G^2}{\pi} \rangle + \cdots ,$$  \hfill (5.19)

where $C_q = -\Lambda^2 + M^2 \ln(\frac{\Lambda^2 + M^2}{M^2})$ in a sharp cut-off procedure. Within dimensional regularization $C_q$ can be written

$$C_q = -M^2 \Gamma(-1 + \epsilon)(4\pi\bar{\mu}^2/M^2)^\epsilon$$  \hfill (5.20)

The equations of $\langle \bar{q}q \rangle^{(0)}$ and $f_\pi^{(0)}$ put restrictions on the parameters of the $\chi Q M$.

In order to have a meaningful convergent expansion in the condensates, the constituent quark mass must not be too small. The equations above define the scale $\Lambda$ and the constituent mass $M$. The quark condensate is numerically determined by QCD sum rules or similar methods. As we approximated $f_\pi^{(0)}$ with $f$ we also do the same for $\langle \bar{q}q \rangle^{(0)}$. Its value is approximately $\langle \bar{q}q \rangle = (-235 \text{MeV})^3$. It is in fact scale dependent and a more refined version is found elsewhere in this thesis. One can draw a function in which the points fulfill the two eqs. 5.18, 5.20 above. However, the result is very much dependent on the parameter $f_\pi$. For $f_\pi = \ln(\Lambda^2/M^2) - 2$ and the value of $\Lambda$ around $800 \text{MeV}$ yields $M$ approximately $260 \text{MeV}$ while the quark condensate becomes $-(235 \text{MeV})^3$. In the dimensional analysis the situation is more complicated.
Figure 5.2: Diagrams contributing to $f_\pi^{(0)}$. The small black circles denotes the vertex of the axial vector.
than for \( f_\pi \). The term corresponding to \( C_q \) above turns out to be \(-M^2 \Gamma(-1 + \epsilon)\) which is divergent both for \( \epsilon = 0,1 \). If the divergence is written as a series in \( \epsilon \) it can not be distinguished from the logarithmic divergence in \( f^{(0)}_\pi \). One can look at \( f^{(0)}_\pi \) and \( C_q \) as book-keeping parameters without ascribing numerical values.

In more complicated calculations one may use a relation between the quark condensate and \( f_\pi \) which reads

\[
C_q = M^2 (f_\pi + 1).
\] (5.22)

One may use this relation to trace back some of the \( 1/\epsilon \) terms to the condensate. However, this must be done with care, as it is not always obvious which terms belong to the quark condensate or to \( f_\pi \).

### 5.2 Vacuum Saturation Approximation

In order to have a theoretical estimate of the \( \Delta I = 1/2 \) selection rule is to have a reliable determination of the short distance effects (Wilson coefficients) and long distance effects (operators). In order to evaluate the hadronic matrix elements one has to deal with four quark operators. They can be dealt with in different models or approximations. One of these approaches is the vacuum saturation approximation (VSA) where one assumes that the main contribution from four quark operators come from the part where one has inserted vacuum states between the currents. This yields the following factorization of the \( Q_2 \) operator:

\[
\langle f | Q_2 | i \rangle = 4 \langle f | \bar{d} \gamma_\mu L u | 0 \rangle \langle 0 | \bar{u} \gamma^\mu L s | i \rangle
\] (5.23)

and its Fierz-transformed version

\[
\langle f | Q^F_2 | i \rangle = 4 \langle f | \bar{d}^i \gamma_\mu L s^k | 0 \rangle \langle 0 | \bar{u}^k \gamma^\mu L u^j | i \rangle
\] (5.24)

Typical matrix elements for a \( K^- \rightarrow \pi^- \) transition are shown in eqs. 3.62, 3.63. Remembering that all possible contractions have to be performed in the \( S \)-matrix expansion we will generally also encounter matrix elements of the form:

\[
\langle 0 | \bar{q} q | 0 \rangle | \pi^- | \bar{d} s | K^- \rangle.
\] (5.25)

To insert only vacuum states between the currents is actually a crude approximation. A more rigorous treatment must include a complete set of intermediate states since it is not obvious that they can be neglected. Such terms are referred to as non-factorizable contributions.

### 5.3 Construction of Weak \( \chi PT \) by using the \( \chi QM \)

Following the procedure of [18] the lowest order terms in the weak sector of \( \chi PT \) can be obtained by inserting a combination of the Gell-Mann \( \lambda \) matrices in between the field operators in a quark current. Typically

\[
\bar{q}_L \gamma^\mu q'_L \rightarrow \bar{q}_L \lambda^m_\mu \gamma^\mu q_L,
\] (5.26)
where $\lambda_i^n$ are appropriate projectors in flavour space. If the same rotation is applied on the quark fields in eq.5.26, they become

$$\bar{q}_L \lambda_i^n \gamma^\mu q_L \rightarrow \bar{Q}_L \xi \lambda_i^n \gamma^\mu \xi^\dagger Q_L$$

(5.27)

The axial vector in eq. 5.17 must be coupled to the rotated quark fields in all possible ways that are consistent with Lorentz invariance and gauge invariance. Note that the vector field transforms differently and will not contribute to gauge invariant terms. This yields an effective theory of mesons when the quark fields are integrated out. The bosonization of the four-quark operators of left-left type is

$$Tr(\xi \lambda_i^n \xi^\dagger A^\mu \lambda_i^n \xi^\dagger A^\mu)$$

(5.28)

$$Tr(\xi \lambda_i^n \xi^\dagger \lambda_i^n \xi^\dagger A^\mu \lambda_i^n \xi^\dagger \lambda_i^n A^\mu)$$

(5.29)

$$Tr(\xi \lambda_i^n \xi^\dagger A^\mu \lambda_i^n \xi^\dagger A^\mu)$$

(5.30)

A similar bosonized composition of four-quark operators of left-right type contain terms of the following type

$$Tr(\lambda_i^n \lambda_i^n \xi^\dagger \lambda_i^n \xi^\dagger \lambda_i^n \lambda_i^n)$$

(5.31)

$$Tr(\lambda_i^n \lambda_i^n D^\mu \xi^\dagger \lambda_i^n \xi^\dagger D^\mu)$$

(5.32)

$$Tr(\lambda_i^n \lambda_i^n D^\mu \xi^\dagger \lambda_i^n \lambda_i^n D^\mu)$$

(5.33)

$$Tr(\lambda_i^n \lambda_i^n \xi^\dagger \lambda_i^n \lambda_i^n D^\mu \xi^\dagger D^\mu)$$

(5.34)

Only for some electroweak penguins all the terms must be applied, while for most of the four-quark operators some of the terms are vanishing. For gluonic penguins only one term is left when summing all quark flavours. This yields the weak chiral Lagrangian in eq. 4.13.

In our way of looking at the $\chi QM$ the renormalization scale $\mu$ is absent. The $\mu$ dependence from the short-distance contribution of QCD is however balanced in principle by a $\mu$ dependence at mesonic level, i.e. due to meson loops in $\chi PT$.

### 5.4 Condensates

Approaching confinement energies from above, the strong coupling constant grows and thus aggravates the perturbative picture and becomes useless when $\alpha_s/\pi \gtrsim 1$. For moderate and small momenta one can calculate deviations from perturbation theory by considering non-vanishing vacuum matrix elements of quark and gluon operators [40]. These operators are local, and can be determined numerically by the QCD sum rule approach, lattice QCD, dilute instanton gas approximation or similar methods. See [41], [40], [15] and references therein. The matrix elements are fundamental in the sense that they characterize the quark-gluon interactions at long distances. To estimate the vacuum corrections is a very difficult task and large numerical uncertainties adheres even to the lowest order operators.
In a low energy calculation each occurrence of a matrix element of dimension $d$ will always be accompanied by a factor $1/M^d$. The mass $M$ is the constituent quark mass for $u,d,s$ and it cannot be too small in order to give a reliable expansion.

In the ordinary penguin diagram the gluon coupled to the effective weak vertex in one end has usually been coupled to a quark current in the other end due to the equations of motion. Due to the non-perturbative vacuum structure the low energy gluons act as a mediator between vacuum and the weak vertex. The gluons attached to a quark line and to vacuum will therefore describe condensate corrections. As we will see more clearly later, we will be faced with factors of gluon tensors times a momentum integration. The product of gluon tensors are most naturally interpreted as a vacuum expectation value. This is how non-perturbative effects are incorporated in a perturbative picture. A Lorentz decomposition gives a sum of products of metric tensors which in turn has to be contracted with the indices in the remaining part of the expression. One should note that in order to have zero momentum in vacuum one must have at least two gluons contributing to a physical process.

By using standard Feynman rules to calculate gluonic vacuum corrections one has to deal with the gauge dependent quantity $A^a_\alpha(x)$. However, the final result will be a function of the gluon tensor and the covariant derivative. Fortunately, one can take advantages by choosing a particular gauge. In the description of the external gluon field we will use the point gauge [42] or sometimes called the Fock-Schwinger gauge.

### 5.4.1 The Point Gauge/Fock-Schwinger Gauge

In order to calculate the effective propagator one uses the so called Fock-Schwinger gauge or point gauge. This gauge shows up to be particularly useful because the vector potential can be written as a series in the gluonic tensor $G_{\mu\nu}$ and its covariant derivative. The gauge is defined by

$$ (x - x_0)^{\alpha} A^a_\alpha(x) = 0 \quad (5.35) $$

where $A^a_\alpha(x)$ is the gluon field. The point $x_0$ is arbitrary and acts as a gauge parameter. This description turns out to be particularly useful because the gluon field can be written as an expansion of the gluon tensor and the covariant derivative. This simplifies the calculation of the gluonic vacuum corrections considerably. In actual calculations the gauge parameter should drop out when summing all relevant diagrams. In order to minimize the number of parameters, we choose a vanishing gauge parameter. To find the vector potential as a function of the gluon tensor, one can start with the partial derivative:

$$ \partial_{\alpha}(z^\beta A^a_\beta(z)) = A^a_\alpha(z) + z^\beta(\partial_{\alpha}A^a_\beta(z)) \quad (5.36) $$

The left hand side is zero due to point gauge. The gluonic tensor is

$$ G^a_{\alpha\beta}(z) = \partial_{\alpha}A^a_\beta(z) - \partial_{\beta}A^a_\alpha(z) + gf^{abc}A^b_\alpha(z)A^c_\beta(z) \quad (5.37) $$
The symbol $f^{abc}$ represents the structure constants for the $SU(3)$ group and $g$ is the strong coupling constant. Solving eq. 5.37 for $\partial_\alpha A^a_\alpha(z)$ and inserting this into eq. 5.36 gives

$$A^a_\alpha(z) + z^{\beta} \partial_\beta A^a_\alpha(z) = z^\beta G^a_{\beta\alpha}(z)$$

(5.38)

The substitution $z = \xi x$ gives a total derivative $\frac{d}{dz}[\xi A(\xi x)]$ on the left hand side. Integrating over $\xi$ from 0 to 1 gives

$$\mathcal{A}'^a_\alpha(z) = \int_0^1 \xi x^\beta G^a_{\beta\alpha}(\xi x) d\xi$$

(5.39)

Expansion around $x = 0$ gives

$$A^a_\alpha(x) = \frac{1}{2} x^\beta G^a_{\beta\alpha}(0) + \frac{1}{3} x^\mu x^\beta [D_\mu G^a_{\beta\alpha}](0) + \frac{1}{8} x^\mu x^\nu x^\rho [D_\mu D_\nu G^a_{\beta\alpha}](0) + \cdots$$

(5.40)

Here $A^a_\alpha = t^a A^a_\alpha$, $G^a_{\beta\alpha} = t^a G^a_{\beta\alpha}$, $D_\mu = \partial_\mu - i g t^a A^a_\mu$ and $t^a$ is the fundamental representation of the $SU(3)$ generators. The index $a$ runs from 1 to 8. This is an expansion in terms of local operators which will be used for the calculation of the fermion propagator. Due to the fixed gauge point $x_0$ there will necessarily emerge terms which are not translationally invariant. This is just temporary, because there will be complete cancellation among these terms in the final result.

5.4.2 The Fermion Propagator in an External Field

Diagrammatically, the fermion propagator can be represented as in figure 5.3 up to order $g^3$. The total vacuum momentum is zero, and there will be no real process where only one external gluon is attached to it. The lowest order fermionic Green's function is defined by the matrix element of the time-ordered product of the spinors $\psi(y)$ and $\bar{\psi}(x)$ taken between vacuum states

$$iS(y - x) = \langle 0 | T[\psi(y)\bar{\psi}(x)] | 0 \rangle$$

(5.41)

This gives rise to the well known expression

$$iS_0(y - x) = \int \frac{d^4p}{(2\pi)^4} \exp[-ip(y-x)][iS_0(p)]$$

(5.42)

where (including colour indices)

$$iS_0(p)_{ij} = \frac{i(\gamma \cdot p + m)\delta_{ij}}{p^2 - m^2}$$

(5.43)

Higher order terms is obtained by attaching gluons to the fermion propagator. In order to find the first correction, we have to integrate over the space-time point $z$:

$$iS_1(y - x) = i \int d^4z [iS_0(y - z)][ig(\gamma^\nu G^a_{\nu\mu})\gamma^\mu][iS_0(z - x)]$$

(5.44)
5.4 The Fermion Propagator in an External Field

Figure 5.3: Contributions to the fermion propagator. The nabla symbols denote contributions from terms in the expansion of $A_a(x)$ which contain covariant derivatives.

where $g$ is the strong coupling constant. The term in the parenthesis ( ) is the first term in the expansion of $A_a$. The calculation is feasible by rewriting the $z$ integral as a partial derivative acting on a delta function:

$$\int \frac{d^4 z}{(2\pi)^4} z^\alpha e^{i z q} \delta(q) = \frac{\partial}{\partial q_\alpha} \delta(q). \quad (5.45)$$

Then, utilize the fact that

$$\frac{\partial S_0(p)}{\partial p_\mu} = -S_0(p)\gamma^\mu S_0(p), \quad (5.46)$$

partial integrate and skip surface terms and translationally non-invariant terms. As mentioned in the introduction the translationally non-invariant terms can be shown that they do not contribute at the end. One obtains the result

$$iS_1(p)_{ij} = -\frac{i}{4} g \sigma^a_{ij} G^a_{\mu \nu} \frac{[\sigma^\mu, (\gamma \cdot p + m)]_+}{(p^2 - m^2)^2} \quad (5.47)$$

where $\sigma^{\mu \nu} = \frac{i}{2} [\gamma^\mu, \gamma^\nu]$. Similarly, for two gluon legs attached to the fermion propagator one uses the first term in eq. 5.40 for each vertex then one gets

$$iS_2(p)_{ij} = -\frac{i}{4} g^2 \epsilon_{ijk} G^a_{\alpha \beta} G^{\beta}_{\mu \nu} \frac{(\gamma \cdot p + m)}{(p^2 - m^2)^3} (f^{a\mu \nu} + f^{a\mu \nu} + f^{a\mu \nu})(\gamma \cdot p + m) \quad (5.48)$$
5.4 The Fermion Propagator in an External Field

The tensor $f^{\alpha\beta\mu\nu} = \gamma^{\alpha}(\gamma \cdot p + m)\gamma^{\beta}(\gamma \cdot p + m)\gamma^{\mu}(\gamma \cdot p + m)\gamma^{\nu}$. The only operator of dimension 4 is $\langle G^2 \rangle = \langle 0 | G^2 | 0 \rangle$ which can be obtained by using the following expressions:

$$\text{Tr}(t^a t^b) = \frac{1}{2} \delta^{ab}$$

(5.49)

$$\langle 0 | G_{\mu\nu}^a G_{\alpha\beta}^a | 0 \rangle = \frac{1}{12} (g_{\mu\alpha} g_{\nu\beta} - g_{\mu\beta} g_{\nu\alpha}) < G^2 >$$

(5.50)

However, these two equations should only be used for correction terms up to dimension 4. This gives for eq. 5.48:

$$i S_2(p, < G^2 >)_{ij} = \frac{i}{4} g^2 < G^2 > \left( \frac{m \gamma \cdot p + p^2}{p^2 - m^2} \right) \delta_{ij}$$

(5.51)

There are two linearly independent operators of dimension 6. One of them is three external gluons and the other one is the square of the quark current. Higher dimension operators will not be included here. This means that one can take the trace in colour space and average over the gluon indices. In order to calculate the power correction one can make use of the following identities:

$$\text{Tr}(t^a t^b t^c) = \frac{1}{4} (d^{abc} + i f^{abc})$$

(5.52)

$$\langle 0 | f^{abc} G_{\mu\nu}^a G_{\alpha\beta}^b G_{\kappa\sigma}^c | 0 \rangle = \frac{1}{24} \left( g_{\mu\omega} g_{\nu\alpha} g_{\beta\rho} + g_{\mu\alpha} g_{\nu\rho} g_{\beta\omega} + g_{\mu\rho} g_{\nu\alpha} g_{\beta\omega} + g_{\mu\alpha} g_{\nu\rho} g_{\beta\omega} - g_{\mu\beta} g_{\nu\sigma} g_{\rho\omega} - g_{\mu\sigma} g_{\nu\beta} g_{\rho\omega} - g_{\mu\rho} g_{\nu\beta} g_{\sigma\omega} - g_{\mu\beta} g_{\nu\sigma} g_{\rho\omega} \right)$$

(5.53)

$$\langle 0 | G_{\mu\nu}^a (D_\sigma D_\rho G_{\alpha\beta})^a | 0 \rangle = 2 C_- g_{\mu\beta} (g_{\nu\alpha} - g_{\nu\omega} g_{\mu\beta})$$

$$+ C_- (g_{\mu\beta} g_{\alpha\sigma} g_{\nu\rho} + g_{\alpha\sigma} g_{\nu\rho} g_{\mu\beta} - g_{\nu\mu} g_{\rho\beta} g_{\sigma\omega})$$

$$+ C_+ (g_{\mu\beta} g_{\alpha\sigma} g_{\nu\rho} + g_{\mu\beta} g_{\alpha\sigma} g_{\nu\rho} - g_{\nu\mu} g_{\rho\beta} g_{\sigma\omega} - g_{\nu\mu} g_{\rho\beta} g_{\sigma\omega})$$

(5.54)

where $C_\pm = \frac{g^2}{72} < j^2 > \pm \frac{1}{48} < G^3 >$ and the short hand notations $< G^3 > = \langle 0 | f^{abc} G_{\mu\nu}^a G_{\alpha\beta}^b G_{\kappa\sigma}^c | 0 \rangle$ and $< j^2 > = \langle 0 | j^a j^a | 0 \rangle$ have been used. Note that only the second term in eq. 5.52 will give a condensate contribution. This is due to the symmetry and antisymmetry of the $d^{abc}$ and $G$ symbols, respectively. In order to extract the $< j^2 >$ dependences one has to use the equation of motion, the Bianchi identity and translation invariance, respectively:

$$(D^a G_{\mu\sigma})^a = g \sum \bar{\psi} \gamma^a t^a \psi = gj^a_\mu,$$

(5.55)

$$D_\mu G_{\mu\nu} + D_\nu G_{\nu\sigma} + D_\sigma G_{\sigma\mu} = 0,$$

(5.56)

$$\langle 0 | D_\mu (j^a G_{\mu\nu}) | 0 \rangle = 0.$$

(5.57)
5.4 The Fermion Propagator in an External Field

<table>
<thead>
<tr>
<th>$T_i$</th>
<th>Expression</th>
</tr>
</thead>
<tbody>
<tr>
<td>$T_1$</td>
<td>$\gamma \cdot p(p^2 - 3m^2) + 2m(2p^2 - m^2)$</td>
</tr>
<tr>
<td>$T_2$</td>
<td>$m(p^2 + 3m^2) [m \gamma \cdot p - p^2]$</td>
</tr>
<tr>
<td>$T_3$</td>
<td>$\gamma \cdot p(p^2 - 2p^2m^2 + 9m^4) + 2m(-p^3 - 6p^2m^2 + 3m^4)$</td>
</tr>
<tr>
<td>$T_4$</td>
<td>$2\gamma \cdot p(p^2 + 3p^2m^2 + 12m^4) + 2m(-7p^4 - 15p^2m^2 + 6m^4)$</td>
</tr>
<tr>
<td>$T_5$</td>
<td>$-3\gamma \cdot p \cdot m^2(p^2 + 7m^2) + 3m(3p^4 + 7p^2m^2 - 2m^4)$</td>
</tr>
<tr>
<td>$T_6$</td>
<td>$4m^2\gamma \cdot p(p^3 + m^2) - m(3p^4 + 4p^2m^2 + m^4)$</td>
</tr>
<tr>
<td>$T_7$</td>
<td>$\gamma \cdot p(p^4 - 33m^4) - 2m(2p^4 - 15p^2m^2 - 3m^4)$</td>
</tr>
</tbody>
</table>

Table 5.3: Functions defined in text.

We do now have the necessary expressions to calculate three gluon condensate corrections in the fermion propagator. By using the first term in eq.5.40 three times gives

$$iS_3(p)_{ij} = \frac{-ig^3 < G^3 > (\gamma \cdot p + m)}{16 (p^2 - m^2)^6} T_1 (\gamma \cdot p + m) \frac{\delta_{ij}}{3} \quad (5.58)$$

where the function $T_i$ and similar functions introduced later are defined in table 5.3. One can now look at the terms involving covariant derivatives. By using the second term in eq. 5.40 for one external gluon, one gets

$$iS_1(p, D)_{ij} = \frac{igt_{ij}}{3} (D_a G_{a\mu}) \frac{\delta_{ij}}{2} [g^{\mu\nu} + f^{\mu\nu\sigma}](\gamma \cdot p + m) \quad (5.59)$$

where $f^{\mu\nu\sigma} = \gamma^{\mu}(\gamma \cdot p + m)\gamma^{\nu}(\gamma \cdot p + m)\gamma^{\sigma}$. Furthermore, the third term in eq. 5.40 gives rise to

$$iS_1(p, D^2)_{ij} = \frac{-gt_{ij}}{8} ((D_a D_{a\beta} + D_{\beta} D_a) G_{a\mu}) \frac{\delta_{ij}}{2} [g^{\mu\nu} + f^{\mu\nu\sigma} + f^{\mu\nu\rho\sigma}](\gamma \cdot p + m) \quad (5.60)$$

where a symmetrization of the $\alpha, \beta$ indices has been used. For two external gluons one may take the first and third term in eq. 5.40 or interchange them. See the last two diagrams in fig. 5.3. This give rise to two different terms and reflects the fact that the point gauge is being used. The two terms are:

$$iS_2(p, D^2)_{ij} = i \left\{ \frac{3}{8} g^3 < G^3 > - \frac{1}{2} g^4 < j^2 > \right\} \frac{(\gamma \cdot p + m)}{(p^2 - m^2)^7} T_2 (\gamma \cdot p + m) \frac{\delta_{ij}}{3} \quad (5.61)$$

$$iS_2(p, D^2)_{ij} = i \left\{ \frac{g^3 < G^3 >}{16} - \frac{g^4 < j^2 >}{12} \right\} \frac{(\gamma \cdot p + m)}{(p^2 - m^2)^7} T_3 (\gamma \cdot p + m) \frac{\delta_{ij}}{3} \quad (5.62)$$

The last term to be presented is the one one gets if one uses the second term in eq. 5.40 twice.

$$iS_2(p, D^2)_{ij} = i \frac{g^2 (\gamma \cdot p + m)}{18 (p^2 - m^2)^7} \left\{ T_4 g^2 < j^2 > + T_5 g < G^3 > \right\} (\gamma \cdot p + m) \frac{\delta_{ij}}{3} \quad (5.63)$$
One can now summarize the terms involving $g^3 < G^3 >$ and $g^4 < j^2 >$ in two effective terms for later use:

$$iS_{eff}(p, g^3 < G^3 >)_{ij} = \frac{ig^3 < G^3 > (\gamma \cdot p + m)}{12(p^2 - m^2)^2} \mathcal{T}_6 (\gamma \cdot p + m)\delta_{ij} \quad (5.64)$$

$$iS_{eff}(p, g^4 < j^2 >)_{ij} = \frac{ig^4 < j^2 > (\gamma \cdot p + m)}{36(p^2 - m^2)^2} \mathcal{T}_7 (\gamma \cdot p + m)\delta_{ij} \quad (5.65)$$

Some of the results above are obtained by using the Form program, which is developed by J. Vermaseren [43]. Eqs. 5.47, 5.48, 5.51, 5.59 and 5.60 are in agreement with [44], while eq. 5.58 differs by a minus sign. Additionally the eqs. 5.61 to 5.65 and especially eqs. 5.64 and 5.65 have been calculated for completeness. They can be used in a calculation of condensate corrections to order $< G^3 >$ and $< j^2 >$ for different processes.

### 5.4.3 The Quark Condensate

It is now possible to calculate the quark condensate in terms of $< G^2 >$, $< G^3 >$ and $< j^2 >$ condensates. Diagrammatically, the quark condensate can be drawn as a loop with a unit vertex. Note that the quadratically divergent term is sometimes thought to be spurious. Attaching external gluon lines to the loop, one can apply the previous techniques in order to obtain condensate corrections. Using the expressions in the former section yield

$$\langle \bar{q}q\rangle = \langle \bar{q}q\rangle_0 - \frac{1}{12} \frac{\alpha_s < G^2 >}{\pi M} - \frac{1}{360} \frac{\alpha_s g < G^3 >}{\pi M^3} - \frac{1}{120} \frac{g^4 < j^2 >}{\pi^2 M^3} \quad (5.66)$$

Eq. 5.66 confirms previous result [45].

### 5.5 Regularization Schemes

Due to higher order loop calculations one has to regularize divergent integrals properly. In this section we review some regularization schemes often used in the literature. The emphasis is put on dimensional regularization but also sharp cut-off and proper time regularization is used in this thesis. There are different types of divergences like logarithmic, quadratic etc. The first one is the mildest and easiest to treat. All the methods below are able to cure logarithmic divergences. After having regularized an integral and separated the divergent part, one can subtract off the noisy part by adding suited counterterms. In some cases this is not an acceptable procedure. In effective field theories, designed to describe the low energy part of a more extensive theory, there is a physical cut-off and loop momenta is naturally regulated by an upper limit. In this case an UV divergence will not disappear whatever regularization one uses. In such cases the divergence is physical and can not be removed. This is the case for the pion decay constant $f_p$ for instance.
The quadratic divergences are more severe and not all regularization methods below can be used. The most robust scheme is the proper time regularization where all types of divergences appearing in field theories can be cured. However, the most used scheme in the literature is dimensional regularization. This is due to its nice properties concerning Lorentz invariance and gauge invariance. Other regularizations not presented here are for instance analytic regularization, point splitting, $\zeta$-function regularization etc.

5.5.1 Sharp Cut-off Regularization

For ultra-violet divergent (UV) integrals, the simplest form of regularization is the sharp cut-off method. In practice, the upper limit of a momentum integration is shifted from $\infty$ to $\Lambda^2$ which then yields a well defined integral. A shift of the momentum integration variable in general changes the result and hence this method does not preserve Lorentz invariance. However, the leading order result is general.

The gauge invariance is also broken in this regularization scheme as seen for instance in the calculation of the vacuum polarization diagram. Using a sharp cut-off does not lead to result proportional to $(k^\mu k^\nu - k^2 g^{\mu\nu})$.

5.5.2 Pauli-Villars Regularization

The method of Pauli-Villars is to regulate the integrals by the propagator substitution

$$
\frac{1}{p^2 - m^2} \rightarrow \frac{1}{p^2 - m^2} - \frac{1}{p^2 - M^2} = \frac{m^2 - M^2}{(p^2 - m^2)(p^2 - M^2)}
$$

This means that a fictitious field of mass $M$ has been introduced. The relative minus sign in the propagator means that the new particle is a ghost. It propagates like a scalar but has Fermi statistics. The modified propagator is $\propto 1/k^4$ while the original propagator is $\propto 1/k^2$. The modified propagator reduces to the original one as $M \rightarrow \infty$. Taking the same limit after loop calculations, the unphysical fermion decouples from the theory.

In this scheme Lorentz invariance is preserved. In quantum electrodynamics and massless Yang-Mills theories, gauge invariance is also preserved. In massive Yang-Mills theories, gauge invariance is not maintained. In [46] the self-energy part of the charged weak boson can not be regularized in a gauge invariant way in the Pauli-Villars regularization scheme.

Different versions of this method uses even more subtractions. These methods have usually some undesired features [47] and are seldom used.
5.5 Dimensional Regularization

5.5.3 Proper Time Regularization

Proper time regularization is defined only in Euclidean space and is defined as a modification of the propagator

\[
\left( \frac{1}{p_{\mu}^2 + m^2} \right)^N = \int_0^\infty \tau^{N-1} e^{-\tau(p^2_{\mu} + m^2)} d\tau \rightarrow \int_\epsilon^\infty \tau^{N-1} e^{-\tau(p^2_{\mu} + m^2)} d\tau
\]  

(5.68)

where \( \epsilon = 1/\Lambda^2 \) and \( N = 1 \) for one propagator. Often associated with Proper Time regularization is the incomplete \( \Gamma \)-function. It is defined as

\[
\Gamma(n - 2, x) = \int_0^\infty z^{n-2} e^{-z} dz
\]

(5.69)

where \( x = M^2/\Lambda^2 \). This regularization preserves gauge invariance [48].

5.5.4 Dimensional Regularization

For divergent integrals it is customary not to be restricted to four dimensions. One introduces the arbitrary dimension \( D \) for the loop momenta. Mathematically, one is then able to extract the divergent part of the loop integrals. This is done only for internal momenta, while external momenta are kept in their original dimension. We will use a regularization procedure invented by 't Hooft and Veltman [49]. It makes use of the standard dimensional regularization where the \( \gamma_5 \) matrix anticommutes with the rest of the Dirac matrices in four dimensions and commutes with them in the remaining dimensions. This is a regularization procedure that respects gauge invariance, Lorentz invariance and Ward-Takahashi identities. The anticommutator

\[
\{\gamma^\alpha, \gamma^\beta\} = 2g^{\alpha\beta}
\]

(5.70)

holds in all dimensions, while the contracted metric tensor is

\[
g^\alpha_\alpha = D
\]

(5.71)

In the naive dimensional regularization scheme (NDR) the \( \gamma_5 \) matrix anticommutes with the rest of the Dirac matrices in all dimensions

\[
\{\gamma^\alpha, \gamma_5\}_{NDR} = 0
\]

(5.72)

The last equation can not be used in the 't Hooft/Veltman regularization scheme (HV). In the HV scheme the \( \gamma_5 \) have different properties in 4 and \( D - 4 \) dimensions. We use a notation where tilde used as topscript refers to 4 dimensions and hatted topscripts refers to \( D - 4 \) dimensions.

\[
\gamma^\alpha = \tilde{\gamma}^\alpha + \hat{\gamma}^\alpha
\]

(5.73)

Since the tilde and hat symbols denote different subspaces they commute

\[
[\tilde{\gamma}^\alpha, \hat{\gamma}^\beta] = 0
\]

(5.74)
The metric tensor is divided up in subspaces as well
\[
g^{\alpha\beta} = \tilde{g}^{\alpha\beta} \oplus \hat{g}^{\alpha\beta}
\]
and
\[
\tilde{g}_\alpha^\alpha = 4 \quad (5.76)
\]
\[
\hat{g}_\alpha^\alpha = D - 4 \quad (5.77)
\]
\[
\tilde{g}_\beta^\alpha \hat{g}_\rho^\beta = 0 \quad (5.78)
\]

The \(\gamma_5\) is defined as anticommuting with the Dirac matrices in 4 dimensions, while it is commuting in the remaining \(D - 4\)
\[
\{\tilde{\gamma}, \gamma_5\} = 0 \quad (5.79)
\]
\[
[\gamma_\alpha, \gamma_5] = 0 \quad (5.80)
\]

When calculating the trace of an even number of Dirac matrices, the following rules apply
\[
Tr(\tilde{\gamma}^{\alpha} \gamma^{\beta}) = 4 \tilde{g}^{\alpha\beta} \quad (5.81)
\]
\[
Tr(\hat{\gamma}^{\alpha} \gamma^{\beta}) = 4 \hat{g}^{\alpha\beta} \quad (5.82)
\]
\[
Tr(\tilde{\gamma}^{\alpha} \hat{\gamma}^{\beta} \gamma^{\gamma} \gamma^{\delta}) = 4 \tilde{g}^{\alpha\beta} \hat{g}^{\gamma\delta} \quad (5.83)
\]

We also use the following rules
\[
\int \frac{d^Dp}{(2\pi)^D} f(p^2) p^\alpha p^\beta = \int \frac{d^Dp}{(2\pi)^D} f(p^2) \frac{p^\alpha p^\beta}{D} g^{\alpha\beta} \quad (5.84)
\]
\[
\int \frac{d^Dp}{(2\pi)^D} f(p^2) p^\alpha p^\beta p^\sigma p^\rho = \int \frac{d^Dp}{(2\pi)^D} f(p^2) \frac{p^4}{D(D+2)} S^{\alpha\beta\sigma\rho} \quad (5.85)
\]

where \(S^{\alpha\beta\sigma\rho} = (g^{\alpha\beta} g^{\sigma\rho} + g^{\alpha\sigma} g^{\beta\rho} + g^{\alpha\rho} g^{\beta\sigma})\). Typical expressions encountered in calculations in the HV scheme give for instance
\[
(-\tilde{g}^{\alpha\beta} + \hat{g}^{\alpha\beta})_{p^\alpha p^\beta} \rightarrow \frac{(D - 8)}{D} p^2 \quad (5.86)
\]
\[
(-\tilde{g}^{\alpha\beta} + \hat{g}^{\alpha\beta})_{p^\alpha p^\beta (p \cdot k)^2} \rightarrow \frac{(D - 10)}{D(D+2)} p^4 k^2 \quad (5.87)
\]

We define the integrals \(I_N\) as follows
\[
I_N = \int \frac{d^Dp}{(2\pi)^D} Q_0^N \quad (5.88)
\]
where \(Q_0 = (p^2 - m^2)^{-1}\) and \(N\) is a positive integer. Momentum dependence in the nominator are reduced to integrals of the type \(I_N\) as follows
\[
I_N^{(k)} = \int \frac{d^Dp}{(2\pi)^D} Q_0^N (p^2 Q_0)^k \quad (5.89)
\]
5.5 Dimensional Regularization

\[ I_N^{(1)} = \frac{D}{2N} I_N \]  
\[ I_N^{(2)} = \frac{D(D + 2)}{4N(N + 1)} I_N \]  
\[ I_N^{(3)} = \frac{D(D + 2)(D + 4)}{8N(N + 1)(N + 2)} I_N \]

The integrals \( I_N \) for \( N \geq 3 \) are finite and well defined. \( I_1 \) and \( I_2 \) corresponds to the divergent integrals for the quark condensate and \( f_\pi \) respectively. They can be written

\[ I_1 = \frac{i}{16\pi^2} \hat{C}_q \]  
\[ = \frac{i(\bar{q}q)^{00}}{4mN_c} \]  
\[ I_2 = \frac{i}{16\pi^2} \hat{f}_\pi \]  
\[ = \frac{if_{\pi}^{00}}{4m^2N_c} \]
Chapter 6

Applications

6.1 The $K^0 - \bar{K}^0$ Transition

In section 3.2 we introduced the $K^0 - \bar{K}^0$ mixing. In the first subsection we will look at effects coming from the so called siamese penguin diagram with momentum dependent penguin vertices. This diagram generates a non-local operator contrary to the local $O(\Delta S = 2)$ operator. Since the structure of $L_{sp}$ is non-local due to the derivatives, a comparison with the box result is not strictly reliable until the matrix element has been calculated.

In the second subsection we calculate nonfactorizable effects from operators with dimension less or equal to six. These effects can be taken into account by dressing the Feynman diagrams with soft external gluons in all possible ways. The contribution with lowest dimension four is $\langle G^2 \rangle$ while for dimension six operators there are two independent condensates, $\langle G^3 \rangle$ and $\langle j^2 \rangle$. We will work with a model where $\langle j^2 \rangle$ will be given as an expansion in gluon condensates of higher order and inverse proportional to the quark mass $M$ [50] i.e.

$$\langle j^2 \rangle \propto \frac{\langle G^4 \rangle}{M^2} + \cdots$$  \hfill (6.1)

To consider $\langle j^2 \rangle$ terms we have to consider direct $\langle G^4 \rangle$ terms as well. Therefore, we will mainly pay attention to the $\langle G^2 \rangle$ and $\langle G^3 \rangle$ terms.

6.1.1 Non-local Effects from the Siamese Penguin Diagram

Two higher order contributions to the $K^0 - \bar{K}^0$ transition are shown in fig.6.1. The upper one is called a double penguin diagram while the lowest one is the siamese penguin diagram. The two contributions are rather different. The double penguin yields a local contribution and hence do not introduce a new operator but merely gives a slight change in the coefficient of the standard $O(\Delta S = 2)$ operator. The double penguin is also of higher order than the siamese penguin contribution with respect to the strong coupling constant. In [51] they have shown that the siamese
6.1 Non-local Effects from the Siamese Penguin Diagram

The Siamese penguin diagram introduces a non-local effective interaction due to derivatives appearing in the effective Lagrangian. There are reasons to believe that the Siamese penguin diagram could give a comfortable contribution to the lowest order result. In [51] and [52] it was estimated to give a contribution below the 10% level. A more refined calculation for the penguin operator in ref.[36] gave roughly a doubling of the penguin operator when momentum variation was taken into account. For the Siamese diagram this would give an increase of a factor 4 and hence its contribution could become highly essential. In the limit where $\Lambda x \rightarrow \infty$, the ratio between the siamese penguin diagram and the standard result could be as large as 50% in a certain approximation (see eqs. 6.13 and 6.16). We decide to investigate this further and take into account momentum dependence of the penguin coefficient. In order to calculate the mixing amplitude we use the $\chi QM$. The momentum variations of the effective interaction will be integrated out.

From the siamese penguin diagram one can construct the following effective Lagrangian [51]:

$$L_{sp} = -\frac{\alpha_s}{18\pi^3} G_F^2 P_\mu P_\nu (d \gamma_\mu L^\alpha \bar{s})(g^{\mu\nu} \partial^2 - \partial^\mu \partial^\nu)(d \gamma_\nu L^\alpha \bar{s})$$

The expression for $P$ is

$$P = \lambda_u(L_u - L_c) - \lambda_t(L_c - L_t)$$

where

$$L_\eta = \frac{6M_W^2}{M_W^2 - m_\eta^2} \left(1 + \frac{m_\eta^2}{2M_W^2}\right) \int_0^1 x(1 - x) \ln\left[\frac{(1 - x)M_W^2 + zm_\eta^2 - x(1 - x)p^2}{m_\eta^2 - x(1 - x)p^2}\right] dx$$
6.1 Non-local Effects from the Siamese Penguin Diagram

This is the exact one-loop result for the penguin diagram with a $W^-$-boson and quark $q$ in the triangular loop. The factor $m_q^2/(2M_W^2)$ in the expression for $L_q$ originates from the unphysical Higgs contribution. This comes about since we have used a gauge where the $W^-$-boson propagator has the form

$$D_F(W)^{\mu\nu} = \frac{-g^{\mu\nu}}{p_W^2 - M_W^2}$$  

(6.5)

The consequence of omitting a term proportional to $p_W^\mu p_W^\nu$ is reflected in the fact that we have to introduce an unphysical Higgs field $\phi$. Its propagator can be found in many textbooks, for instance in [53] and [5]. It reads

$$D_F(\phi) = \frac{1}{p_\phi^2 - M_\phi^2}$$  

(6.6)

We notice that the $W^-$-mass is connected to the unphysical Higgs as well. This simplifies the calculation since the answer will have a similar form as for the $W^+$-propagator. The vertex factor for the unphysical Higgs is

$$\frac{-ig_{\mu\nu}}{2M_W} \{m_{\nu} L - m_{\nu} R\} V_{\nu\nu'}$$  

(6.7)

We will primarily consider CP-conserving effects and thus look at the terms proportional to $\lambda_u$. In a first approximation one can use $\mathcal{P}$ as a constant

$$\mathcal{P}_{uc} = L_u - L_c = \ln \left( \frac{m_u^2}{\lambda_x^2} \right) + \frac{5}{3}$$  

(6.8)

where $\lambda_x$ has been chosen as renormalization point $\mu$. We have kept the next to leading order term and we see that it is numerically as important as the leading logarithmic term. This elucidates a problem connected to the penguin interaction. The logarithmic factor in eq.6.8 appears due to a strong cancellation between large logarithmic terms. Due to the GIM mechanism, the penguin diagram is proportional to \( \ln \left( \frac{M_W^2}{\Lambda_x^2} \right) - \ln \left( \frac{M_W^2}{m_c^2} \right) = \ln \left( \frac{m_c^2}{\Lambda_\xi^2} \right) \). We see that scales above $m_c$ drops out. The awkward problem is partially avoided if we use a more general momentum dependent version of the $\mathcal{P}$ which is

$$\mathcal{P}(p^2) = 6 \int_0^1 x(1-x) \int_{m_2^2}^{\rho} \frac{1}{\rho - x(1-x)p^2} dx dp.$$  

(6.9)

Due to the momentum dependence in the coupling the loops are coupled and we must perform a two-loop calculation. See fig.6.2.

Our model for calculating the $K^0 \rightarrow \bar{K}^0$ mixing contains the following terms

$$\mathcal{L}_{eff} = \mathcal{L}_{QCD} + \mathcal{L}_{bos} + \mathcal{L}_{sp} + \Delta \mathcal{L}_{X} + \cdots$$  

(6.10)

where the dots denote higher order corrections which are neglected in our approach. It should be noted that the RGE analysis is not performed in the case of momentum dependent penguin coefficients. However, in the RGE analysis the momentum
dependence is lost. The renormalization point \( \mu \) corresponds to some combination of external quark momenta in our approach.

A rough estimate of the amplitude corresponding to the siamese penguin interaction \( \mathcal{L}_{sp} \) will give \((\bar{d} \gamma \mu L s)^2 \rightarrow \mathcal{O}(f_K m_K)^2\) as in eq. 3.58. The derivatives will correspond to some scale \( \Lambda_x \), defined in eq. 5.5, or lower and the penguin interaction \( \mathcal{P} \) taken as a constant. This yields a contribution

\[
\mathcal{M}_{sp} = (\sqrt{2} f_K m_K)^2 \cdot (G_F^2 \frac{\alpha_s}{18\pi^3}) \cdot \mathcal{P}^2 \cdot \Lambda_x^2 \cdot H. \tag{6.11}
\]

The factor \( H \) is a number which has to be determined in a more refined treatment of the siamese penguin. If we assume \( H \) to be equal to one and taking \( \mathcal{P} \) as in eq. 6.8, with or without the next to leading order contribution, we get a contribution around \( 10 - 20\% \).

Calculating the \( K^0 - \bar{K}^0 \) mixing, one has to deal with the lowest diagram of fig. 6.1 which in the \( \chi QM \) becomes a two loop diagram shown in fig. 6.2. We work in the chiral limit and put the current quark masses to zero. With a momentum dependent penguin coefficient we have to handle the following expression

\[
\mathcal{M}_{sp} = G_F^2 \frac{\alpha_s}{3\pi^3} \left( \frac{M^2}{f_\pi} \right)^2 \text{Tr}(t^a t^a) k^2 \int \frac{d^4 t}{(2\pi)^4} \int \frac{d^4 r}{(2\pi)^4} \frac{P^2 [P^2]^2}{(t^2 - M^2)^2 (r^2 - M^2)^2} \tag{6.12}
\]

calculated to lowest order in momentum \( k \) of the kaon \( (k^2 = m_K^2) \) due to \( SU(3) \) symmetry. The gluon momentum is \( p = t - r \) while \( t \) and \( r \) are the momenta of the strange and down quark, respectively. The gluon momentum makes the integrals in

---

Figure 6.2: Diagrammatic representation of the siamese penguin diagram in the \( \chi QM \).
6.1 Non-local Effects from the Siamese Penguin Diagram

eq. 6.12 connected. The momentum variation of the penguin coefficients $\mathcal{P} \sim 1/p^2$ for large momentum transfers makes the integrals finite in the limit $\Lambda_x \to \infty$. We find a result of the following form

$$\mathcal{M}_{sp}^{uc} = 48 \frac{G_F^2 \alpha_s}{4\pi^2} \left( \frac{M^2}{4\pi^2 f_\pi} \right)^2 m_K^2 m_c^2 \ln \left( \frac{m_c^2}{M^2} \right) F_{uc}. \quad (6.13)$$

where $F_{uc}$ is a quadruple integral over Feynman parameters

$$F_{uc} = \int_0^1 dx \int_0^1 dy \int_0^1 du \int_0^{1-u} dv \left( \frac{y(1-y)}{v} \ln(1 + \frac{vx(1-x)}{uy(1-y)}) + \frac{x(1-x)}{u} \ln(1 + \frac{uy(1-y)}{vx(1-x)}) \right) \quad (6.14)$$

$$\approx 0.25. \quad (6.15)$$

We did two different numerical calculations in order to check the result. First we used a program in the Cernlib library developed at CERN [54], then we tried the software package Mathematica [55]. Both calculations gave the same result above. In order to find an analytic expression for the eq. 6.12 we had to use the following approximation:

$$u\left( \frac{\rho}{x(1-z)} + M^2 \right) + v\left( \frac{\sigma}{y(1-y)} + M^2 \right) \gg M^2 \quad (6.16)$$

where $\rho$ and $\sigma$ are variables which will be integrated over up to the c-quark mass $m_c^2$. This approximation is not always good and even fails for instance for $u = v = 0$ and $x, y \neq 0, 1$. However, one can extract the leading behaviour where $\rho, \sigma \to m_c^2$. If we use the strong coupling constant at the c-quark scale ($\alpha_s(m_c) \approx 1/3$), the result obtained is of the order of 50% of the lowest order box diagram which is exorbitantly large. It can be understood since we put $\Lambda_x \to \infty$ and also used a rather crude approximation.

To improve this result we must incorporate a finite cut-off. Taking $\mathcal{P}$ as a constant we get a more refined version of eq. 6.11:

$$\mathcal{M}_{sp}^{uc} = \frac{2 G_F^2 \alpha_s}{3 \pi^2} \left( \frac{M^2}{4\pi^2 f_\pi} \right)^2 m_K^2 \Lambda_x^2 \tilde{C} \quad (6.17)$$

where $\tilde{C}$ is a dimensionless factor depending on the way the cut-off is introduced. We introduced the cut-off in different ways, but all the numerical integrations gave $\tilde{C}$ of order 0.1 even with a momentum dependent penguin coefficient. If we use $\alpha_s \approx 1/2$ at the $\Lambda_x$ scale we find that the siamese penguin diagram is only 0.2% of the box diagram. An analysis with a running coupling constant could be done, but will not give a drastic change in the result and therefore there will be no need to pursue this further.
6.1.2 Non-factorizable Condensate Effects to $B_K$

The $\mathcal{O}(\Delta S = 2)$ operator gives rise to two diagrams pictorially given as

$$\langle K^0|\mathcal{O}(\Delta S = 2)|\bar{K}^0\rangle = \text{"} \cdots \bigcirc \bigcirc \cdots \text{"} + \text{"} \cdots \bigcirc \bigcirc \cdots \text{"} \quad (6.18)$$

The first diagram is $\propto N_c^2$ while the second diagram is colour suppressed and is $\propto N_c$.

In some calculations it is customary to Fierz transform the operator $\mathcal{O}$. We can take some advantages of the transformed operator as some of the generated diagrams cancel. Consider the Fierz transformed operator:

$$\mathcal{O}_F = \frac{1}{N_c} (\bar{d}\gamma_\mu L s)(\bar{d}\gamma^\mu L s) + 2(\bar{d}\gamma_\mu L t^a s)(\bar{d}\gamma^\mu L t^a s) \quad (6.19)$$

The matrix element can now be written symbolically:

$$\langle K^0|\mathcal{O}_F|\bar{K}^0\rangle = \frac{1}{N_c} \{\text{"} \cdots \bigcirc \bigcirc \cdots \text{"} \} (\text{vertex} : \gamma^\mu L) + 2\{\text{"} \cdots \bigcirc \bigcirc \cdots \text{"} \} (\text{vertex} : \gamma^\mu L t^a) \quad (6.20)$$

The Fierz transformed operator gives rise to four diagrams. The symbolic descriptions of $\mathcal{O}$ and $\mathcal{O}_F$ are of course identical. The first term in eq.6.18 corresponds to the second and the fourth term of eq.6.20. Similarly, the second term of eq.6.18 corresponds to the first and the third term of eq.6.20. Examples below show that this is indeed the case whether non-perturbative gluon effects are included or not. One can in fact choose to consider only "factorizable" contributions with different vertices and prefactors (first and third term of eq.6.20 and first term of eq.6.18).

If the external field is turned off, one sees from the colour structure that the first term in eq.6.18 corresponds to the second and fourth term in eq.6.20:

$$N_c^2 = \frac{1}{N_c} N_c + 2 Tr(t^a t^a). \quad (6.21)$$

Similarly, the second term in eq.6.18 corresponds to the first term (and a vanishing third term) in eq.6.20.

In order to calculate non-perturbative contributions to $B_K$, we need fermion propagators in an external gluon field. Thus, we use the technique developed in a former chapter.

When calculating the off-diagonal $K^0 - \bar{K}^0$ amplitude with external gluons, one can take advantage of the diagrams already included in the expression for $f_\pi$,

$$f_\pi^2 = \frac{N_c M^2}{4\pi^2} \left[ f_\pi + \frac{\pi^2}{6 N_c M^4} \left( \frac{\alpha_s}{\pi} G^2 \right) + \frac{1}{360 N_c} \frac{g_s^2 G^3}{M^6} + \cdots \right] \quad (6.22)$$

where the dots denotes higher order condensates [15]. During the calculation we need some useful trace expressions of the $SU(3)$ generators:

$$Tr(t^a t^b) = \frac{1}{2} \delta^{ab}, \quad (6.23)$$
6.1 Non-factorizable Condensate Effects to $B_K$

$$Tr(t^a t^b t^c) = \frac{1}{4}(d^{abc} + if^{abc})$$

(6.24)

and

$$Tr(t^a t^b t^c t^d) = \frac{1}{4N_c}\delta^{ab}\delta^{cd} + \frac{1}{8}(d^{abe} + if^{abe})(d^{cde} + if^{cde}).$$

(6.25)

When dressing the "blobs" from the $O_F$ operator with two external gluons ending in vacuum, one can realize that the two diagrams having only one single colour line, i.e. the second and fourth term of eq.6.20, indeed cancels in the two gluon case (one gluon on each loop) as seen from the colour structure

$$\frac{1}{N_c}Tr(t^a t^b) + 2Tr(t^a t^b t^c t^d) = 0.$$ (6.26)

In addition, the first term in eq.6.20 vanishes due to traceless colour matrices. Thus, we only have to consider the diagrams coming from the third term in eq.6.20 with external gluons attached. This is a non-factorizable contribution due to the vertex structure $\gamma^L t^a$. Non-perturbative gluons to order $(G^2)$ contributes to $B_K$ as

$$B_K = \frac{3}{4}\left\{1 + \frac{1}{N_c}\left[1 - \frac{N_c(2\pi G^2)}{32\pi^2 f^4}\right]\right\}.$$ (6.27)

This is in accordance with [15]. The terms with two gluons on each loop are only contributing to $f_\pi$ and therefore bring no effect into $B_K$. Numerically, this result change very much the next to leading $1/N_c$ contribution.

In the case of three gluons, one again realizes that the second and fourth term in eq.6.20 cancels in the case of two external gluons on one loop and one gluon on the other. The cancellation is again seen by looking at the colour structure. One encounters the sum

$$\frac{1}{N_c}Tr(t^a t^b t^c) + 2Tr(t^a t^b t^c t^d) = 0$$

(6.28)

where we have used the commutator and completeness relation of two $SU(3)$ generators and applied the trace expressions for up to four $SU(3)$ generators. The diagrams with all three gluons attached to one loop are already included in the physical pion decay constant. Thus, we are left only with the third term in eq.6.20 which we dress with gluons. Due to the different "G", "DG" and "DDG" parts in eq.5.40 there are totally 20 diagrams to calculate.

A useful tool for the calculation is to use a software package for algebraic manipulation. We use the Form program, invented by J. Vermaseren [43], which is well suited for our purpose. By summing up all diagrams, we find a total cancellation among the different contributions. Especially, contributions with one external gluon with one derivative on each loop give directly zero while contributions with two derivatives for one gluon field and none derivatives at the other completely cancel contributions with three gluons. This result is a generalization of a result obtained in the paper of W. Hubschmid and S. Mallik [56] where they show that there are no contribution of $(G^3)$ to two-point functions. A more complicated structure is encountered in our case.
when calculating the $B_K$ parameter and their result is therefore not automatically applicable in our case.

As a cross-check of our result, we also performed the calculation with just the original operator $O$. This was a more time consuming operation, but the same vanishing result was obtained.

We have also calculated the contributions proportional to $(j^2)$ and obtained a non-vanishing constant term (constant with respect to the kaon momentum) which is not acceptable. However, $(G^4)$ type contributions should be included as well according to eq.6.1. In order to complete this part of the calculation would require a major effort.

### 6.2 $K \rightarrow 2\pi$ Transitions

In $K \rightarrow 2\pi$ transitions we calculate in the first subsection the $L_5$ coefficient as defined in $\chi PT$. In the second subsection we calculate the effect of a momentum dependent penguin coefficient while in the third subsection we calculate self-energy effects due to the off-diagonal $s \rightarrow d$ transition where we take into account both CP-conserving and CP-violating contributions and also allow a heavy top quark. The fourth subsection contains no results but indicates some contributions which could be important in the explanation of the $\Delta I = 1/2$ rule.

#### 6.2.1 Bosonization of $Q_6$

In this section we will calculate the coefficient $L_5$ from off-shell $K \rightarrow \pi$ transitions by use of eq. 4.20. Within the dimensional regularization scheme we have calculated the different building blocks associated with the off-shell $K^- \rightarrow \pi^-$ process both in the NDR and in the HV scheme. We will make use of the scale factor $\Lambda_\chi$ which was defined earlier as

$$\Lambda_\chi = 2\pi f_\pi \sqrt{\frac{6}{N_c}}.$$  \hspace{1cm} (6.29)

It will correspond to the scale of chiral symmetry breaking. First of all, a subtlety one should be aware of is that the HV regularization introduces anomalous terms. They do not cause serious problems since they can be removed by adding suitable counterterms [57] and [20]. An anomalous term occurs in the $f_+$ form factor for the matrix element

$$\langle \pi^- (r) | \bar{d} \gamma^\mu s | K^- (k) \rangle = f_+^{HV} q_+^\mu + f_-^{HV} q_-^\mu$$  \hspace{1cm} (6.30)

where $q_\pm = k \pm r$. Direct calculation in the $\chi QM$ gives

$$\langle \pi^- (r) | \bar{d} \gamma^\mu s | K^- (k) \rangle_{HV} = \frac{4iM^2 N_c}{3f^2} \left\{ q_+^\mu \left[ i_3 (D - 7) + 4M^2 I_3 + \cdots \right] 
+ q_-^\mu \left[ 0 + \cdots \right] \right\}$$

$$= q_+^\mu \left[ \frac{f(0)}{j} + \frac{8M^2}{\Lambda_\chi^2} + \cdots \right] + q_-^\mu \left[ 0 + \cdots \right].$$  \hspace{1cm} (6.31)
where the dots denotes higher order corrections in terms of $q^2$. We see that in the zero momentum limit the form factors become ($f^{(0)}_\pi = f$)

$$f^{HV}_+(0) = 1 + 8\frac{M^2}{\Lambda^2}$$  \hspace{1cm} (6.33)

$$f^{HV}_-(0) = 0$$  \hspace{1cm} (6.34)

The last term in $f^{HV}_+(0)$ is the anomalous term and it can be cured by adding counterterms [57]. In a naive dimensional regularization scheme (NDR) the anomalous term is absent. A drawback with NDR is that the Fierz transformation is not allowed in $D \neq 4$ dimensions. In the calculation of $f_\pi$, there is no discrepancy between the two regularization schemes. The result is

$$(0|\bar{u}\gamma^\mu\gamma_5 s|K^-(k))_{HV,NDR} = -\sqrt{2}ik^\mu f^{(0)}_\pi.$$  \hspace{1cm} (6.35)

In order to determine $L_5$, defined in eq. 4.10, we have calculated the following matrix elements both in NDR and HV

$$(0|\bar{u}\gamma_5 s|K^-(k))_{HV} = \frac{4\sqrt{2}MN_c}{f} \left\{ \frac{D-8}{2}I_1 + M^2I_2 ight\}$$

$$+ k^2 \left\{ I_2\left( \frac{5}{6} - \frac{D}{12} \right) - \frac{M^2}{3}I_3 \right\}$$  \hspace{1cm} (6.36)

$$= \sqrt{2}i \left\{ -\frac{\langle qq\rangle_0^{(0)}}{f} + Mf^{(0)}_\pi - 6f^{(0)}_\pi \right\}$$

$$+ \frac{k^2}{2M} \left\{ f^{(0)}_\pi + 4f^{(0)}_\pi \right\}$$  \hspace{1cm} (6.37)

$$= \sqrt{2}i \left\{ -\frac{\langle qq\rangle_0^{(0)}}{f} - 12f^{(0)}_\pi + \frac{k^2}{2M} \left\{ f^{(0)}_\pi + 4f^{(0)}_\pi \right\} \right\}$$  \hspace{1cm} (6.38)

$$= (0|\bar{u}\gamma_5 s|K^-(k))_{NDR} + 12\sqrt{2}iM^3\frac{f^{(0)}_\pi}{\Lambda^2} \left\{ 1 - \frac{k^2}{6M^2} \right\}$$  \hspace{1cm} (6.39)

$$(\pi^-(r)|\bar{d}s|K^-(k))_{HV} = \frac{8iM^3N_c}{f^2} \left\{ \frac{3D-16}{4}I_2 + M^2I_3 ight\}$$

$$+ k^2 \left\{ I_3\left( \frac{11}{6} - \frac{D}{4} \right) - \frac{M^2}{2}I_4 \right\}$$  \hspace{1cm} (6.40)

$$= \frac{2Mf^{(0)}_\pi}{f} + 24M^3\frac{f^{(0)}_\pi}{\Lambda^2} + 6M^2k^2$$  \hspace{1cm} (6.41)

$$= (\pi^-(r)|\bar{d}s|K^-(k))_{NDR} + \frac{24M^3}{\Lambda^2}$$  \hspace{1cm} (6.42)

In order to have a result consistent with a sharp cut-off regularization we have used the following relation, valid in dimensional regularization only,

$$\frac{\langle qq\rangle_0^{(0)}}{f} = Mf^{(0)}_\pi + 6f^{(0)}_\pi \frac{M^3}{\Lambda^2}$$  \hspace{1cm} (6.43)
to arrive at eq. 6.38 for only one of the two $\langle \bar{q}q \rangle_0^{(0)} f$ factors. There are three types of diagrams involved in this $K \to \pi$ transition. They are shown in fig. 6.3. The last two contributions have to be multiplied by a factor two due to either a $d$ or an $s$ quark in the condensate loop. For $D = 4-2e$ we indeed find a cancellation among the diagrams for the part which is not proportional to momentum. To see the cancellation one must make use of the relation between the quark condensate and $f_\pi$. The remaining part contribute to the $L_5$ factor. $L_5$ can be expressed by the $K \to \pi$ matrix element of the $Q_6$ operator as in eq. 4.20. The calculated diagrams give $L_5$ in the NDR scheme

$$L_5^{NDR} = -\frac{f_\pi^{(0)}}{8M\langle \bar{q}q \rangle_0} \left[ 1 - 6 \frac{M^2}{\Lambda^2} \frac{f_\pi^{(0)}}{f} \right]$$

(6.44)

while in the HV scheme the result reads

$$L_5^{HV} = -\frac{f_\pi^{(0)}}{8M\langle \bar{q}q \rangle_0} \left[ f_\pi^{(0)} \frac{f_\pi^{(0)}}{f} - 2 \frac{M^2}{\Lambda^2} + 12 \frac{M^3 f_\pi^{(0)}}{\Lambda^2 \langle \bar{q}q \rangle_0^{(0)}} + 48 \frac{M^5 f^2}{\Lambda^4 \langle \bar{q}q \rangle_0^{(0)}} \right]$$

(6.45)

The different results in the two schemes stem from the fact that in the HV scheme there are anomalous terms. However, there should be no scheme dependence in $L_5$. In ref.[57] and [20] they have added counterterms and find the same result as in the NDR case. Thus, $L_5^{NDR}$ is the correct one. We see that this coefficient is very sensitive for values of the constituent quark mass and the chiral symmetry breaking scale. Numerically this corresponds to $L_5 = 1.3 \cdot 10^{-3}$ for $M = 250 MeV$ and $\langle \bar{q}q \rangle = (-0.235 GeV)^3$.

We have also calculated $L_5$ within a proper time regularization scheme and obtained the result

$$L_5^{PT} = -\frac{f_\pi^{(0)}}{8M\langle \bar{q}q \rangle} \left[ 1 - \frac{1}{\Gamma(0,x)} \right]$$

(6.46)

where $\Gamma(0,x)$ is the incomplete $\Gamma$-function defined as

$$\Gamma(0,x) = \int_x^\infty \frac{e^{-z}}{z} dz$$

(6.47)
6.2 Effects of a Momentum Dependent Penguin Coefficient

with \( z = M^2/\Lambda^2 \). In order to see the logarithmic behaviour of \( L_5^{NDR} \) we have kept \( f^{(0)}_\ell \) different from \( f \). The same logarithmic behaviour is expressed by the incomplete \( \Gamma \)-function in the case of proper time regularization. The two expressions are equivalent if we use eq.5.5, \( f_x = \Gamma(0, z) \) and eq.5.18 without condensates.

The two terms in \( L_5 \) correspond to the "eight" and "keyhole" diagram, respectively. If we put \( f^{(0)}_\ell = f \), the expression for \( L_5 \) is very sensitive to the value of \( M \). Numerically, there is a strong cancellation between the two terms especially for \( M > 250 \text{MeV} \). In our approach we are restricted by \( M < 330 \text{MeV} \), otherwise we would get a change of sign in \( L_5 \).

By using the \( L_5^{PT} \) expression for values of \( M \) less than 250\text{MeV}, one would find smaller values than in the \( NDR \) case.

In [15] the expression in eq.6.44 was not used in the calculation of \( g(Q_6) \). Instead the value \( L_5 = 1.8 \times 10^{-3} \) was extracted from other arguments. Moreover, using a low \( \mu \approx M \), and the scale independent quark condensate \( \langle \bar{q}q \rangle = (-194 \text{MeV})^3 \), it is found that \( g(Q_6) = 0.26 \). In general, larger values are obtained for the quark condensate if one uses the expression used in [14, 36, 58],

\[
\langle \bar{q}q \rangle(\mu) = -\frac{f_\pi^2 m_\pi^4}{m_u(\mu) + m_d(\mu)} .
\]

Alternatively, using the values [59] \( \langle \bar{q}q \rangle = (-235 \text{MeV})^3 \), \( L_5 = 1.4 \times 10^{-3} \) at the scale 1 \( \text{GeV} \) (This corresponds to \( M \approx 245 \text{MeV} \) if eq.6.44 is used), and using [60, 61] \( C_6(\mu = 1 \text{GeV}) = -0.026 \) for \( \Lambda_{QCD} = 400 \text{ MeV} \), one obtains \( g(Q_6) = 0.15 \). However, larger values might be obtained. If one uses \( \mu \approx 0.8 \text{GeV} \), \( \Lambda_{QCD} = 350 \text{MeV} \), \( C_6 \approx -0.1 \) and in addition [57, 18] \( M \approx 200 \text{MeV} \), one obtains \( g(Q_6) \approx 0.8 \). (Note that, from \( C_6 = -\alpha_s/(12\pi)(\ln(m_d^2/\Lambda_\chi^2) + 5/3) \) one obtains \( C_6 \approx -0.05 \) for \( \mu \approx \Lambda_\chi \).

6.2.2 Effects of a Momentum Dependent Penguin Coefficient

The leading penguin term of the Lagrangian can be written as

\[
\mathcal{L}_6^{\Delta S=1} = -\frac{G_F}{\sqrt{2}} \lambda_u C_6(-8) \sum_{q=u,d,s} (\bar{d}Rq)(\bar{q}Ls) + h.c.
\]

It is common to use a coefficient \( C_6 \) dependent on the renormalization parameter \( \mu \). However, this procedure does not separate between different quark momenta but only their overall scale. However, the momentum dependence should not be ignored in order to take into account potentially important contributions. A thorough study of the renormalization group equation gives \( C_6 \) as given by Bardeen et al. [14]. The eq. 6.49 together with the \( \chi QM \) gives rise to the three diagrams in fig. 6.3. Each diagram has a constant part and only the first two have momentum dependence.

We will in this section concentrate on the momentum dependence of the penguin interaction and we will write the \( C_6 \) coefficient as follows [62]

\[
C_6(p^2) = \frac{C_6(p^2)}{4} = -\frac{\alpha_s}{12\pi} P(p^2)
\]
6.2 Effects of a Momentum Dependent Penguin Coefficient

When using dimensional regularization $P(p^2)$ takes the following form

$$P(p^2) = 6\Gamma(\epsilon)\tau(\epsilon) \int_0^1 x(1-x)(B^{\epsilon} - A^{\epsilon}) dx$$  \hspace{1cm} (6.51)

where

$$A = m^2 - x(1-x)p^2$$  \hspace{1cm} (6.52)

$$B = m^2 - x(1-x)p^2$$  \hspace{1cm} (6.53)

and $\tau(\epsilon) = (-4\pi\mu^2)^\epsilon$.

First we will investigate the eight diagram. Its amplitude can be written as

$$M_a(K^+ \rightarrow \pi^-; Q_6) = \sqrt{2} G_F \lambda_u (-8) (\frac{N_c m^2}{f_\pi})^2 I_x$$  \hspace{1cm} (6.54)

where

$$I_x = \int \frac{d^D r}{(2\pi)^D} F(s',s) C_p(p^2) F(r',r).$$  \hspace{1cm} (6.55)

where $s$ and $r$ are the loop momenta in the eight diagram, while $p = s - r$ is the momentum of the penguin gluon. We have performed the trace calculation in naive dimensional regularization. The function $F$ is defined as

$$F(r',r) = \frac{r \cdot r' - M^2}{(r^2 - M^2)(r'^2 - M^2)}$$  \hspace{1cm} (6.56)

Since $C_p$ is momentum dependent, this means that the two loops are coupled. In the limit of constant $C_p$ the loops decouple and we are back at the standard approach for $K^+ \rightarrow \pi^-$ amplitude. However, when $P(p^2)$ is written as in eq.6.51 we find that $I_x$ can be calculated analytically within dimensional regularization. We use the Feynman parametrization to calculate $I_x$. Notice that the momentum dependent penguin coefficient has a form similar to a propagator. One has to apply Feynman parameter technique both in the ordinary sense and in reversed form in order to disentangle the integrals after the momentum integrations. Some details are collected in the appendix. The divergent part is transferred to $\Gamma$-functions while the momentum integration is left finite. The leading contribution shows up to be proportional to $\Gamma(-1 + 3\epsilon)\Gamma(-\epsilon) F(\epsilon)$ where $F(\epsilon)$ is non-singular as $\epsilon \rightarrow 0$. This should be compared with the result with a constant penguin coefficient which is proportional to $\Gamma(-1 + \epsilon)\Gamma(\epsilon)$. Using the momentum dependent penguin coefficient shown above, the matrix element for $K^+ \rightarrow \pi^-$ can in principle be calculated analytically to any order in mass. Within the approximation $m_c^2 \gg M^2$, we find the result:

$$I_x = (-\frac{\alpha_s}{3\pi})(\frac{i}{16\pi^2})^2 k^2 (-2) (\frac{1}{\epsilon})^2 m^2 \left[ 1 + O\left(\frac{M^2}{m^2} \right) + O(\epsilon) \right],$$  \hspace{1cm} (6.57)

The keyhole diagram is proportional to $M^2$ rather than $m_c^2$ and is therefore neglected here.
One can interpret the $\epsilon$'s in light of the divergence in a cut-off procedure. If the divergence was logarithmic one has to interpret $1/\epsilon$ as $f_n$ and similarly $1/\epsilon$ is interpreted as $\tilde{C}_q$ if the divergence is quadratic. This interpretation leads to the following $g_6^{(1/2)}(Q_6)$ contribution:

$$g(Q_6)_{DR} = \frac{\alpha_s \langle \bar{q}q \rangle m_c^2}{3\pi} \left[ 1 + \mathcal{O}\left( \frac{M^2}{\Lambda^2} \right) + \mathcal{O}\left( \frac{M^2}{m_c^2} \right) \right]. \quad (6.58)$$

This should be compared with the standard result

$$g(Q_6) = -\frac{\alpha_s}{3\pi} \left( \ln \frac{m_c^2}{\Lambda^2} + \frac{5}{3} \right) \frac{\langle \bar{q}q \rangle}{2Mf_n^2} \left[ 1 - \frac{6M^2}{\Lambda^2} \right], \quad (6.59)$$

obtained from eqs.4.22, 6.8, 6.44 and 6.50. We see that there is a substantial enhancement in eq.6.58 compared to the standard procedure due to the $m_c^2$ term. This yields a contribution of order 5 which, together with other contributions as for instance $Q_-$, will give a too large large value for the $g_6^{(1/2)}$ coefficient. Note that we have only kept the leading term in mass. There will also be sub-leading terms in $\epsilon$. These terms are ambiguous for the following reason: The amplitude is proportional to $(m_c^2)^{1-3\epsilon}$. If we expand this term in order of $\epsilon$, we will encounter terms like

$$\epsilon \Gamma(-1 + 3\epsilon) \Gamma(-\epsilon) \ln(m_c^2) \quad (6.60)$$

These terms are difficult to interpret since the cancellation of the $\epsilon$ pole is either from the first or the second (or a combination) $\Gamma$ function. Possible correction terms may have the following form

$$\frac{M^2}{\Lambda^2} \ln \frac{m_c^2}{\Lambda^2}, \quad \frac{M^3 f_n^2}{\Lambda^2 < \bar{q}q >} \ln \frac{m_c^2}{\Lambda^2}, \quad \text{and} \quad \frac{M^2}{\Lambda^2} \ln \frac{\Lambda^2}{M^2}. \quad (6.61)$$

One should however have in mind that the mentioned next to leading order terms in $\epsilon$ have large coefficients of order 10. Though at first sight it seems like the eight-diagram is substantially enhanced due to the $m_c^2$ factor. Unfortunately, the correction terms are difficult to pin down. Due to large coefficients they may be of the same order as the leading result. This should not come as a surprise since we know the difficulties which arise when treating the penguin interaction perturbatively as we have seen in the case of a constant penguin coefficient. In eq.6.8 the next to leading term was numerically equally important as the leading logarithmic term.

The quadratic divergence is only appearing in left-right operators like $Q_6$. From penguin operators containing two left-handed currents, one obtains a product of two logarithmic divergences, and there is no ambiguity as for quadratic divergences. The standard contribution from $Q_4$ is (compare with (6.59))

$$g(Q_4) = \frac{\alpha_s}{12\pi} \left( \ln \frac{m_c^2}{\Lambda^2} + \frac{5}{3} \right). \quad (6.62)$$
Using the same method as for the \( Q_6 \), we obtain an analytical result for a variable penguin coefficient within dimensional regularization. In this case only the eight diagram contributes with axial vector currents acting at the weak vertices.

\[
g(Q_4)_{\text{DR}} = \frac{\alpha_s}{12\pi} \left( \ln \frac{m_c^2}{M^2} + \frac{2}{3} - \frac{9}{\Lambda_x^2} \left( \ln \frac{m_c^2}{M^2} \ln \frac{m_c^2}{\Lambda_x^2} + \Delta \right) \right), \tag{6.63}
\]

where \( \Delta \) parametrizes some calculable non-leading terms, and we have put \( \mu = \Lambda_x \) in eq.6.51. Using the values \( m_c = 1.4\text{GeV} \), \( M \simeq 200\text{MeV} \) and \( \Lambda_x \simeq 830\text{MeV} \), we obtain an increase by almost 70\% of the leading term with respect to eq.6.62. However, this is partially compensated by the non-leading term \( M^2/\Lambda_x^2 \), and the overall result is an increase of order 20\% of \( g(Q_4)_{\text{DR}} \) with respect to \( g(Q_4)_s \). Even if \( g(Q_4) \) is small (\( \simeq 0.05 \)), eq.6.63 gives an idea of how the penguin contributions behave when the variation of the penguin coefficients are taken into account. However, \( g(Q_4)_{\text{DR}} \) is rather sensitive to variations in \( M \).

In order to check the reliability of our leading order result in the case of the \( Q_6 \) operator, we perform a proper time regularization of the same amplitude. With this method we do not get an analytically answer, but a numerical calculation should reveal whether or not next to leading order terms are important in the dimensional analysis. The results are tabulated in table 6.1. We write \( g(Q_6)_P \) for proper time results with a momentum dependent penguin coefficient and \( g(Q_6)_S \) for the proper time regularized standard result (i.e. constant penguin coefficient). In the last case we use \( \mu = \Lambda \). Since for various values for \( \Lambda \) and \( M \) the proper time regularized integrals for \( \langle \bar{q}q \rangle \) and \( f_\pi \) are either over- or under-estimated, the result for the \( g \) factor will therefore also be over- or under-estimated. We correct for this in the last two columns of table 6.1 by multiplication of the ratio between the physical values and their corresponding proper time results. The phenomenologically modified \( g \)-factors are written as follows:

\[
g(Q_6)^M_{\text{S}}(P) = \frac{\langle \bar{q}q \rangle(\mu = \Lambda)}{\langle \bar{q}q \rangle(\Lambda, M)} \frac{f_\pi}{f_\pi(\Lambda, M)} g(Q_6)_{\text{S}}(P). \tag{6.64}
\]

The physical quark condensate is determined from eq. 3.75 with \( \langle \bar{q}q \rangle(\mu = 1\text{GeV}) \sim (\text{-235MeV})^3 \) which corresponds to \( (m_u + m_d) \sim 12\text{MeV} \) at \( \mu = 1\text{GeV} \). From the \( L_5 \) coefficient discussed earlier, one sees that there is a strong cancellation between the eight- and the keyhole-diagram for certain values of \( \Lambda \) and \( M \). This is reflected in the proper time calculation. In table 6.1, one sees that for \( \Lambda = 0.7\text{GeV} \) and \( M = 0.33\text{GeV} \) the \( L_5 \)-factor becomes negative. The same happens for the \( L_5 \) expression calculated in dimensional regularization. We see that the modified \( g \)-factors are very much below the result found in the previous dimensional analysis. This means that there are large cancellations if the next to leading order contributions could be determined. The proper time regularization will have an effective exponential damping. A typical factor encountered in proper time regularization is \( e^{-p^2/\Lambda^2} \). In light of this we empirically damp the dimensional regularization result by a factor \( e^{-\beta \frac{m_c^2}{\Lambda_x^2}} \) where \( \beta \) can be determined by a comparison of the two calculations of \( g_8^{(1/2)}(Q_6) \).
6.2 Effects of a Momentum Dependent Penguin Coefficient

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Table 6.1: Values of $g(Q_6)^S_5$ and $g(Q_6)^P_5$, for constant and momentum dependent penguin coefficient respectively, calculated for different values of $\Lambda$ and $M$. The corresponding modified values obtained when compensating for wrong values obtained for $f_\pi$ and $\langle \bar{q}q \rangle$ are also given.
6.2 Non-diagonal Self-energy $s \rightarrow d$ Effects

By comparing the results, one can determine $\beta$ to be around 2. Thus, the damped, leading order (in $\epsilon$), matrix element in dimensional regularization becomes

$$
\langle \pi^- | \mathcal{L}_{\Delta S=1} | K^- \rangle = \frac{G_F}{\sqrt{2}} \frac{\alpha_s}{12\pi} \frac{8(gq)(p_K \cdot p_\pi)}{M^3} m_\pi^2 e^{-\beta \frac{m_\pi^2}{\Lambda^2}}
$$

(6.65)

where $\beta$ parametrizes the subleading terms. The result obtained from the proper time method also give a value for $g(Q_6)$ at the same order as other authors have concluded. A replacement of the constant penguin coefficient by a momentum dependent one gives a $10 - 20\%$ increase. This is surely not enough to explain the $\Delta I = 1/2$ rule.

6.2.3 Non-diagonal Self-energy $s \rightarrow d$ Effects

For non-leptonic weak radiative decays we have written down an effective Lagrangian which contains 6 four-quark operators and their attending Wilson coefficients to order $G_F$. Within this procedure, and to the same order in the coupling constant, we have omitted terms containing operators of the form $(i\gamma \cdot D - m_q)$ by appealing to the equations of motion for the quark fields

$$
(i\gamma \cdot D - m_q)\psi_q = 0
$$

(6.66)

where $D_\mu$ is the covariant derivative containing the gluon field and $m_q$ is the appropriate current quark mass. This is valid in perturbative QCD but at low energies the quark contents of mesons or baryons can not be treated perturbatively due to confinement. This means that the quarks are not on-shell. The off-shell effects should be investigated with respect to the sensitive CP-violating ratio $\epsilon'/\epsilon$ but also with respect to possible effects connected to the $\Delta I = 1/2$ rule.

The off-shell effect is related to the binding energy and it is probably strongest in the meson-octet and especially for the pions and kaons due to their relatively small mass. The binding energy must be stronger for a stable particle made up of light quarks than a corresponding stable particle made up of heavier quarks. This is also partially supported by the Bag Model where the quarks are considered as free within the bag. The model gives a fair description of baryons but it does not give especially good results for light meson physics so the on-shell approach can be questioned [63].

In a series of papers [64, 65, 66, 67, 68, 69], it has been pointed out that operators containing $(i\gamma \cdot D - m_q)$ are not zero in general and that they will contribute to processes like $K \rightarrow 2\pi$, $K \rightarrow 2\gamma$ and $B \rightarrow 2\gamma$. When a typical bound state parameter is turned off, the genuinely off-shell effects will vanish. In the $\chi Q M$ this parameter is the constituent quark mass $M$.

The Standard Model Lagrangian induces no direct $s \rightarrow d$ transition. However, from higher order corrections it is possible to construct an effective interaction containing its effect. In order to include effects of this operator it must be constructed such that in the on-shell limit it must vanish after regularization. This defines the physical $s$ and $d$ quarks in the presence of weak interaction.
6.2 Non-diagonal Self-energy $s \rightarrow d$ Effects

The unrenormalized $s \rightarrow d$ transition corresponds to an effective Lagrangian

$$L_{sd}^U = \bar{d}(ai\gamma \cdot DL + bL + cR)s,$$

where $a, b, c$ are divergent quantities given by the loop integrations, and $L, R$ are the left- and right-handed chiral operators in Dirac space. The quantities $b$ and $c$ have dimension one while $a$ is dimensionless. Thus, the effective Lagrangian has dimension four. The masses and wavefunctions are multiplicatively renormalizable and the renormalized non-diagonal $s \rightarrow d$ self-energy corresponds to an effective Lagrangian of the form [70]

$$L_{sd}^R = -A \bar{d}(i\gamma \cdot D - m_d)(i\gamma \cdot DR + M_R R + M_L L)(i\gamma \cdot D - m_s)s,$$

which has dimension six as the standard four-quark operators since $A$ is dimensionless and $M_L, M_R$ have dimension one. In momentum space, $A$ is a finite, slowly varying function of the external quark momentum ($k$) squared. Explicitly, in an expansion of $a$ in terms of $k^2$, $A$ becomes the coefficient of $k^2$. Hence, $A$ is the derivative of $a$ in eq.6.67 with respect to $k^2$. The constant term is removed by counterterms. $M_R$ and $M_L$ depend on the mass of the $s$- and $d$-quarks.

In the limit $m_s, m_d \rightarrow 0$, which we will work, the Lagrangian becomes

$$L_{sd}^R(m_s, m_d \rightarrow 0) = -A \bar{d}(i\gamma \cdot D)^3 Ls.$$

One should emphasize that $L_{sd}^R$ is obtained within perturbation theory and can therefore not be applied on off-shell quarks strongly bound in pions and kaons.

To lowest order, i.e. when QCD effects are turned off, there is a strong suppression of the CP-conserving amplitude due to the GIM-mechanism [24]. One finds that $A \propto G_F m_c^2 / M_W^4$. In the CP-violating case the suppression is much weaker due to a large top quark mass. In Feynman gauge the top quark contribution is dominated by the unphysical Higgs. See fig.6.4 where $H$ is substituted for the $W$ boson. In the extreme limit where $m_c \rightarrow 0$ and $m_t \rightarrow \infty$, we find that the top quark contribution exactly cancels the charm contribution:

$$\mathcal{M}(s \rightarrow d, m_c \rightarrow 0, m_t \rightarrow \infty) \rightarrow \frac{1}{6M_W^4} - \frac{m_t^2}{2M_W^4} \frac{1}{3m_c^2} = 0.$$

When QCD effects are introduced, one gets diagrams to calculate similar to fig.6.4. It is called the self-penguin due to the gluon from the penguin loop diagram is reabsorbed by the external $s$- or $d$-quarks. In the CP-conserving case one obtains an unsuppressed charm-quark contribution $\propto G_F \alpha_s (\log m_c)^2$ [71, 72, 65]. This shows that the GIM-cancellation do not appear in the same way to different orders. The suppression lies now in the strong coupling constant.

The perturbative Dirac equation has to be modified in order to incorporate bound state effects. This yields

$$[i\gamma \cdot (iD - eA^{pert} - eA^{bind}) - m] \psi = 0$$

(6.71)
The expectation value of the perturbative Dirac operator is thus proportional to the binding energy of the hadronic state.
\[
\langle \psi_h | (i\gamma \cdot D - m) | \psi_h \rangle = \langle \psi_h | \gamma_0 V(x) | \psi_h \rangle = \int d^3 x \bar{\psi}(x)_h V(x) \psi(x)_h. \tag{6.73}
\]

A QED analogy to the $s \rightarrow d$ self-energy transition is the electron self-energy which is zero when the electron is on-shell. However, when the electron is bound to a proton in the hydrogen atom, off-shell effects gives an important contribution to Lamb shift where the energy difference between the $2s_{1/2}$ and $2p_{1/2}$ is measured to be $(1057.8 \pm 0.1)\text{MHz}$ obtained by Triebwasser, Dayhoff and Lamb in 1953. If the $s$ and $d$ quarks were bound by electromagnetic forces, the contribution to $\bar{d}s \rightarrow 2\gamma$ would be proportional to the binding energy of the system, that is, of the order $\alpha_{\text{em}}/r_B$, where $r_B$ is the Bohr radius of the system. Even though this is a small effect in QED, one expects that off-shell effects are larger at low energy QCD scales where confinement is very important.

We consider the $s \rightarrow d$ off-diagonal self-energy expression and in the massless case it can be written as
\[
\mathcal{M}(s \rightarrow d) = -i\delta \Sigma_{ds} s = -ik^2 \gamma \cdot k s A(k^2) \tag{6.74}
\]
in agreement with eq.6.69 where $k$ is the four momentum of the external $s$- and $d$-quark.

To determine the function $A(k^2)$ exactly in the pure electroweak case, we use the Feynman gauge and find
\[
A(k)_{\text{EW}} = 2\lambda_1 \frac{g_W^2}{16\pi^2} (1 + \frac{m^2}{2M_W^2}) \int_0^1 dx \frac{x}{k^2} \ln \left[ 1 - \frac{x(1-x)k^2}{M_W^2} \right] \tag{6.75}
\]
where $\bar{M}_q^2 = xM^2_W + (1-x)m^2$ where $m$ is the mass of the quark running in the loop. The term $\propto m^2/(2M^2_W)$ is due to the unphysical Higgs. For small $k^2$ we obtain

\[ A_{EW} = \frac{G_F}{\sqrt{2} 2\pi^2} \left[ \lambda_u(r_u - r_c) - \lambda_t(r_c - r_t) \right] \]

(6.76)

where the quantities $r_q$ are given by

\[ r_q = -\left(1 + \frac{m^2_q}{2M^2_W}\right) \int_0^1 dx x^2(1-x) \frac{M^2_W}{M^2_q} \]

(6.77)

where $q = u, c, t$. For a finite top quark mass $m_t = 180 GeV$ the pure electroweak CP-conserving case gives us numerically

\[ r_c - r_t = -0.167 - (-0.124) = -0.043 \]

(6.78)

To order $\alpha_sG_F$ the GIM suppression is logarithmic. This was shown originally by Shabalin. To obtain the two loop self-penguin we need the penguin ($s \rightarrow d + gluon$) loop for arbitrary external momenta. By direct calculation this is divergent, and before we insert it into the next loop to get the self-penguin, it has to be properly regularized. We impose that the resulting expression must fulfill the generalized continuity equation for charge. For this purpose we consider the Ward-identity

\[ q_u \Gamma^\nu_{ds} = \Sigma_{ds}(k + q) - \Sigma_{ds}(k) \]

(6.79)

in order to determine the necessary counterterms or remove troublesome divergent terms which are of no relevance to us. In our case $\Gamma^\nu$ corresponds to the penguin one loop. Note that the strong coupling $g_s$ is not included. We define the tensor $T^{\sigma\nu}$ in the following way

\[ \Gamma^\nu(k, q) = T^{\sigma\nu}\gamma_\sigma L \]

(6.80)

The most general tensor structure compatible with the Ward-Takahashi identity must be made of the accessible momenta, the metric tensor and the generalized antisymmetric Levi-Civita symbol:

\[ T^{\sigma\nu} = \left[ (k + q)^2A(k + q) - k^2A(k) \right] \frac{k^\sigma q^\nu}{q^2} + (k + q)^2A(k + q) g^{\sigma\nu} \]

\[ + B_0[q^\sigma q^\nu - q^2g^{\sigma\nu}] + B_1[q^\sigma k^\nu - q.kg^{\sigma\nu}] \]

\[ + B_2[k^\sigma k^\nu - \frac{q.k}{q^2}k^\sigma q^\nu] + B_3 i\gamma^\nu \gamma^\alpha \gamma_\sigma q_\alpha \]

(6.81)

We have used the following decomposition of the product of three $\gamma$-matrices:

\[ \gamma^\alpha \gamma^\beta \gamma^\gamma = g^{\alpha\beta}\gamma^\gamma - g^{\alpha\gamma}\gamma^\beta + g^{\beta\gamma}\gamma^\alpha - i\epsilon^{\alpha\beta\gamma} \gamma_\mu \gamma_5 \]

(6.82)

where $\epsilon^{0123} = 1$ and $\gamma_5 = i\gamma_0\gamma_1\gamma_2\gamma_3$. Comparing this with the raw result for $\Gamma^\nu$ we find the regularized expressions (For $W$- exchange only):

\[ B_j = 2g^jW g_t^e \int_0^1 dx \int_0^{1-x} dy f_j \left[ \frac{M^2_Z}{x(1-x)q^2} - y(1-y)k^2 + 2xyq \cdot k \right]^{-1} \]

(6.83)
6.2 Non-diagonal Self-energy $s \rightarrow d$ Effects

where $M_y$ is defined similar to $M_x$ earlier and $t^s$ is a colour matrix. Furthermore, the $f_i'$s are given by $f_0 = 2x(1 - x); f_1 = y(2x - 1); f_2 = -2y^2$ and $f_3 = -y$, respectively.

We will write a lower index $k$ in the tensor symbol $T_{x}^{\sigma y}$ to indicate different powers of external momentum ($s$ or $d$ quark momentum). This yields

\begin{align*}
T_0^{\sigma \nu} &= \tilde{I}_0 g^{\sigma \nu} + C_0 [q^\sigma q^\nu - q^2 g^{\sigma \nu}] \tag{6.84} \\
T_1^{\sigma \nu} &= 2 \tilde{I}_2 (k \cdot q) g^{\sigma \nu} + \tilde{I}_0 \frac{q^\sigma k^\nu}{q^2} + F_0 (k^\nu q^\sigma - (k \cdot q) g^{\sigma \nu}) \\
&+ H_0 (k \cdot q) [q^\sigma q^\nu - q^2 g^{\sigma \nu}] + J_0 k_\alpha q_\beta \varepsilon^{\sigma \nu \alpha \beta} \tag{6.85} \\
T_2^{\sigma \nu} &= 2 \tilde{I}_2 (k \cdot q) \frac{k^\sigma q^\nu}{q^2} + (\tilde{I}_3 k^2 + \tilde{I}_5 (k \cdot q)^2) g^{\sigma \nu} + Q_0 [(k \cdot q) k^\sigma q^\nu - q^2 k^\nu k^\sigma] \\
&+ R_0 (k \cdot q) [q^\sigma k^\nu - (k \cdot q) g^{\sigma \nu}] + T_0 k^2 [q^\sigma q^\nu - q^2 g^{\sigma \nu}] \\
&+ W_0 (k \cdot q)^2 [q^\sigma q^\nu - q^2 g^{\sigma \nu}] + P_0 (k \cdot q) k_\alpha q_\beta \varepsilon^{\sigma \nu \alpha \beta} \tag{6.86} \\
T_3^{\sigma \nu} &= (\tilde{I}_2 k^2 + \tilde{I}_3 (k \cdot q)^2 - \tilde{I}_2 (q = 0) k^2) \frac{k^\sigma q^\nu}{q^2} + (\tilde{I}_3 k^2 + \tilde{I}_5 (k \cdot q)^2)(k \cdot q) g^{\sigma \nu} \\
&+ \alpha_2 k^2 [q^\sigma k^\nu - (k \cdot q) g^{\sigma \nu}] + \alpha_3 (k \cdot q)^2 [q^\sigma k^\nu - (k \cdot q) g^{\sigma \nu}] \\
&+ \beta_0 k^2 (k \cdot q) [q^\sigma q^\nu - q^2 g^{\sigma \nu}] + \beta_0 (k \cdot q)^3 [q^\sigma q^\nu - q^2 g^{\sigma \nu}] \\
&+ \alpha_1 (k \cdot q) [k^\sigma k^\nu - \frac{(k \cdot q)^2 q^\nu}{q^2}] + \beta_1 (k \cdot q) [k^\sigma k^\nu - (k \cdot q) k^\nu q^\nu] \\
&+ \alpha_4 k^2 k_\alpha q_\beta \varepsilon^{\sigma \nu \alpha \beta} + \alpha_8 (k \cdot q)^2 k_\alpha q_\beta \varepsilon^{\sigma \nu \alpha \beta} \tag{6.87}
\end{align*}

where the coefficients are functions of $q^2$ only. We collect the relevant coefficients in table 6.2.

We obtain the following expression for the renormalized self-penguin contribution:

\begin{equation}
M(s \rightarrow d, g_2^2) = -2i g_s^2 g_W^2 d t^s t^s \int \frac{d^4 q}{(2\pi)^4} \frac{\gamma^\nu}{q^2} S(k + q) \Gamma_\nu(k, q) s \tag{6.88}
\end{equation}

To obtain the result compatible with eq.6.69 we expand the $d$- (or $s$-) quark propagator $S(k + q)$ and $\Gamma_\nu$ in powers of the external momentum $k$ and keep in total the order $(k)^2$ only.

\begin{equation}
M(s \rightarrow d, g_2^2) = -2i g_s^2 g_W^2 (t^2 t^s) k^2 k' d \gamma_\mu L s (\frac{i}{16\pi^2})^2 (\zeta_W + \frac{m^2}{2M_W^2} \zeta_H) \tag{6.89}
\end{equation}

where

\begin{equation}
\zeta_W = \int d(q_E^2) \left\{ \frac{(1 - z)}{4} \tilde{I}_3 - \frac{(1 - z)}{4} \tilde{I}_5 + C_0 \frac{F_0}{q_E^2} - \frac{H_0}{4} - \frac{iJ_0}{2q_E^2} + R_0 - \frac{W_0 q_E^2}{4} \right. \\
+ \frac{iP_0}{2} - \frac{3a_2}{2} + \frac{\alpha_4 q_E^2}{2} + \frac{3q_E^2 \beta_0}{4} - \frac{3q_E^2 \beta_6}{8} - \frac{3i \alpha_4}{2} + \left. \frac{i q_E^2 \alpha_8}{4} \right\} \tag{6.90}
\end{equation}

and $\zeta_H$ is found by changing the sign of the following terms in $\zeta_W$: $J_0, P_0, \alpha_4$ and $\alpha_8$. We have skipped a divergent term $\alpha \int d(q_E^2) / q_E^2$ which vanishes in the dimensional
### 6.2 Non-diagonal Self-energy $s \rightarrow d$ Effects

<table>
<thead>
<tr>
<th>Coefficient</th>
<th>function dependence</th>
</tr>
</thead>
<tbody>
<tr>
<td>$I_0$</td>
<td>$\ln(\xi/\xi(q = 0))$</td>
</tr>
<tr>
<td>$I_1$</td>
<td>$-z(1-z)/\xi$</td>
</tr>
<tr>
<td>$I_2$</td>
<td>$-2z^2(1-z)^2/\xi^2$</td>
</tr>
<tr>
<td>$I_3$</td>
<td>$-8z^3(1-z)^3/(3\xi^3)$</td>
</tr>
<tr>
<td>$C_0$</td>
<td>$2z(1-z)/A$</td>
</tr>
<tr>
<td>$F_0$</td>
<td>$y(1-2z)/A$</td>
</tr>
<tr>
<td>$H_0$</td>
<td>$4y(z^3-z^4)/A^2$</td>
</tr>
<tr>
<td>$J_0$</td>
<td>$iy/A$</td>
</tr>
<tr>
<td>$R_0$</td>
<td>$2y^2z(-1+2z)/A^2$</td>
</tr>
<tr>
<td>$W_0$</td>
<td>$8y^2(z^3-z^4)/A^3$</td>
</tr>
<tr>
<td>$P_0$</td>
<td>$-2iy^2z/A^2$</td>
</tr>
<tr>
<td>$\alpha_4$</td>
<td>$(y^2-y^3)(1-2z)/A^2$</td>
</tr>
<tr>
<td>$\alpha_4$</td>
<td>$i(y^2-y^3)/A^2$</td>
</tr>
<tr>
<td>$\alpha_7$</td>
<td>$4y^3z^4(1-2z)/A^3$</td>
</tr>
<tr>
<td>$\alpha_8$</td>
<td>$4iy^3z^2/A^3$</td>
</tr>
<tr>
<td>$\beta_0$</td>
<td>$8(y^2-y^3)(z^3-z^2)/A^3$</td>
</tr>
<tr>
<td>$\beta_6$</td>
<td>$16y^3(z^3-z^4)/A^4$</td>
</tr>
</tbody>
</table>

Table 6.2: Coefficients related to the self penguin diagram. Each term includes (not written) integration over the coupled Feynman parameters $y$ (from 0 to 1 — $z$) and $z$ (from 0 to 1). $A$ is defined as $A = m^2 + y(M_W^2 - m^2) + q_E^2 z(1-z)$ and $\xi = M_W^2 - z(M_W^2 - m^2) + q_E^2 z(1-z)$. 
regularization scheme. After the integration of momentum $q_g^2$, a great simplification occurs where all the contributions in $\zeta_W$ coming from $T_3\gamma^u$ vanishes. Then we obtain (for $q = c, t$):

$$\mathcal{M}(s \rightarrow d,g_x^2) = -i \frac{g^2 g_W}{3 M_W^2} x k^\mu d^a \gamma_\mu L s \left( \frac{i}{16 \pi^2} \right)^2 F_q \tag{6.91}$$

where $F_q$ is the dimensionless quantity

$$F_q = 6 M_W^2 \int_0^1 dx \int_0^{1-x} dy \left\{ \frac{\left[ (-1/2 + 2x) y + x(1-x)] ln(\frac{M_y^2}{[M_y^2 - x(1-x) \Lambda^2]}) \right)}{[M_y^2 - x(1-x) \Lambda^2]} \right\} + \frac{y x}{M_y^2} + \frac{y^2(3x - 1)}{(1-x)M_y^2} \frac{x(1-x)}{3 M_y^2} \right\} + 3 m_t^2 \int_0^1 dx \int_0^{1-x} dy \left\{ \frac{\left[ (-3/2 + 2x) y + x(1-x)] ln(\frac{M_y^2}{x(1-x) \Lambda^2}) \right)}{[M_y^2 - x(1-x) \Lambda^2]} \right\} + \frac{y(x + 3y)}{M_y^2} - \frac{3 y^2(1-y)}{x(1-x)M_y^2} + \frac{2 y^3}{(1-x)^2 M_y^2} - \frac{x(1-x)}{3 M_y^2} \right\} \right\} \tag{6.92}$$

where the unphysical Higgs contribution $\propto 3m^2$ is included. Since we have omitted any current mass for the $s$ and $d$ quark and external momenta, we get some infrared divergent terms which have been cured by introducing the lower cut-off scale $\Lambda$. It will later be identified with the renormalization point. Its magnitude will be taken as the chiral symmetry breaking scale $\Lambda = 0.83 \text{ GeV}$. Note that the second last term is divergent in the $x$ parameter and in order to obtain numerical results we regularize it the following way

$$\int_0^1 \frac{dx}{x} \rightarrow \int_1^{1} \frac{dx}{x} \tag{6.93}$$

This problem is due to a $\ln k^2$ term in eq.6.88 which is problematic in our expansion and we have to compensate this with a regularization where $c^{-1} = ln(m_t^2/\Lambda^2)$. For arbitrary $k^2$ we have terms like $k^2 \ln k^2$ which is the origin of the problem. We thus had to introduce an ultraviolet cut-off in loop integrations which we have taken to be $m_t$.

It should be noted that in a previous calculation [65], only the case $m^2 \ll M_W^2$ was considered. In this case mainly the $B_0$ term in eq.6.81 contributes to order $G_F$. Contributions from the other $B_i$'s are suppressed by $m_i^2/M_W^2$ with respect to the $B_0$ contribution. This can be seen from the $y$-integration where a proper change of variable yields a $1/M_W^2$ suppression compared to $B_0$. Including the heavy top quark, there are no obvious suppressions and all contributions have to be taken into account. In order to obtain the total result we must also add the contribution where the gluon is absorbed by the $d$-quark. This yields the same result as in eq.6.92.
Combining the eqs.6.91 and 6.92, we find the self-penguin contribution to the quantity $A$ in (6.69):

$$A_{SP} = \frac{G_F}{\sqrt{2}} \frac{\alpha_s}{24\pi^2} \left[ \lambda_u (F_u - F_c) - \lambda_t (F_c - F_t) \right],$$  

(6.94)

where $h_c = Tr(t^a t^a)/N_c = (N_c^2 - 1)/(2N_c)$ is a colour factor. We have also multiplied by a factor 2 due to the second self-penguin diagram. Numerically, we find for $m = m_c = 1.4$ GeV and $\Lambda = 0.83$ GeV:

$$F_c \simeq 49.0$$  

(6.95)

Similarly, for $m = m_t = 180$ GeV and $\Lambda = 0.83$ GeV, we obtain

$$F_t \simeq -10.7$$  

(6.96)

For the $c$-quark case one obtains the analytical result within the leading logarithmic approximation

$$F_c = \frac{1}{2} \left[ \ln \left( \frac{M^2}{\Lambda^2} \right) \right]^2 - \frac{1}{2} \left[ \ln \left( \frac{m_c^2}{\Lambda^2} \right) \right]^2 + \left( \frac{5}{6} - \frac{4}{6} \right) \ln \left( \frac{M^2}{m_c^2} \right) - \frac{5}{6} \ln \left( \frac{m_c^2}{\Lambda^2} \right).$$  

(6.97)

The leading part of this result was obtained in [65] for the charm quark case, while the general result in eq.6.92 is new. Taking into account the leading logarithmic QCD corrections by folding in [73, 74] the well known Wilson coefficients $C_{\pm}$ [11] our result takes the form

$$\frac{\alpha_s}{\pi} F_c \to \tilde{F}_c = \int_{m_c^2}^{M_W^2} \frac{dp^2}{p^2} \rho(p^2) \left\{ \ln \left( \frac{p^2}{\Lambda^2} \right) + \left( \frac{5}{6} - \frac{4}{6} \right) \right\} - \frac{5}{6} \rho(m_c^2) \ln \left( \frac{m_c^2}{\Lambda^2} \right),$$  

(6.98)

where $\rho(p^2) = [C_+(p^2) + C_-(p^2)]\alpha_s(p^2)/(2\pi)$. Because the operator in (6.69) has zero anomalous dimension there is no extra factor attached to $\rho(p^2)$. The anomalous dimension of all operators containing covariant derivatives are zero when perturbation theory is used. Using the equations of motion the covariant derivative can be reduced to mass factors and therefore no anomalous dimension is associated with this type of operators. The linear combination $(C_+ + C_-)/2$ is the colour octet combination of the coefficients which has to be used because the gluon within the self-penguin belongs to a colour octet. The $C_{\pm}$'s are given by ratios of $\alpha_s$ at different scales to a power determined by the anomalous dimension of the corresponding four quark operators $Q_{\pm}$. Thus the integral in eq.6.98 can be performed analytically and the result obtained in terms of $\alpha_s$ at the scales $\Lambda, m_c, m_t$, and $M_W$, respectively. Numerically, we find $\tilde{F}_c \simeq 3.25$. For momenta above the $W$ mass, we expect no QCD corrections except for the running of $\alpha_s$ and we take

$$\tilde{F}_t = \frac{\alpha_s(m_t^2)}{\pi} F_t \simeq -0.37$$  

(6.99)
6.2 Non-diagonal Self-energy $s \rightarrow d$ Effects

Again, because our operator in (6.69) has no anomalous dimension, we can just add the results for $A_{EW}$ and $A_{SP}$. We find that the pure EW result is numerically approximately the same as the suppressed top quark contribution in eq.6.99.

A non-vanishing result for the self-penguin diagram will affect the $K \rightarrow 2\pi$ transitions. In the $\chi QM$ the basic diagram with strong interaction effects are shown in fig. 6.5 and fig.6.6 for one and two pion decay, respectively. There are also two other diagrams for $K^0 \rightarrow \pi^+\pi^-$ as shown in fig.6.7. We calculate the amplitude in the effective $\chi QM$ both for $K^- \rightarrow \pi^-$ and also $\bar{K}^0 \rightarrow \pi^+\pi^-$. As a check we also found that the $K \rightarrow \text{vacuum}$ was zero such that chiral symmetry could be used to relate $K^- \rightarrow \pi^-$ to the two pion decay. This means that eq.6.69 gives an octet contribution that can simply be added to the $Q_6$ contribution up to $O(p^2)$. To order $O(p^4)$ we might obtain different contributions from $Q_6$ and $C_6^b$ in eq.6.69 due to the extra derivatives and in principle they could give contributions to different octets.

We find the following result:

$$ M(K^0 \rightarrow \pi^+\pi^-)_\Sigma = \frac{\sqrt{2}}{4f_\pi^2}(m_K^2 - m_\pi^2)A_M M^2 \left[ \frac{<\bar{q}q>}{M} + 2f_\pi^2 \left(1 - \frac{3M^2}{2\Lambda_\chi^2}\right) \right] \quad (6.100) $$

where $A$ is the quantity in (6.69). We want to compare this with the corresponding amplitude for the penguin operator $Q_6$

$$ M(K^0 \rightarrow \pi^+\pi^-)_{Q_6} = \frac{2\sqrt{2}}{f_\pi}(m_K^2 - m_\pi^2)C_6 \frac{<\bar{q}q>}{M} \left[ 1 - \frac{3M^2}{2\Lambda_\chi^2}\right] \quad (6.101) $$

where we will take $\mu = \Lambda_\chi$ in $C_6$. We observe that the ratio between the amplitudes in (6.100) and (6.101) goes to zero in the limit $M \rightarrow 0$, as expected.

Numerically, the ratio between the self-penguin and the $Q_6$ contribution is less than 5 % in the CP-conserving case. On the other hand for $\epsilon'/\epsilon$, the situation is
Figure 6.6: A typical diagram for $K \rightarrow 2\pi$ with a selfpenguin insertion.

Figure 6.7: Two diagrams with two-meson vertex for $K \rightarrow 2\pi$ with a selfpenguin insertion.
6.2 Penguin Condensation

Figure 6.8: The condensed penguin diagram.

different because there is a large cancellation between the gluonic and electroweak penguin contributions\([75, 58, 57, 60, 61]\), i.e. between the \(Q_6\) and the \(Q_8\) operators. We have compared the self-penguin contribution with the standard \(Q_6\) contribution in the CP-violating case. Using \(C_i = -(G_F/\sqrt{2})(\lambda_u z_i - \lambda_t y_i)\), with \([60, 61]\) \(y_6 = -0.137\); \(M \approx 220\) MeV and \(<\bar{q}q>^{1/3} \approx -220\) MeV \([57]\), we find that the self-penguin contribution is 10 – 15\% of the \(Q_6\) contribution, that is of the order \(\epsilon'/\epsilon\) itself, after the partial cancellation between the \(Q_6\) and the \(Q_8\) contributions has been taken into account.

6.2.4 Penguin Condensation

In this section we will consider a contribution to the penguin interaction which by our knowledge has not been considered previously. Within the standard approach the equation of motion is used in the penguin interaction in order to obtain four-quark operators. We will now consider a new possibility where the gluon condense in vacuum. When the penguin gluon is not attached to another quark line but to vacuum, we call the corresponding diagram the condensed penguin diagram. See fig. 6.8. Similar to what we did in previous calculations, the penguin gluon can also form a vacuum condensate when combined with another gluon emitted from some quark leg.

The penguin diagram induces an effective interaction which have been described earlier. The new operator will be written in a form that vanishes in the standard
6.2 Penguin Condensation

approach by using the equation of motion. A possible form is

$$\Delta L_{P} = \sqrt{2} G_F \lambda_u C_\Delta Q_\Delta$$

(6.102)

where $C_\Delta = C_P$ to leading order is the Wilson coefficient of the operator

$$Q_\Delta = \left\{ -\frac{1}{g_s} (D^\mu G_{\mu\nu})^s - j_\nu^s \right\} \left( \partial^\nu L t^a s \right).$$

(6.103)

It is written in a form that vanishes in the standard approach by using the equation of motion, i.e. $(D^\mu G_{\mu\nu})^s = -g_s j_\nu^s$

It can be shown that one, two or three gluons without derivatives (first term in eq. 5.40) give no contribution. The reason is that either the integrand is odd or due to helicity conservation ($RL = 0$). This means that the first non-zero contribution comes from the second term in the expansion eq. 5.40. Since the operator $Q_\Delta$ already contains $(D^a G_{a\mu})^s$, one may convince oneself that the only condensate which contributes is:

$$\langle (D_\alpha G_{\alpha})^s (D^a G_{a\mu})^s \rangle = (g_s^2 j^2).$$

(6.104)

Contributions of this kind to $K \to \pi$ come from the diagrams shown in fig.6.9. Note that in the rest of this section we only look at the first part of the $Q_\Delta$ operator in order to find rough an estimate because there are contributions from $(G^4)$ terms which we have omitted, cfr. eq. 6.1. A more rigorous treatment would also include mixing between operators. In our incomplete calculation there are constant terms. They have to cancel when taking into account $(G^4)$ terms. We find a contribution which is

$$g_8^{(1/2)}(\Delta L; \langle (DG)^2 \rangle) = -\frac{g_s^2 (j^2)}{16\pi^2 f_w^4} Tr(t^a t^a) \frac{7C_\Delta}{60 M^2}.$$

(6.105)

In [76] an estimate of the condensate is given: $(g_s^4 j^2) = -\rho^6$, with $\rho = 0.52 GeV$. The value of $\rho$ has large uncertainties. This makes our estimate of $g_8^{(1/2)}$ very sensitive. If we use $M = 230 MeV$ we find a contribution to $g_8^{(1/2)}$ which is around 0.5.

In lattice gauge calculations of hadronic matrix elements of the operators $Q_i$, the so called eye diagram is found to be very important. An example of this diagram dressed with gluons condensing in the vacuum is represented in fig.6.10. In this case there is a four-quark operator at the cross. This corresponds to a $Q_\pm$ operator acting at the weak vertex. The lowest order non-zero contribution is again $\alpha (j^2)$. This contribution is in some sense a non-perturbative version of eq.6.105 in the sense that the perturbative loop with c- and u-quarks giving $C_P$ is replaced by a u-quark loop within the $\chi QM$. This means that the $\ln(m_c^2/\mu^2)$ term in $C_P$ (with $\mu = \Lambda$) is replaced by $\ln(\Lambda^2/M^2)$. Furthermore, $\ln(\Lambda^2/M^2)$ will be interpreted as $4\pi^2 f_w^2/(N_c M^2)$, cfr. eq.5.18 without condensates. The colour octet part of $Q_\pm$ contributes by $(C_+ + C_-)/2$. This yields the following contribution for the eye diagram

$$g_8^{(1/2)}(Q_\pm, (j^2)) = -\frac{(g_s^4 j^2)}{16\pi^2 f_w^2 M^4} Tr(t^a t^a) \frac{7(C_+ + C_-)}{360 N_c}.$$

(6.106)
Figure 6.9: Condensate contributions in $K \rightarrow \pi$ transitions.
This result is very sensitive to the constituent quark mass $M$. However, for $M = 250 \text{MeV}$ and $C_\pm = 0.7, 2.1$, respectively, we find the contribution to be 0.3. Correspondingly, a constituent quark mass $M = 220 \text{MeV}$ yields a contribution of 0.5. Thus, the contributions from the higher order gluon condensates can be sizable. One must note that the numerical values given in this section are only rough estimates.

The standard effects should be subtracted as indicated by the construction of the operator $Q_\Delta$ and also direct $(G^4)$ terms must be included since we are using the $\chi Q M$. Including these terms one must have a vanishing constant term. The results are also very sensitive to $M$. It is also important that the total contribution must vanish in the limit where we apply the equations of motion. This is easy to see at the level of the Lagrangian but is not obvious from the results given in this section. A typical parameter which will tend to zero at high energies is the constituent quark mass $M$. Unless the condensate $(j^2)$ is proportional to some power of $M$, one cannot take the limit $M \to 0$.

Note that eq.6.106 contributes only to the CP-conserving amplitude while eq.6.105 also includes a CP-violating part of $C_P$ which contributes to $\epsilon'/\epsilon$.

The message of this section is to point out that contributions of the kind above exist and that one has to do further investigations in order to get a complete result.
Chapter 7

Resumé

In order to do calculations at low energies of QCD one needs effective models since the Standard Model does not describe mesonic decays but merely describes the interactions at quark level. This is due to the growth of the strong coupling constant when approaching lower and lower energies and hence the quarks become confined. This is why we can not treat the interactions perturbatively at low energies. Both the $\chi QM$ and $\chi PT$ describe the low energy behaviour of QCD.

Previously, there were no direct connection between QCD which can be used perturbatively down to energies of 1 GeV and $\chi PT$ valid up to a few hundred MeV. This energy gap has been bridged by the $\chi QM$ for instance.

The $\chi QM$ is a phenomenological model that can be regarded as having integrated out the heavy degrees of freedom in QCD. It contains quarks, mesons and soft gluons which can condense and thus make gluon condensates. Such corrections are one of the most important ingredients to describe the $\Delta I = 1/2$ rule. In the $\chi QM$ there are interaction terms between quarks and mesons which means that hadronic matrix elements can be written as loop diagrams. The model at work has no propagating pion fields, i.e. the kinetic terms for mesons appear only after having integrated out the quarks. If the quarks are integrated out, it can describe $\chi PT$ to a high order. The $\chi QM$ has been applied to energies below the chiral symmetry breaking scale of order 1 GeV down to 200 MeV. The $\chi QM$ is a QCD inspired model while $\chi PT$ is an effective theory of QCD.

Three of the papers which this thesis is based upon were made together with my supervisor and one paper as a single author. A short review of the results is presented:

The main result of the first paper was that a very small contribution to $K^0 - \bar{K}^0$ was found for the siamese penguin diagram with a momentum dependent penguin coefficient. The calculation was done with different regularizations.

The same momentum dependent penguin interaction were used in the second paper. In dimensional regularization this enabled us to calculate analytical results for $K \to \pi$ and a relatively small $g_6^{1/2}$ factor was found due to large subleading terms. Effects of the subleading terms from the density operator $Q_6$ had to be determined by a proper time regularization of the same result. This was not necessary for calculations involving the $Q_4$ operator. The subleading terms were found explicitly to be large.
However, a 10 – 20% increase in the amplitude was observed relative to a constant penguin approach.

In the third paper nonperturbative effects on the $B_K$ parameter were obtained. To order $\langle G^3 \rangle$ a vanishing result appeared due to a complete cancellation among the 20 contributing diagrams.

In the fourth paper we did a calculation of $K \rightarrow 2\pi$ which included non-diagonal self-energy effects due to the $s \rightarrow d$ transition. This calculation also enabled us to include a heavy top quark. The calculation was done in two ways. First, we calculated the unphysical $K \rightarrow \pi$ transition. The result was then related to the physical $K \rightarrow 2\pi$ decay due to chiral symmetry. Second, the same result was obtained by a direct calculation of $K \rightarrow 2\pi$. In the CP-conserving case the contribution were found to be small while the CP-violating part were found to be sizable. Due to a large cancellation between the operators $Q_6$ and $Q_8$ the contribution were found to be of the same size as $\epsilon'/\epsilon$ itself.
8.1 Clebsch-Gordan Coefficients Applied on a Member of the 27-plet

For instance a member of the 27-plet with hypercharge $-1$, isospin $1/2$ and $z$-component $+1/2$ formed by two octet states can be written:

$$|27; Y = -1, I = 1/2, I_z = +1/2)$$

$$= -\frac{\sqrt{5}}{10} |\underline{8}; Y_1 = -1, I_1 = 1/2, Y_2 = 0, I_2 = 1)$$

$$+ \frac{\sqrt{5}}{10} |\underline{8}; Y_1 = 0, I_1 = 1) Y_2 = -1, I_2 = 1/2)$$

$$+ \frac{3\sqrt{5}}{10} |\underline{8}; Y_1 = -1, I_1 = 1/2) Y_2 = 0, I_2 = 0)$$

$$+ \frac{3\sqrt{5}}{10} |\underline{8}; Y_1 = 0, I_1 = 0) Y_2 = -1, I_2 = 1/2)$$

(8.1)

$$= -\frac{\sqrt{5}}{10} (-1) \frac{\sqrt{5}}{\sqrt{3}} |\underline{8}; Y_1 = -1, I_1 = 1/2, I_{1z} = -1/2) Y_2 = 0, I_2 = 1, I_{2z} = +1)$$

$$- \frac{\sqrt{5}}{10} (-1) \frac{1}{\sqrt{3}} |\underline{8}; Y_1 = -1, I_1 = 1/2, I_{1z} = +1/2) Y_2 = 0, I_2 = 1, I_{2z} = 0)$$

$$+ \frac{\sqrt{5} \sqrt{2}}{10 \sqrt{3}} |\underline{8}; Y_1 = 0, I_1 = 1, I_{1z} = +1) Y_2 = -1, I_2 = 1/2, I_{2z} = -1/2)$$

$$+ \frac{\sqrt{5}}{10} (-1) \frac{1}{\sqrt{3}} |\underline{8}; Y_1 = 0, I_1 = 1, I_{1z} = 0) Y_2 = -1, I_2 = 1/2, I_{2z} = +1/2)$$

$$+ \frac{3\sqrt{5} \sqrt{2}}{10 \sqrt{3}} |\underline{8}; Y_1 = -1, I_1 = 1/2, I_{1z} = +1/2) Y_2 = 0, I_2 = 0, I_{2z} = 0)$$

$$+ \frac{3\sqrt{5} \sqrt{2}}{10 \sqrt{3}} |\underline{8}; Y_1 = 0, I_1 = 0, I_{1z} = 0) Y_2 = -1, I_2 = 1/2, I_{2z} = +1/2).$$

(8.2)
We can now associate the different states with the pseudoscalar particles.

\[ |27; Y = -1, I = 1/2, I_\pi = +1/2 \rangle = \frac{1}{\sqrt{30}} K^- \pi^+ - \frac{1}{\sqrt{60}} K^0 \pi^0 + \frac{1}{\sqrt{30}} K^- \pi^- - \frac{1}{\sqrt{60}} K^0 \pi^- + \frac{3}{\sqrt{20}} K^0 \eta_8 + \frac{3}{\sqrt{20}} \eta_8 K^0. \]  

(8.3)

### 8.2 Technical Details of Dimensional Analysis of \( K \to \pi \)

We will here give some details from the calculation of \( I_x \) in eq.6.55. We expand the propagators up to second order in the external virtual meson momentum \( k \). Then the propagators \((r^2 - M^2)^{-1}\) and \(((r - p)^2 - M^2)^{-1}\) occur in some powers. We need one Feynman parameter \( (x) \) to perform the integral. We use the general formula

\[
I_x = \int \frac{d^D r}{(2\pi)^D} \left[ r^2 - A \right]^{-n} = i J_n(D)[(-A)]^{(D/2-n)},
\]  

(8.4)

where

\[
J_n(D) = \frac{(-\pi)^{D/2} \Gamma(n-D/2)}{(2\pi)^D \Gamma(n)}.
\]  

(8.5)

Note that \( J_{n+1}(D) = J_n(D)(n-D/2)/n \). Dropping the constant term \( \sim k^0 \), we obtain

\[
I_x = i J_3(D) k^2 \int_0^1 dx \left\{ S_c(3) \left[ \frac{6}{D} (1 - x(1 - x)) - 2 \right.ight.
\]
\[
- \left. \left( \frac{6-D}{D} \right)(1 - 2x(1 - x)) \right] + \left( \frac{6-D}{D} \right) x(1 - x) M^2 S_c(4) \} \right.
\]
\[
- (c \to u),
\]  

(8.6)

where the quantity \( S_q(n) \) depends on the current quark masses for \( q = c, u \):

\[
S_q(n) = \int \frac{d^D p}{(2\pi)^D} \left[ C_p(p^2) \right]_q [-A]^{(D/2-n)},
\]  

(8.7)

where \( [C_p(p^2)] = [C_p(p^2)]_u - [C_p(p^2)]_c \). In this expression \( A = M^2 - x (1 - x) p^2 \).

To find \( S_q(n) \), we have to use the dimensional regularization version of \([C_p(p^2)]\) given in 6.51, and we need the generalized Feynman parametrization

\[
\frac{1}{(-A)^{(l-D_q)c}} = \frac{1}{B(l, c)} \int_0^1 dy \frac{(1 - y)^{l-1} y^{l-1}}{N^l_q},
\]  

(8.8)
where \( l = n - D/2 \) and \( N_q = (1 - y)(-A) + y(-D_q) = h(y^2 - \overline{m}_q^2) \). Here, \( h = yt(1 - t) + (1 - y)x(1 - x) \) and \( \overline{m}_q^2 = (ym^2 + (1 - y)M^2)/h \). The quantity \( B \) is given by

\[
B(a, b) = \int_0^1 dx x^{a-1} (1 - x)^{b-1} = \frac{\Gamma(a) \cdot \Gamma(b)}{\Gamma(a + b)} \quad (8.9)
\]

Using again (8.4), we obtain

\[
S_\ell(n) = iJ_{\ell+\epsilon}(D) \frac{\Gamma(\epsilon)}{B(l, \epsilon)} \int_0^1 dt 6t(1 - t) \int_0^1 dy \frac{(1 - y)^{l-1}y^{\ell-1}}{h^{l+\epsilon}} (-\overline{m}_q^2)^u \quad , (8.10)
\]

where \( u = D/2 - (l + \epsilon) \). Now, for the charm quark case \( q = c \), we use the approximation \( \overline{m}_c^2 = ym_c^2/h \). Then we obtain

\[
S_c(n) = iJ_{\ell+\epsilon}(D) \frac{\Gamma(\epsilon)}{B(l, \epsilon)} (-\overline{m}_c^2)^u \int_0^1 dt 6t(1 - t) I(x, t) \quad (8.11)
\]

Then the clue is to recognize the integral

\[
I(x, t) = \int_0^1 dy \frac{(1 - y)^{l-1}y^{\ell+u-1}}{h^{l+\epsilon}} = \frac{B(l, u + \epsilon)}{[x(1 - x)]^l[6t(1 - t)]^{u+\epsilon}} \quad (8.12)
\]

which gives

\[
S_c(n) = \overline{S}_c(n) [x(1 - x)]^{-l} \quad (8.13)
\]

with

\[
\overline{S}_c(n) = iJ_{\ell+\epsilon}(D) \frac{\Gamma(\epsilon)}{B(l, \epsilon)} (-\overline{m}_c^2)^u B(l, u + \epsilon) 6 B(2 - u - \epsilon, 2 - u - \epsilon) \quad (8.14)
\]

Then, to leading order in \( m_c^2 \), we find

\[
I_\chi = -iJ_3(D)k^2 \overline{S}_c(3) B(D/2 - 2, D/2 - 2) \quad (8.15)
\]

Here the factor \( B(D/2 - 2, D/2 - 2) \) corresponds to the logarithmic divergence connected to \( f_\tau \), and \( J_{\ell+\epsilon}(D) \) in \( S_c(3) \) to the quadratic divergence connected to the quark condensate.
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