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# INVESTIGATION OF THE PERIODIC SYSTEMS BY MEANS OF GENERALIZED HILL METHOD



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Key words: periodic system, Hill equation, plasma wave, parametric instability, infinite determinant, lower hybrid wave, Shrődinger equation, periodic potential, plasma waveguide.

We propose the new method of investigation of infinite periodic determinants, which is a generalized Hill method. This method is applicable to the solution of the following problems: electromagnetic wave propagation in a periodic media; a parametric instability of a large amplitude plasma waves; an electron beam stability in the periodic systems and external fields, a solid state band structure and others. The solution of the problems mentioned above reduces to the searching of the infinite periodic determinant roots and can be expressed as infinite series with the coefficients being determined through the auxiliary determinants independent from a frequency and a wave number. These expressions are appropriate both for a numerical and an analytic solution of these problems and our method permits the solution to be obtained in the situation when other methods aren't adequate. This method has been used for finding of the characteristic value for the Hill equation, calculation of the parametric instability growth rate in the case of a large amplitude plasma wave, finding the band structure of the one-dimensional periodic potential and obtaining of the dispersion equation for the electromagnetic wave propagation in the waveguide filled by plasma with arbitrary periodic density modulation.

## 1. Introduction

Various problems of plasma physics, radio physics, solid state physics concern periodicity of the system both in time and in the space [1-5]. They are electromagnetic wave propagation in the periodic media, electron beam instability in the periodic media or in the periodic external fields, parametric instability of the electromagnetic and plasma waves, band structure of solids and superlattises and others. One of the main questions in investigations of these problems is obtaining of the dispersion equation and its solution. Harnessing of the Floguet theorem for the periodic systems permits this problem to be reduced to the searching of the infinite determinant's roots [1-4]. These roots are the such values of the frequency and the wave number so that the infinite determinant is equal to zero. In a general case this problem can't be solved analytically. Different methods based on the existence of a small parameter have been developed for solution of such problems. There are various perturbation methods [6], or the infinite determinant can be replaced to the finite one or to the infinite series [3,4]. Also there are complicated problems with a numerical calculation of the infinite determinant roots, when a small parameter is absent.

We shall consider the different problems of plasma physics, solid state physics and radio physics which are reduced to the searching of the infinite periodic determinant roots. For the problems mentioned above to be solved we have developed the new method which is the generalization of the Hill determinant method [1,7]. In this method the infinite determinant is considered as a complex function of two variables (usually a frequency and a wave number). Regularization of this determinant permits it to be expressed as the infinite series with the coefficients of this series being dependent from auxiliary determinants in the primary determinant poles. This expression can be used both for a numerical calculation and for obtaining the analytic solutions by means of perturbation technique if a small parameter exists. This method permit to obtain most known results by unified way and is appropriated when other methods aren't adequate.

## 2. Hill equation

### 2.1. Hill determinant method

The second order differential equation with periodic coefficient, is named as Hill equation, is the most famous object which is reduced to the infinite periodic determinant [7-9]. Such classical problems as a stationary Schrödinger equation for a particle in the periodic potential, wave propagation in the media with periodic density modulation are reduced to this equation [1,4,5]. Its general solution can't be expressed through the elementary functions and different numerical and approximate analytic methods have been developed to solve this equation [4,6-9].

According to the Floquet theorem the solution of the Hill equation

$$X'' + U(t)\,X = 0, \qquad (1)$$

where U(t) is a real periodic function with the period T, has the form:

$$X(t) = \exp(i\omega t)\,F(t) = \exp(i\omega t)\sum_{m=-\infty}^{\infty} F_m \exp(im\Omega t), \qquad (2)$$

where F(t) is a periodic function with the period T, $\Omega = 2\pi/T$, $\omega$ is the characteristic value which depends on the parameters of the system and determines the solution properties. One of the main problems with investigation on (1) is the finding of the stability regions, where $\omega$ is real. Substituting (2) in (1) and equating coefficients with equal $\exp(i\omega t+im\Omega t)$ the following infinite system is obtained [7]:

$$F_m\,(\omega + m\,\Omega\,)^2 - \sum_{n=-\infty}^{\infty} U_{m-n}\,F_n = 0, \qquad (3)$$

where $U_n$ are the Fourier coefficients of U(t). This system has a nontrivial solution when the infinite determinant of the system (3), is named as Hill determinant, equals to zero, with $\omega$ being corresponded to this condition is a characteristic value.

The finite determinants, obtaining from infinite ones by deleting of rows with high values m, are usually used for the numerical calculation of the characteristic value, the number of garmonics being determined by desired accuracy [4]. After finding of the characteristic value the unknown coefficients $F_m$ can be obtained by direct solving of (3). Another standard method for the numerical calculation of the characteristic value is the Floquet method, demanding of the numerical integration of the Hill equation [7-9].

An analytic calculation of the characteristic value and of the boundaries of stable regions is usually obtained by using different perturbation techniques [6].

The method of considerable promise for solution of this equation was suggested by Hill [1,7]. He transform the determinant of system (3) by dividing of all the rows by the correspondent diagonal elements and considered the received determinant D

$$D_{mn} = 1 \text{ for } m=n, \text{ and } D_{mn} = -\frac{U_{m-n}}{(\omega+m\Omega)^2 - U_0} \text{ for } m \neq n \qquad (4)$$

as an analytic function of the characteristic value ω. If the determinant of the system (3) equals to zero, the determinant (4) equals to zero too. If the series $U_n$ absolutely converges, D(ω) is as an analytic periodic meromorphic function of ω and has a simple poles for $\omega = \pm\,\omega_0 - m\Omega$, where $\omega_0 = \sqrt{U_0}$ [1,7]. D(ω) can be represented in a following form:

$$D(\omega) = N(\omega) + \frac{\pi \sin(2\pi\omega_0/\Omega)\cdot D^0(\omega_0)}{2\Omega\,\omega_0(\sin^2(\pi\omega/\Omega) - \sin^2(\pi\omega_0/\Omega))}\;, \qquad (5)$$

where $D^0(\omega_0)$ - is the infinite determinant obtaining from $D(\omega_0)$ by regularization of the row with m = 0.

$$\text{if } m\neq 0:\ D^0_{mm} = 1 \text{ and } D^0_{mn} = -\frac{U_{m-n}}{2m\Omega(\ \omega_0 + \frac{m\Omega}{2}\ )} \quad \text{if } m \neq n; \tag{6}$$

$$D^0_{00} = 0,\ D^0_{0n} = -U_{-n}\ .$$

It is easy to verify that $D(\omega)$ have the same principal parts as the fraction in the right-hand side of (5) at all the poles of $D(\omega)$ and $N(\omega)$ hasn't poles anywhere in the complex $\omega$ plane. Consequently $N(\omega)$ is an entire periodic function of $\omega$. Also $N(\omega{=}i\infty) = 1$ and according to Liouville theorem $N(\omega)\equiv 1$ [7]. The condition $D(\omega) = 0$ will be

$$\cos(2\pi\omega/\Omega) = \cos(2\pi\omega_0/\Omega) - \frac{\pi \sin(2\pi\omega_0/\Omega)\ D^0(\omega_0)}{\Omega\ \omega_0}, \tag{7}$$

Therefore the analytic properties of the Hill determinant permit it to be expressed through the simple periodic $\omega$ function and through the auxiliary infinite determinant $D^0(\omega_0)$, which is independent from the characteristic value. The main complexity implies that the expression (7) includes the infinite determinant $D^0(\omega_0)$ and the analytic solution can be obtained only if a small parameter exists. A numerical calculation of $\omega$ can be done with an arbitrary accuracy determining by the accuracy of the calculation of $D^0(\omega_0)$ (6). The expression (7) differs from the Hill expression [1,7], where $\omega$ is calculated through $D(\omega{=}0)$. As will be shown later the expression (7) is more preferential.

## 2.2. Second regularization of the Hill determinant

The main difficulties in the analysis of (7) is the presence of infinite determinant $D^0(\omega_0)$, and although analytic expressions can be obtained with an arbitrary accuracy, this method isn't appropriate because the obtained expressions are too cumbersome. Besides that $D^0(\omega_0)$ has singularities for such $\omega_0$ when the resonant interaction of garmonics exists, the solution in the vicinity of this points being most interesting.

For a further simplification we shall regularize $D^0(\omega_0)$ being considered as a complex $\omega_0$ function. $D^0(\omega_0)$ is an even analytic meromorphic function, having the simple poles at $\omega_0 = -m\Omega/2$, where $m = \pm 1, \pm 2, \pm 3 \ldots$, and according to Mittag - Lefleure theorem $D^0(\omega_0)$ can be presented in the following form [7] :

$$D^0(\omega_0) = R(\omega_0) + \sum_{j=1}^{j=\infty} \frac{K_j}{\omega_0^2 - (j\Omega/2)^2}, \tag{8}$$

where $R(\omega_0)$ is an entire function. Equating principle parts of $D^0(\omega)$ and right-hand side of (8) at the poles $\omega_0 = -j\Omega/2$, we obtain

$$K_j = - D^{0j}(\omega_0 = -j\Omega/2)/2, \tag{9}$$

where $D^{0j}$ is a determinant of the following matrix

$$D^{0j}_{mn} = -\frac{U_{m-n}}{\Omega^2 m(m-j)} \text{ for } m \neq n, 0, j; \quad D^{0j}_{mm} = 1, \text{ for } m=n,\ m \neq 0, j;$$

$$D^{0j}_{00} = D^{0j}_{jj} = 0; \quad D^{0j}_{0n} = -U_{0n},\ D^{0j}_{jn} = -U_{j-n} \text{ for } n \neq j. \tag{IO}$$

With such $K_j$ function $R(\omega_0)$ has no poles and is bounded at the infinity. Consequently $R(\omega_0)$ = const = $D^0(\omega_0 = i\infty) = 0$ and equation $D(\omega) = 0$ will be:

$$\cos(2\pi\omega/\Omega) - \cos(2\pi\omega_0/\Omega) + \frac{\pi \sin(2\pi\omega_0/\Omega)}{2\Omega\, \omega_0} \sum_{j=1}^{\infty} \frac{D^{0j}}{\omega_0^2 - (j\Omega/2)^2} = 0. \tag{11}$$

As a result the regularization of the Hill determinant D in the two variables complex space ($\omega$ and $\omega_0$) comes to the expression of this determinant as an infinite series with the coefficients of this series being expressed through the auxiliary determinants. This expression can be used both for numerical calculation and, if a small parameter exists, for obtaining of an analytic expressions by means of perturbation techniques. Proposed method permits to obtain most known results by unified way.

### 3.3. Boundaries of stability regions

Let's consider obtaining of the stable region boundaries. At these boundaries $\mathrm{Im}\,\omega = 0$ and, as can be shown from (11), $\omega = 0$ or $\omega = \Omega/2$, the solution of (1) being periodic with the period $2\pi/\Omega$ ($\omega=0$) or $4\pi/\Omega$ ($\omega=\Omega/2$). These results are in accordance with Floquet theory [7-9].

For example, let's obtain stable region boundaries for the following equation

$$X'' + (\delta + 2\,\varepsilon\,(\cos 2t + \alpha\,\cos(4t))\,X=0. \qquad (12)$$

In this case $\Omega = 2$, $\omega_0^2 = U_0 = \delta$, $U_1 = U_{-1} = \varepsilon$, $U_2 = U_{-2} = \alpha\,\varepsilon$.

If $\varepsilon \ll 1$, the infinite determinants can be approximated by finite ones and its substitution in the (11) gives an approximate analytic expressions for the boundaries. Coefficients for $\varepsilon^2$ in the expansion series of $D^{0j}$ are obtained from rows 0 and j, coefficients for $\varepsilon^3$ are obtained from a finite set of rows: -1, 0, 1, and j-1, j, j+1. It hasn't been possible to calculate highest degree coefficients exactly, because they depend on all of the determinant's rows. However a contribution of the row is inversely proportional to its number and this causes the fast convergence of the series to the exact value.

An analytic investigation on (12) needs the calculations of high order functional determinants that is a rather difficult problem. We have used the system of analytic calculations MATEMATICA for calculations of these determinants and subsequent analysis. This system permits these determinants to be calculated in an analytic form. In the case of $\varepsilon \ll 1$ the expansion series of the determinants $D^{0j}$ are:

$$D^{01} = -\,\varepsilon^2 - \alpha\,\varepsilon^3/2 + O(\varepsilon^4),$$

$$D^{02} = -\,\alpha^2\varepsilon^2 + \alpha\,\varepsilon^3/2 + O(\varepsilon^4), \qquad (13)$$

$$D^{0j} = O(\varepsilon^4),\ j \geqslant 3.$$

It is known, that an infinite sequence of unstable regions exists in the vicinity of $\delta = n^2$ , where $n = 0, 1, 2 \ldots$ [7-9]. This result is directly followed from analysis of (11), because in the vicinity of $\delta = n^2$ we have $|\cos(\pi\omega_0)| \approx 1$ and the condition of the stable region boundary $|\cos(\pi\omega)| = 1$ is satisfied. Let's take $\delta = n^2 + a_1\varepsilon + a_2\varepsilon^2 + \ldots$ and substitute it to the (11), then perform power series expansion of obtaining equation in $\varepsilon$ and then equate the coefficients for the equal degrees of $\varepsilon$ to zero. Solution of the obtaining system of equations permits to obtain $a_n$ and find expressions for the stable region boundaries

$$\begin{aligned}
n=0: \quad & \delta_0 = -\varepsilon^2(1/2+\alpha^2/8). \\
n = 1 \quad & \delta_{1u} = 1 + \varepsilon - \varepsilon^2( 1/8 - \alpha/4 + \alpha^2/6 ) + O(\varepsilon^3), \\
& \delta_{1d} = 1 - \varepsilon - \varepsilon^2( 1/8 + \alpha/4 + \alpha^2/6 ) + O(\varepsilon^3). \\
n = 2: \quad & \delta_{2u} = 4 + \alpha\varepsilon - \varepsilon^2( 1/12 + \alpha^2/32 ) + O(\varepsilon^3), \\
& \delta_{2d} = 4 - \alpha\varepsilon + \varepsilon^2(5/12 - \alpha^2/32 ) + O(\varepsilon^3).
\end{aligned} \qquad (14)$$

The first index denotes the zone number, u and d denote the upper and the lower boundaries respectively. The expressions for another zones can be obtained similarly. It's interesting to note that for $\varepsilon = \alpha/4$ the zone width is equal to zero.

For the particular case $\alpha = 0$ (12) is reduced to the Mathieu equation, and the expressions (14) are transformed to the well known expressions for the stable region boundaries of Mathieu equation, which can be obtained by means of perturbation technique too (see, for example, [6]).

### 3. Parametric instability of large amplitude plasma wave.

Generally, the problem of finding growth rates of a parametric instability of an electrostatic plasma wave $\vec{E}(t) = \vec{E}_0 \sin(\Omega t)$ comes to the calculation of such values frequency $\omega$, at which the following infinite determinant vanishes [3]

$$D_N = \left| \begin{array}{c|c} I_{mn} & D^{e}_{mn} \\ \hline D^{1}_{mn} & I_{mn} \end{array} \right| . \qquad (15)$$

Here I is the unit matrix ($I_{mn} = \delta_{mn}$, $\delta_{mn} = \begin{cases} 1 & \text{for } m=n \\ 0 & \text{for } m \neq n \end{cases}$ is the Kronecer delta), $D^{e}_{mn} = R_{em} J_{n-m}(\mu)$, $D^{1}_{mn} = R_{1m} J_{m-n}(\mu)$, $\mu = \vec{k}\,\vec{r}$, $\vec{k}$ is the wave vector of an unstable wave, $\vec{r}$ is the electron oscillation amplitude in the pump field, $R_{\alpha m} = \dfrac{\delta\varepsilon_\alpha(\omega+m\Omega,\vec{k})}{(1+\delta\varepsilon_\alpha(\omega+m\Omega,\vec{k}))}$, $\delta\varepsilon_\alpha(\omega+m\Omega,\vec{k})$ is the permittivity of plasma particles of species $\alpha$, $J_n$ is the the Bessel function of the first kind.

An analytic solution of this problem can be obtained for the small values $\mu$, when only garmonics with the numbers $m = 0, \pm 1$ are taken into consideration. Otherwise, it becomes necessary to study the determinant of a large-rank matrix; that problem can't be handled analytically. A numerical calculation of $\omega$ is the rather complex problem too.

We propose to use the Hill determinant method for the solution of this problem [11]. For the case of a cold magnitoactive plasma the determinant (15) have similar features with the Hill determinant (4): $D(\omega)$ is an even periodic function of $\omega$ with a period $\Omega$ and its poles are the roots of equations $1+\delta\varepsilon_\alpha(\omega+n\Omega,\vec{k})=0$. For n = 0 this biquadratic equation have four roots (branches)

$$\omega_{\alpha m0} = \pm \left[ \frac{\omega^2_{p\alpha} + \Omega^2_{H\alpha}}{2} \pm \left[ \frac{(\omega^2_{p\alpha}+\Omega^2_{H\alpha})^2}{4} - \Omega^2_{H\alpha}\, \omega^2_{p\alpha} \cos^2\theta \right]^{1/2} \right]^{1/2} , \qquad (16)$$

and the determinant poles are

$$\omega_{\alpha mn} = \omega_{\alpha m0} + n\Omega, \quad n=0,\pm1,\pm2,\ldots\ldots , \qquad (17)$$

where $\theta$ is the angle between the wave vector $\vec{k}$ and the external magnetic field, $\omega_{p\alpha}$, $\Omega_{H\alpha}$ are the Langmuire and cyclotron frequencies of particles of species $\alpha$ and m denotes the branches number: for m = 1 - the signs in (16) are: (+,+), for m = 2 - (+,-), for m = 3 - (-,+), for m = 4 - (-,-).

Let's present $D(\omega)$ in the following form:

$$D(\omega)= N(\omega) + \sum_{\alpha}\sum_{m=1,2} \frac{K_{\alpha m0}}{\sin^2(\pi\omega_{\alpha m0}/\Omega) - \sin^2(\pi\omega/\Omega)}, \qquad (18a)$$

where

$$K_{\alpha m0}=-\frac{\pi \sin(2\pi\omega_{\alpha m0}/\Omega)}{\Omega} \lim_{\delta\to 0} \delta\, D(\omega_{\alpha m0}+\delta)= \frac{\pi D_{\alpha m0}\sin(2\pi\omega_{\alpha m0}/\Omega)}{\Omega \left.\frac{\partial\, \delta\varepsilon_\alpha(\omega,k)}{\partial \omega}\right|_{\omega=\omega_{\alpha m0}}}, \qquad (18b)$$

$D_{\alpha m0}$ is the determinant of the matrix $D(\omega=\omega_{\alpha m0})$ with the regularized row containing singularities for $\omega=\omega_{\alpha m0}$: 1 in this row is replaced by 0 and $R_{\alpha m}$ by 1. With such $K_{\alpha m}$ the double sum in the right side of (18) has the same principal parts as $D(\omega)$ anywhere in the complex $\omega$ plane, and, consequently, $N(\omega)$ haven't poles on the whole complex plate $\omega$, i.e. $N(\omega)$ is a entire periodic function of $\omega$. Also $N(\omega)$ is the bounded function too, because $N(\omega=i\infty) = D(\omega=i\infty) = 1$ and, according to the Liouville theorem, it is identically equal to a constant ( $N(\omega) \equiv 1$ ) [7]. The dispersion equation $D(\omega) = 0$ then will be [11]:

$$1 + \sum_{\alpha}\sum_{m=1,2} \frac{K_{\alpha m0}}{\sin^2(\pi\omega_{\alpha m0}/\Omega) - \sin^2(\pi\omega/\Omega)} = 0, \qquad (19)$$

where $K_{\alpha m0}$ is given by (18b). This equation reduces to a polynomial in $\sin(\pi\omega/\Omega)$ and it can be easily numerically solved.

An essential simplification can be done for zero or infinite magnetic fields values, when only two solutions of the equation $1+\delta\varepsilon_\alpha(\omega+n\Omega,k)=0$ exist. In this case the dispersion equation is biquadratic in $\sin(\pi\omega/\Omega)$ and can be easily solved for $\omega$:

$$\omega = \pm \frac{\Omega}{\pi} \arcsin\left[ \left[ \frac{A \pm (A^2-4B)^{1/2}}{2} \right]^{1/2} \right], \qquad (20)$$

where $A = K_{e0} + K_{10} + \sin^2(\pi\omega_{e0}/\Omega) + \sin^2(\pi\omega_{10}/\Omega)$,

$B = K_{e0} \sin^2(\pi\omega_{10}/\Omega) + K_{10} \sin^2(\pi\omega_{e0}/\Omega) + \sin^2(\pi\omega_{e0}/\Omega)$ $\sin^2(\pi\omega_{10}/\Omega)$.

The equations (19) or (20) can be used both for numerical and analytic calculations of the parametric instability growth rate. For example let's consider two problems: dispersion properties of the high frequency wave in the unmagnetized plasma and modulational instability of the lower hybrid wave.

If the pump wave frequency $\Omega \gg \omega_{p\alpha}, \Omega_{H\alpha}$, we can neglect the elements of the determinants in the (20), containing $\Omega$ in the denominators. Then these determinants can be easily calculated

$$D_e = - J_0^2(\mu)\, R_1(\omega_{pe}), \quad D_1 = - J_0^2(\mu)\, R_e(\omega_{p1}),$$

and

$$A = \frac{\pi^2}{\Omega^2} ( \omega_{pe}^2 + \omega_{p1}^2 ), \; B = \frac{\pi^4 \omega_{pe}^2 \omega_{p1}^2}{\Omega^4} ( 1 - J_0^2(\mu) ). \qquad (21)$$

Therefore using (20) and (21) we obtain the well known dispersion equation for waves in such a system [3].

Further we shall consider a modulational instability of a wave at lower hybrid frequency $\Omega = \omega_{p1}/(1+\omega_{pe}^2/\Omega_{He}^2)^{1/2}$ [10,11]. Using our method to the solution of this problem is interesting because the instability growth rate of the large amplitude pump wave is close to the pump wave frequency $\Omega$ and is realized for $\mu \geq 1$ [11]. In this case a small parameter isn't exists and high garmonics of $\omega$ must be taken into consideration in order to obtain the correct solution . Further we shall consider the instability of the wave with the wave vector $\vec{k}$ almost perpendicular to the direction of magnetic field $(k_z/k \ll \omega_{p1}/\omega_{pe})$. Under the conditions

$$\Omega_{Hi} \ll \omega \ll \Omega_{He},\ \omega_{pe} \ll \Omega_{He},\ \frac{\omega}{k_z v_{Te}} \gg 1,\ kr_{d_{e,i}} \ll 1,$$

where $r_{d_{e,i}}$ are the electron and ion Debye length, the electron and ion permittivities for the waves with $k_z/k \sim \omega_{pi}/\omega_{pe}$ are: $\delta\varepsilon_e = -\omega_{pi}^2 y^2/\omega^2$, $\delta\varepsilon_i = -\omega_{pi}^2/\omega^2$, where $y = k_z\omega_{pe}/(k\omega_{pi})$. The poles of the determinant $D(\omega)$ are: $\omega_{en} = \pm\,\omega_{pi}y + n\,\Omega$, $\omega_{in} = \pm\,\omega_{pi} + n\,\Omega$.

In this paper we shall demonstrate obtaining of the analytic solution in the case of small amplitude pump wave, when $\mu \ll 1$. Also we suppose $y \ll 1$ and $\Omega = \omega_{pi}(1 + \delta)$, $\delta \ll 1$ and obtain dispersion relation as a power series in $\mu$. If $\mu = 0$ determinant $D(\omega)$ (15) can be calculated exactly:

$$D(\omega,\mu=0) = \prod_{n=-\infty}^{\infty}\left(1 - \frac{\delta\varepsilon_e(\omega+n\Omega,k)\ \delta\varepsilon_i(\omega+n\Omega,k)}{(1+\delta\varepsilon_e(\omega+n\Omega,k))\ (1+\delta\varepsilon_i(\omega+n\Omega,k))}\right) =$$

$$= \prod_{n=-\infty}^{\infty}\frac{(\ 1+\delta\varepsilon_e(\omega+n\Omega,k)+\delta\varepsilon_i(\omega+n\Omega,k)\ )}{(1+\delta\varepsilon_e(\omega+n\Omega,k))\ (1+\delta\varepsilon_i(\omega+n\Omega,k))} = \tag{22}$$

$$= \frac{\sin^2(\pi\omega/\Omega)\ (\sin^2\pi\omega/\Omega - \sin^2(\pi\omega_L/\Omega))}{(\sin^2(\pi\omega_e/\Omega) - \sin^2(\pi\omega/\Omega))(\sin^2(\pi\omega_i/\Omega) - \sin^2(\pi\omega/\Omega))},$$

where $\omega_L = (\omega_e^2 + \omega_i^2\ )^{1/2}$. We account for that

$$\prod_{n=-\infty}^{\infty}(1 + \delta\varepsilon_\alpha(\omega+n\Omega,k)) = \prod_{n=-\infty}^{\infty}\frac{(\omega+n\Omega-\omega_\alpha)\ (\omega+n\Omega+\omega_\alpha)}{(\omega+n\Omega)^2} =$$

$$= \frac{\omega^2 - \omega_\alpha^2}{\omega^2}\prod_{n=1}^{\infty}\frac{\left(1 - \frac{(\omega-\omega_\alpha)^2}{n^2\,\Omega^2}\right)\left(1 - \frac{(\omega+\omega_\alpha)^2}{n^2\,\Omega^2}\right)}{(1 - (\omega/n\Omega)^2\ )} =$$

$$= \frac{\sin^2\pi\omega/\Omega - \sin^2(\pi\omega_\alpha/\Omega)}{\sin^2\pi\omega/\Omega} .$$

Dispersion relation is given by

$$\sin^2(\pi\omega/\Omega)\ (\sin^2\pi\omega/\Omega - \sin^2(\pi\omega_L/\Omega) - K_{e2} - K_{12}) +$$

$$K_{e2}\sin^2(\pi\omega_1/\Omega) + K_{12}\sin^2(\pi\omega_e/\Omega) = 0, \qquad (22)$$

where $K_{\alpha\mu}$ is the power series expansion of $K_\alpha$ in a variable $\mu$ without zero order term. If $\mu = 0$ we have $K_{\alpha\mu} = 0$ and equation (22) describes lower hybrid wave. In the case of $\mu$, $\delta$, $y \ll 1$ power series expansions of $K_{e,1}$ to orders $\mu^2$, $\delta^2$, $y^4$ can be obtained

$$K_{e\mu} = \frac{\pi\,\omega_{p1}y\,\sin(2\pi\omega_{p1}y/\Omega)\ D_{e2}}{2\,\Omega} = \frac{\mu^2\,\pi^2\,y^2(\delta/4 - y^2/8)}{(\delta^2 - y^2)} + \frac{3\pi^2\mu^2y^2}{8}$$

$$K_{1\mu} = \frac{\pi\,\omega_{p1}\sin(2\pi\omega_{p1}/\Omega)\ D_{1\mu}}{2\,\Omega} = -\frac{\mu^2\,\pi^2\,y^2(\delta/4 - y^2/8)}{(\delta^2 - y^2)}. \qquad (23)$$

Substitution (23) to (22) with accounting for that $\omega/\Omega \ll 1$ gives dispersion relation for the parametric instability of lower hybrid wave in the supersonic regime for the pump wave frequency near by $\omega_{p1}$

$$\omega^2(\omega^2 - \omega_{p1}^2y^2) + \frac{\mu^2y^2}{4}\beta\,\Omega^4 = 0, \qquad (24)$$

where $\beta = \delta - y^2/2$.

This equation was obtained by Porcolab [10] and investigated in the regimes of modulational instability ($\beta < 0$) [11] and modified decay ($\beta > 0$) [3,10].

If y isn't small the expressions for $K_{e,1}$ are rather complex and the number of garmonics to be accounted for is increased with increasing y. If $\mu$ isn't small the analytic solution of the problem can't be obtained because of lacking of small parameters. The numerical calculation of growth rate with using of (19) have been performed for the case of $\Omega = \Omega_{LH}$ in [11]. For $\mu \ll 1$ the results obtained from (19) and from dispersion relation

$$\frac{1}{1+\delta\varepsilon_e(\omega,k)} + \frac{1}{\delta\varepsilon_i(\omega,k)} + \frac{\mu^2}{4}\Big\{\frac{1}{1+\delta\varepsilon_e(\omega+\Omega,k)+\delta\varepsilon_i(\omega+\Omega,k)} + \frac{1}{1+\delta\varepsilon(\omega-\Omega,k)+\delta\varepsilon(\omega-\Omega,k)}\Big\} = 0, \qquad (25)$$

which is ordinary used to analyze such instabilities [3,10] are practically identical. For $\mu > 1$ equation (25) generated incorrect results and, consequently can't be used. The number of garmonics which must be taken into consideration is increased with increasing $\mu$. For several values of $y = \pm\delta$ the $K_{e,i}$ have the singularities, ones being compensated with the analytic power series expansion of A and B (23). The presence of these peculiarities involves the sacrifice of sensitivity during the numerical calculations in the vicinity of such values $y$.

4. Band structure of the one dimensional periodic potential.

Let's consider the one-dimensional stationary Shrödinger equation for an electron in the periodic potential U(z)

$$\frac{\hbar^2}{2m_e}\frac{d^2\phi(z)}{d\,z^2} + (E - U(z))\,\phi(z) = 0. \qquad (26)$$

Here $\hbar$ is the Plank constant, $m_e$, $E$ are the mass and energy of electron. The main problem in the investigation of (26) is a finding of the dependence energy E from momentum p. The equation (26) reduces to the Hill equation and we can use (11) in order to obtain dispersion relation

$$\cos\left(\frac{2\pi p}{\tau}\right) = \cos\left(\frac{2\pi\sqrt{2m_eE}}{\tau}\right) - \frac{\pi m_e \sin(2\pi\sqrt{2m_eE}/\tau)}{\tau\sqrt{2m_eE}}\sum_{j=1}^{\infty}\frac{D^{0j}}{E - j^2\tau^2/8m}, \qquad (27)$$

where $\tau = 2\pi/a$, a is the period of U(z), $D^{0j}$ is the determinant of a following matrix

$$D^{0j}_{mn} = \frac{2\, m_e\, U_{m-n}}{\tau^2\, m(m-j)} \text{ for } m \neq n,0,j; \quad D^{0j}_{mm} = 1, \text{ for } m = n,\ m \neq 0,j;$$

$$D^{0j}_{00} = D^{0j}_{jj} = 0; \qquad D^{0j}_{jn} = U_{j-n} \text{ for } n \neq j, \tag{28}$$

where $U_n = U^*_{-n}$ are the Fourier coefficients of $U(t)$ and we suppose, that $U_0 = 0$.

After normalization of E and $U_n$ by $\tau^2/8m_e$ and p by $\tau/2$ (27) is transformed to

$$\cos(\pi p) = \cos(\pi E^{1/2}) - \frac{\pi \sin(\pi E^{1/2})}{4\, E^{1/2}} \sum_{j=1}^{\infty} \frac{D^{0j}}{E - j^2} = 0, \tag{29}$$

where $D^{0j}_{mn} = U_{m-n}/(4m(m-j))$; $D^{0j}_{00} = D^{0j}_{jj} = 0$; $D^{0j}_{jn} = U_{j-n}$ for $n \neq j$.

We express p through an infinite series, the elements of infinite matrix $D^{0j}$ being independent from p and E, haven't singularities and expressed only through $U_n$. The determinants $D^{0j}$ are the constants for each concrete periodic potential U(z). The equation (29) is a simple transcendental function p(E), which can be easily calculated. This equation can be used for obtaining of the approximated analytic expressions E(p). It is well known, that an infinite sequence of the energy gaps exists in the vicinity of $E_n = n^2$ for the particle in a periodic potential, where real values of p aren't exist. An analytic solutions in the vicinity of $E_n$ can be obtained for the case of a small potential $U_n << 1$. Then $D^{0j} << 1$ and $E = n^2 + E'$, $p = |\sin(\pi n/2)| + p'$, $E', p' << 1$ we obtain from (29)

$$p^2 = E^2 - \sum_{j=1}^{\infty} D^{0j}/(2j^2) \qquad \text{for } n = 0,$$

$$p'^2 = \frac{D^{0n}}{4n^2} + \frac{E'^2}{4n^2} + \frac{E'}{4n^2} \sum_{\substack{j=1 \\ j\neq n}}^{\infty} \frac{D^{0j}}{n^2 - j^2} \qquad \text{for } n > 0. \tag{30}$$

The boundaries of the energy gaps $E_n^{u,d}$ can be obtained from (30) by substitution p' = 0

$$E_n^{u,d} = E_n - \frac{1}{2} \sum_{\substack{j=1 \\ j \neq n}}^{\infty} \frac{D^{0j}}{n^2 - j^2} \pm (-D^{0n})^{1/2} \quad \text{for} \quad n > 1, \qquad (31)$$

where u,d denote the upper and lower boundaries respectively.

Effective electron masses $\mu_n^{-1} = \frac{d^2E'}{dp'^2}\Big|_{p'=0}$ in the vicinity of the $E_n$ are expressed through $D^{0n}$:

$$\mu_n = (-D^{0n})^{1/2}/4n^2. \qquad (32)$$

Power series expansion of determinants $D^{0j}$ in $U_n$ can be obtained. Such calculations are rather cumbersome and have been done with the analytic computer system MATEMATICA. Calculations show, that this expansion to second order in $U_n$ is $D^{0n} = -|U_n|^2$. From (31) we find, that width of n - th energy gap equal to $2|U_n|$ and a shift of the energy gap center with respect to the $E_n$ is expressed through the sum of $U_j^2$. A more exact analytic expressions for $E_n^{u,d}$ and $\mu_n$ can be derived by power series expansion of determinants $D^{0j}$ to the highest orders in $U_n$. An exact power series expansion of $D^{0j}$ to the order $U_n^3$ is obtained by using of reduced finite-rank determinants and the approximate expressions can be founded by using the expansion to higest orders in $U_n$. The generalization of the proposed method for the case of the Shrodinger equation with three-dimensional periodic potential will permit to develop efficient both the numerical and analytic method for calculations the band structure of real solids.

## 5. Electromagnetic wave propagation in the waveguide filled by plasma with periodic density modulation.

Waveguides filled by a periodically modulated plasma are widely used as a slow - wave structures in the powerful generators of the electromagnetic waves. A phase velocity of the electromagnetic wave in a vacuum waveguide exceeds light speed and resonant Cherencov interaction of this wave with an electron beam is impossible. A periodic modulation of the plasma density makes this interaction possible [12].

Let's consider cylindrical waveguide filled by the plasma with periodic density modulatation. A very strong external magnetic field ($\omega_{pe} \ll \Omega_{He}$) and azimuthal symmetry is also assumed. Longitudinal dielectric constant $\varepsilon_{zz}$ in this case is

$$\varepsilon_{zz}(\vec{r},z) = 1 + \sum_{n=-\infty}^{\infty} \delta\varepsilon_n \exp(in\tau z), \tag{33}$$

where $\delta\varepsilon_n = \delta\varepsilon^*_{-n}$ are the coefficients of the Fourier series expansion of the electron plasma permittivity, $\tau = 2\pi/a$, a is the period of modulation. According to the Floquet theorem the electric field $E_z$ of this wave is

$$E_z(\vec{r},t) = \sum_{n=-\infty}^{\infty} E_{z_n} J_0(k_\perp r) \exp(-i\omega t + i(k_z - n\tau)z). \tag{34}$$

Here $J_0$ is the Bessel function of the first kind, $k_\perp = \nu_0/R$, $\nu_0$ is the root of the Bessel function, R is the waveguide radius. It is known, that dispersion relation for the waves in such a waveguide comes to the infinite system [12]

$$\left( k_\perp^2/(k_m^2 - \omega^2/c^2) + \varepsilon_0 \right) E_{z_m} + \sum_{n=-\infty}^{\infty} \delta\varepsilon_n E_{z_{m-n}} = 0, \tag{35}$$

where $k_m = k_z - m\tau$, $\varepsilon_0 = 1 + \delta\varepsilon_0$. This system has a nontrivial solution when the following infinite determinant $D(\omega,k_m)$ vanishes

$$D_{mm} = 1, \quad D_{m,m-n} = \frac{\delta\varepsilon_n(k_m^2 - \omega^2/c^2)/\varepsilon_0}{(k_m^2 - (\omega^2/c^2 - k_\perp^2/\varepsilon_0))}. \tag{36}$$

For this determinant to be considered as analytic function of $k_z$ the series $D_{m,m-n}$ must converges absolutely [1,7]. In the case of cold plasma it isn't so, because $\lim_{m\to\infty} D_{m,m-n} = \delta\varepsilon_n/\varepsilon_0 \neq 0$. In order to gain convergence of $D(\omega,k_z)$ we account for plasma temperature and choose plasma permittivity $\delta\varepsilon_n = \dfrac{\omega_{pn}^2}{\omega^2 - k_m^2 v_T^2}$. Under this condition $D(\omega, k_z)$ is a periodic analytic function of $k_z$ with the period $\tau$ and it has the simple poles when $k_m^2 - (\omega^2/c^2 - k_\perp^2/\varepsilon_0) = 0$, or $k_z = \pm k_0 + m\tau$, where $\kappa_0 = (\omega^2/c^2 - k_\perp^2/\varepsilon_0)^{1/2}$, $m = 0, \pm 1, \pm 2, \pm 3, \dots\dots\dots$ . It is easy to verify that the determinant poles correspond to the dispersion curves of the electromagnetic waves and a spatial garmonics thereof in the waveguide filled by a plasma with the uniform permittivity $\varepsilon_0$.

After the first regularization of the poles in the complex $k_z$ plane, as we have done in the case of Hill equation, we obtain the dispersion function in the form

$$1 + \sum_\alpha \frac{\pi \sin(2\pi k_\alpha(\omega))\, D^0(k_z = k_\alpha)}{2\,\tau\, k_\alpha(\omega)\, (\sin^2(\pi k_z/\tau) - \sin^2(\pi k_\alpha(\omega))} = 0, \qquad (37)$$

Father we shall consider $\omega \gg k_z v_T$ and account for only $\alpha = 0$, or $k_z = k_0(\omega)$. $D^0(k_z = k_0(\omega))$ is the determinant of the following matrix

$$D^0_{00} = 0,\ D^0_{0,-n} = \delta\varepsilon_n\, k_\perp^2/\varepsilon_0,$$
$$D^0_{mm} = 1,\ D^0_{m,m-n} = -\frac{\delta\varepsilon_n((k_0 - m\tau)^2 - \omega^2/c^2)/\varepsilon_0}{2\,\tau\, m\,(k_0 - m\tau/2)}. \qquad (38)$$

$D^0$ is independent from $k_z$ and is an even analytic function of $\omega$. This determinant has the simple poles at the $\omega$ satisfying to the equation $k_0(\omega) = j\tau/2$, where $j = \pm 1, \pm 2, \dots$ . This poles correspond to the intersections of the dispersion curves $k_z = k_0(\omega)$ and $k_z = -k_0(\omega) + j\tau$. $D^0$ can be represented in the following form

$$D^0(\omega) = \sum_{\substack{j=-\infty \\ j\neq}}^{j=\infty} \frac{K_j}{\omega - \omega_{\alpha j}} = \sum_{\substack{j=-\infty \\ j\neq}}^{j=\infty} \frac{D^{\alpha j}\, V_{\alpha j}}{2j\tau(\omega - \omega_{\alpha j})} =$$

$$-\sum_{j=1}^{j=\infty} \frac{D^{\alpha j}\, V_{\alpha j}}{j\tau(\omega^2 - \omega_{\alpha j}^2)}\,, \qquad (39)$$

where $V_{\alpha j} = V_\alpha(k_z = j\tau/2,\ \omega = \omega_{\alpha j})$, $\omega_{\alpha j} = \omega_\alpha(j\tau/2)$, $V_\alpha = \dfrac{d\omega_\alpha(k_z)}{dk_z}$ is the group velocity of the electromagnetic waves in the waveguide filled by plasma with uniform permittivity $\varepsilon_0$ with following dispersion relation $\omega_\alpha(k_z)$

$$\omega_{1,2}^2 = \frac{1}{2}\left[\omega_{p0}^2 + (k_z^2 + k_\perp^2)\, c^2 \pm \left[(\omega_{p0}^2 + k_z^2 c^2 + k_\perp^2 c^2)^2 + 4 k_z^2 c^2 \omega_{p0}^2\right]^{1/2}\right], \qquad (40)$$

$D^{\alpha j}$ is the determinant of the matrix $D^0(\omega = \omega_{\alpha j})$ with regularized $j$-th row

$$D_{jj}^{\alpha j} = D_{00}^{\alpha j} = 0,\ D_{mm}^{\alpha j} = 1 \text{ for } m \neq 0, j,$$

$$D_{m,m-n}^{\alpha j} = \frac{\delta\varepsilon_n((m-j/2)^2\tau^2 - \omega_{\alpha j}^2/c^2)/\varepsilon_0}{m\tau^2(m-j)} \quad \text{for } m \neq j,\ n \neq 0; \qquad (41)$$

$$D_{j,j-n}^{\alpha j} = D_{0,-n}^{\alpha j} = \delta\varepsilon_n\, k_\perp^2/\varepsilon_0,\quad \omega_{\alpha j}^2 = (j^2\tau^2/4 + k_\perp^2/\varepsilon_0)c^2.$$

After second regularization of the auxiliary determinant $D^0$ in the complex $\omega$-plate, we can obtain dispersion relation in the form of the double infinite series

$$\cos\left(\frac{2\pi k_z}{\tau}\right) = \cos\left(\frac{2\pi k_0(\omega)}{\tau}\right) - \frac{\pi \sin(2\pi k_0(\omega)/\tau)}{\tau^2 k_0(\omega)} \sum_{\alpha=1,2} \sum_{j=1}^{\infty} \frac{D_\alpha^{\alpha j}\, \omega_{\alpha j}\, V_{\alpha j}}{j\,(\omega^2 - \omega_{\alpha j}^2)}\,.$$

This dispersion relation is analogous to the dispersion relations (11),(27) and can be investigated in a similar manner.

## 6. Conclusion

Therefore we have developed the new method of investigation on infinite periodic determinants. A large number of physical problem, such as an electromagnetic wave propagation in the periodic media; a parametric instability of the large amplitude plasma waves; an electron beam stability in the periodic systems and external fields, band structure of the solids and superlattices and other problems can be investigated by means of this method. If the matrix elements have no singularities other then simple poles, the infinite determinant being an analytic, meromorphic, periodic function, can be expressed as the infinite series with the coefficients are determined through the auxiliary determinants independent from the frequency and the wave number. Both numerical and analytic results can be obtained from these single expression by conventional way.

This method have been applied to the investigation on the Hill equation, the parametric instability problem, Shrodinger equation with periodic potential, an electromagnetic wave propagation in the waveguide filled by a plasma with the periodic density modulation. The expression for the characteristic value has been obtained not only Mathieu equation but for the Hill equation with arbitrary continuous periodic function. In the case of parametric instability of the plasma waves the characteristic determinant is a periodic function of frequency. We obtain unique dispersion equation for all types of parametric instabilities without invoking small parameters. The growth rate of modulational instability of the large amplitude lower hybrid wave have been obtained in the analytic form. Band structure of electron in the periodic potential and an electromagnetic wave propagation in the periodic waveguide have been investigated by means of our method too.

Our method permit to obtain the solutions of the problems mentioned above and another problems which are reduced to the investigation on the infinite periodic determinants in the unified form . Therefore this method can be easy realized with an analytic computer system. We have used MATHEMATICA program for these purposes. By using this system the calculations of large symbolic determinants and simplification of obtained expressions have been done.

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